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GEOMETRIC FLOW ON COMPACT LOCALLY CONFORMALLY KÄHLER MANIFOLDS

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Abstract. We study two kinds of transformation groups of a compact locally conformally Kähler (l.c.K.) manifold. First, we study compact l.c.K. manifolds by means of the existence of holomorphic l.c.K. flow (i.e., a conformal, holomorphic flow with respect to the Hermitian metric.) We characterize the structure of the compact l.c.K. manifolds with parallel Lee form. Next, we introduce the Lee-Cauchy-Riemann (LCR) transformations as a class of diffeomorphisms preserving the specific G-structure of l.c.K. manifolds. We show that compact l.c.K. manifolds with parallel Lee form admitting a non-compact holomorphic flow of LCR transformations are rigid: such a manifold is holomorphically isometric to a Hopf manifold with parallel Lee form.

1. Introduction. Let (M, g, J) be a connected, complex Hermitian manifold of complex dimension $n \geq 2$. We denote its fundamental 2-form by ω, which is defined by $\omega(X, Y) = g(X, JY)$. If there exists a real 1-form θ satisfying the integrability condition

$$d\omega = \theta \wedge \omega \quad \text{with} \ \ d\theta = 0\,,$$

then g is said to be a *locally conformally Kähler* (l.c.K.) metric. A complex manifold M endowed with a l.c.K. metric is called a l.c.K. manifold. The conformal class of a l.c.K. metric g is said to be a l.c.K. structure on M. The closed 1-form θ is called *the Lee form* and it encodes the geometric properties of such a manifold. The vector field $\theta^\sharp$, defined by $\theta(X) = g(X, \theta^\sharp)$, is called the Lee field.

The purpose of this paper is to study two kinds of transformation groups of a compact l.c.K. manifold (M, g, J). We first consider $\mathrm{Aut}_{\text{l.c.K.}}(M)$, the group of all conformal, holomorphic diffeomorphisms. We discuss its properties in §2. A holomorphic vector field Z on (M, g, J) generates a 1-dimensional complex Lie group $\mathcal{C}$. (The universal covering group of $\mathcal{C}$ is $\boldsymbol{C}$.) We call $\mathcal{C}$ a holomorphic flow on M.

DEFINITION 1.1. If a holomorphic flow $\mathcal{C}$ (resp. holomorphic vector field Z) belongs to $\mathrm{Aut}_{\text{l.c.K.}}(M)$ (resp. Lie algebra of $\mathrm{Aut}_{\text{l.c.K.}}(M)$), then $\mathcal{C}$ (resp. Z) is said to be a *holomorphic l.c.K. flow* (resp. *holomorphic l.c.K. vector field*).

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A nontrivial subclass of l.c.K. manifolds is formed by those (M, g, J) having parallel Lee form with respect to the Levi-Civita connection ∇^g (i.e., $\nabla^g\theta = 0$). We observe that a compact non-Kähler l.c.K. manifold (M, g, J) with parallel Lee form θ supports a holomorphic vector field $Z = \theta^\sharp - \mathrm{i}J\theta^\sharp$ which generates holomorphic isometries of g. (Compare [18], [19], [6].) We shall prove that the converse is also true:

THEOREM A. *Let (M, g, J) be a compact, connected, l.c.K. non-Kähler manifold, of complex dimension at least 2. If $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ contains a holomorphic l.c.K. flow, then there exists a metric with parallel Lee form in the conformal class of g.*

COROLLARY A_1. *With the same hypothesis, M admits a l.c.K. metric with parallel Lee form if and only if it admits a holomorphic l.c.K. flow.*

In §3, we discuss the existence of l.c.K. metrics with parallel Lee form on the Hopf manifold. (Compare with [7].) Let $\Lambda = (\lambda_1, \ldots, \lambda_n)$ with the λ_i's complex numbers satisfying $0 < |\lambda_n| \leq \cdots \leq |\lambda_1| < 1$. By a *primary Hopf manifold* M_Λ of type Λ we mean the compact quotient manifold of $\boldsymbol{C}^n - \{0\}$ by a subgroup Γ_Λ generated by the transformation $(z_1, \ldots, z_n) \mapsto (\lambda_1 z_1, \ldots, \lambda_n z_n)$. Note that a primary Hopf manifold of type Λ of complex dimension 2 is a primary Hopf surface of Kähler rank 1. We prove the following:

THEOREM B. *The primary Hopf manifold M_Λ of type Λ supports a l.c.K. metric with parallel Lee form.*

See §3 which is devoted to the construction of such a metric. More generally, we prove the existence of a l.c.K. metric with parallel Lee form on the Hopf manifold (cf. Theorem 3.1).

In the second half of the paper we adopt the viewpoint of G-structure theory in order to study a non-compact, non-holomorphic, transformation group of a compact l.c.K. manifold (M, g, J) with parallel Lee form. Locally, the 2-form ω defines the real 1-forms $\theta, \theta\circ J$ and $n-1$ complex 1-forms θ^α and their conjugates $\bar\theta^\alpha$, where $\theta\circ J$ is called the *anti-Lee form* and is defined by $\theta\circ J(X) = \theta(JX)$. We consider the group $\mathrm{Aut}_{\mathrm{LCR}}(M)$ of transformations of M preserving the structure of unitary coframe fields $\mathcal{F} = \{\theta, \theta\circ J, \theta^1, \ldots, \theta^{n-1}, \bar\theta^1, \ldots, \bar\theta^{n-1}\}$. More precisely, an element f of $\mathrm{Aut}_{\mathrm{LCR}}(M)$ is called a *Lee-Cauchy-Riemann* (LCR) transformation if it satisfies the equations:

$$\begin{aligned} f^*\theta &= \theta\,, \\ f^*(\theta\circ J) &= \lambda\cdot(\theta\circ J)\,, \\ f^*\theta^\alpha &= \sqrt{\lambda}\cdot\theta^\beta U^\alpha_\beta + (\theta\circ J)\cdot v^\alpha\,, \\ f^*\bar\theta^\alpha &= \sqrt{\lambda}\cdot\bar\theta^\beta \overline{U^\alpha_\beta} + (\theta\circ J)\cdot\overline{v}^\alpha\,. \end{aligned}$$

Here λ, v^α, U^α_β are smooth functions with values, respectively, in $\boldsymbol{R}^+$, $\boldsymbol{C}$ and $\mathrm{U}(n-1)$. Obviously, if $\mathrm{I}(M, g, J)$ is the group of holomorphic isometries, then both $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ and $\mathrm{Aut}_{\mathrm{LCR}}(M)$ contain $\mathrm{I}(M, g, J)$.

As the main result of this part we exhibit the rigidity of compact l.c.K. manifolds under the existence of a non-compact LCR flow:

THEOREM C. *Let* (M, g, J) *be a compact, connected, l.c.K. non-Kähler manifold of complex dimension at least* 2, *with parallel Lee form* θ. *Suppose that* M *admits a closed subgroup* $\boldsymbol{C}^* = S^1 \times \boldsymbol{R}^+$ *of Lee-Cauchy-Riemann transformations whose* S^1 *subgroup induces the Lee field* $\theta^\sharp$. *Then* M *is holomorphically isometric, up to scalar multiple of the metric, to the primary Hopf manifold* M_Λ *of type* Λ *endowed with the canonical l.c.K. metric as stated in Theorem B.*

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2. Locally conformally Kähler transformations.

PROPOSITION 2.1. *Let* (M, g, J) *be a compact l.c.K. manifold with* $\dim_{\boldsymbol{C}} M \geq 2$. *Then* $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ *is a compact Lie group.*

PROOF. Note that $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ is a closed Lie subgroup in the group of all conformal diffeomorphisms of (M, g). If $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ were noncompact, then by the celebrated result of Obata and Lelong-Ferrand ([15], [14]), (M, g) would be conformally equivalent with the sphere S^{2n}, $n \geq 2$. Hence M would be simply connected. It is well-known that a compact simply connected l.c.K. manifold is conformal to a Kähler manifold (cf. [6]), which is impossible because the sphere S^{2n} has no Kähler structure. □

From now on, we shall suppose that the l.c.K. manifold we work with is compact, non-Kähler and, moreover, that the Lee field is nowhere vanishing. In particular, such a manifold is not simply connected (cf. [6]). Given a l.c.K. manifold (M, g, J), let $\tilde{M}$ be the universal covering space of M, let $p : \tilde{M} \to M$ be the canonical projection and denote also by J the lifted complex structure on $\tilde{M}$. We can associate to the fundamental 2-form ω a canonical Kähler form on $\tilde{M}$ as follows. Since the Lee form θ is closed, its lift to $\tilde{M}$ is exact, hence $p^*\theta = d\tau$ for some smooth function τ on $\tilde{M}$. We put $h = e^{-\tau} \cdot p^*g$ (resp. $\Omega = e^{-\tau} \cdot p^*\omega$). It is easy to check that $d\Omega = 0$, thus h is a Kähler metric on $(\tilde{M}, J)$. In particular g *is locally conformal to the Kähler metric* h (compare with [6] and the bibliography therein). Let $f \in \mathrm{Aut}_{\mathrm{l.c.K.}}(M)$. By definition, $f^*\omega = e^\lambda \cdot \omega$ for some function λ on M. Differentiate this equality to yield that $(f^*\theta - \theta - d\lambda) \wedge \omega = 0$. As ω is nondegenerate and $\dim_{\boldsymbol{C}} M > 1$, $f^*\theta = \theta + d\lambda$. Since $p^*\theta = d\tau$, for any lift $\tilde{f}$ of f to $\tilde{M}$ we have $d\tilde{f}^*\tau = d(\tau + p^*\lambda)$, thus $-\tilde{f}^*\tau + p^*\lambda = -\tau + c$ for some constant c. We can write $\tilde{f}^*\Omega = e^c \cdot \Omega$. If $c \neq 0$, $\tilde{f}$ is a holomorphic homothety with respect to h; when $c = 0$, $\tilde{f}$ will be an isometry.

We denote by $\mathcal{H}(\tilde{M}, \Omega, J)$ the group of all holomorphic, homothetic transformations of $\tilde{M}$ with respect to the Kähler structure (h, J). If $f_1, f_2 \in \mathcal{H}(\tilde{M}, \Omega, J)$, there exist some constants $\rho(f_i)$ $(i = 1, 2)$ satisfying $f_i^*\Omega = \rho(f_i) \cdot \Omega$ as above. It is easy to check that

$\rho(f_1 \circ f_2) = \rho(f_1) \cdot \rho(f_2)$. We obtain a continuous homomorphism:

$$\rho : \mathcal{H}(\tilde{M}, \Omega, J) \longrightarrow \boldsymbol{R}^+. \tag{2.1}$$

Let $\pi_1(M)$ be the fundamental group of M. Then we note that $\pi_1(M) \subset \mathcal{H}(\tilde{M}, \Omega, J)$. For this, if $\gamma \in \pi_1(M)$, then $\gamma^*\Omega = e^{-\gamma^*\tau} \cdot \gamma^* p^*\omega = e^{-\gamma^*\tau} \cdot p^*\omega = e^{-\gamma^*\tau+\tau} \cdot \Omega$. Since Ω is a Kähler form ($n \geq 2$), $e^{-\gamma^*\tau+\tau}$ must be constant $\rho(\gamma)$.

Let $\mathcal{C}$ be a holomorphic l.c.K. flow on M. If we denote by $\tilde{\mathcal{C}}$ a lift of $\mathcal{C}$ to $\tilde{M}$, then $\tilde{\mathcal{C}} \subset \mathcal{H}(\tilde{M}, \Omega, J)$. If V is a vector field which generates a one-parameter subgroup of $\tilde{\mathcal{C}}$, then so does JV with V and JV together generating $\tilde{\mathcal{C}}$. We define a smooth function $s : \tilde{M} \to \boldsymbol{R}$ to be $s(x) = \Omega(JV_x, V_x)$. Since $\tilde{\mathcal{C}}$ centralizes each element γ of $\pi_1(M)$, it follows that $s(\gamma x) = \Omega(JV_{\gamma x}, V_{\gamma x}) = \Omega(\gamma_* JV_x, \gamma_* V_x) = \rho(\gamma)s(x)$. If every element γ satisfies $\rho(\gamma) = 1$, i.e., $\gamma^*\Omega = \Omega$, then $\pi_1(M)$ acts as holomorphic isometries of h so that Ω would induce a Kähler metric on M. By our hypothesis, this does not occur. There exists at least one element γ such that $\rho(\gamma) \neq 1$. In particular, we note that:

(2.2) The function s is not constant on $\tilde{M}$.

