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GEOMETRIC FOURIER ANALYSIS

by Antonio CORDOBA

In this paper we present several results related to maximal and square functions whose proofs have a similar flavour: after some algebraic manipulation and the use of the uncertainty principle they are reduced to certain properties of the geometry of "rectangles" in $\mathbf{R}^n$.

A. In $\mathbf{R}^2$ let us consider the angles

$$\omega_j = \frac{2\pi j}{N}, \; j = 0, 1, \ldots, N-1, \; N \in Z^+$$

and let us denote by H_j the Hilbert transform in the direction ω_j and by S_j the projection, at the Fourier transform side, over the angles

$$\Delta_j = \{\xi, \, 2\pi j/N \leqslant \arg(\xi) \leqslant 2\pi(j+1)/N\}$$

$$\text{i.e. } \widehat{S_j f}(\xi) = \chi_{\Delta_j}(\xi)\, \hat{f}(\xi).$$

THEOREM 1. — *There exist constants independent of* N, $0 < a$, $c < \infty$, *so that*

i) $$\left\| \left[\sum_{j=1}^{N} |H_j f_j|^2 \right]^{1/2} \right\|_4 \leqslant C(\log N)^a \left\| \left(\sum |f_j|^2 \right)^{1/2} \right\|_4$$

ii) $$\left\| \left(\sum_{j=1}^{N} |S_j f|^2 \right)^{1/2} \right\|_4 \leqslant C[\log N]^a \; \|f\|_4 .$$

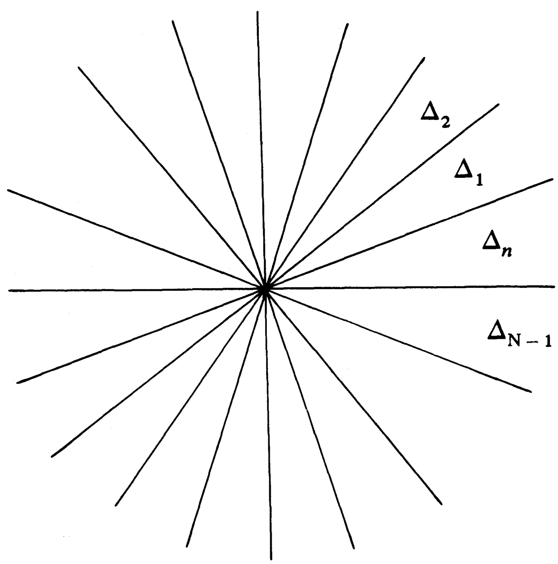
B. Let $\gamma : [0,1] \longrightarrow S^{n-1}$ be a smooth curve crossing a finite number of times each hyperplane of $\mathbf{R}^n$. Given a real number $N \gg 1$ let us consider the family $\mathcal{B}_N$ of cylinders of $\mathbf{R}^n$ having eccentricity = height/radius = N and direction in the curve γ. With a locally integrable function f we may consider its maximal function Mf given by the formula

$$Mf(x) = \sup_{x \in R \in \mathcal{B}_N} \frac{1}{\mu\{R\}} \int_R |f(y)|\, d\mu(y)$$

where μ denotes Lebesgue's measure in $\mathbf{R}^n$.

THEOREM 2. – *There exists a constant* C_γ, *independent of* N, *such that* $\|Mf\|_2 \leqslant C_\gamma [\log N]^2 \ \|f\|_2$.

A. The square function.



$$Sf(x) = \left(\sum |S_j f(x)|^2 \right)^{1/2},$$

$$\widehat{S_j f}(\xi) = \chi_{\Delta_j}(\xi)\ \hat{f}(\xi),$$

$$\Delta_j = \{\xi : 2\pi j/N \leqslant \arg(\xi) \leqslant 2\pi(j+1)/N\}.$$

Part (i) of theorem 2 was proved in ref. [4] and, therefore, we shall concentrate in part (ii). Although we have not made a careful analysis of the nature of the best constant a, it has to be strictly positive, as an adequate Kakeya's set argument can show. On the other hand, interpolating with the L^2-result, one may obtain $\|Sf\|_p \leqslant C[\log N]^{a(p)} \|f\|_p$, $2 \leqslant p \leqslant 4$, which it is the best range of p's where such an inequality can hold. We shall proceed proving a previous lemma.

