GEOMETRIC HAMILTON–JACOBI THEORY

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April 21, 2006

Abstract

The Hamilton–Jacobi problem is revisited bearing in mind the consequences arising from a possible bi-Hamiltonian structure. The problem is formulated on the tangent bundle for Lagrangian systems in order to avoid the bias of the existence of a natural symplectic structure on the cotangent bundle. First it is developed for systems described by regular Lagrangians and then extended to systems described by singular Lagrangians with no secondary constraints. We also consider the example of the free relativistic particle, the rigid body and the electron-monopole system.

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1   Introduction

Hamilton–Jacobi theory provides important physical examples of the deep connection between first-order partial differential equations and systems of first-order ordinary differential equations. In this respect, it is also a stepping stone to the Schrödinger wave equation in quantum mechanics, and takes us as close as possible, within classical theory, to the notions of wave function and state in quantum theory. As a matter of fact, we obtain Hamilton–Jacobi-type equations whenever we consider a short-wave approximation for the solutions of wave-type equations, i.e., hyperbolic-type equations (this includes the classical limit of quantum mechanics in the Schrödinger picture by means of eikonal coordinates and the geometrical optics limit of wave optics) [17]. Within the framework of wave mechanics, a complete solution of the Hamilton–Jacobi equation allows us to reconstruct an approximate solution of the Schrödinger equation by providing us with the phase of the wave function and the amplitude via the van Vleck determinant constructed out of the Hessian of the complete solution itself (see, for instance, [17], p. 172).

As regards the Hamiltonian formulation of geometrical optics, one may recall that the origin of the whole method of canonical transformations in analytical mechanics can be traced back to the famous memoirs on Optics presented by Hamilton to the Royal Irish Academy. There, Hamilton showed that the propagation of wavefronts can be entirely characterized by the knowledge of a single function called the characteristic function. He also showed that the characteristic function obeys a first order partial differential equation, the so-called eikonal equation which is strictly related to the Hamilton–Jacobi equation.

The name Hamilton–Jacobi is justified by the contribution given by Jacobi that the dynamical problem (the ordinary differential equation) is completely solved once a complete solution of the associated partial differential equation is known.

Taking into account the fact that these equations were discovered almost two centuries ago, one may believe that everything must be known for them. As a matter of fact, we will argue and show in this paper, and in the forthcoming ones, that there are several aspects which have so far not been considered. Our own interest in reconsidering the Hamilton–Jacobi theory was generated by the existence of bi-Hamiltonian descriptions for completely integrable dynamical systems and the desire to unveil and understand the quantum counterpart of bi-Hamiltonian systems. In particular, due to the relevant role of the Hamilton–Jacobi theory in the Schrödinger picture, it seems appropriate to achieve a proper understanding of the Hamilton–Jacobi formulation for bi-Hamiltonian systems as a preliminary step toward the the possibility of a better understanding of the corresponding quantum situation.

Vinogradov [37] has exhibited a deep relation between the commutation relations of differential operators acting on functions over the configuration space Q and the canonical Poisson brackets of their principal symbols on the cotangent bundle T^*Q. This connection seems to rule out the possibility of considering the Hamilton–Jacobi version of the bi-Hamiltonian systems. To escape this apparent impossibility we find convenient to formulate the Hamilton–Jacobi theory on the tangent bundle TQ with the help of a regular Lagrangian function and the associated Lagrangian two-form. Thus we remove the bias of a natural symplectic structure on our carrier space, unlike in the case of the cotangent bundle. Working with Lagrangians on TQ we have

the possibility of dealing more directly with relativistic aspects and with dynamical systems described by degenerate Lagrangians (gauge theories), and therefore with the classical limit of their corresponding quantum systems.

In this paper we will not address the problem of constrained Lagrangian dynamics in full generality; this will be done in a forthcoming paper. Subsequently we also shall consider the classical limit of quantum bi-Hamiltonian systems and extend our Hamilton–Jacobi picture to classical field theories which allow for bi-Hamiltonian descriptions.

As a spin-off from our tangent bundle formulation of the Hamilton–Jacobi problem, we will identify two main geometric aspects of the classical formulation: the first one consists of finding a foliation transverse to the fibers (of TQ or T^*Q) and invariant under the dynamical evolution, while the second one requires that the foliation be Lagrangian with respect to a dynamically preserved symplectic structure. In this approach the dynamics (ordinary differential equation) plays a prominent role because we consider alternative Lagrangian or Hamiltonian descriptions. Therefore, our generalization is to search for invariant foliations of the carrier space with leaves having the same dimension as the configuration space Q, since we drop the requirement of "Lagrangianity". Thus the partial differential equation associated with our problem (equation (5) in Section 2) will be a partial differential equation for a vector valued function rather than for a scalar valued function as in the standard formulation. The transition from the vector valued function to the scalar valued one takes place with the help of the symplectic structure which allows us to associate a closed 1-form (and therefore locally a function) with our vector field by requiring the foliation to be Lagrangian.

When considering geodetical motions on Lie groups, an interesting situation arises in which the first step is accomplished but the second one is problematic, as we will see in Section 7. Similar aspects also emerge when dealing with bi-Hamitonian systems or systems described by equivalent Lagrangians: Here we find invariant foliations; they may be Lagrangian with respect to one symplectic structure but not Lagrangian with respect to some other invariant symplectic structure. We shall discuss a few very simple examples to illustrate what is taking place.

The paper is organized as follows. In Sections 2 and 3 we state the Lagrangian and Hamiltonian geometrical formulations of the Hamilton–Jacobi problem, respectively, showing how the standard classical problem is a particular case of the extended one, and clarifying the geometrical meaning of particular and complete solutions [27, 35]. The relation between both formalisms is also discussed. Section 4 is devoted to extending the theory to the particular case of singular dynamical systems: those where there are no Lagrangian constraints or, what is equivalent, when secondary Hamiltonian constraints do not appear. As an application of the above case, the Hamilton–Jacobi problem for non-autonomous Lagrangian and Hamiltonian systems is discussed in Section 5. Finally, as examples, we apply our theory to the free relativistic particle in Section 6, to the free motion on a Lie group, to the rigid body, and to the electron-monopole system in Section 7.

Notation: Throughout this paper Q is a n-dimensional differentiable manifold representing the configuration space of a dynamical system, and $\tau_Q \colon TQ \to Q$ and $\pi_Q \colon T^*Q \to Q$ are its tangent and cotangent bundles, representing the phase spaces of velocities and momenta, respectively.