On the other hand, we prove the following lemma. (The proof of the lemma is almost the same as that of [10].)

LEMMA 2.1. *$\rho(\tilde{\mathcal{C}}) = \boldsymbol{R}^+$, i.e., the group $\tilde{\mathcal{C}}$ acts by holomorphic, non-trivial homotheties with respect to the Kähler metric h on $\tilde{M}$.*

PROOF. Since $\tilde{\mathcal{C}}$ is connected, if $\rho(\tilde{\mathcal{C}}) \neq \boldsymbol{R}^+$, it must be trivial. By reduction to absurdity, suppose that $\rho(\tilde{\mathcal{C}}) = \{1\}$. Then $\tilde{\mathcal{C}}$ leaves Ω invariant. As $\{V, JV\}$ generates $\tilde{\mathcal{C}}$, it follows that $\mathcal{L}_V\Omega = \mathcal{L}_{JV}\Omega = 0$. In particular, $Vs = (JV)s = 0$. For any distribution D on $\tilde{M}$, denote by $D^\perp$ the orthogonal complement to D with respect to the metric h, where $h(\tilde{X}, \tilde{Y}) = \Omega(J\tilde{X}, \tilde{Y})$. Since $0 = (\mathcal{L}_V\Omega)(JV, \tilde{X}) = V\Omega(JV, \tilde{X}) - \Omega([V, JV], \tilde{X}) - \Omega(JV, [V, \tilde{X}])$, if $\tilde{X} \in \{V, JV\}^\perp$, then $\Omega(JV, [V, \tilde{X}]) = 0$, similarly $\Omega(V, [JV, \tilde{X}]) = 0$. The equality

$$\begin{aligned} 0 = 3d\Omega(\tilde{X}, V, JV) &= \tilde{X}\Omega(V, JV) - V\Omega(\tilde{X}, JV) + JV\Omega(\tilde{X}, V) \\ &\quad - \Omega([\tilde{X}, V], JV) - \Omega([V, JV], \tilde{X}) - \Omega([JV, \tilde{X}], V) \end{aligned}$$

implies that $\tilde{X}\Omega(V, JV) = 0$, i.e., $\tilde{X}s = 0$ for any $\tilde{X} \in \{V, JV\}^\perp$. Therefore, s becomes constant, being a contradiction to (2.2). □

2.1. The submanifold W and its pseudo-Hermitian structure. As $\mathrm{Ker}\rho$ has one dimension, denote by $-J\xi$ the vector field whose one-parameter subgroup $\{\psi_t\}_{t\in\boldsymbol{R}}$ acts as holomorphic isometries on $\tilde{M}$.

$$\psi_t^*\Omega = \Omega\,, \quad t \in \boldsymbol{R}\,. \tag{2.3}$$

Since $-J\xi$ and ξ together generate the group $\tilde{\mathcal{C}}$, the 1-parameter subgroup $\{\varphi_t\}_{t\in\boldsymbol{R}}$ generated by ξ acts as nontrivial holomorphic homotheties with respect to Ω by Lemma 2.1. In particular, the group $\{\varphi_t\}_{t\in\boldsymbol{R}}$ is isomorphic to $\boldsymbol{R}$. Since $\varphi_t^*\Omega = \rho(\varphi_t) \cdot \Omega$ ($t \in \boldsymbol{R}$, $\rho(\varphi_t) \in \boldsymbol{R}^+$) from

(2.1) and ρ is a continuous homomorphism, $\rho(\varphi_t) = e^{at}$ for some constant $a \neq 0$. We may normalize $a = 1$ so that:

$$\varphi_t^* \Omega = e^t \cdot \Omega\,, \quad t \in \boldsymbol{R}\,. \tag{2.4}$$

LEMMA 2.2. *The group $\{\varphi_t\}_{t\in\boldsymbol{R}}$ acts properly and hence freely on $\tilde{M}$. In particular, $\xi \neq 0$ everywhere on $\tilde{M}$.*

PROOF. Recall that $\mathcal{C}$ lies in $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ by definition. As $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ is a compact Lie group, its closure $\overline{\mathcal{C}}$ in $\mathrm{Aut}_{\mathrm{l.c.K.}}(M)$ is also compact and so isomorphic to a k-torus ($k \geq 2$). Therefore, the lift H of $\overline{\mathcal{C}}$ to $\tilde{M}$ acts properly on $\tilde{M}$. The lift H is isomorphic to $\boldsymbol{R}^l \times T^m$, where $l + m = k$. Note that $l \geq 1$ because ρ maps any compact subgroup of H to $\{1\}$, but the group $\{\varphi_t\}_{t\in\boldsymbol{R}} \subset H$ satisfies $\rho(\{\varphi_t\}) = \boldsymbol{R}^+$. Hence the group $\{\varphi_t\}_{t\in\boldsymbol{R}}$ has a nontrivial summand in $\boldsymbol{R}^l$, which implies that $\{\varphi_t\}_{t\in\boldsymbol{R}}$ is closed in H. Thus, the group $\{\varphi_t\}_{t\in\boldsymbol{R}}$ acts properly on $\tilde{M}$. If we note that $\{\varphi_t\}_{t\in\boldsymbol{R}}$ is isomorphic to $\boldsymbol{R}$, then it acts freely on $\tilde{M}$. □

PROPOSITION 2.2. *Let $s : \tilde{M} \to \boldsymbol{R}$ be the smooth map defined as $s(x) = \Omega(J\xi_x, \xi_x)$. Then 1 is a regular value of s, and hence $s^{-1}(1)$ is a codimension one, regular submanifold of $\tilde{M}$.*

PROOF. As φ_t is holomorphic, $s(\varphi_t x) = \Omega(J\xi_{\varphi_t x}, \xi_{\varphi_t x}) = \Omega(\varphi_{t*}J\xi_x, \varphi_{t*}\xi_x) = e^t \cdot s(x)$. Hence,

$$\mathcal{L}_\xi s = \lim_{t\to 0} \frac{\varphi_t^* s - s}{t} = s\,.$$

We also note that

$$\mathcal{L}_\xi \Omega = \Omega\,. \tag{2.5}$$

By Lemma 2.2, notice that $\xi \neq 0$ everywhere on $\tilde{M}$. Since $s(x) \neq 0$, $s^{-1}(1) \neq \emptyset$. For $x \in s^{-1}(1)$, $ds(\xi_x) = (\mathcal{L}_\xi s)(x) = s(x) = 1$. This proves that $ds : T_x\tilde{M} \to \boldsymbol{R}$ is onto and so $s^{-1}(1)$ is a codimension one smooth regular submanifold of $\tilde{M}$. □

Let now $W = s^{-1}(1)$. We can prove:

LEMMA 2.3. *The submanifold W is connected and the map $H : \boldsymbol{R} \times W \to \tilde{M}$, defined by $H(t, w) = \varphi_t w$, is an equivariant diffeomorphism.*

PROOF. Let W_0 be a component of $s^{-1}(1)$ and $\boldsymbol{R} \cdot W_0$ the set $\{\varphi_t w\ ;\ w \in W_0, t \in \boldsymbol{R}\}$. As $\boldsymbol{R} = \{\varphi_t\}$ acts freely and $s(\varphi_t x) = e^t s(x)$, we have $\varphi_t W_0 \cap W_0 = \emptyset$ for $t \neq 0$. Thus $\boldsymbol{R} \cdot W_0$ is an open subset of $\tilde{M}$. We prove that it is also closed. Let $\overline{\boldsymbol{R} \cdot W_0}$ be the closure of $\boldsymbol{R} \cdot W_0$ in $\tilde{M}$. We choose a limit point $p = \lim \varphi_{t_i} w_i \in \overline{\boldsymbol{R} \cdot W_0}$. Then $s(p) = \lim s(\varphi_{t_i} w_i) = \lim e^{t_i} s(w_i) = \lim e^{t_i}$. Put $t = \log s(p)$. Then $t = \lim t_i$, so $\varphi_t^{-1}(p) = \lim \varphi_{t_i}^{-1}(\lim \varphi_{t_i} w_i) = \lim w_i$. Since $s^{-1}(1)$ is regular (i.e., closed with respect to the relative topology induced from $\tilde{M}$), its component W_0 is also closed. Hence $\varphi_t^{-1} p \in W_0$. Therefore $p = \varphi_t(\varphi_t^{-1} p) \in \boldsymbol{R} \cdot W_0$, proving that $\boldsymbol{R} \cdot W_0$ is closed in $\tilde{M}$. In conclusion, $\boldsymbol{R} \cdot W_0 = \tilde{M}$. Now, if W_1 is another component of $s^{-1}(1)$, the same argument shows $\boldsymbol{R} \cdot W_1 = \tilde{M}$. As $\boldsymbol{R} \cdot W_0 = \boldsymbol{R} \cdot W_1$ and $s(W_1) = 1$, this implies $W_0 = W_1$, in other words, W is connected. □

Let $i : W \to \tilde{M}$ be the inclusion and $\pi : \tilde{M} \to W$ the canonical projection. Define a 1-form η on W to be

$$\eta = i^* \iota_\xi \Omega \,. \tag{2.6}$$

Here ι_ξ denotes the interior product with ξ. From the definition of $\{\psi_t\}_{t\in\boldsymbol{R}}$ (see the beginning of § 2.1) we have

$$\left.\frac{d\psi_t}{dt}(x)\right|_{t=0} = -J\xi_x \,. \tag{2.7}$$

By (2.3), $s(\psi_t w) = s(w) = 1$ $(w \in W)$ so that the group $\{\psi_t\}_{t\in\boldsymbol{R}}$ leaves W invariant. Hence, the vector field $-J\xi$ restricts to a vector field A to W. If $\{\psi'_t\}_{t\in\boldsymbol{R}}$ is the one-parameter subgroup generated by A, then

$$\psi_t = i \circ \psi'_t \,. \tag{2.8}$$

LEMMA 2.4. *The* 1*-form η is a contact form on W for which A is the characteristic vector field* (*Reeb field*).

PROOF. First note that $\eta(A_w) = \iota_\xi\Omega(-J\xi_w) = \Omega(J\xi_w, \xi_w) = s(w) = 1$ $(w \in W)$. Moreover, from (2.5), $d\eta = i^* d\iota_\xi \Omega = i^*(d\iota_\xi\Omega + \iota_\xi d\Omega) = i^*\mathcal{L}_\xi \Omega = i^*\Omega$. Hence, $\eta \wedge d\eta^{n-1} \neq 0$ on W showing that η is a contact form. Noting (2.3), (2.8) and that both φ_t and ψ_θ commutes with each other, it is easy to see that

$$\begin{aligned} {\psi'}^*_t \iota_\xi \Omega = \iota_\xi \quad &\text{on} \ \ \tilde{M} \,. \\ {\psi'}^*_t \eta = \eta \quad &\text{on} \ \ W \,. \end{aligned} \tag{2.9}$$

Let Null $\eta = \{X \in TW \mid \eta(X) = 0\}$ be the contact subbundle. Since $\mathcal{L}_A\eta(X) = A\eta(X) - \eta([A, X])$ and $\mathcal{L}_A\eta = 0$ from (2.9), if $X \in$ Null η, then $\eta([A, X]) = 0$. Moreover, $d\eta(A, X) = (A\eta(X) - X\eta(A) - \eta([A, X]))/2 = 0$, which implies that $d\eta(A, X) = 0$ for all $X \in TW$, showing that A is the characteristic vector field. □

Recall that $\boldsymbol{R} \to \tilde{M} \xrightarrow{\pi} W$ is a principal fiber bundle with $T\boldsymbol{R} = \langle\xi\rangle$. By Lemma 2.3, each point $x \in \tilde{M}$ can be described uniquely as $x = \varphi_t w$. By (2.8),

$$\begin{aligned} \pi \circ \psi_\theta(x) &= \pi \circ \psi_\theta(\varphi_t w) = \pi \circ \varphi_t(\psi_\theta w) \\ &= \pi \circ i{\psi'}_\theta(w) = {\psi'}_\theta(w) = {\psi'}_\theta \circ \pi(x) \,, \end{aligned} \tag{2.10}$$

and hence, $\pi_*(-J\xi) = A$. As $i_*\pi_*X_x - X_x = a \cdot \xi_x$ for some function a, by (2.6), π maps $\{\xi, J\xi\}^\perp$ isomorphically onto Null η. Since $\{\xi, J\xi\}^\perp$ is J-invariant, there exists an almost complex structure J on Null η such that the following diagram is commutative:

$$\begin{array}{ccc} \{\xi, J\xi\}^\perp & \xrightarrow{\pi_*} & \mathrm{Null}\,\eta \\ \downarrow J & & \downarrow J \\ \{\xi, J\xi\}^\perp & \xrightarrow{\pi_*} & \mathrm{Null}\,\eta \,. \end{array} \tag{2.11}$$