In $\mathbf{R}^n$ let us consider a cubic lattice $\mathcal{L} = \{Q_\nu\}_{\nu \in Z^n}$ i.e. the Q_ν's are congruent cubes with disjoint interiors and such that $\mathbf{R}^n = \cup Q_\nu$. Define, for each ν, the operators $\widehat{P_\nu f} = \chi_{Q_\nu} \cdot \hat{f}$ and the square function $Gf(x) = (\Sigma\, |P_\nu f(x)|^2)^{1/2}$.

Lemma. – *For each $s > 1$ there exists a finite constant C_s so that for every f, $\omega \in C_0(\mathbf{R}^n)$ we have:*

$$\int_{\mathbf{R}^n} |Gf(x)|^2\, \omega(x)\, dx \leqslant C_s \int_{\mathbf{R}^n} |f(x)|^2\, A_s\omega(x)\, dx ,$$

where $A_s\omega = [(\omega^s)^*(x)]^{1/s}$ and $*$-denotes the Hardy-Littlewood maximal function.

Proof. – Without lack of generality we may assume that $\mathcal{L}$ is the unit lattice i.e. Q_ν is centered at the point $\nu \in Z^n$ and has volume equal to one. Let ψ be a smooth function with compact support and equal to 1 in Q_0. For each $y \in Q_0$ let us consider the Fourier multiplier $m_y(z) = \sum_\nu e^{2\pi i \nu . y}\, \psi(z - \nu)$, $z \in \mathbf{R}^n$. Then the kernel $\mu_y = \hat{m}_y$ is a measure of finite total variation uniformly in $y \in Q_0$. More concretely: $\mu_y = \sum_\nu \hat{\psi}(y + \nu)\, \delta_{y+\nu}$ where, as usual, δ_x denotes Dirac's function translated to the point x. Therefore, $|\mu_y * f(x)|^2 \leqslant C \sum_\nu (1 + |\nu|)^{2n}\ |\hat{\psi}(y + \nu)|^2\ |f(x - y - \nu)|^2$ and, since $\hat{\psi}$ is rapidly decreasing, we have:

$$\int_{\mathbf{R}^n} |\mu_y * f(x)|^2\, \omega(x)\, dx \leqslant C_N \sum_\nu (1 + |\nu|)^{2n-N} \int_{\mathbf{R}^n} |f(x)|^2\, \omega(x + y + \nu)\, dx$$

(we may assume that $\omega \geqslant 0$).

Thus,

$$\int_{Q_0}\int_{\mathbf{R}^n} |\mu_y * f(x)|^2 \omega(x)\,dx$$
$$\leqslant C_N \sum_{\nu} (1+|\nu|)^{2n-N} \int_{\mathbf{R}^n} |f(x)|^2 \left| \int_{Q_0} \omega(x+y+\nu)\,dy \right| .\,dx$$
$$\leqslant C \int_{\mathbf{R}^n} |f(x)|^2 \omega^*(x)\,dx, \quad \text{taking } N \geqslant 4n+1 .$$

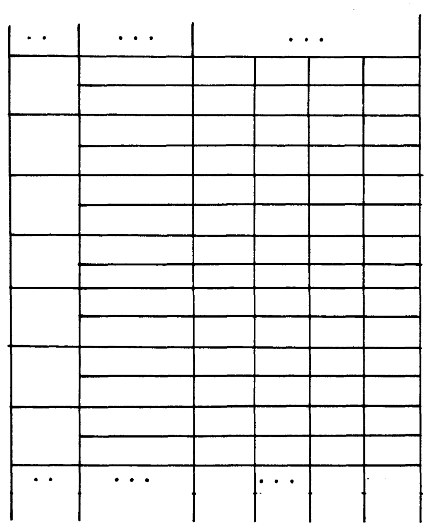
On the other hand, if $\widehat{T_\nu f}(\xi) = \psi(\xi - \nu)\,\hat{f}(\xi)$, we have:

$$\mu_y * f(x) = \sum_{\nu} e^{2\pi i \nu . y}\, T_\nu f(x)$$

and therefore,

$$\int_{Q_0}\int_{\mathbf{R}^n} |\mu_y * f(x)|^2 \omega(x)\,dxdy = \sum_{\nu} \int_{\mathbf{R}^n} |T_\nu f(x)|^2 \omega(x)\,dx.$$

To finish we observe that $P_\nu f = P_\nu T_\nu f$ and we may apply the weighted inequality of reference [5].