On the cotangent bundle there is a canonical symplectic form $\omega = -d\theta$, where θ is the 1-form θ with coordinate expression $\theta = p_i dq^i$. Here and in the rest of this paper, sum over paired covariant and contravariant indices is understood. This symplectic form associates a vector field Z_H to every function $H \in C^\infty(T^*Q)$, as the solution of the equation $i(Z_H)\omega = dH$ (see e.g. [1] for the details).

On the tangent bundle, the canonical object is the vertical endomorphism S, with coordinate expression $S = (\partial/\partial v^i) \otimes dq^i$. Given a Lagrangian function $L \in C^\infty(TQ)$, we define the Cartan 1-form $\theta_L = d_S L = (\partial L/\partial v^i)dq^i$ and the Cartan 2-form $\omega_L = -d\theta_L$. The dynamical vector fields associated to the Lagrangian are the solutions of the dynamical equation $i(\Gamma)\omega_L = dE_L$, where $E_L = \Delta(L) - L \in C^\infty(TQ)$ is the Lagrangian energy function and $\Delta \in \mathfrak{X}(TQ)$ is the Liouville vector field (see e.g. [23, 12] and references therein).

We also remark that $T(TQ)$ has two different vector bundle structures over TQ, given respectively by $\tau_{TQ} \colon T(TQ) \to TQ$, i.e. considering TQ as new configuration space, and $T\tau_Q \colon T(TQ) \to TQ$. Maps $X \colon Q \to TQ$ that are sections for τ_Q are the vector fields in Q, and the set of such vector fields will be denoted by $\mathfrak{X}(Q)$. Correspondingly, maps $X \colon TQ \to T(TQ)$ that are sections for τ_{TQ} are the vector fields in TQ, and those which are also sections for $T\tau_Q$ are said to be second order differential equation fields (hereafter referred to as SODE vector fields). This means that their integral curves, which are the trajectories of the system, are holonomic. A vector field $X \in \mathfrak{X}(Q)$ can be lifted to TQ producing the so called complete or tangent lift of X and denoted by $X^T \in \mathfrak{X}(TQ)$. More details can be found in [13, 30].

2   Lagrangian formulation of the Hamilton–Jacobi problem

In this section we formulate the Hamilton–Jacobi problem on the tangent bundle. In this setting we are able to handle dynamical systems which admit alternative Lagrangian descriptions, and we clearly show how the search for solutions of the Hamilton–Jacobi problem splits in two steps.

We recall that, in the standard formulation, the Hamilton–Jacobi problem consists in finding a function $S(t, q)$, known as the principal function, such that the partial differential equation (PDE)

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0\,,$$

is satisfied. If we put $S(t, q) = W(q) - t\,E$, where E is a constant, then the function W, known as the characteristic function, has to satisfy

$$H\left(q, \frac{\partial W}{\partial q}\right) = E\,. \tag{1}$$

Both of the above PDE are known as the Hamilton–Jacobi equation. However we will always refer to the second one.

In geometric terms, equation (1) can be written as $(dW)^*H = E$, where we understand dW as a section of the cotangent bundle. In other words, we look for a section α of T^*Q such that $\alpha^*H = E$ and α is a closed 1-form, $d\alpha = 0$, and hence locally exact, $\alpha = dW$. The second condition, $d\alpha = 0$, can alternatively be expressed in terms of the canonical symplectic form on

T^*Q in the form $\alpha^*\omega = 0$, so that one can reformulate the Hamilton–Jacobi equation in the form [1]

$$\alpha^* H = E, \qquad \alpha^*\omega = 0. \tag{2}$$

Consider now the Lagrangian formalism. Let $L \in C^\infty(TQ)$ be the Lagrangian function and θ_L, ω_L be the associated Cartan forms. A literal translation of the above coordinate-free formulation of the Hamilton–Jacobi equations from the cotangent bundle to the tangent bundle would be [34]

$$X^*(E_L) = E, \qquad X^*(\omega_L) = 0, \tag{3}$$

where $X : Q \to TQ$ is the unknown "vector valued" function, and the second equation states that the vector field X is associated (at least locally) with a function W by means of the relation $X^*(\theta_L) = dW$, which is a stronger version of $X^*(\omega_L) = 0$.

Among the many important consequences that may be deduced from the existence of a solution of the Hamilton–Jacobi equation, let us recall the following. Let $P(q) = \frac{\partial W}{\partial q}(q)$, and consider the vector field $X = \frac{\partial H}{\partial q}(q, P(q))$. If $q = \gamma(t)$ is a solution of the differential equation $\dot{q} = X(q)$, then $\lambda(t) = (\gamma(t), P(\gamma(t)))$ is a solution of the Hamilton equations. The Lagrangian counterpart of this property reads as follows. If X is a solution of (3) and $q = \gamma(t)$ is a solution of the differential equation $\dot{q} = X(q)$, then $\xi(t) = (\gamma(t), X(\gamma(t)))$ is a solution of the Euler–Lagrange equations.

This fact will be our starting point in the study of the Hamilton–Jacobi equation and its generalization. We will look for the implications of this property and its relation with equations (3).

2.1 Statement of the problem and solutions

We will assume first that the Lagrangian L is regular, and we will leave for Section 4 the analysis of the unconstrained singular case. The regularity of the Lagrangian is equivalent to the regularity of the Cartan 2-form, so that ω_L is symplectic. It follows that there exists a unique solution $\Gamma_L \in \mathfrak{X}(TQ)$ of the Lagrangian dynamical equation

$$i(\Gamma_L)\omega_L = dE_L . \tag{4}$$

Γ_L is called the Lagrangian vector field of the Lagrangian system. It is well known [12] that Γ_L is a second order differential equation.

Generalized Lagrangian Hamilton–Jacobi problem. *Let $L \in C^\infty(TQ)$ be a Lagrangian function. The* generalized Lagrangian Hamilton–Jacobi problem *consists in finding a vector field $X \colon Q \to TQ$ such that, if $\gamma \colon \mathbb{R} \to Q$ is an integral curve of X, i.e. $\dot{\gamma} = X \circ \gamma$, then $\dot{\gamma} \colon \mathbb{R} \to TQ$ is an integral curve of Γ_L; that is,*

$$X \circ \gamma = \dot{\gamma} \implies \Gamma_L \circ \dot{\gamma} = \overline{X \circ \gamma} .$$

X is said to be a solution of the generalized Lagrangian Hamilton–Jacobi problem.

As we will see in a moment, in geometrical terms, this requirement means that the image of X, as a map from Q to TQ, is a Γ_L-invariant submanifold of TQ. Let us show first an example.