PROPOSITION 2.3. *The pair (η, J) is a strictly pseudoconvex, pseudo-Hermitian structure on W.*

PROOF. Let $\Psi : \mathrm{Null}\,\eta \times \mathrm{Null}\,\eta \to \boldsymbol{R}$ be the bilinear form defined by $\Psi(X, Y) = d\eta(JX, Y)$. There exist $\tilde{X}, \tilde{Y} \in \{\xi, J\xi\}^{\perp}$ such that $\pi_*\tilde{X} = X$, $\pi_*\tilde{Y} = Y$. Then it is easy to see that $i_*JX \equiv J\tilde{X}$, $i_*Y \equiv \tilde{Y} \bmod \xi$. Using $d\eta = i^*\Omega$ as above, $\Psi(X, Y) = i^*\Omega(JX, Y) = \Omega(J\tilde{X}, \tilde{Y}) = h(\tilde{X}, \tilde{Y})$, and hence Ψ is positive definite. By definition, η is strictly pseudoconvex. Let $\{\xi, J\xi\}^{\perp} \otimes \boldsymbol{C} = B^{1,0} \oplus B^{0,1}$ be the canonical splitting of J. Then we prove that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Let $\tilde{X}, \tilde{Y} \in B^{1,0}$. Since $T^{1,0}\tilde{M} = \{\xi - \mathrm{i}J\xi\} \oplus B^{1,0}$ (where $\mathrm{i} = \sqrt{-1}$) and J is integrable on $\tilde{M}$, $[\tilde{X}, \tilde{Y}] \in T^{1,0}\tilde{M}$. Put $[\tilde{X}, \tilde{Y}] = a(\xi - \mathrm{i}J\xi) + \tilde{Z}$ for some function a and $\tilde{Z} \in B^{1,0}$. As $\pi_*(-J\xi) = A$ from (2.10), $\pi_*([\tilde{X}, \tilde{Y}]) = a\mathrm{i}A + \pi_*\tilde{Z}$. By definition, $2d\eta(\pi_*\tilde{X}, \pi_*\tilde{Y}) = -\eta([\pi_*\tilde{X}, \pi_*\tilde{Y}]) = -a\mathrm{i}$. On the other hand, since Ω is J-invariant, $\Omega(\tilde{X}, \tilde{Y}) = 0$ for any $\tilde{X}, \tilde{Y} \in B^{1,0}$. As above, $i_*\pi_*\tilde{X} \equiv \tilde{X} \bmod \xi$, similarly for $\tilde{Y}$, we obtain that $d\eta(\pi_*\tilde{X}, \pi_*\tilde{Y}) = \Omega(i_*\pi_*\tilde{X}, i_*\pi_*\tilde{Y}) = \Omega(\tilde{X}, \tilde{Y}) = 0$. Hence, $a = 0$ and so $[\tilde{X}, \tilde{Y}] = \tilde{Z} \in B^{1,0}$. If we note that $\pi_* : \{\xi, J\xi\}^{\perp} \otimes \boldsymbol{C} \to \mathrm{Null}\,\eta \otimes \boldsymbol{C}$ is J-isomorphic by (2.11), then $\mathrm{Null}\,\eta \otimes \boldsymbol{C} = \pi_*B^{1,0} \oplus \pi_*B^{0,1}$ is the splitting for J, in which we have shown $[\pi_*B^{1,0}, \pi_*B^{1,0}] \subset \pi_*B^{1,0}$. Therefore J is a complex structure on $\mathrm{Null}\,\eta$. □

Consider the group of pseudo-Hermitian transformations on (W, η, J):

$$\mathrm{PSH}(W, \eta, J) = \{f \in \mathrm{Diff}(W) \mid f^*\eta = \eta,\ f_* \circ J = J \circ f_* \text{ on } \mathrm{Null}\,\eta\}. \tag{2.12}$$

COROLLARY 2.1. *The characteristic vector field A generates the subgroup $\{\psi'_t\}_{t\in\boldsymbol{R}}$ consisting of pseudo-Hermitian transformations.*

PROOF. By (2.3) and (2.9), ψ_t (resp. ψ'_t) preserves $\{\xi, J\xi\}^{\perp}$ (resp. $\mathrm{Null}\,\eta$). Then the equality $\pi \circ \psi_\theta = \psi'_\theta \circ \pi$ from (2.10) with diagram (2.11) implies that $\psi'_{t*}J = J\psi'_{t*}$ on $\mathrm{Null}\,\eta$. Therefore

$$\{\psi'_t\}_{t\in\boldsymbol{R}} \subset \mathrm{PSH}(W, \eta, J)\,. \tag{2.13}$$ □

Proof of Theorem A

2.2. Parallel Lee form. Let again φ_t be the 1-parameter subgroup generated by ξ. According to the notation in Lemma 2.3, let $Y_{\varphi_t w} \in T_{\varphi_t w}\tilde{M}$ be any vector. We have $\pi_*Y_{\varphi_t w} \in T_wW$, and hence $i_*\pi_*Y_{\varphi_t w} - \varphi_{-t\,*}Y_{\varphi_t w} = \lambda\xi_w$ for some number λ. Then,

$$\begin{aligned}\iota_\xi\Omega(i_*\pi_*Y_{\varphi_t w}) &= \Omega(\xi_w, i_*\pi_*Y_{\varphi_t w}) = \Omega(\xi_w, \varphi_{-t\,*}Y_{\varphi_t w}) + \Omega(\xi_w, \lambda\xi_w)\\ &= \varphi^*_{-t}\Omega(\varphi_{t*}\xi_w, Y_{\varphi_t w}) = e^{-t}\Omega(\xi_{\varphi_t w}, Y_{\varphi_t w}) = e^{-t}\iota_\xi\Omega(Y_{\varphi_t w})\,.\end{aligned}$$

By the definition (2.6),

$$\pi^*\eta = \pi^*i^*\iota_\xi\Omega = e^{-t}\iota_\xi\Omega, \quad \text{equivalently,} \quad e^t\pi^*\eta = \iota_\xi\Omega\,. \tag{2.14}$$

As $\Omega = \mathcal{L}_\xi\Omega = d\iota_\xi\Omega$ from (2.5), we obtain that

$$d(e^t\pi^*\eta) = \Omega \quad \text{on } \tilde{M}. \tag{2.15}$$

For the given l.c.K. metric g, the Kähler metric h is obtained as $h = e^{-\tau} \cdot p^*g$ where $d\tau = \tilde{\theta}$. As ω is the fundamental 2-form of g, note that $\Omega = e^{-\tau} \cdot p^*\omega$.

We now consider on $\tilde{M}$ the 2-form:

$$\bar{\Theta} = 2e^{-t} \cdot d(e^t \pi^* \eta) \; (= 2e^{-t} \cdot \Omega)\,. \tag{2.16}$$

Then $\bar{g}(X, Y) = \bar{\Theta}(JX, Y)$ is a l.c.K. metric. Put $\bar{\theta} = -dt$. Then, as $d\bar{\Theta} = -2e^{-t} dt \wedge d(e^t\pi^*\eta) = -dt \wedge \bar{\Theta}$, we see that $\bar{\theta}$ is the Lee form of $\bar{g}$.

LEMMA 2.5. *$\bar{\theta}$ is parallel with respect to $\bar{g}$* ($\nabla^{\bar{g}}\bar{\theta} = 0$).

PROOF. First we determine the Lee field $\bar{\theta}^\sharp$ (where $\bar{\theta}(X) = \bar{g}(X, \bar{\theta}^\sharp)$). We start from:

$$\begin{aligned}\bar{g}(\xi, Y) &= \bar{\Theta}(J\xi, Y) = 2e^{-t}(e^t dt \wedge \pi^*\eta + e^t d\pi^*\eta)(J\xi, Y)\\ &= 2(dt \wedge \pi^*\eta + d\pi^*\eta)(J\xi, Y) = 2(dt \wedge \pi^*\eta)(J\xi, Y),\end{aligned}$$

because $A = -\pi_* J\xi$ is the characteristic vector field of the contact form η. As before, a point $x \in \tilde{M}$ can be described uniquely as $\varphi_t w$ for some $w \in W$. In particular, by Lemma 2.3, the t-coordinate of x is t. Noting that $\psi_\theta(x) = \varphi_t \psi_\theta w$ and $\psi_\theta w \in W$, by the uniqueness of the t-coordinate of $\psi_\theta(x)$, $t(\psi_\theta(x)) = t$. From (2.7),

$$dt(-J\xi_x) = dt\left(\frac{d\psi_\theta}{d\theta}(x)\bigg|_{\theta=0}\right) = \frac{dt}{d\theta}\bigg|_{\theta=0} = 0\,. \tag{2.17}$$

The above formula becomes:

$$\begin{aligned}\bar{g}(\xi, Y) &= 2(dt \wedge \pi^*\eta)(J\xi, Y) = -dt(Y)\eta(-A)\\ &= dt(Y) = -\bar{\theta}(Y) = -\bar{g}(Y, \bar{\theta}^\sharp)\,,\end{aligned} \tag{2.18}$$

proving that $\bar{\theta}^\sharp = -\xi$. Next we observe that the flow $\{\varphi_s\}_{s\in\boldsymbol{R}}$ acts by isometries with respect to $\bar{g}$. As φ_s is holomorphic, it is enough to prove that each φ_s leaves $\bar{\Theta}$ invariant, but

$$\varphi_s^*\bar{\Theta} = 2e^{-\varphi_s^* t} d(e^{\varphi_s^* t}\varphi_s^*\pi^*\eta) = 2e^{-(s+t)}d(e^{s+t}\pi^*\eta) = 2e^{-t}d(e^t\pi^*\eta) = \bar{\Theta}\,.$$

Thus $\mathcal{L}_{\bar{\theta}^\sharp}\bar{g} = -\mathcal{L}_\xi \bar{g} = 0$. Now we put $\sigma = \bar{\theta}$ in the equality $(\mathcal{L}_{\sigma^\sharp}\bar{g})(X, Y) + 2d\sigma(X, Y) = 2\bar{g}(\nabla^{\bar{g}}_X \sigma^\sharp, Y)$, valid for any 1-form σ, take into account $d\bar{\theta} = 0$ and obtain $\nabla^{\bar{g}}\bar{\theta}^\sharp = 0$, which is equivalent with $\nabla^{\bar{g}}\bar{\theta} = 0$, so $\bar{\theta}$ is parallel with respect to $\bar{g}$ as announced. □

By the equation (2.16), $\bar{g}$ is conformal to the lifted metric p^*g:

$$\bar{\Theta} = \mu \cdot p^*\omega \;\; (\text{equivalently } \bar{g} = \mu \cdot p^* g)\,, \tag{2.19}$$

where $\mu = 2e^{-(t+\tau)} : \tilde{M} \to \boldsymbol{R}^+$ is a smooth map. We finally prove:

LEMMA 2.6. *$\pi_1(M)$ acts by holomorphic isometries of $\bar{g}$. In particular, $\pi_1(M)$ leaves $\bar{\theta}$ invariant.*

PROOF. We prove the following two facts:

1. $\gamma^*\pi^*\eta = \pi^*\eta$ for every $\gamma \in \pi_1(M)$.
2. $\gamma^* e^t = \rho(\gamma) \cdot e^t$, where $\rho : \pi_1(M) \to \boldsymbol{R}^+$ is the restriction of the homomorphism defined in (2.1).