Let us consider for each $k \in Z$ a decomposition of the strip $2^k \leqslant x_n \leqslant 2^{k+1}$ into congruent disjoint parallelepipeds $\{Q_\nu^k\}$ whose sides are paralled to the coordinate axis. Define:

$$Sf(x) = \Big(\sum_{k,\nu} |P_\nu^k f(x)|^2 \Big)^{1/2}$$

where $\widehat{P_\nu^k f}(\xi) = \chi_{Q_\nu^k}(\xi)\,\hat{f}(\xi)$. Combining the Littlewood-Paley theorem with the previous result we obtain:

COROLLARY 1. – *For each* p, $2 \leqslant p < \infty$,. *there exists a finite constant* C_p *so that* $\| Sf \|_p \leqslant C_p \|f\|_p$, *for every* $f \in C_0(\mathbf{R}^n)$.

Proof of theorem 1. – We may assume, without lack of generality, that $0 \leqslant j \leqslant \frac{N}{8}$ so that $0 \leqslant \frac{2\pi j}{N} \leqslant \frac{\pi}{4}$.

We define

$$\Delta_j = \left\{ \xi = \xi_1 + i\xi_2 \,,\; 1 \leqslant \xi_1 \leqslant 2 \,,\; \frac{2\pi j}{N} \leqslant \arg(\xi) \leqslant \frac{2\pi(j+1)}{N} \right\}$$

$$j = 0, 1, \ldots, \frac{N}{8},$$

$$\widehat{P_j f} = \chi_{\Delta_j} \cdot \hat{f}$$

and we want to compute:

$$\sum_{j,k} \int_{\mathbf{R}^2} |P_j f(x)\, P_k f(x)|^2 \, dx = \sum_{|j-k|<N^{1/2}} \int_{\mathbf{R}^2} |P_j f(x)\, P_k f(x)|^2 \, dx$$

$$+ \sum_{|j-k| \geqslant N^{1/2}} \int_{\mathbf{R}^2} |P_j f(x)\, P_k f(x)|^2 \, dx = \mathrm{I} + \mathrm{II}.$$

We decompose further each sector Δ_j into $N^{1/2}$ subsectors $\Delta_j^1, \ldots, \Delta_j^{N^{1/2}}$, where

$$\Delta_j^\alpha = \{ \xi = \xi_1 + i\xi_2 \in \Delta_j \mid \alpha N^{-1/2} \leqslant \xi_1 - 1 \leqslant (\alpha + 1) N^{-1/2} \}.$$

It happens that if $|j - k| \geqslant N^{1/2}$ the overlapping of the sets $\Delta_j^\alpha + \Delta_k^\beta$, $\alpha, \beta = 1, \ldots, N^{1/2}$, is finite (uniformly on N).

Therefore,

$$\mathrm{II} \leqslant \sum_{|j-k| \geqslant N^{1/2}} \sum_{\alpha,\beta} \int_{\mathbf{R}^2} |P_j^\alpha f(x)\, P_k^\beta f(x)|^2 \, dx \leqslant \left\| \left(\sum_{j,\alpha} |P_j^\alpha f|^2 \right)^{1/2} \right\|_4^4$$

where the operators P_j^α have the obvious definition $\widehat{P_j^\alpha f} = \chi_{\Delta_j^\alpha} \cdot \hat{f}$.
We claim that

$$\left\| \left(\sum_{j,\alpha} |P_j^\alpha f|^2 \right)^{1/2} \right\|_4 \leqslant C[\log N]^{1/4} \; \|f\|_4$$

for some universal constant C.