Example 1 The dynamics of the free particle in $\mathbb{R}^2$ is given by the regular Lagrangian function

$$L(q^1, q^2, v^1, v^2) = \frac{1}{2} \left[(v^1)^2 + (v^2)^2 \right] ,$$

with associated geometrical objects

$$
\begin{aligned}
\theta_L &= \frac{\partial L}{\partial v^1} \, dq^1 + \frac{\partial L}{\partial v^2} \, dq^2 = v^1 \, dq^1 + v^2 \, dq^2 \\
E_L &= \frac{1}{2} \left[(v^1)^2 + (v^2)^2 \right] \\
\omega_L &= dq^1 \wedge dv^1 + dq^2 \wedge dv^2 \\
\Gamma_L &= v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2}
\end{aligned}
$$

The vector field

$$X = k \frac{\partial}{\partial q^1} + \frac{k \, q^2 - l}{q^1} \frac{\partial}{\partial q^2} , \qquad k, l \in \mathbb{R} ,$$

defines a two-parameter family of vector fields on $Q = \mathbb{R}^2$ which are generalized solutions. We also find that

$$
\begin{aligned}
X^*(\omega_L) &= -\frac{k \, q^2 - l}{q^1} \, dq^1 \wedge dq^2 , \\
X^*(E_L) &= \frac{1}{2} \left[k^2 + \left(\frac{k \, q^2 - l}{q^2} \right)^2 \right] .
\end{aligned}
$$

Thus, the simple translation of the geometrical relations from T^*Q to TQ would be violated. Now we can formulate on TQ a PDE which replaces the PDE for the characteristic function W. We find that it must be stated in terms of a vector valued function.

Proposition 1 *X is a solution of the generalized Lagrangian Hamilton–Jacobi problem if, and only if, X and Γ_L are X-related; that is,*

$$\Gamma_L \circ X = TX \circ X . \tag{5}$$

Proof X is a solution of the generalized Lagrangian Hamilton–Jacobi problem if, for every $\gamma \colon \mathbb{R} \to Q$ such that $X \circ \gamma = \dot{\gamma}$, then

$$\Gamma_L \circ \dot{\gamma} = \overline{\overset{\cdot}{X \circ \gamma}} = TX \circ \dot{\gamma} = TX \circ X \circ \gamma .$$

But $\Gamma_L \circ \dot{\gamma} = \Gamma_L \circ X \circ \gamma$, and as X has integral curves through every point $q \in Q$, this is equivalent to $TX \circ X = \Gamma_L \circ X$.

The proof of the converse is straightforward. ∎

This equation for a given SODE Γ_L defines a PDE for X and replaces the PDE for W in the standard formulation of the Hamilton–Jacobi problem.

In addition we have:

Proposition 2 *X is a solution of the generalized Lagrangian Hamilton–Jacobi problem if, and only if, the submanifold* $\operatorname{Im} X \subset TQ$ *is invariant by the Lagrangian vector field* Γ_L *(that is,* Γ_L *is tangent to the submanifold* $X(Q)$*).*

Proof For the direct implication, it suffices to show that, for every $q \in Q$, $\Gamma_L(X_q)$ is tangent to $\operatorname{Im} X$, and it holds because, by proposition 1, $\Gamma_L(X_q) = T_q X(X_q)$.

Conversely, if Γ_L leaves $\operatorname{Im} X$ invariant, then $\Gamma_L(X_q) \in T_{X_q} \operatorname{Im} X$. Therefore, there exists $u \in T_q Q$ such that $\Gamma_L(X_q) = T_q X(u)$; hence

$$X_q = (T\tau_Q \circ \Gamma_L)(X_q) = (T_q \tau_Q \circ T_q X)(u) = T_q(\tau_Q \circ X)(u) = u \ ,$$

because $\tau_Q \circ X = \operatorname{Id}_Q$, and Γ_L being a SODE, it is a section of the projection $T\tau_Q$, so $T\tau_Q \circ \Gamma_L = \operatorname{Id}_{TQ}$. Thus $\Gamma_L(X_q) = T_q X(X_q)$ for every $q \in Q$; that is, $\Gamma_L \circ X = TX \circ X$, and X is a solution of the generalized Lagrangian Hamilton–Jacobi problem by proposition 1. ∎

If X is a solution of the generalized Lagrangian Hamilton–Jacobi problem, then the integral curves of X are the τ_Q-projection of integral curves of Γ_L contained in $\operatorname{Im} X$.

Observe that we have not used that Γ_L is the Lagrangian vector field, so these results actually hold for every SODE $\Gamma \in \mathfrak{X}(TQ)$. Using the fact that Γ_L is the Lagrangian vector field of a Lagrangian system, the above results can be related with the energy Lagrangian function E_L in the following way, which avoids the explicit calculation of the dynamical Lagrangian vector field.

Theorem 1 *X is a solution of the generalized Lagrangian Hamilton–Jacobi problem if, and only if,*

$$i(X)(X^* \omega_L) = d(X^* E_L) \tag{6}$$

Proof From the Lagrangian dynamical equation (4) we obtain

$$X^* i(\Gamma_L) \omega_L = X^* dE_L = d(X^* E_L) \ ,$$

but, as X and Γ_L are X-related (proposition 1), we have that

$$X^* i(\Gamma_L) \omega_L = i(X)(X^* \omega_L) \ ,$$

which yields (6).

Conversely, suppose that X satisfies (6). The deviation D_L from the relatedness

$$D_L = \Gamma_L \circ X - TX \circ X : Q \to TTQ \ ,$$

is a vector field along X. We have to prove that $D_L = 0$. First we have that D_L is τ_Q-vertical. In fact, $\tau_Q \circ X = \operatorname{Id}_Q$, and Γ_L being a SODE, it is a section of the projection $T\tau_Q$, so $T\tau_Q \circ \Gamma_L = \operatorname{Id}_{TQ}$, hence

$$T\tau_Q \circ D_L = T\tau_Q \circ \Gamma_L \circ X - T\tau_Q \circ TX \circ X = X - X = 0 \ .$$

Furthermore, from the Lagrangian dynamical equation (4) we have $X^*i(\Gamma_L)\omega_L = X^*dE_L = d(X^*E_L)$, which combined with the hypothesis, $i(X)(X^*\omega_L) = d(X^*E_L)$, leads to $X^*i(\Gamma_L)\omega_L - i(X)(X^*\omega_L) = 0$. Therefore, for every $q \in Q$ and $Y_q \in T_qQ$, we have

$$
\begin{aligned}
0 &= (X^*i(\Gamma_L)\omega_L - i(X)(X^*\omega_L))_q(Y_q) = (\omega_L)_{X(q)}(\Gamma_L(q), T_qX(Y_q)) - (X^*\omega_L)_q(X_q, Y_q) \\
&= (\omega_L)_{X(q)}(\Gamma_L(q), T_qX(Y_q)) - (\omega_L)_{X(q)}(T_qX(X_q), T_qX(Y_q)) \\
&= (\omega_L)_{X(q)}(D_L(q), T_qX(Y_q)) \ .
\end{aligned}
$$