First note that as $\boldsymbol{R} = \{\varphi_t\}$ centralizes $\pi_1(M)$, $\gamma_*\xi = \xi$ for $\gamma \in \pi_1(M)$. As γ is holomorphic, $\gamma_* J\xi = J\xi$. Since $\pi_1(M)$ acts on $\tilde{M}$ as holomorphic homothetic transformations, (i.e., $\gamma^*\Omega = \rho(\gamma)\cdot\Omega$), $\pi_1(M)$ preserves $\{\xi, J\xi\}^{\perp}$. If we recall that $\pi_* : \{\xi, J\xi\}^{\perp} \to \text{Null}\ \eta$ is isomorphic, then for $X \in \{\xi, J\xi\}^{\perp}$, $\gamma^*\pi^*\eta(X) = \eta(\pi_*\gamma_* X) = 0$. As $-\pi_* J\xi = A$ is the characteristic field of η, it follows that $\gamma^*\pi^*\eta(J\xi) = \eta(\pi_*\gamma_* J\xi) = \eta(\pi_* J\xi) = -1$. This shows that $\gamma^*\pi^*\eta = \pi^*\eta$ on $\tilde{M}$. On the other hand, if we note $\gamma_*\xi = \xi$, then

$$\begin{aligned}\gamma^*(\iota_\xi\Omega)(X) &= \Omega(\xi, \gamma_* X) = \Omega(\gamma_*\xi, \gamma_* X) = \gamma^*\Omega(\xi, X)\\ &= \rho(\gamma)\cdot\Omega(\xi, X) = \rho(\gamma)\cdot\iota_\xi\Omega(X)\,,\end{aligned}$$

where $\rho(\gamma)$ is a positive constant. Applying γ^* to $\pi^*\eta = e^{-t}\cdot\iota_\xi\Omega$ from (2.14), we obtain $\gamma^* e^{-t}\cdot\rho(\gamma) = e^{-t}$. Equivalently, $\gamma^* e^t = \rho(\gamma)\cdot e^t$. This shows 1 and 2. From (2.16),

$$\begin{aligned}\gamma^*\bar{\Theta} &= \gamma^*(2e^{-t}\cdot d(e^t\pi^*\eta)) = 2\rho(\gamma)^{-1}\cdot e^{-t} d(\rho(\gamma)\cdot e^t\gamma^*\pi^*\eta)\\ &= 2e^{-t}\cdot d(e^t\pi^*\eta) = \bar{\Theta}\,.\end{aligned}$$

Since $\bar{g}(X, Y) = \bar{\Theta}(JX, Y)$, $\pi_1(M)$ acts through holomorphic isometries of $\bar{g}$. We have that $\bar{\theta}(Y) = \bar{g}(Y, \bar{\theta}^\sharp) = -\bar{g}(Y, \xi)$ $(Y \in T\tilde{M})$ from (2.18). Then,

$$\gamma^*\bar{\theta}(Y) = -\bar{g}(\gamma_* Y, \xi) = -\bar{g}(\gamma_* Y, \gamma_*\xi) = -\bar{g}(Y, \xi) = \bar{\theta}(Y)\,. \qquad \square$$

From this lemma, the covering map $p : \tilde{M} \to M$ induces a l.c.K. metric $\hat{g}$ with parallel Lee form $\hat{\theta}$ on M such that $p^*\hat{g} = \bar{g}$ and $p^*\hat{\theta} = \bar{\theta}$ with $\nabla^{\hat{g}}_{p_* X}\hat{\theta}(p_* Y) = \nabla^{\bar{g}}_X\bar{\theta}(Y)$. Applying γ^* to both sides of (2.19), we derive

$$\gamma^*\bar{g} = \bar{g} = \mu\cdot p^*\,, \quad \gamma^*\mu\cdot\gamma^* p^* g = \gamma^*\mu\cdot p^* g\,.$$

Therefore $\gamma^*\mu = \mu$, which implies that μ factors through a map $\hat{\mu} : M \to \boldsymbol{R}^+$ so that $p^*\hat{g} = p^*(\hat{\mu}\cdot g)$. We have $\hat{\mu}\cdot g = \hat{g}$. The conformal class of g contains a l.c.K. metric $\hat{g}$ with parallel Lee form $\hat{\theta}$. This ends the proof of Theorem A. $\square$

As to Corollary A_1 in the Introduction, we recall the following. (Compare [18], [6, p. 37].) Let (M, g, J) be a compact, connected, non-Kähler, l.c.K. manifold with parallel Lee form θ. Then the following results hold: $g(\theta^\sharp, \theta^\sharp)$ =const,

$$\mathcal{L}_{\theta^\sharp} J = \mathcal{L}_{J\theta^\sharp} J = 0\,, \quad \mathcal{L}_{\theta^\sharp} g = \mathcal{L}_{J\theta^\sharp} g = 0\,.$$

Then $Z = \theta^\sharp - \mathrm{i}J\theta^\sharp$ is a holomorphic vector field because $[\theta^\sharp, J\theta^\sharp] = 0$ (cf. [12]). By Definition 1.1, $Z = \theta^\sharp - \mathrm{i}J\theta^\sharp$ is a holomorphic l.c.K. vector field.

PROPOSITION 2.4. *The real vector fields $\theta^\sharp$ and $J\theta^\sharp$ satisfy the following*:

1. *A flow generated by the Lee field $\theta^\sharp$ lifts to a one-parameter subgroup of nontrivial homothetic holomorphic transformations with respect to Ω.*
2. *A flow generated by the anti-Lee field $-J\theta^\sharp$ lifts to a one-parameter subgroup consisting of holomorphic isometries with respect to Ω.*

PROOF. Let $\{\hat{\varphi}_t\}_{t\in\boldsymbol{R}}$ be the flow generated by $\theta^\sharp$ on M and $\{\varphi_t\}_{t\in\boldsymbol{R}}$ its lift to $\tilde{M}$. Denote by ξ the vector field on $\tilde{M}$ induced by $\{\varphi_t\}$. Then, $p_*\xi = \theta^\sharp$. Because θ is parallel, $\{\hat{\varphi}_t\}$ (resp. $\{\varphi_t\}$) acts by holomorphic isometries with respect to g (resp. $p^* g$). In particular, $\{\varphi_t\}$ preserves $p^*\omega$. Then, for $\Omega = e^{-\tau}p^*\omega$, we have $\varphi_t^*\Omega = e^{-(\varphi_t^*\tau - \tau)}\Omega$. As $\rho : \{\varphi_t\}_{t\in\boldsymbol{R}} \to \boldsymbol{R}^+$

is a homomorphism and $\rho(\varphi_t) = e^{-(\varphi_t^*\tau-\tau)}$ is constant for each $t \in \boldsymbol{R}$ ($\dim_{\boldsymbol{C}} M \geq 2$), we can describe as $-(\varphi_t^*\tau - \tau) = c \cdot t$ for some constant c. Recall that h is the Kähler metric associated to Ω. If $\{\varphi_t\}$ acts as holomorphic isometries with respect to h, then the above equation implies that $c = 0$, i.e., $\varphi_t^*\tau - \tau = 0$ for every t, and so $\mathcal{L}_\xi\tau = 0$. On the other hand, as $d\tau = p^*\theta$, we have:

$$0 = \mathcal{L}_\xi\tau = d\tau(\xi) = \theta(p_*\xi) = \theta(\theta^\sharp) = \text{const.} > 0,$$

a contradiction. Thus, $\varphi_t^*\Omega = \rho(\varphi_t)\Omega = e^{c\cdot t}\Omega$ with $c \neq 0$. Hence, $\{\varphi_t\}_{t\in\boldsymbol{R}}$ is a group of nontrivial homothetic holomorphic transformations isomorphic to $\boldsymbol{R}$. On the other hand, let $\{\hat{\psi}_t\}_{t\in\boldsymbol{R}}$ (resp. $\{\psi_t\}_{t\in\boldsymbol{R}}$) be the flow generated by $-J\theta^\sharp$ on M (resp. $-J\xi$ on $\tilde{M}$). As $p_*(J\xi) = Jp_*\xi = J\theta^\sharp$,

$$\mathcal{L}_{J\xi}\tau = d\tau(J\xi) = p^*\theta(J\xi) = \theta(J\theta^\sharp) = g(J\theta^\sharp, \theta^\sharp) = 0\,,$$

and hence $\psi_t^*\tau = \tau$ for every $t \in \boldsymbol{R}$. By the fact that $\mathcal{L}_{J\theta^\sharp}g = 0$, $\mathcal{L}_{J\theta^\sharp}\omega = 0$. This implies that $\psi_t^*\Omega = \psi_t^*e^{-\tau}\psi_t^*p^*\omega = e^{-\tau}p^*\hat{\psi}_t^*\omega = e^{-\tau}p^*\omega = \Omega$. □

Let $\boldsymbol{R} \to \tilde{M} \stackrel{\pi}{\to} W$ be the principal bundle, where $\boldsymbol{R} = \{\varphi_t\}_{t\in\boldsymbol{R}}$ (cf. Lemma 2.2). Define the centralizer of $\boldsymbol{R}$ in $\mathcal{H}(\tilde{M}, \Omega, J)$ to be:

DEFINITION 2.1. $\mathcal{C}_{\mathcal{H}}(\boldsymbol{R}) = \{f \in \mathcal{H}(\tilde{M}, \Omega, J) \mid f \circ \varphi_t = \varphi_t \circ f$ for *all* $t \in \boldsymbol{R}\}$.

As $\tilde{\mathcal{C}}$ centralizes the fundamental group $\pi_1(M)$, noting the remark below (2.1), we have

$$\pi_1(M) \subset \mathcal{C}_{\mathcal{H}}(\boldsymbol{R})\,. \tag{2.20}$$

LEMMA 2.7. *There exists a homomorphism* $\nu : \mathcal{C}_{\mathcal{H}}(\boldsymbol{R}) \to \mathrm{PSH}(W, \eta, J)$ *for which* $\pi : \tilde{M} \to W$ *becomes* ν*-equivariant. Moreover, there exists a splitting homomorphism* q : $\mathrm{PSH}(W, \eta, J) \to \mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$.

PROOF. By definition, any element $f \in \mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$ satisfies $f_*\xi = \xi$. As $f^*\Omega = \rho(f)\Omega$, choosing $e^s = \rho(f)$, put $\gamma = \varphi_{-s} \circ f$. Then, $\gamma^*\Omega = \Omega$. In particular, γ leaves W invariant. Let γ' be the restriction of γ to W (i.e., $i \circ \gamma' = \gamma$). Using (2.6) and $\gamma_*\xi = \xi$, we have that $\gamma'^*\eta = \gamma^*\mathcal{L}_\xi\Omega = \mathcal{L}_\xi\Omega = \eta$. Hence $\gamma' \in \mathrm{PSH}(W, \eta, J)$. If we define $\nu(f) = \gamma'$, then it is easy to see that ν is a well-defined homomorphism. Let $x = \varphi_t w$ be a point in $\tilde{M}$. As $\pi(x) = w$, $\pi(fx) = \pi(\varphi_s\gamma(\varphi_t w)) = \pi(\varphi_s\varphi_t i\gamma' w) = \pi(i\gamma' w) = \gamma' w = \nu(f)\pi(x)$, so π is ν-equivariant.