To see this we take $\omega \geqslant 0$ in $L^2(\mathbf{R}^n)$ and we consider:

$$\sum_{j,\alpha} \int_{\mathbf{R}^2} |P_j^\alpha f(x)|^2 \omega(x)\, dx = \sum_{l=1}^{\frac{1}{8}N^{1/2}} \sum_{\alpha=1}^{N^{1/2}} \sum_{j=lN^{1/2}}^{(l+1)N^{1/2}} \int_{\mathbf{R}^2} |P_j^\alpha f(x)|^2 \omega(x)\, dx$$

$$\leqslant C_s \sum_{l=1}^{\frac{1}{8}N^{1/2}} \sum_{\alpha=1}^{N^{1/2}} \int_{\mathbf{R}^2} |Q_l^\alpha f(x)|^2 M_s \omega(x)\, dx$$

where $Q_l^\alpha f$ is given, at the Fourier transform side, as multiplication by the characteristic function of a rectangle, with sides parallel to the coordinates axis, and dimensions $N^{-1/2} \times 2N^{-1/2}$, $M_s\omega = (M\omega^s)^{1/s}$, $1 < s < \infty$, and M denotes the maximal function associated to the base of rectangles with directions in the set $2\pi j/N$, $j = 0, 1, \ldots, N/8$ (see ref. [5]).

In establishing the last estimate we have made a repeat use of the lemma. Using Holder's inequality together with the known estimates for M, we get: $II \leqslant C[\log N]\ \|f\|_4^4$.

We estimate I in the following manner:

$$I = \sum_{\nu=0}^{\frac{1}{2}\log N} \sum_{2^{-\nu}N^{1/2} \leqslant |j-k| < 2^{-\nu+1}N^{1/2}} \int_{\mathbf{R}^2} |P_j f(x)\, P_k f(x)|^2\, dx + \sum_j \|P_j f\|_4^4 .$$

Since we always have $\sum_{j=1}^{N} \|P_j f\|_4^4 \leqslant C\, \|f\|_4^4$ and we want an estimate with a factor of $(\log N)^a$, we may estimate each block of the preceeding sum independently:

For each ν we decompose the secteur Δ_j into subsectors

$$\Delta_j^\alpha = \{\xi = \xi_1 + i\xi_2 \in \Delta_j \mid \alpha 2^\nu N^{-1/2} \leqslant \xi_1 - 1 \leqslant (\alpha+1) 2^\nu N^{-1/2}\}$$

and we repeat the same arguments used in the estimation of II.

To finish we observe that, by homogeneity, we have proved the following: $\left\| \left(\sum_j |P_{j,n} f|^2 \right)^{1/2} \right\|_4 \leqslant C(\log N)^{1/2}\ \|f\|_4$, uniformly on n, where, for each $n \in Z$

$$\Delta_{j,n} = \{\xi = \xi_1 + i\xi_2 \in \Delta_j \mid 2^n \leqslant \xi_1 \leqslant 2^{n+1}\}$$

$$\widehat{P_{j,n} f} = \chi_{\Delta_{j,n}} \cdot \hat{f}.$$

We decompose $\Delta_j = \bigcup_{l=1}^{\log N} \bigcup_{n \equiv l (\mathrm{mod}\,[\log N])} \Delta_{j,n}$ which gives us the decomposition

$$\Big(\sum_j |P_j f|^2\Big)^{1/2} \leqslant \sum_{l=1}^{\log N} \Big(\sum_j |P_j^l f(x)|^2\Big)^{1/2} ;$$

here P_j^l is given by the multiplier $\bigcup_{n \equiv l (\mathrm{mod}\,[\log N])} \Delta_{j,n}$.

The point is that if $n_1 \equiv n_2 (\mathrm{mod}\,[\log N])$ and, says, $n_1 > n_2$, then $2^{n_1} \geqslant N\, 2^{n_2}$. That is: the smaller side $2^{n_1}/N$ of the rectangles corresponding to Δ_{j,n_1}, $j = 1, 2, \dots, N$ is bigger than the diameter of the set $\bigcup_{\substack{n \equiv n_1 \mathrm{mod}([\log N]) \\ n < n_1}} \Delta_{j,n}$.

Furthermore, we decompose each $\Delta_{j,n}$ into N "squares" $\{\Delta_{j,n}^{\alpha}\}$ of side $\simeq 2^n N^{-1}$ and following our convention we shall define the corresponding multiplier operators $P_{j,n}^{\alpha}$.