Moreover, for every τ_Q-vertical vector field $V \in \mathfrak{X}(TQ)$ we have that

$$
(\omega_L)_{\alpha(q)}(D_L(q), V(X(q))) = -d(\theta_L)_{\alpha(q)}(D_L(q), V(X(q))) \ ,
$$

which vanishes for every $q \in Q$. We recall that if α is a semibasic form, then $d\alpha(V_1, V_2) = 0$ for every pair of vertical fields V_1 and V_2. But θ_L is a τ_Q-semibasic form, and $T_{X(q)}TQ = T_{X(q)}(\text{Im } X) \oplus V_{X(q)}(\tau_Q)$, thus we have proved that

$$
(\omega_L)_{\alpha(q)}(D_L(q), Z(X(q))) = 0 \ , \text{ for every } q \in Q, Z \in \mathfrak{X}(TQ) \ ,
$$

and hence $D_L(q) = 0$, for every $q \in Q$, since ω_L is nondegenerate. ∎

To solve the generalized Lagrangian Hamilton–Jacobi problem is, in general, a hard task; it amounts to finding Γ_L-invariant submanifolds of TQ which are transverse to the fibers. Thus, it is convenient to consider a less general problem, which constitutes the standard version of the Lagrangian Hamilton–Jacobi problem:

Lagrangian Hamilton–Jacobi problem *Given a Lagrangian function $L \in C^\infty(TQ)$, the Lagrangian Hamilton–Jacobi problem consists in finding solutions X of the generalized Lagrangian Hamilton–Jacobi problem satisfying that $X^*\omega_L = 0$.*

As $0 = X^*\omega_L = -X^*d\theta_L = -d(X^*\theta_L)$, we have that every point has an open neighborhood $U \subset Q$ where there is a function $W \in C^\infty(U)$ such that $X^*\theta_L = dW$ (in U).

Remark: In the example of the free particle in $\mathbb{R}^2$ given above, the pull-back of the symplectic 2-form ω_L by the vector field X is different from zero. Hence, X does not provide a solution of the Lagrangian Hamilton–Jacobi problem because it is not associated with a closed 1-form on the configuration space.

A straightforward consequence of the last theorem is:

Corollary 1 *If X is a solution of the Lagrangian Hamilton–Jacobi problem, then $d(X^*E_L) = 0$.*
∎

Observe that if X is a solution of the Lagrangian Hamilton–Jacobi problem, then $\text{Im } X$ is a Lagrangian submanifold of (TQ, ω_L) contained in a level set of E_L. In fact, $\dim \text{Im } X = n$ and, if $j_X : \text{Im } X \hookrightarrow TQ$ denotes the natural embedding, we have that $j_X^*\omega_L = 0$, due to $X^*(\omega_L) = 0$.

We can summarize the above results in the following:

Proposition 3 *Let $X \in \mathfrak{X}(Q)$ satisfy $X^* \omega_L = 0$. Then, the following assertions are equivalent:*

1. *X is a solution of the Lagrangian Hamilton–Jacobi problem.*

2. *$d(X^* E_L) = 0$*

3. *$\mathrm{Im}\, X$ is a Lagrangian submanifold of TQ invariant by Γ_L.*

4. *The integral curves of Γ_L with initial conditions in $\mathrm{Im}\, X$ project onto the integral curves of X.*

∎

Coordinate expressions   Let us show the local expressions of the objects so far presented. Consider coordinates (q^i) on Q, and the corresponding natural coordinates (q^i, v^i) on its tangent bundle.

Consider an arbitrary vector field $\Gamma \in \mathfrak{X}(TQ)$ satisfying the second-order condition, $\Gamma(q, v) = (q, v; v, a(q, v))$, and a vector field $X \in \mathfrak{X}(Q)$: $X(q) = (q, w(q))$. Then we have

$$(TX \circ X - \Gamma \circ X)(q) = \left(q, w(q); 0, \frac{\partial w}{\partial q} w - a(q, w(q)) \right), \tag{7}$$

which is a vertical vector field along X. Its vanishing is the necessary and sufficient condition for X and Γ to be X-related:

$$\frac{\partial w^i}{\partial q^j} w^j(q) - a^i(q, w(q)) = 0.$$

This equation is the PDE for the vector valued function $w^j(q)$ which replaces the standard PDE for the scalar function W.

When Γ is the Lagrangian vector field Γ_L, its components satisfy $W_{ij} a^j = \dfrac{\partial L}{\partial q^i} - \dfrac{\partial^2 L}{\partial v^i \, \partial q^j} v^j$, where $W_{ij} = \dfrac{\partial^2 L}{\partial v^i \partial v^j}$ is the Hessian matrix of L.

Then we can compute the 1-form

$$-i_X X^*(\omega_L) + X^*(dE_L) = \left(\frac{\partial^2 L}{\partial v^i \partial v^j} \frac{\partial w^j}{\partial q^k} w^k + \frac{\partial^2 L}{\partial v^i \partial q^j} w^j - \frac{\partial L}{\partial q^i} \right) \Bigg|_{v=w(q)} dq^i, \tag{8}$$

whose vanishing also expresses that X is a solution of the generalized Lagrangian Hamilton–Jacobi problem.

Looking carefully at the local expressions one can find a relation between $-i_X X^*(\omega_L) + X^*(dE_L)$ and $TX \circ X - \Gamma \circ X$, which is given by the Hessian, as we are going to show.

To this end, let us first recall that, for any vector bundle $E \to Q$, we have the vertical lift map $\mathrm{vl}_E \colon E \times_Q E \to VE \subset TE$, an isomorphism which in fiber coordinates reads $\mathrm{vl}(q, u, v) = (q, u; 0, v)$. With $E = TQ$, this gives an isomorphism $\mathrm{vl} \colon TQ \times_Q TQ \to V(TQ) \subset T(TQ)$.

Associated with the Lagrangian L, we have the Legendre transformation $\mathcal{F}L\colon TQ \to T^*Q$, which in coordinates reads $\mathcal{F}L(q,v) = (q^i, \dfrac{\partial L}{\partial v^i})$. In other words, $\mathcal{F}L$ is the fibre derivative of L.

Moreover, we can define the fiber Hessian $\mathcal{F}^2L\colon TQ \to T^*Q \otimes T^*Q$ which defines, if the Lagrangian is regular, another isomorphism $\widehat{\mathcal{F}^2L}\colon TQ \times_Q TQ \to TQ \times_Q T^*Q$. In coordinates, $\mathcal{F}^2L(q,v) = (q^i, W_{ij})$ and $\widehat{\mathcal{F}^2L}(q^i, v^i, u^i) = (q^i, v^i, W_{ij}u^j)$.