For $\gamma \in \mathrm{PSH}(W, \eta, J)$, we define a diffeomorphism $\tilde{\gamma} : \tilde{M} \to \tilde{M}$ to be

$$\tilde{\gamma}(x) = \tilde{\gamma}(\varphi_t w) = \varphi_t\gamma w\,. \tag{2.21}$$

By definition, $\pi \circ \tilde{\gamma} = \gamma \circ \pi$ and the t-coordinate satisfies that $\tilde{\gamma}^*t = t$. By (2.15) and $\gamma^*\eta = \eta$, it follows that $\tilde{\gamma}^*\Omega = d(e^{\gamma^*t}\pi^*\gamma^*\eta) = d(e^t\pi^*\eta) = \Omega$. To see that $\tilde{\gamma} : \tilde{M} \to \tilde{M}$ is holomorphic, notice that $\tilde{\gamma}_*\xi = \xi$. As $\tilde{\gamma}(\psi_\theta x) = \tilde{\gamma}(\psi_\theta\varphi_t w) = \tilde{\gamma}(\varphi_t i\psi'_\theta w) = \varphi_t i\gamma\psi'_\theta w$, and $\gamma_* A = A$,

$$\begin{aligned}\tilde{\gamma}_*(-J\xi_x) &= \tilde{\gamma}_*\left(\frac{d\psi_\theta}{d\theta}(x)\Big|_{\theta=0}\right) = \left(\frac{d\varphi_t i\gamma(\psi'_\theta w)}{d\theta}\Big|_{\theta=0}\right)\\ &= \varphi_{t*} i_* \gamma_*\left(\frac{d\psi'_\theta}{d\theta}(w)\Big|_{\theta=0}\right) = \varphi_{t*} i_* \gamma_* A_w = \varphi_{t*} i_* A_{\gamma w}\\ &= \varphi_{t*}(-J\xi_{\gamma w}) = -J\xi_{\tilde{\gamma}x}\,.\end{aligned} \tag{2.22}$$

Hence, $\tilde{\gamma}$ preserves $\{\xi, J\xi\}^{\perp}$. Since the complex structure J : Null $\eta \to$ Null η is defined by the commutative diagram (2.11), $J\gamma_*(\pi_* X) = \gamma_* J(\pi_* X)$ for $X \in \{\xi, J\xi\}^{\perp}$ by definition. Then $\pi_*\tilde{\gamma}_* J(X) = J\gamma_*\pi_*(X) = J\pi_*\tilde{\gamma}_*(X) = \pi_* J\tilde{\gamma}_*(X)$. As a consequence, $\tilde{\gamma}_* \circ J = J \circ \tilde{\gamma}_*$ on $\tilde{M}$. Hence, $\tilde{\gamma} \in \mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$. It is easy to check that $q(\gamma) = \tilde{\gamma}$ is a homomorphism of $\mathrm{PSH}(W, \eta, J)$ into $\mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$ such that $\nu \circ q = \mathrm{id}$. □

REMARK 2.1. From this lemma, there is an isomorphism $\mathcal{C}_{\mathcal{H}}(\boldsymbol{R}) \approx \boldsymbol{R} \times \mathrm{PSH}(W, \eta, J)$, where each element of $\mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$ is described as $\varphi_s \cdot q(\alpha)$ for $s \in \boldsymbol{R}$, $\alpha \in \mathrm{PSH}(W, \eta, J)$. It acts on $\tilde{M}$ as

$$\varphi_s \cdot q(\alpha)(\varphi_t \cdot w) = \varphi_{s+t} \cdot \alpha w\,,$$

for which there is an equivariant principal bundle:

$$\boldsymbol{R} \longrightarrow (\mathcal{C}_{\mathcal{H}}(\boldsymbol{R}), \tilde{M}) \xrightarrow{(\nu,\pi)} (\mathrm{PSH}(W, \eta, J), W)\,.$$

2.3. Central group extension. The material in this subsection and, in particular, Proposition 2.5, will be needed in Section 4.

Consider the exact sequence:

$$1 \longrightarrow \boldsymbol{R} \longrightarrow \mathcal{C}_{\mathcal{H}}(\boldsymbol{R}) \xrightarrow{\nu} \mathrm{PSH}(W, \eta, J) \longrightarrow 1\,. \tag{2.23}$$

Suppose that $\boldsymbol{R} \cap \pi_1(M)$ is nontrivial. Then it is an infinite cyclic subgroup $\boldsymbol{Z}$ such that the quotient group $\boldsymbol{R}/\boldsymbol{Z}$ is a circle S^1. Put $Q = \nu(\pi_1(M)) \subset \mathrm{PSH}(W, \eta, J)$. We have a central group extension:

$$1 \longrightarrow \boldsymbol{Z} \longrightarrow \pi_1(M) \xrightarrow{\nu} Q \longrightarrow 1\,. \tag{2.24}$$

The above principal bundle restricts to the following one:

$$(\boldsymbol{Z}, \boldsymbol{R}) \longrightarrow (\pi_1(M), \tilde{M}) \xrightarrow{(\nu,\pi)} (Q, W). \tag{2.25}$$

As both $\boldsymbol{R}$ and $\pi_1(M)$ act properly on $\tilde{M}$, Q acts also properly discontinuously (but not necessarily freely) on W such that the quotient Hausdorff space W/Q is compact. Since $\rho(\boldsymbol{Z}) \subset \rho(\boldsymbol{R}) = \boldsymbol{R}^+$ from § 2.1, $\rho(\boldsymbol{Z})$ is an infinite cyclic subgroup of $\boldsymbol{R}^+$. We need the following lemma. (Compare [10], [5].)

LEMMA 2.8. *Let* $1 \to \boldsymbol{Z} \to \pi_1(M) \xrightarrow{\nu} Q \to 1$ *be the central extension as given in* (2.24). *Then,* $\pi_1(M)$ *has a splitting subgroup* π' *of finite index*:

$$1 \longrightarrow \boldsymbol{Z} \longrightarrow \pi' \xrightarrow{\nu} Q' \longrightarrow 1\,.$$

In particular, there exists a subgroup H' of π' which maps isomorphically onto a subgroup Q' of finite index in Q.

PROOF. Consider the homomorphism $\rho' = \rho|_{\pi_1(M)} : \pi_1(M) \to \boldsymbol{R}^+$ from (2.1). Then, $\rho'(\pi_1(M))$ is a free abelian group of rank $k \geq 1$. If we note that $\rho'(\boldsymbol{Z})$ is an infinite cyclic subgroup of $\rho'(\pi_1(M))$, then we can choose a subgroup G of finite index in $\rho'(\pi_1(M))$ such that $\rho'(\boldsymbol{Z})$ is a direct summand in G; $G = \rho'(\boldsymbol{Z}) \times \boldsymbol{Z}^{k-1}$. Put $\pi' = \rho'^{-1}(G)$ and $H' = \rho'^{-1}(\boldsymbol{Z}^{k-1})$. Then, π' has finite index in $\pi_1(M)$. Obviously, ν maps H' isomorphically onto $\nu(H') = Q'$, which is of finite index in Q. □

PROPOSITION 2.5. *The subgroup Q' acts freely on W so that the orbit space W/Q' is a closed strictly pseudoconvex pseudo-Hermitian manifold induced from the pseudo-Hermitian structure (η, J) on W.*

PROOF. Let $f = \nu'^{-1} : Q' \to H'$ be the inverse isomorphism. For each $\alpha' \in Q'$ there exists a unique element $\lambda(\alpha') \in \boldsymbol{R}$ such that $f(\alpha') = \varphi_{\lambda(\alpha')} \cdot q(\alpha')$. As we know that Q acts properly discontinuously on W from the remark below (2.25), the stabilizer at each point is finite. Suppose that $\alpha' w = w$ for some point $w \in W$. As $\alpha' \in Q_w$, $(\alpha')^l = 1$ for some l. Since φ_t is a central element and q is a homomorphism, $1 = f((\alpha')^l) = \varphi_{l\lambda(\alpha')} \cdot q((\alpha')^l) = \varphi_{l\lambda(\alpha')}$. Thus, $\lambda(\alpha') = 0$, i.e., $f(\alpha') = q(\alpha')$. By the definition of the action $(\pi', \tilde{M})$, $f(\alpha')(\varphi_t w) = q(\alpha')(\varphi_t w) = \varphi_t \alpha' w = \varphi_t w$. As π' acts freely on $\tilde{M}$, $f(\alpha') = 1$ and so $\alpha' = 1$. If we note that $Q' \subset \mathrm{PSH}(W, \eta, J)$, then (η, J) induces a pseudo-Hermitian structure $(\hat{\eta}, J)$ on W/Q'. Here we use the same notation J for the complex structure on Null $\hat{\eta}$. □

3. Examples of l.c.K. manifolds with parallel Lee form. In this section we present an explicit construction for the Hopf manifolds.

Let $S^{2n-1} = \{(z_1, \dots, z_n) \in \boldsymbol{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 = 1\}$ be the sphere endowed with its standard contact structure

$$\eta_0 = \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)\,, \tag{3.1}$$

where $z_j = x_j + \sqrt{-1}\, y_j$. Let J_0 be the restriction of the standard complex structure of $\boldsymbol{C}^n$ to $\boldsymbol{C}^n - \{0\}$. It is known that the group of pseudo-Hermitian transformations, $\mathrm{PSH}(S^{2n-1}, \eta_0, J_0)$ is isomorphic with $\mathrm{U}(n)$ (see [21], for example). We define a 1-parameter subgroup $\{\psi_t\}_{t \in \boldsymbol{R}} \subset \mathrm{PSH}(S^{2n-1}, \eta_0, J_0)$ by the formula:

$$\psi_t(z_1, \dots, z_n) = (e^{\mathrm{i}ta_1} z_1, \dots, e^{\mathrm{i}ta_n} z_n)\,,$$

where $\mathrm{i} = \sqrt{-1}$ and $a_1, \dots, a_n \in \boldsymbol{R}$. The vector field induced by this action is

$$A = \sum_{j=1}^{n} a_j \left(x_j \frac{d}{dy_j} - y_j \frac{d}{dx_j} \right)$$

and satisfies $\eta_0(A) = a_1 |z_1|^2 + \cdots + a_n |z_n|^2$.

Now we require that $\eta_0(A) > 0$ everywhere on S^{2n-1}. Then the numbers a_k must satisfy (up to rearrangement):

$$0 < a_1 \leq \cdots \leq a_n \,. \tag{3.2}$$

Define a new contact form η_A on the sphere by

$$\eta_A = \frac{1}{\sum_{j=1}^{n} a_j |z_j|^2} \cdot \eta_0 \,.$$

The contact distributions of η_0 and η_A coincide, but the characteristic field of η_A is A: $\eta_A(A) = 1$, $\iota_A d\eta_A = 0$. As A generates the flow $\{\psi_t\}_{t\in \boldsymbol{R}} \subset \mathrm{PSH}(S^{2n-1}, \eta_0, J_0)$, note that $\psi_{t*} \circ J_0 = J_0 \circ \psi_{t*}$ on $\mathrm{Null}\, \eta_A$. Define a 2-form on the product $\boldsymbol{R} \times S^{2n-1}$ by:

$$\Omega_A = 2d(e^t \mathrm{pr}^* \eta_A) \,, \quad t \in \boldsymbol{R} \,.$$

Here $\mathrm{pr} : \boldsymbol{R} \times S^{2n-1} \to S^{2n-1}$ is the projection. If $\boldsymbol{R} = \{\varphi_s\}_{s\in\boldsymbol{R}}$ acts on $\boldsymbol{R} \times S^{2n-1}$ by left translations: $\varphi_s(t, z) = (s + t, z)$, then the group $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ acts by homothetic transformations with respect to Ω_A:

$$(\varphi_s \times \alpha)^* \Omega_A = e^s \cdot \Omega_A \,, \quad \alpha \in \mathrm{PSH}(S^{2n-1}, \eta_A, J_0) \,. \tag{3.3}$$

In general, $\mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ is the centralizer of $\{\psi_t\}_{t\in\boldsymbol{R}}$ in $\mathrm{U}(n)$. In view of the formula of ψ_t, $\mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ contains at least the maximal torus of $\mathrm{U}(n)$:

$$T^n \subset \mathrm{PSH}(S^{2n-1}, \eta_A, J_0) \,. \tag{3.4}$$

(For example, if all a_j are distinct, $\mathrm{PSH}(S^{2n-1}, \eta_A, J_0) = T^n$.)