To simplify notation we shall keep l fixed in the following and we shall assume that the index n ranges in the set of integers congruent with l mod([log N]). We have:

$$\Big\|\Big(\sum_j |P_j^l f|^2\Big)^{1/2}\Big\|_4^4 \sim \Big\|\Big(\sum_{j,n} |P_{j,n} f|^2\Big)^{1/2}\Big\|_4^4$$

$$= \sum_n \Big\|\Big(\sum_j |P_{j,n} f|^2\Big)^{1/2}\Big\|_4^4 + 2 \sum_{\substack{j,k \\ n_1 > n_2}} \int_{\mathbf{R}^2} |P_{j,n_1} f(x)\, P_{k,n_2} f(x)|^2\, dx$$

$$\leqslant C(\log N)^2\, \|f\|_4^4 + 2 \sum_{\substack{j,k \\ n_1 > n_2}} \int_{\mathbf{R}^2} \Big|\sum_{\alpha} P_{j,n_1}^{\alpha} f\, P_{k,n_2} f\Big|^2\, dx .$$

we have,

$$\sum_{\substack{j,k \\ n_1 > n_2}} \int \Big|\sum_{\alpha} P_{j,n_1}^{\alpha} f\, P_{k,n_2} f\Big|^2 dx = \sum_{\substack{j,k \\ n_1 > n_2}} \int \Big|\sum_{\alpha} \widehat{P_{j,n_1}^{\alpha} f} * \widehat{P_{k,n_2} f}\Big|^2 d\xi$$

$$\leqslant C \sum_{\substack{j,k \\ n_1 > n_2}} \sum_{\alpha} \int |\widehat{P_{j,n_1}^{\alpha} f} * \widehat{P_{k,n_2} f}|^2\, d\xi$$

$$= C \sum_{\substack{j,k \\ n_1 > n_2}} \sum_{\alpha} \int |P_{j,n_1}^{\alpha} f(x)|^2\ |P_{k,n_2} f(x)|^2\, dx$$

$$\leqslant C \left\| \Big(\sum_{j,k,\alpha} |P^{\alpha}_{j,n} f|^2 \Big)^{1/2} \right\|_4^2 \left\| \Big(\sum_{k,n} |P_{k,n} f|^2 \Big)^{1/2} \right\|_4^2$$

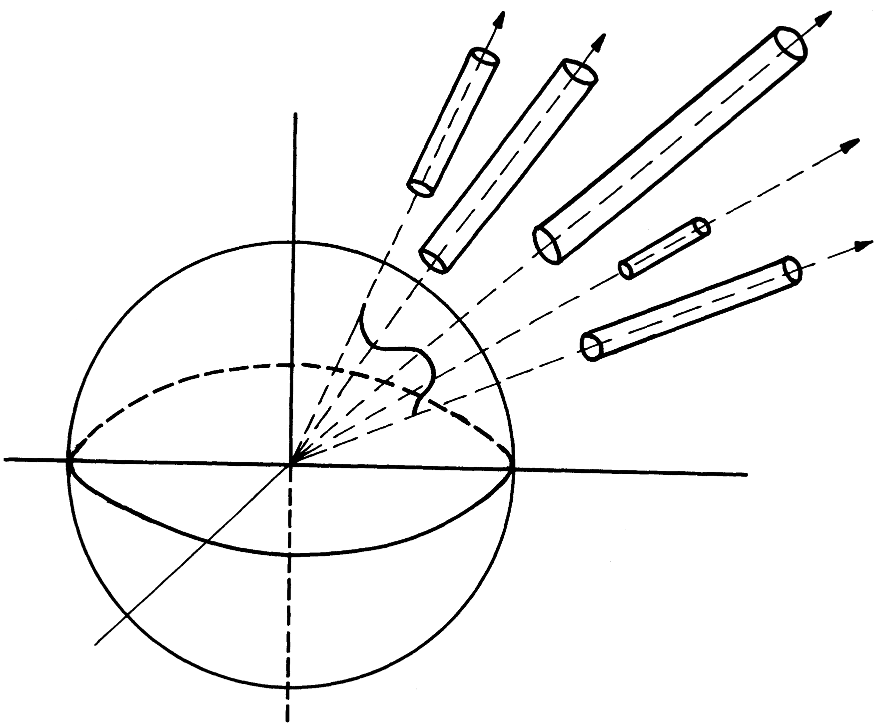
$$\leqslant C \, \|f\|_4^2 \left\| \Big(\sum_j |P_j^l f|^2 \Big)^{1/2} \right\|_4^2 .$$

That is, we have obtained the inequality:

$$\left\| \Big(\sum_j |P_j^l f|^2 \Big)^{1/2} \right\|_4^2 \leqslant C(\log N)^2 \, \|f\|_4^4 + C \, \|f\|_4^2 \left\| \Big(\sum_j |P_j^l f|^2 \Big)^{1/2} \right\|_4^2 .$$

From which the desired result follows very easily.