With these ingredients, we achieve an alternative understanding of Theorem 1:

Proposition 4 *Let* vl *be the vertical lift map of the tangent bundle TQ. For any vector field X on Q, we have*

$$(X, -i_X X^*(\omega_L) + X^*(dE_L)) = \widehat{\mathcal{F}^2L} \circ \mathrm{vl}^{-1} \circ (TX \circ X - \Gamma_L \circ X). \tag{9}$$

Therefore, $-i_X X^*(\omega_L) + X^*(dE_L)$ vanishes if and only if $TX \circ X - \Gamma_L \circ X$ vanishes too. ∎

2.2 Complete solutions

The most useful and essential idea in the standard Hamilton–Jacobi theory consists in finding, not only one particular solution as we have used in the previous subsection, but rather a complete solution of the problem. This may be defined as follows.

Definition 1 *Consider a solution X_λ depending on n additional parameters $\lambda \in \Lambda$, where $\Lambda \subseteq \mathbb{R}^n$ is an open set, and suppose that the map $\Phi\colon Q \times \Lambda \to TQ$ given by $\Phi(q, \lambda) = X_\lambda(q)$ is a local diffeomorphism. In this case $\{X_\lambda; \lambda \in \Lambda\}$ is said to be a complete solution of the generalized Lagrangian Hamilton–Jacobi problem.*

From the definition, it follows that a complete solution provides TQ with a foliation transverse to the fibers, and that the Lagrangian vector field Γ_L is tangent to the leaves.

If $\{X_\lambda; \lambda \in \Lambda\}$ is a complete solution, the integral curves of X_λ, for different $\lambda \in \Lambda$, will provide all the integral curves of the Lagrangian vector field Γ_L. This means that, if $(q_0, v_0) \in \mathrm{Im}\, X$, then there is $\lambda_0 \in \Lambda$ such that $X_{\lambda_0}(q_0) = v_0$, and the integral curve of X_{λ_0} through q_0, lifted by X_{λ_0} to TQ, gives the integral curve of Γ_L through (q_0, v_0). This justifies the name of "complete solution".

Remark: We may use instead a fiber bundle P over Λ, such that Γ_L projects onto the null vector field; i.e. Λ is a space of constants of the motion and fibers have the same dimension as the configuration space Q. Thus we may take into account the nontriviallity of P as a bundle. On the other hand, if Λ were contractible the bundle would be trivial, and we would revert to the previous situation.

Furthermore, different transversal foliations of TQ, with Γ_L tangent to the leaves, are different ways to collect solutions of Γ_L smoothly and such that they project onto Q in a coherent way: integral curves of Γ_L in $\mathrm{Im}\, X_\lambda$ project onto integral curves of the associated vector field X_λ.

The relation between Γ_L and complete solutions is the following:

- If we have a family of n first integrals $f_1, \ldots, f_n$ of Γ_L such that $d_S f_1 \wedge \ldots \wedge d_S f_n \neq 0$, then $f_i = c_i$, $c_i \in \mathbb{R}$, for $i = 1, \ldots, n$, define a transversal foliation. Thus we can locally isolate the velocities as functions of the coordinates q^i and the constants c_i. Now, replacing in the expression of Γ_L these velocities and projecting to the basis, we obtain a local complete solution $X_{(c_1,\ldots,c_n)}$.

- Conversely, if $\Phi \colon Q \times \Lambda \to TQ$ is a complete solution, then the functions defining locally the foliation give us the above family of integrals of motion of Γ_L. More explicitly, the components of the map $F \colon TQ \to \Lambda$ given by $F = \mathrm{pr}_2 \circ \Phi^{-1}$, are constants of the motion.

Moreover, if the foliation is Lagrangian in (TQ, ω_L), then we have a complete solution of the Lagrangian Hamilton–Jacobi problem. In this case the above family of first integrals are in involution.

In our previous example of the free particle, varying the parameters $(k, l) \in \mathbb{R}^2$ we obtain a complete solution.

All these considerations are shown in the following example.

Example 2 Let us consider the example of the two-dimensional standard harmonic oscillator described by

$$L = \frac{1}{2}((v^1)^2 + (v^2)^2 - (q^1)^2 - (q^2)^2) \ .$$

The dynamical vector field is

$$\Gamma_L = v^1 \frac{\partial}{\partial q^1} + v^2 \frac{\partial}{\partial q^2} - q^1 \frac{\partial}{\partial v^1} - q^2 \frac{\partial}{\partial v^2} \ ,$$

and the standard Lagrangian symplectic 2-form is $\omega_L = dq^1 \wedge dv^1 + dq^2 \wedge dv^2$.

We know that the functions

$$f_1 = v^1 v^2 + q^1 q^2, \ \ f_2 = (v^1)^2 + (q^1)^2, \ \ f_3 = (v^2)^2 + (q^2)^2, \ \ f_4 = q^1 v^2 - q^2 v^1$$

are constants of the motion. Of course, not all of them are functionally independent. Suppose their values are $f_1 = C$, $f_2 = 2E_1$, $f_3 = 2E_2$, $f_4 = l$. We can use, for instance, f_2 and f_3 to express v^1 and v^2 as functions of the base coordinates and the two parameters E_1 and E_2, and using these expressions in the dynamical vector field we find a vector field on the base Q depending on the two energies:

$$X_{E_1, E_2} = \left(\pm \sqrt{2\,E_1 - (q^1)^2} \frac{\partial}{\partial q^1} \pm \sqrt{2\,E_2 - (q^2)^2} \frac{\partial}{\partial q^2} \right) \ .$$

Note that the two functions we have used are in involution, $\{f_2, f_3\} = 0$, and that

$$(X_{E_1, E_2})^* \omega_L = dq^1 \wedge d(\sqrt{2\,E_1 - (q^1)^2}) + dq^2 \wedge d(\sqrt{2\,E_2 - (q^2)^2}) = 0 \ .$$

This is a 2-parameter family of vector fields, for which the images are Lagrangian submanifolds with respect to ω_L.