Let $N = d/dt$ be the vector field induced on $\boldsymbol{R} \times S^{2n-1}$ by the $\boldsymbol{R}$-action. Taking into account that $T(\boldsymbol{R} \times S^{2n-1}) = N \oplus A \oplus \mathrm{Null}\, \eta_A$, we define an almost complex structure J_A on $\boldsymbol{R} \times S^{2n-1}$ by

$$J_A N = -A \,, \quad J_A A = N \,,$$
$$J_A | \mathrm{Null}\, \eta_A = J_0 \,,$$

and show its integrability. Indeed, let

$$T(\boldsymbol{R} \times S^{2n-1}) \otimes \boldsymbol{C} = \{T^{1,0} + (A - \mathrm{i}N)\} \oplus \{T^{0,1} + (A + \mathrm{i}N)\}$$

be the splitting corresponding to J_A (here $T^{1,0} + T^{0,1} = \mathrm{Null}\, \eta_A \otimes \boldsymbol{C}$). As $J_A|\mathrm{Null}\, \eta_A = J_0$, $[T^{1,0}, T^{0,1}] \subset T^{1,0}$. Recalling that A is the characteristic field of η_A, we see that $[X, A] \in \mathrm{Null}\, \eta_A$ for any $X \in \mathrm{Null}\, \eta_A$. If $X \in T^{1,0}$, then $[X, A - \mathrm{i}N] = [X, A] = \lim_{t\to 0}(X - \psi_{-t*}X)/t$. Noting that $\psi_t \in \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ (i.e., $\psi_{t*}J_0 = J_0\psi_{t*}$),

$$J_A[X, A - \mathrm{i}N] = J_0[X, A] = \lim_{t\to 0} \frac{J_0 X - \psi_{-t*} J_0 X}{t} = [J_0 X, A]$$
$$= [\mathrm{i}X, A] = \mathrm{i}[X, A] = \mathrm{i}[X, A - \mathrm{i}N] \,.$$

Thus $[X, A - \mathrm{i}N] \in \{T^{1,0} + (A - \mathrm{i}N)\}$. Hence J_A is integrable. By the definition of J_A, it is easy to check that the elements of $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ are holomorphic with respect to J_A. Moreover, Ω_A is J_A-invariant. Hence, Ω_A is a Kähler form on the complex manifold

$(\boldsymbol{R} \times S^{2n-1}, J_A)$ on which $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ acts as the group of holomorphic homothetic transformations. Define a Hermitian metric $\tilde{g}_A$ and its fundamental 2-form $\tilde{\omega}_A$ by setting

$$\begin{aligned} &\tilde{\omega}_A = 2e^{-t} \cdot \Omega_A\,. \\ &\tilde{g}_A(X, Y) = \tilde{\omega}_A(J_A X, Y)\,, \quad X, Y \in T(\boldsymbol{R} \times S^{2n-1})\,. \end{aligned} \tag{3.5}$$

(Compare (2.16).) By (3.3), $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ acts as holomorphic isometries of $(\tilde{g}_A, J_A)$. When we choose a properly discontinuous group $\Gamma \subset \boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ acting freely on $\boldsymbol{R} \times S^{2n-1}$, $\tilde{g}_A$ (resp. $\tilde{\omega}_A$) induces a Hermitian metric g_A (resp. the fundamental 2-form ω_A) on the quotient complex manifold $(\boldsymbol{R} \times S^{2n-1}/\Gamma, \hat{J}_A)$, where the complex structure $\hat{J}_A$ is induced from J_A. We have to check that g_A is a l.c.K. metric with parallel Lee form. Let $p : \boldsymbol{R} \times S^{2n-1} \to \boldsymbol{R} \times S^{2n-1}/\Gamma$ be the projection so that $p^*\omega_A = \tilde{\omega}_A$. Since $\tilde{\omega}_A = e^{-t} \cdot \Omega_A$, we have $d\tilde{\omega}_A = -dt \wedge \tilde{\omega}_A$. Thus $\tilde{g}_A$ is a l.c.K. metric with Lee form $d(-t)$ on $\boldsymbol{R} \times S^{2n-1}$. If we note that the group $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ leaves $d(-t)$ invariant, i.e., $(\varphi_s \times \alpha)^* d(-t) = d(-(s+t)) = d(-t)$, then $d(-t)$ induces a 1-form θ on $\boldsymbol{R} \times S^{2n-1}/\Gamma$ such that $p^*\theta = d(-t)$. The equation $d\tilde{\omega}_A = -dt \wedge \tilde{\omega}_A$ implies that $d\omega_A = \theta \wedge \omega_A$ on $\boldsymbol{R} \times S^{2n-1}/\Gamma$. As $d\theta = 0$, g_A is a l.c.K. metric with Lee form θ. For the rest, the same argument as in the proof of Lemma 2.5 can be applied to show that θ is the parallel Lee form of g_A. Finally, we examine the complex structure $\hat{J}_A$ on $\boldsymbol{R} \times S^{2n-1}/\Gamma$.

Let $H : \boldsymbol{R} \times S^{2n-1} \to \boldsymbol{C}^n - \{0\}$ be the diffeomorphism defined by

$$H(t, (z_1, \ldots, z_n)) = (e^{-a_1 t} z_1, \ldots, e^{-a_n t} z_n),$$

where $\{a_1, \ldots, a_n\}$ satisfies the condition (3.2). We shall show that H is (J_A, J_0)-biholomorphic. We have:

$$\begin{aligned} H_*(N_{(s,z)}) &= \left.\frac{dH(t+s, z)}{dt}\right|_{t=0} = (-a_1 \cdot e^{-a_1 s} \cdot z_1, \ldots, -a_n \cdot e^{-a_n s} \cdot z_n)\,; \\ H_*(J_A N_{(s,z)}) &= H_*(-A_{(s,z)}) = -H_*\left(\left(s, \left.\frac{d}{dt}(e^{\mathrm{i}ta_1} z_1, \ldots, e^{\mathrm{i}ta_n} z_n)\right|_{t=0}\right)\right) \\ &= -(\mathrm{i}a_1 e^{-a_1 s} z_1, \ldots, \mathrm{i}a_n e^{-a_n s} z_n) = J_0 H_*(N_{(s,z)})\,. \end{aligned}$$

From $H_*(A_{(s,z)}) = -J_0 H_*(N_{(s,z)})$, we derive $J_0 H_*(A_{(s,z)}) = H_*(N_{(s,z)}) = H_*(J_A A)$. Now let $X \in \mathrm{Null}\,\eta_A \subset TS^{2n-1}$ and let $\sigma(t)$ be an integral curve of X on S^{2n-1}: $\dot{\sigma}(t) = X$, $\dot{\sigma}(0) = X_z$. We can view X as a pair: $X_{(s,z)} = (s, \dot{\sigma}(0))$. Then

$$H_*(X_{(s,z)}) = \frac{d}{dt} H(s, \sigma(t))|_{t=0} = (e^{-a_1 s}\dot{\sigma}_1(0), \ldots, e^{-a_n s}\dot{\sigma}_n(0))\,.$$

From this we obtain

$$\begin{aligned} H_*(J_A X_{(s,z)}) &= H_*((s, J_0 \dot{\sigma}(0))) = H_*((s, (\mathrm{i}\dot{\sigma}_1(0), \ldots, \mathrm{i}\dot{\sigma}_n(0)))) \\ &= (\mathrm{i}e^{-a_1 s}\dot{\sigma}_1(0), \ldots, \mathrm{i}e^{-a_n s}\dot{\sigma}_n(0)) \\ &= J_0(e^{-a_1 s}\dot{\sigma}_1(0), \ldots, e^{-a_n s}\dot{\sigma}_n(0)) = J_0 H_*(X_{(s,z)})\,. \end{aligned}$$

Therefore $H : (\boldsymbol{R} \times S^{2n-1}, J_A) \to (\boldsymbol{C}^n - \{0\}, J_0)$ is biholomorphic.

Let $\mathrm{Hol}(\boldsymbol{C}^n - \{0\}, J_0)$ be the group of all biholomorphic transformations. We can obtain a faithful homomorphism $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0) \to \mathrm{Hol}(\boldsymbol{C}^n - \{0\}, J_0)$ by associating to each $\gamma \in \boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ the biholomorphic map $H \circ \gamma \circ H^{-1}$. Let Γ_H be the image of Γ in $\mathrm{Hol}(\boldsymbol{C}^n - \{0\}, J_0)$.

DEFINITION 3.1. The quotient complex manifold $(\boldsymbol{C}^n - \{0\})/\Gamma_H$ is called a Hopf manifold.

Since our map H induces a holomorphic diffeomorphism $\hat{H} : (\boldsymbol{R} \times S^{2n-1})/\Gamma \to (\boldsymbol{C}^n - \{0\})/\Gamma_H$, letting $\hat{H}^* g = g_A$ for the *l.c.K.* metric g_A on $(\boldsymbol{R} \times S^{2n-1})/\Gamma$, we have shown:

THEOREM 3.1. *The Hopf manifold $(\boldsymbol{C}^n - \{0\})/\Gamma_H$ admits a l.c.K. metric g with parallel Lee form θ.*

By (3.4), $T^n \subset \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$. Choose $s \in \boldsymbol{R} - \{0\}$ and n complex numbers $c_1, \dots, c_n \in S^1$. Let $(s, (c_1, \dots, c_n)) \in \boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_A, J_0)$ and consider an infinite cyclic subgroup $\boldsymbol{Z}$ generated by this element. Then the corresponding group $\boldsymbol{Z}_H$ is generated by the element $(e^{-a_1 s} \cdot c_1, \dots, e^{-a_n s} \cdot c_n)$ acting on $\boldsymbol{C}^n - \{0\}$. Let $\Lambda = (\lambda_1, \dots, \lambda_n)$, with $\lambda_j = e^{-a_j s} \cdot c_j$ and so $\boldsymbol{Z}_H = \langle(\lambda_1, \dots, \lambda_n)\rangle$. The condition (3.2) ensures that the complex numbers λ_j satisfy

$$0 < |\lambda_n| \le \dots \le |\lambda_1| < 1 .$$

Put $M_\Lambda = (\boldsymbol{C}^n - \{0\})/\boldsymbol{Z}_H$. We call M_Λ a *primary Hopf manifold of type Λ*. Indeed, for $n = 2$, one recovers the primary Hopf surfaces of Kähler rank 1. In particular, we derive Theorem B in the Introduction.

REMARK 3.1. Note that the manifolds M_Λ are all diffeomorphic with $S^1 \times S^{2n-1}$ and that for $c_1 = \dots = c_n = 1$ and $a_1 = \dots = a_n$, we obtain the standard Hopf manifold, the first known example of a l.c.K. manifold with parallel Lee form, cf. [18].

In [7] a l.c.K. metric with parallel Lee form is constructed on the primary Hopf surface $M_{\lambda_1,\lambda_2} = (\boldsymbol{C}^2 - \{0\})/\Gamma$, $\Gamma \cong \boldsymbol{Z}$ being generated by $(z_1, z_2) \mapsto (\lambda_1 z_1, \lambda_2 z_2)$, $|\lambda_1| \ge |\lambda_2| > 1$. There the diffeomorphism between M_{λ_1,λ_2} and $S^1 \times S^3$ is used to construct a potential for the Kähler metric h (in the notation of the present paper) on the universal cover. The same diffeomorphism is then used to transport the l.c.K. structure on $S^1 \times S^3$ and to show that the induced Sasakian structure on S^3 is a deformation of the standard Sasakian structure of the 3-sphere. See also [1] where a complete list of compact, complex surfaces admitting l.c.K. metrics with parallel Lee form is provided.

4. Lee-Cauchy-Riemann transformations. In this section, we study the group $\mathrm{Aut}_{\mathrm{LCR}}(M)$ described in the Introduction.

Let $\{\theta, \theta \circ J, \theta^\alpha, \bar{\theta}^\alpha\}_{\alpha=1,\dots,n-1}$ be a unitary, local coframe field adapted to a l.c.K. manifold (M, g, J) with parallel Lee form. Consider the subgroup G of $GL(2n, \boldsymbol{R})$ consisting of the following elements:

$$\left\{ \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & u & v^\alpha & \bar{v}^\alpha \\ 0 & 0 & \sqrt{u}\, U^\alpha_\beta & 0 \\ 0 & 0 & 0 & \sqrt{u}\, \bar{U}^\alpha_\beta \end{array} \right) \;\middle|\; u \in \boldsymbol{R}^+, v^\alpha \in \boldsymbol{C}, U^\alpha_\beta \in \mathrm{U}(n-1) \right\}.$$

Let $G \to P \to M$ be the principal bundle of the G-structure consisting of the above coframes $\{\theta, \theta \circ J, \theta^\alpha, \bar{\theta}^\alpha\}$. If we note that G is isomorphic to the semidirect product $\boldsymbol{C}^{n-1} \rtimes (\mathrm{U}(n-1) \times \boldsymbol{R}^+)$, then the Lie algebra $\mathfrak{g}$ is isomorphic to $\boldsymbol{C}^{n-1} \rtimes \mathfrak{u}(n-1) + \boldsymbol{R}$. Note that the subgroup $\boldsymbol{C}^{n-1}$ is of even rank, while $\mathfrak{u}(n-1) + \boldsymbol{R}$ is of order 2. In particular, the matrix group $\mathfrak{g} \subset \mathfrak{gl}(2n, \boldsymbol{R})$ has no element of rank 1, i.e., it is *elliptic* (cf. [11]). As M is assumed to be compact, it is known that the group of automorphisms $\mathcal{U}$ of P is a finite dimensional Lie group.

DEFINITION 4.1. The group of all diffeomorphisms of M onto itself which preserve the above G-structure is denoted by $\mathrm{Aut}_{\mathrm{LCR}}(M, g, J, \theta)$ (or simply by $\mathrm{Aut}_{\mathrm{LCR}}(M)$). We call $\mathrm{Aut}_{\mathrm{LCR}}(M)$ the group of Lee-Cauchy-Riemann transformations on a l.c.K. manifold (M, g, J) adapted to the parallel Lee form θ.