B. The maximal function.



Our hypothesis over γ means that for each $\omega \in S^{n-1}$ and $b \in \mathbf{R}$ the function: $t \longrightarrow \gamma(t) . \omega - b$ has a finite number of changes of signum, uniformly in ω and b. It should be noted that, in general, C_γ grows to infinity with this number. However, an estimate of the form $\| Mf \|_n \leqslant C(\log N)^a \| f \|_n$, should be true with C independent of γ. This is an interesting open problem.

For every positive integer m let us consider the points of γ given by $\omega_m^j = \gamma(j/2^m)$, $j = 1, 2, \ldots, 2^m$. Let $\Psi \geqslant 0$ be a smooth function on the real line, supported on $|t| \leqslant 2$ and equal to 1 on $|t| \leqslant 1$.

We define $A_m^j f(x) = \int_{-\infty}^{+\infty} f(x - t\omega_m^j)\, \Psi(t)\, dt$

and
$$A_m f(x) = \sup_{j=1,2,\ldots,2^m} |A_m^j f(x)|.$$

Claim. – $\| A_m f \|_2 \leqslant C\, m \, \| f \|_2$, for every $f \in L^2(\mathbf{R}^n)$, where C is independent of m.

We shall prove the claim by induction. The case $m = 1$ is a consequence of the Hardy-Littlewood maximal theorem. Let us assume that the result is true for $k \leqslant m - 1$. It is very easy to check that $A_m f(x) \leqslant A_{m-1} f(x) + B_m f(x)$ where

$$B_m f(x) = \Big(\sum_{j=1}^{2^m} |A_m^j f(x) - A_m^{j-1} f(x)|^2 \Big)^{1/2}.$$

Therefore our claim is a consequence of the estimate: $\| B_m f \|_2 \leqslant C \| f \|_2$, uniformly on m. To see this we use Plancherel's theorem:

$$\int_{\mathbf{R}^n} |B_m f(x)|^2\, dx = \sum_{j=1}^{2^m} \int_{\mathbf{R}^n} |A_m^j f(x) - A_m^{j-1} f(x)|^2\, dx$$
$$= \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 \sum_{j=1}^{2^m} |\hat{\Psi}(\xi \cdot \omega_m^j) - \hat{\Psi}(\xi \cdot \omega_m^{j-1})|^2\, d\xi ;$$

and we observe that, because of our hypotheses on γ, we have: $\sum_{j=1}^{2^m} |\hat{\Psi}(\xi \cdot \omega_m^j) - \hat{\Psi}(\xi \cdot \omega_m^{j-1})|^2 \leqslant C_\gamma < \infty$ uniformly on m.

To continue let us observe that, in order to prove theorem 2, we can, without lack of generality, restrict to the case $r = 2^n$, $n \in Z$ and, because of the fixed eccentricity, we may also consider cylinders with direction in the set $\gamma\left(\frac{j}{N}\right)$, $j = 1, \ldots, N$. Finally we may take N of the form $N = 2^m$, $m \in Z^+$.

Let us define:

i) $T^j_{2^\nu} f(x) = \sup_{x \in R} \frac{1}{\mu\{R\}} \int_R |f(y)| \, d\mu(y)$, where the supremum is taken over all cylinders with dimensions $(2^\nu)^{n-1} \times N\, 2^\nu$ and direction $\gamma\left(\frac{j}{N}\right)$.

ii) $T_{2^\nu} f(x) = \sup_j T^j_{2^\nu} f(x)$

$T^j f(x) = \sup_\nu T^j_{2^\nu} f(x)$

$Mf(x) = \sup_j T^j f(x) = \sup_\nu T_{2^\nu} f(x)$.