We can also choose the functions f_1 and f_4 for obtaining expressions of the velocities in terms of positions, when $q^1 v^2 + q^2 v^1 \neq 0$, because

$$d_S f_1 \wedge d_S f_4 = (v^2 \, dq^1 + v^1 \, dq^2) \wedge (-q^2 \, dq^1 + q^1 \, dq^2) = (q^1 v^2 + q^2 v^1) \, dq^1 \wedge dq^2 \; ,$$

and in this case,

$$v^1 = \frac{-l \pm \sqrt{l^2 + 4\, q^1 \, q^2 (C - q^1 \, q^2)}}{2\, q^2} \, , \qquad v^2 = \frac{l \pm \sqrt{l^2 + 4\, q^1 \, q^2 (C - q^1 \, q^2)}}{2\, q^1} = \frac{l + q^2 v^1}{q^1} \, ,$$

and we have the vector field in Q

$$
\begin{aligned}
X_{C,l}(q^1, q^2) \;=\; & \left(\frac{-l \pm \sqrt{l^2 + 4\, q^1 \, q^2 (C - q^1 \, q^2)}}{2\, q^2} \right) \frac{\partial}{\partial q^1} \\
& + \left(\frac{l \pm \sqrt{l^2 + 4\, q^1 \, q^2 (C - q^1 \, q^2)}}{2\, q^2} \right) \frac{\partial}{\partial q^2} \; .
\end{aligned}
$$

However, notice that because of $\{f_1, f_4\} = f_2 - f_3$, we find $(X_{C,l})^* \omega_L \neq 0$. Therefore, $X_{C,l}$ is a complete solution for the generalized problem, but not for the standard Hamilton–Jacobi problem.

3   Formulation of the Hamilton–Jacobi problem on T^*Q

3.1   Statement of the problem and solutions

We now consider the Hamiltonian formalism in the cotangent bundle. Let $H \in C^\infty(T^*Q)$ be a Hamiltonian function, and denote by $\omega = -d\theta \in \Omega^2(T^*Q)$ the canonical symplectic form. There exists a unique vector field $Z_H \in \mathfrak{X}(T^*Q)$ whose integral curves are the trajectories of the system; that is, the solutions of the Hamilton equation. Geometrically this means that Z_H is the solution of the Hamiltonian dynamical equation

$$i(Z_H)\omega = dH \; . \tag{10}$$

Z_H is called the Hamiltonian vector field of the system.

As in the Lagrangian formalism, let us start with the generalized version of the Hamilton–Jacobi problem, which can be stated as follows:

Generalized Hamiltonian Hamilton–Jacobi problem   *Given a Hamiltonian vector field $Z_H \in \mathfrak{X}(T^*Q)$, the* generalized Hamiltonian Hamilton–Jacobi problem *consists in finding a vector field $X \colon Q \to TQ$ and a 1-form $\alpha \colon Q \to T^*Q$ such that, if $\gamma \colon \mathbb{R} \to Q$ is an integral curve of X, then $\alpha \circ \gamma \colon \mathbb{R} \to T^*Q$ is an integral curve of Z_H. That is,*

$$X \circ \gamma = \dot{\gamma} \implies \overline{\dot{\alpha \circ \gamma}} = Z_H \circ (\alpha \circ \gamma) \; . \tag{11}$$

The first result is:

Proposition 5 *Given a vector field $Z_H \in \mathfrak{X}(T^*Q)$, (X, α) satisfies the condition (11) if, and only if, the vector fields X and Z_H are α-related; that is,*

$$Z_H \circ \alpha = T\alpha \circ X . \tag{12}$$

Proof If (X, α) satisfies the condition (11) then, for every $\gamma \colon \mathbb{R} \to Q$ such that $X \circ \gamma = \dot{\gamma}$, we have

$$Z_H \circ \alpha \circ \gamma = \overline{\dot{\alpha \circ \gamma}} = T\alpha \circ \dot{\gamma} = T\alpha \circ X \circ \gamma ;$$

but, as X has integral curves through every point $q \in Q$, this condition is equivalent to $Z_H \circ \alpha = T\alpha \circ X$.

The proof of the converse is straightforward. ∎

In fact, both elements (X, α) satisfying the condition (11) are related, since, by composing both sides of the above equation (12) with $T\pi_Q$, and taking into account that $\pi_Q \circ \alpha = \mathrm{Id}_Q$, we have the following immediate consequence:

Corollary 2 *If (X, α) satisfies the condition (11) then*

$$X = T\pi_Q \circ Z_H \circ \alpha.$$

It is interesting to remark that we also have the following relation between X and α:

$$X = \mathcal{F}H \circ \alpha ,$$

where $\mathcal{F}H \colon T^*Q \to TQ$ is the fiber derivative of the Hamiltonian function.

In terms of our previous geometrical formulation, this amounts to $X^*(\theta_L) = \alpha$ when L is the Lagrangian function associated with H.

As X is determined by α, we introduce the following:

Definition 2 *A solution of the generalized Hamiltonian Hamilton–Jacobi problem for Z_H is a 1-form $\alpha \in \Omega^1(Q)$ such that, if $\gamma \colon \mathbb{R} \to Q$ is an integral curve of $X = T\pi_Q \circ Z_H \circ \alpha$, then $\alpha \circ \gamma \colon \mathbb{R} \to T^*Q$ is an integral curve of Z_H; that is,*

$$T\pi_Q \circ Z_H \circ \alpha \circ \gamma = \dot{\gamma} \implies \overline{\dot{\alpha \circ \gamma}} = Z_H \circ (\alpha \circ \gamma) .$$

Then $X = T\pi_Q \circ Z_H \circ \alpha$ is said to be the vector field associated with α.

Example 1 (continued)

Consider the Hamiltonian function for a free particle in $\mathbb{R}^2$

$$H(q^1, q^2, p_1, p_2) = \frac{1}{2} \left(p_1{}^2 + p_2{}^2 \right).$$

The 1-form $\alpha = \dfrac{1}{q^1} dq^2$ and its associated vector field $X = \dfrac{1}{q^1} k_1 \dfrac{\partial}{\partial q^2}$ provide a solution for the generalized Hamiltonian Hamilton–Jacobi problem, but we will see later that they do not give rise to any solution of the standard Hamilton–Jacobi problem.

Proposition 6 *Given a vector field $Z_H \in \mathfrak{X}(T^*Q)$, a 1-form $\alpha \in \Omega^1(Q)$ is a solution of the generalized Hamiltonian Hamilton–Jacobi problem if, and only if, the submanifold $\operatorname{Im} \alpha \subset T^*Q$ is invariant under the flow of the vector field Z_H (that is, Z_H is tangent to the submanifold $\operatorname{Im} \alpha$).*

Proof If $\alpha \in \Omega^1(Q)$ is a solution of the Hamiltonian Hamilton–Jacobi problem, and $X = T\pi_Q \circ Z_H \circ \alpha$, then $Z_H \circ \alpha = T\alpha \circ X$, and thus $Z_H(\alpha(q)) = T\alpha(X(q))$, for every $q \in Q$. Hence, Z_H is tangent to $\operatorname{Im} \alpha$.