By definition, if $f \in \mathrm{Aut}_{\mathrm{LCR}}(M)$, then $f^* : P \to P$ is a bundle automorphism satisfying

$$\begin{aligned} & f^*\theta = \theta\,, \\ & f^*(\theta \circ J) = \lambda \cdot (\theta \circ J) \text{ for some positive, smooth function } \lambda\,, \\ & f^*\theta^\alpha = \sqrt{\lambda} \cdot \theta^\beta V^\alpha_\beta + (\theta \circ J) \cdot w^\alpha\,, \\ & f^*\bar{\theta}^\alpha = \sqrt{\lambda} \cdot \bar{\theta}^\beta \bar{V}^\alpha_\beta + (\theta \circ J) \cdot \bar{w}^\alpha \end{aligned} \tag{4.1}$$

for functions V^α_β, w^α with values in $\mathrm{U}(n-1)$, respectively in $\boldsymbol{C}$. Note that the group of holomorphic isometries $\mathrm{I}(M, g, J)$ is contained in $\mathrm{Aut}_{\mathrm{LCR}}(M)$. In fact, an element $f \in \mathrm{I}(M, g, J)$ satisfies $f^*\theta = \theta$, $f^*(\theta \circ J) = (\theta \circ J)$ and $f^*\omega = \omega$. Let $\{\theta^\sharp, J\theta^\sharp\}^\perp$ be the orthogonal complement of the complex plane field $\{\theta^\sharp, J\theta^\sharp\}$ with respect to g. It is obviously J-invariant. If we observe that $\omega|\{\theta^\sharp, J\theta^\sharp\}^\perp = -\mathrm{i} \sum_{\alpha,\beta} \delta_{\alpha\beta}\theta^\alpha \wedge \bar{\theta}^\beta$, then $f^*\theta^\alpha = \theta^\beta U^\alpha_\beta$, $f^*\bar{\theta}^\alpha = \bar{\theta}^\beta \bar{U}^\alpha_\beta$ for some $\mathrm{U}(n-1)$-valued function U^α_β.

LEMMA 4.1. *Any element $f \in \mathrm{Aut}_{\mathrm{LCR}}(M)$ preserves $\{\theta^\sharp, J\theta^\sharp\}^\perp$ and is holomorphic on it.*

PROOF. Let $X \in \{\theta^\sharp, J\theta^\sharp\}^\perp$. The equations $f^*\theta = \theta$, $f^*(\theta \circ J) = \lambda \cdot (\theta \circ J)$ show that

$$\begin{aligned} g(f_*X, \theta^\sharp) &= \theta(f_*X) = \theta(X) = g(X, \theta^\sharp) = 0\,, \\ g(f_*X, J\theta^\sharp) &= -g(Jf_*X, \theta^\sharp) = -\theta(Jf_*X) = -\theta \circ J(f_*X) \\ &= -\lambda \cdot \theta \circ J(X) = -g(X, (\theta \circ J)^\sharp) = g(X, J\theta^\sharp) = 0\,. \end{aligned} \tag{4.2}$$

Thus f_* applies $\{\theta^\sharp, J\theta^\sharp\}^\perp$ onto itself. Moreover, if $\theta^\sharp_\alpha$ is a dual frame field to θ^α (similarly for $\bar{\theta}^\alpha$), then the frame $\{\theta^\sharp_\alpha, \bar{\theta}^\sharp_\alpha\}_{\alpha=1,\ldots,n-1}$ spans $\{\theta^\sharp, J\theta^\sharp\}^\perp \otimes \boldsymbol{C}$. The equation $f^*\theta^\alpha =$

$\sqrt{\lambda}\cdot\theta^\beta V^\alpha_\beta + (\theta\circ J)\cdot w^\alpha$ implies that $f_*\theta^\sharp_\alpha = \sqrt{\lambda}\cdot\theta^\sharp_\beta V^\beta_\alpha$ (similary for $f_*\bar{\theta}^\sharp_\alpha$). Therefore $f_*\circ J = J\circ f_*$ on $\{\theta^\sharp, J\theta^\sharp\}^\perp$. □

When a noncompact LCR flow exists on a compact l.c.K. manifold M with parallel Lee form, we shall prove a rigidity similar to the one implied by a noncompact CR-flow on a compact CR-manifold (cf. [15], [9]).

Proof of Theorem C

4.1. Existence of spherical CR-structure on W/Q'. Let $1\to \boldsymbol{Z}\to\pi' \xrightarrow{\nu} Q'\to 1$ be the split central group extension from Lemma 2.8. Put $M' = \tilde{M}/\pi'$. Then it is easy to see that the Lee form θ, the LCR-action $\boldsymbol{C}^*$ lift to those of M', so we retain the same notation for M'. We put $\boldsymbol{C}^* = S^1\times\boldsymbol{R}^+$, where $\boldsymbol{R}^+ = \{\hat{\phi}_t\}_{t\in\boldsymbol{R}}$ is a LCR flow on M'. By hypothesis, $S^1 = \{\hat{\varphi}_t\}_{t\in\boldsymbol{R}}$ induces the Lee field $\theta^\sharp$. From 1 of Proposition 2.4, S^1 lifts to a nontrivial holomorphic homothetic flow $\boldsymbol{R} = \{\varphi_t\}_{t\in\boldsymbol{R}}$ on $\tilde{M}$ with respect to Ω. We obtain a LCR-action of $\boldsymbol{R}\times\boldsymbol{R}^+$ on $\tilde{M}$ for which $\boldsymbol{R}$ acts properly as before. Consider the commutative diagram of principal bundles:

$$\begin{array}{ccccc}
\boldsymbol{Z} & \longrightarrow & \pi' & \xrightarrow{\nu} & Q' \\
\downarrow & & \downarrow & & \downarrow \\
\boldsymbol{R} & \longrightarrow & (\boldsymbol{R}\times\boldsymbol{R}^+, \tilde{M}) & \xrightarrow{(\tilde{\nu},\pi)} & (\boldsymbol{R}^+, W) \\
\downarrow & & \downarrow p & & \downarrow p \\
S^1 & \longrightarrow & (S^1\times\boldsymbol{R}^+, M') & \xrightarrow{(\hat{\nu},\hat{\pi})} & (\boldsymbol{R}^+, W/Q') .
\end{array} \tag{4.3}$$

From the bottom line, the projection $\hat{\nu}$ maps the group $\boldsymbol{R}^+ = \{\hat{\phi}_t\}_{t\in\boldsymbol{R}}$ onto a group $\boldsymbol{R}^+ = \{\bar{\phi}_t\}_{t\in\boldsymbol{R}}$ acting on W/Q'.

LEMMA 4.2. *The group $\boldsymbol{R}^+ = \{\bar{\phi}_t\}_{t\in\boldsymbol{R}}$ acts by CR-transformations on W/Q' with respect to the CR-structure induced from the strictly pseudoconvex, pseudo-Hermitian structure $(\hat{\eta}, J)$.*

PROOF. As ξ generates the flow $\boldsymbol{R} = \{\varphi_t\}_{t\in\boldsymbol{R}}$, $p_*\xi = \theta^\sharp$ on M' by hypothesis and so $p:\tilde{M}\to M'$ maps the complex plane field $\{\xi, J\xi\}$ onto $\{\theta^\sharp, J\theta^\sharp\}$. By Lemma 4.1, each $\hat{\phi}_t\in \mathrm{Aut}_{\mathrm{LCR}}(M')$ preserves $\{\theta^\sharp, (\theta\circ J)^\sharp\}^\perp$. So its lift ϕ_t preserves the J-invariant distribution $\{\xi, J\xi\}^\perp$. Since $\pi_*: (\{\xi, J\xi\}^\perp, J)\to(\mathrm{Null}\ \eta, J)$ is J-isomorphic and each ϕ_t is holomorphic on $\{\xi, J\xi\}^\perp$, $\hat{\pi}_*: (\{\theta^\sharp, (\theta\circ J)^\sharp\}^\perp, J)\to(\mathrm{Null}\ \hat{\eta}, J)$ is also J-isomorphic through the commutative diagram and thus each $\bar{\phi}_t$ is holomorphic on Null $\hat{\eta}$; $(\bar{\phi}_{t*}\circ J = J\circ\bar{\phi}_{t*})$. Therefore, $\boldsymbol{R}^+ = \{\bar{\phi}_t\}_{t\in\boldsymbol{R}}$ is a closed, noncompact subgroup of CR-transformations of W/Q' with respect to $(\mathrm{Null}\ \hat{\eta}, J)$. □

By this lemma, we obtain a compact strictly pseudoconvex CR-manifold W/Q' admitting a closed, noncompact CR-transformations $\boldsymbol{R}^+$. Then we apply the result of [9] to show that W/Q' is CR-equivalent to the sphere S^{2n-1} with the standard CR-structure. In particular

$Q' = \{1\}$ and thus Q is a finite subgroup of $\mathrm{PSH}(W, \eta, J)$ from Lemma 2.8. By the definition of spherical CR-structure (cf. [13], [8]), there exists a developing pair:

$$(\mu,\ \mathrm{dev}) : (\mathrm{Aut}_{\mathrm{CR}}(W), W) \to (\mathrm{PU}(n, 1), S^{2n-1})$$

for which dev is a CR-diffeomorphism and $\mu : \mathrm{Aut}_{\mathrm{CR}}(W) \to \mathrm{PU}(n, 1)$ is the holonomy isomorphism. Here $\mathrm{PU}(n, 1) = \mathrm{Aut}_{\mathrm{CR}}(S^{2n-1})$ and $\mathrm{Aut}_{\mathrm{CR}}(W)$ is the group of all CR-automorphisms of W containing the groups $\boldsymbol{R}^+$ and $\mathrm{PSH}(W, \eta, J) \supset Q$.

As S^1 ($\subset \boldsymbol{C}^*$) acts on M without fixed points (but not necessarily freely, i.e., with possible subset of exceptional orbits $S^1 \cdot x$ for which the stabilizer S^1_x is a non-trivial cyclic subgroup of S^1; cf. [3]), the quotient space $M/S^1 = W/Q (\approx S^{2n-1}/\mu(Q))$ is an orbifold, so such a finite subgroup Q may exist.

On the other hand, we recall some facts from the theory of hyperbolic groups (cf. [4]). The noncompact closed $\mu(\boldsymbol{R}^+)$-action on S^{2n-1} is characterized as whether it is either loxodromic ($= \boldsymbol{R}^+$) or parabolic ($= \mathcal{R}$) for which $\boldsymbol{R}^+$ has exactly two fixed points $\{0, \infty\}$ or $\mathcal{R}$ has the unique fixed point $\{\infty\}$ on S^{2n-1}. Moreover, the centralizer $\mathcal{C}_{\mathrm{PU}(n,1)}(\mu(\boldsymbol{R}^+))$ of $\mu(\boldsymbol{R}^+)$ in $\mathrm{PU}(n, 1)$ is one of the following groups up to conjugacy:

$$\mathcal{R} \times \mathrm{U}(n-1) \quad \text{or} \quad \boldsymbol{R}^+ \times \mathrm{U}(n-1)\,. \tag{4.4}$$