Given $\alpha > 0$ we obtain, for each j, a sequence of disjoint cylinders $\{R^j_\lambda\}_{\lambda = 1,2,\dots}$ with direction $\gamma(j/N)$ and such that: $E^j_\alpha = \{x : T^j f(x) > \alpha\} \subset \bigcup_\lambda \widetilde{R}^j_\lambda$ where $\widetilde{R}$ denotes the result of expanding R by the factor two. We have,

$$E_\alpha = \{x : Mf(x) > \alpha\} = \bigcup_{j=1}^{N} E^j_\alpha .$$

The heights of the N collections of cylinders, $\{R^j_\lambda\}$, $j = 1, \dots, N$, are bounded from above. By induction we may obtain, for each k, a familiy of cylinders B_k with dimensions $(2^{\nu k})^{n-1} \times 2^{\nu k} N$, $\nu_0 > \nu_1 > \dots$ in such a way that:

1) No cylinder of B_k is contained in the double of another cylinder of B_j, $j \leqslant k$.

2) If $R \in \bigcup_{j=1}^{N} \{R^j_\lambda\}$ and if

$$\dim(R) = (2^\nu)^{n-1} \times 2^\nu . N, \quad \nu_{k-1} > \nu \geqslant \nu_k ,$$

then either $R \in B_k$ or R is contained in the double of a cylinder in $\bigcup_{j \leqslant k} B_j$. Obviously: $E_\alpha \subset \bigcup_{R \in \cup B_k} \approx{R}$.

Let us denote by Δ_k the union of the families B_j's where $\nu_0 - k \log N \geqslant \nu_j \geqslant \nu_0 - (k+1) \log N$ and let $E_i = \bigcup_{R \in \Delta_i} R$, $\approx{E}_i = \bigcup_{R \in \Delta_i} \approx{R}$. We know that $E_\alpha \subset \bigcup \approx{E}_i$

We can now observe that the family $\{E_i\}$ is almost disjoint; more concretely, if $|i - j| \geqslant 2$ then $E_i \cap E_j = \emptyset$. This is true because if $R_i \in \Delta_i$, $R_j \in \Delta_j$, $i - j \geqslant 2$, then the radius of R_i is greater than the height of R_j ans, therefore, if $R_i \cap R_j \neq \phi$ then $R_j \subset \widetilde{R}_i$ which it is impossible.

Let $f_i = f/E_i$, $i = 0, 1, \ldots$ and let S_i be the maximal function given in the following way: $S_i g(x) = \sup\limits_{x \in R} \frac{1}{\mu\{R\}} \int_R |g(y)|\, d\mu(y)$, where the sup is taken over the set of cylinders of dimensions $(2^\nu)^{n-1} \times 2^\nu N$, where $\nu_0 + 2 - i \log N \geqslant \nu \geqslant \nu_0 + 2 - (i+1) \log N$.

The previously obtained estimate $\| A_m f \|_2 \leqslant C\, m \| f \|_2$ implies that S_i is bounded on $L^2(\mathbf{R})$ with bound less than $C_\gamma (\log N)^{3/2}$.

If $x \in \stackrel{\approx}{E}_i$ there exists a cylinder $R \in \Delta_i$ so that $x \in \stackrel{\approx}{R}$ and, therefore:

$$\begin{aligned} S_i f_i(x) &\geqslant \frac{1}{\mu\{\stackrel{\approx}{R}\}} \int_R |f_i(y)|\, d\mu(y) \\ &\geqslant \left(\frac{1}{4}\right)^n \frac{1}{\mu\{R\}} \int_R |f_i(y)|\, d\mu(y) \geqslant \left(\frac{1}{4}\right)^n \alpha . \end{aligned}$$

That is, $\stackrel{\approx}{E}_i \subset \{x : S_i f_i(x) \geqslant 4^{-n}\alpha\}$, which implies

$$\begin{aligned} \mu\{E_\alpha\} \leqslant \sum_i \mu\{\stackrel{\approx}{E}_i\} &\leqslant C_\gamma (\log N)^3\, \alpha^{-2} \sum_j \| f_i \|_2^2 \\ &\leqslant C_\gamma (\log N)^3 \alpha^{-2} \| f \|_2^2 . \end{aligned}$$

A standard use of the Marcinkiewicz interpolation theorem would yield the strong type inequality of Theorem 1. q.e.d.

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