Conversely, if $\operatorname{Im} \alpha$ is invariant by Z_H then $Z_H(\alpha(q)) \in T_{\alpha(q)} \operatorname{Im} \alpha$, which implies that there exists $u \in T_q Q$ such that $Z_H(\alpha(q)) = T_q \alpha(u)$. Defining X by $T_q \alpha(X_q) = Z_H(\alpha(q))$, then X is differentiable since $X = T\pi_Q \circ Z_H \circ \alpha$. Hence X is a vector field in Q which satisfies $Z_H \circ \alpha = T\alpha \circ X$, and then α is a solution of the Hamiltonian Hamilton–Jacobi problem. $\blacksquare$

If $\alpha \in \Omega^1(Q)$ is a solution of the generalized Hamiltonian Hamilton–Jacobi problem, taking into account Corollary 2, we can conclude that the π_Q-projection of the integral curves of Z_H contained in $\operatorname{Im} \alpha$ are the integral curves of X.

Observe also that until now we have not used that Z_H is a Hamiltonian vector field, so these results actually hold for every vector field $Z \in \mathfrak{X}(T^*Q)$. When Z_H is the Hamiltonian vector field of a Hamiltonian system, the above results can be expressed in terms of the Hamiltonian function.

As in the Lagrangian case, we can obtain an equation not involving directly the dynamical vector field:

Theorem 2 *Given the Hamiltonian vector field $Z_H \in \mathfrak{X}(T^*Q)$, a 1-form $\alpha \in \Omega^1(Q)$ is a solution of the generalized Hamiltonian Hamilton–Jacobi problem if, and only if,*

$$i(X)d\alpha = -d(\alpha^* H) , \tag{13}$$

where $X = T\pi_Q \circ Z_H \circ \alpha$ is the vector field associated with α by means of the Hamiltonian vector field Z_H.

Proof From the Hamiltonian dynamical equation (10) for Z_H we obtain

$$\alpha^* i(Z_H)\omega = \alpha^* dH = d(\alpha^* H) .$$

Furthermore, θ is the canonical form of T^*Q, so $\alpha^* \theta = \alpha$, and then

$$\alpha^* \omega = -\alpha^* d\theta = -d(\alpha^* \theta) = -d\alpha , \tag{14}$$

therefore, as X and Z_H are α-related, we have

$$\alpha^* i(Z_H)\omega = i(X)\alpha^* \omega = -i(X)d\alpha ,$$

which yields (13).

To prove the converse, first let us define

$$D_H = Z_H \circ \alpha - T\alpha \circ X \colon Q \to TT^*Q \,,$$

which is a vector field along α. We have to prove that $D_H = 0$. First we have that D_H is π_Q-vertical; in fact, as $\pi_Q \circ \alpha = \mathrm{Id}_Q$,

$$
\begin{aligned}
T\pi_Q \circ D_H &= T\pi_Q \circ (Z_H \circ \alpha - T\alpha \circ X) = T\pi_Q \circ (Z_H \circ \alpha - T\alpha \circ T\pi_Q \circ Z_H \circ \alpha) \\
&= T\pi_Q \circ Z_H \circ \alpha - T\pi_Q \circ Z_H \circ \alpha = 0 \,.
\end{aligned}
$$

Furthermore, from the Hamiltonian dynamical equation (10) and the hypothesis, as $\alpha^*\omega = -d\alpha$, we have the following relations:

$$
\begin{aligned}
\alpha^* i(Z_H)\omega = \alpha^* dH &= d(\alpha^* H) \,, \\
i(X)\alpha^*\omega = -i(X)d\alpha &= d(\alpha^* H) \,,
\end{aligned}
$$

and hence $\alpha^* i(Z_H)\omega - i(X)\alpha^*\omega = 0$. Therefore, for every $q \in Q$ and $Y_q \in T_q Q$, we have

$$
\begin{aligned}
0 &= (\alpha^* i(Z_H)\omega - i(X)\alpha^*\omega)_q(Y_q) = \omega_{\alpha(q)}(Z_H(\alpha(q)), T_q\alpha(Y_q)) - \omega_{\alpha(q)}(T_q\alpha(X_q), T_q\alpha(Y_q)) \\
&= \omega_{\alpha(q)}(D_H(q), T_q\alpha(Y_q)) \,.
\end{aligned}
$$

Moreover, as $\mathrm{V}(\pi_Q)$ (the π_Q-vertical subbundle of TT^*Q) is a Lagrangian distribution in (T^*Q, ω), for every π_Q-vertical vector field $V \in \mathfrak{X}(T^*Q)$ we have that

$$\omega_{\alpha(q)}(D_H(q), V(\alpha(q))) = 0 \,; \,,$$

for every $q \in Q$. But $T_{\alpha(q)}T^*Q = T_{\alpha(q)}(\mathrm{Im}\,\alpha) \oplus \mathrm{V}_{\alpha(q)}(\pi_Q)$, hence we have proved that

$$\omega_{\alpha(q)}(D_H(q), Z(\alpha(q))) = 0$$

for every $q \in Q$ and $Z \in \mathfrak{X}(T^*Q)$; since ω is non-degenerate, we conclude that $D_H = 0$, or what is equivalent, X and Z_H are α-related, and thus α is a solution of the generalized Hamiltonian Hamilton–Jacobi ∎

As in the Lagrangian case, in general, to solve the generalized Hamiltonian Hamilton–Jacobi problem is a difficult task. So it is convenient to consider the following less general problem, which constitutes the standard version of the Hamiltonian Hamilton–Jacobi problem:

Hamiltonian Hamilton–Jacobi problem   *Given a vector field $Z_H \in \mathfrak{X}(T^*Q)$, the* Hamiltonian Hamilton–Jacobi problem *consists in finding a solution $\alpha \in \Omega^1(Q)$ of the generalized Hamiltonian Hamilton–Jacobi problem which is moreover* closed, *$d\alpha = 0$.*

As a consequence, every point has an open neighbourhood $U \subset Q$, where there is a function $W \in C^\infty(U)$ such that $\alpha = dW$.

Notice also that, because of (14), the closeness condition $d\alpha = 0$ is equivalent to $\alpha^*\omega = 0$.

A straightforward consequence of the previous theorem is:

Corollary 3 *A closed 1-form α is a solution of the Hamiltonian Hamilton–Jacobi problem if, and only if, $d(\alpha^* H) = 0$.* ∎

Observe that, if $\alpha \in \Omega^1(Q)$ is a solution of the Hamiltonian Hamilton–Jacobi problem, as $\alpha^* \omega = 0$, then $\operatorname{Im} \alpha$ is a Lagrangian submanifold of $(T^* Q, \omega)$, contained in a level set of H, because (13) implies that $d(\alpha^* H) = 0$. In fact, $\dim \operatorname{Im} \alpha = n$ and, if $j \colon \operatorname{Im} \alpha \hookrightarrow T^* Q$ denotes the natural embedding, we have that $j^* \omega = 0$. Thus we recover some geometrical aspects of the classical Hamiltonian Hamilton–Jacobi theory.