Since $\pi_1(M)$ centralizes $\boldsymbol{R} \times \boldsymbol{R}^+$, note that Q centralizes $\boldsymbol{R}^+$ (cf. (2.24)). The holonomy group $\mu(Q)$ belongs to $\mathcal{C}_{\mathrm{PU}(n,1)}(\mu(\boldsymbol{R}^+))$. As $\mu(Q)$ is a finite subgroup, (4.4) implies that

$$\mu(Q) \subset \mathrm{U}(n-1)\,. \tag{4.5}$$

4.2. Rigidity of (M, g, J) under the LCR action of $\boldsymbol{R}^+$. Let (η_0, J_0) be the standard strictly pseudoconvex pseudo-Hermitian structure on S^{2n-1} (cf. (3.1)). By definition, there exists a positive function u on W such that

$$\mathrm{dev}^* \eta_0 = u \cdot \eta\,. \tag{4.6}$$

By Lemma 2.4, we know that A is the characteristic CR-vector field on W for (η, J). If $\{\psi'_t\}$ is the flow generated by A, then note from (2.1.3) that $\{\psi'_t\} \subset \mathrm{PSH}(W, \eta, J)$. Because W is compact, $\mathrm{PSH}(W, \eta, J)$ is compact. As $\mathrm{PSH}(W, \eta, J) \subset \mathrm{Aut}_{\mathrm{CR}}(W)$, the closure of the holonomy image $\mu(\{\psi'_t\})$ (which is a connected abelian group) lies in the maximal torus T^n of the maximal compact subgroup $\mathrm{U}(n)$ in $\mathrm{PU}(n, 1)$ up to conjugacy. We can describe it as

$$\mu(\psi'_t) = (e^{ia_1 \cdot t}, \ldots, e^{ia_n \cdot t})\,, \quad t \in \boldsymbol{R}$$

for some $a_i \in \boldsymbol{R}$ ($i = 1, \ldots, n$). On the other hand, let $\mathcal{A} = \mathrm{dev}_*(A)$. Since dev is equivariant, $\mathrm{dev}(\psi'_t w) = \mu(\psi'_t)\mathrm{dev}(w)$ on $S^{2n-1} = \{z = (z_1, z_2, \ldots, z_n) \in \boldsymbol{C}^n \mid |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 = 1\}$, we have

$$\mathcal{A}_z = \frac{d\mu(\psi'_t)}{dt} = \sum_{j=1}^{n} a_j \left(x_j \frac{d}{dy_j} - y_j \frac{d}{dx_j} \right), \quad z = \mathrm{dev}(w), \ \ z_j = x_j + \mathrm{i} y_j\,. \tag{4.7}$$

As $\eta(A) = 1$, we have

$$u(w) = \mathrm{dev}^* \eta_0(A) = \eta_0(\mathcal{A}_z) = \sum_{j=1}^{n} a_j \cdot |z_j|^2 . \tag{4.8}$$

Since $u > 0$ from (4.6), we can assume that, up to rearranging the order of indices

$$0 < a_1 \leq \cdots \leq a_n . \tag{4.9}$$

As dev^{-1} maps the pseudo-Hermitain structure (η, J) on W to $(\mathrm{dev}^{-1*}\,\eta, J_0)$ on S^{2n-1}, we put

$$\eta_{\mathcal{A}} = \mathrm{dev}^{-1*}\eta . \tag{4.10}$$

Using (4.8), we obtain

$$\eta_{\mathcal{A}} = \frac{1}{\sum_{j=1}^{n} a_j \cdot |z_j|^2} \cdot \eta_0 \quad \text{on} \ \ S^{2n-1} . \tag{4.11}$$

When we note that $\eta_0 = u' \cdot \eta_{\mathcal{A}}$ where $u' = u \circ \mathrm{dev}^{-1}$, and $T(\boldsymbol{R} \times S^{2n-1}) = \{d/dt, \mathcal{A}\} \oplus$ Null η_0, denote the complex structure $J_{\mathcal{A}}$ on $\boldsymbol{R} \times S^{2n-1}$ by

$$\begin{aligned} &J_{\mathcal{A}}\frac{d}{dt} = -\mathcal{A}, \quad J_{\mathcal{A}}\mathcal{A} = \frac{d}{dt} , \\ &J_{\mathcal{A}}|\mathrm{Null}\ \eta_0 = J_0 . \end{aligned} \tag{4.12}$$

(Compare §3.) Let $\Pr : \boldsymbol{R} \times S^{2n-1} \to S^{2n-1}$ be the canonical projection. In view of (3.5), setting

$$\begin{aligned} &\Omega_{\mathcal{A}} = d(e^t \cdot \Pr^* \eta_{\mathcal{A}}) , \quad \tilde{\omega}_{\mathcal{A}} = 2e^{-t} \cdot \Omega_{\mathcal{A}} , \\ &\tilde{g}_{\mathcal{A}}(X, Y) = \tilde{\omega}_{\mathcal{A}}(J_{\mathcal{A}}X, Y) , \end{aligned} \tag{4.13}$$

we obtain a l.c.K. structure $(\Omega_{\mathcal{A}}, J_{\mathcal{A}})$ on the product $\boldsymbol{R} \times S^{2n-1}$ endowed with the group $\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0)$ of holomorphic homothetic transformations.

PROPOSITION 4.1. *There exists an equivariant holomorphic isometry between the l.c.K. manifolds* $(\mathcal{C}_{\mathcal{H}}(\boldsymbol{R}), \tilde{M}, \Omega, J)$ *and* $(\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0), \boldsymbol{R} \times S^{2n-1}, \Omega_{\mathcal{A}}, J_{\mathcal{A}})$.

PROOF. Let $G : \tilde{M} \to \boldsymbol{R} \times S^{2n-1}$ be a diffeomorphism defined by $G(\varphi_t w) = (t, \mathrm{dev}(w))$. Note that $\Pr \circ G = \mathrm{dev} \circ \pi$ on $\tilde{M}$. As every element of $\mathcal{C}_{\mathcal{H}}(\boldsymbol{R})$ is described as $\varphi_s \cdot q(\alpha)$ from Remark 2.1, define a homomorphism $\Psi : \mathcal{C}_{\mathcal{H}}(\boldsymbol{R}) \to \boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0)$ by setting

$$\Psi(\varphi_s \cdot q(\alpha)) = (s, \mu(\alpha)) .$$

Recall that the action $q(\alpha)(\varphi_t w) = \varphi_t \alpha w$ from (2.21). Then,

$$\begin{aligned} G(\varphi_s \cdot q(\alpha)(\varphi_t w)) &= G(\varphi_{s+t} \cdot \alpha w) = (s + t, \mathrm{dev}(\alpha w)) = (s + t, \mu(\alpha)\,\mathrm{dev}(w)) \\ &= (s, \mu(\alpha))(t, \mathrm{dev}(w)) = \Psi(\varphi_s \cdot q(\alpha))G(\varphi_t w) . \end{aligned}$$

Hence, $(\Psi, G) : (\mathcal{C}_{\mathcal{H}}(\boldsymbol{R}), \tilde{M}) \to (\boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0), \boldsymbol{R} \times S^{2n-1})$ is equivariantly diffeomorphic. Next, since $G^*t = t$ for the t-coordinate of $\boldsymbol{R} \times S^{2n-1}$ and $\mathrm{dev}^*\,\eta_{\mathcal{A}} = \eta$ from

(4.10), it follows that

$$G^*\Omega_{\mathcal{A}} = G^*d(e^t \cdot \mathrm{Pr}^*\eta_{\mathcal{A}}) = d(e^{G^*t} \cdot G^*\mathrm{Pr}^*\eta_{\mathcal{A}}) = d(e^t \cdot \pi^*\eta) = \Omega\,. \tag{4.14}$$

By definition, $G_*\xi = d/dt$. Moreover, when $x = \varphi_s w$,

$$G(\psi_t(x)) = G(\varphi_s\psi_t w) = G(\varphi_s i\psi'_t w) = (s, \mathrm{dev}(\psi'_t w)) = (s, \mu(\psi'_t)\,\mathrm{dev}(w))\,.$$

By (2.7) and (4.7),

$$G_*(-J\xi_x) = \frac{dG\psi_t}{dt}(x)\bigg|_{t=0} = \mathcal{A}_{Gx} = -J_{\mathcal{A}}\left(\frac{d}{dt}\right)_{Gx}\,.$$

Thus $G_*(J\xi) = J_{\mathcal{A}}G_*\xi$. As $G^*\Omega_{\mathcal{A}} = \Omega$ from (4.14), G maps $\{\xi, J\xi\}^\perp$ onto $\{d/dt, \mathcal{A}\}^\perp$. Consider the commutative diagram:

$$\begin{array}{ccc} (\{\xi, J\xi\}^\perp, J) & \xrightarrow{\pi_*} & (\mathrm{Null}\,\eta, J) \\ \big\downarrow G_* & & \big\downarrow \mathrm{dev}_* \\ (\{d/dt, \mathcal{A}\}^\perp, J_{\mathcal{A}}) & \xrightarrow{\mathrm{Pr}_*} & (\mathrm{Null}\,\eta_0, J_0)\,. \end{array} \tag{4.15}$$

Here note that $J_{\mathcal{A}} = J_0$ on $\mathrm{Null}\,\eta_{\mathcal{A}} = \mathrm{Null}\,\eta_0$. For $X \in \{\xi, J\xi\}^\perp$,

$$\mathrm{Pr}_*G_*J(X) = \mathrm{dev}_*(J\pi_*X) = J_0\mathrm{dev}_*\pi_*(X) = J_{\mathcal{A}}\mathrm{Pr}_*G_*(X) = \mathrm{Pr}_*J_{\mathcal{A}}G_*(X)\,,$$

thus, $G_*J(X) = J_{\mathcal{A}}G_*(X)$. Hence, G is $(J, J_{\mathcal{A}})$-biholomorphic. Moreover, as $G^*\tilde{\omega}_{\mathcal{A}} = G^*(2e^{-t}\Omega_{\mathcal{A}}) = 2e^{-t}\Omega = \bar{\Theta}$ and $\bar{g}(X,Y) = \bar{\Theta}(JX, Y)$, we obtain that $G^*\tilde{g}_{\mathcal{A}} = \bar{g}$. Therefore, (Ψ, G) induces a holomorphic isometry from $(M, \hat{g}, J)$ onto $(\boldsymbol{R} \times S^{2n-1}/\Psi(\pi_1(M)), \hat{g}_{\mathcal{A}}, \hat{J}_{\mathcal{A}})$. □

4.3. The Hopf manifold $\boldsymbol{R} \times S^{2n-1}/\Psi(\pi_1(M))$. We prove that $\boldsymbol{R}\times S^{2n-1}/\Psi(\pi_1(M))$ is a primary Hopf manifold M_Λ for some Λ obtained in §3. Each element of $\pi_1(M)$ is of the form $\gamma = \varphi_s \cdot q(\alpha)$ for some $s \in \boldsymbol{R}$, where $\nu(\gamma) = \alpha \in Q = \nu(\pi_1(M))$. By the definition of Ψ, $\Psi(\gamma) = (s, \mu(\alpha))$. We show that $\Psi(\pi_1(M))$ has no torsion element. For this, if $\Psi(\gamma)$ is of finite order (say, l), then $1 = (0, 1) = \Psi(\gamma^l) = (ls, \mu(\alpha^l))$. Then, $s = 0$ so that $\Psi(\gamma) = (0, \mu(\alpha))$. On the other hand, recall from (4.5) that $\mu(Q) \subset \mathrm{U}(n-1)$ up to conjugacy, and so $\mu(Q)$ has a fixed point $w_0 \in S^{2n-1}$. Since $\Psi(\pi_1(M))$ acts freely on $\boldsymbol{R} \times S^{2n-1}$, while $\Psi(\gamma)(t, w_0) = (t, \mu(\alpha)w_0) = (t, w_0)$, it follows that $\Psi(\gamma) = 1$. Moreover, if $\gamma_1 = \varphi_{s_1} \cdot q(\alpha_1)$, $\gamma_2 = \varphi_{s_2} \cdot q(\alpha_2)$, then $\Psi([\gamma_1, \gamma_2]) = (0, \mu([\alpha_1, \alpha_2])$. For the same reason, $\Psi([\pi_1(M), \pi_1(M)]) = \{1\}$. Hence, $\pi_1(M)$ is a finitely generated torsionfree abelian group. If we recall from (2.24) that $1 \to \boldsymbol{Z} \to \pi_1(M) \xrightarrow{\nu} Q \to 1$ is the central group extension where Q is finite, then $\pi_1(M)$ itself is an infinite cyclic group. Since $\Psi(\pi_1(M)) \subset \boldsymbol{R} \times \mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0)$ and the projection maps $\Psi(\pi_1(M))$ onto $\mu(Q)$ in $\mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0)$, $\mu(Q)$ is a finite cyclic group. As $\mathrm{PSH}(S^{2n-1}, \eta_{\mathcal{A}}, J_0)$ has the maximal torus T^n (cf. (3.4)), we obtain that $\Psi(\pi_1(M)) \subset \boldsymbol{R} \times T^n$ up to conjugacy. A generator of $\Psi(\pi_1(M))$ is described as $(s, (c_1, \ldots, c_n)) \in \boldsymbol{R} \times T^n$. Noting (4.9), let $\lambda_j = e^{-a_j s}c_j$ and

$\Lambda = (\lambda_1, \ldots, \lambda_n)$. By Theorem 3.1 and the remark below it, $\boldsymbol{R} \times S^{2n-1}/\Psi(\pi_1(M))$ is a primary Hopf manifold M_Λ of type Λ. This finishes the proof of Theorem C in the Introduction.

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