We can summarize the above results in the following:

Proposition 7 *Let $\alpha \in \Omega^1(Q)$ be a closed 1-form. Then, the following assertions are equivalent:*

1. *α is a solution of the Hamiltonian Hamilton–Jacobi problem.*

2. *$d(\alpha^* H) = 0$.*

3. *$\operatorname{Im} \alpha$ is a Lagrangian submanifold of $T^* Q$ invariant by Z_H.*

4. *The integral curves of Z_H with initial conditions in $\operatorname{Im} \alpha$ project onto the integral curves of $X = T\pi_Q \circ Z_H \circ \alpha$.*

If moreover $\alpha = dW$, then these conditions can also be written as

5. *$H \circ dW$ is locally constant.* ∎

Coordinate expressions Let us see how all the objects presented appear when we consider coordinates (q^i) on Q, and the corresponding natural coordinates $(q^i, \dot{q}^i)$ and (q^i, p_i) on its tangent and cotangent bundles.

First, the coordinate expression of the Hamiltonian vector field Z_H is given by

$$Z_H(q, p) = (q, p; \partial H/\partial p, -\partial H/\partial q).$$

Consider a 1-form $\alpha \in \Omega^1(Q)$ and a vector field $X \in \mathfrak{X}(Q)$. In coordinates they read $\alpha = a_i \, dq^i$ and $X = X^i \, \partial/\partial q^i$. Then $T\alpha \circ X$ and $Z_H \circ \alpha$ are vector fields along α, which read

$$(T\alpha \circ X)(q) = \left(q^i, a_i(q); X^i(q), \frac{\partial a_i}{\partial q^j} X^j(q) \right),$$

$$(Z_H \circ \alpha)(q) = \left(q^i, a_i(q); \frac{\partial H}{\partial p_i}(q, a(q)), -\frac{\partial H}{\partial q^i}(q, a(q)) \right).$$

Therefore their difference is

$$(T\alpha \circ X - Z_H \circ \alpha)(q) = \left(q^i, a_i(q); X^i(q) - \frac{\partial H}{\partial p_i}(q, a(q)), \frac{\partial a_i}{\partial q^j} X^j(q) + \frac{\partial H}{\partial q^i}(q, a(q)) \right).$$

Its vanishing determines $X = \mathcal{F}H \circ \alpha$, that is:

$$X^i(q) = \frac{\partial H}{\partial p_i}(q, a(q)). \tag{15}$$

With this X, the preceding difference becomes

$$(T\alpha \circ X - Z_H \circ \alpha)(q) = \left(q^i, a_i(q); 0, \frac{\partial a_i}{\partial q^j}(q)\frac{\partial H}{\partial p_j}(q, a(q)) + \frac{\partial H}{\partial q^i}(q, a(q)) \right). \tag{16}$$

So, the condition for α to be a solution of the generalized Hamiltonian Hamilton–Jacobi problem is

$$\frac{\partial a_i}{\partial q^j}(q)\frac{\partial H}{\partial p_j}(q, a(q)) + \frac{\partial H}{\partial q^i}(q, a(q)) = 0.$$

Now let us consider $d\alpha = \dfrac{\partial a_i}{\partial q^j}dq^j \wedge dq^i$. Then $i(X)d\alpha = X^j\left(\dfrac{\partial a_i}{\partial q^j} - \dfrac{\partial a_j}{\partial q^i}\right)dq^i$. On the other hand, $dH = \dfrac{\partial H}{\partial q^i}dq^i + \dfrac{\partial H}{\partial p_i}dp_i$, so $\alpha^*(dH) = \left(\dfrac{\partial H}{\partial q^i}(q, a(q)) + \dfrac{\partial H}{\partial p_j}\dfrac{\partial a_j}{\partial q^i}(q, a(q))\right)dq^i$. Therefore we have

$$i(X)d\alpha + \alpha^*(dH) = \left(X^j\frac{\partial a_i}{\partial q^j} + (\frac{\partial H}{\partial p_j} - X^j)\frac{\partial a_j}{\partial q^i} + \frac{\partial H}{\partial q^i}\right)\bigg|_{p=a(q)} dq^i.$$

Again, with X given as $\mathcal{F}H \circ \alpha$, this expression becomes

$$i(X)d\alpha + \alpha^*(dH) = \left(\frac{\partial H}{\partial p_j}\frac{\partial a_i}{\partial q^j} + \frac{\partial H}{\partial q^i}\right)\bigg|_{p=a(q)} dq^i. \tag{17}$$

Finally, if $\alpha = dW$, then $a_i = \partial W/\partial q^i$, and the last condition in Proposition 7 reads

$$H(q^i, \partial W/\partial q^i) = \text{const},$$

which is the classical form of the time-independent Hamilton–Jacobi equation.

A careful look at the local expressions (16) and (17) gives an alternative understanding of Theorem 2. Using the vertical lift map again, we have:

Proposition 8 *Let* vl *be the vertical lift map of the cotangent bundle T^*Q. Given a 1-form α on Q, and the vector field $X = \mathcal{F}H \circ \alpha$ on Q, we have*

$$\text{vl}\left(\alpha, i(X)d\alpha + \alpha^*(dH)\right) = T\alpha \circ X - Z_H \circ \alpha. \tag{18}$$

Therefore, $T\alpha \circ X - Z_H \circ \alpha$ vanishes if, and only if, $i(X)d\alpha + \alpha^*(dH)$ also does. ∎

3.2   Complete solutions

As in the Lagrangian case, we are interested in finding not only a particular solution as described in the preceding section, but a complete solution to the problem. In this way, we define:

Definition 3 *Consider a solution α_λ depending on n additional parameters $\lambda \in \Lambda$, where $\Lambda \subseteq \mathbb{R}^n$ is an open set, and suppose that the map $\Phi \colon Q \times \Lambda \to T^*Q$ given by $\Phi(q, \lambda) = \alpha_\lambda(q)$ is a local diffeomorphism. In this case $\{\alpha_\lambda; \lambda \in \Lambda\}$ is said to be a* complete solution *of the generalized Hamiltonian Hamilton–Jacobi problem.*

From the definition it follows that a complete solution provides T^*Q with a foliation transverse to the fibers, and that the Hamiltonian vector field Z_H is tangent to the leaves.

If $\{\alpha_\lambda; \lambda \in \Lambda\}$ is a complete solution, the integral curves of the vector fields provide all the integral curves of the Hamiltonian vector field Z_H. This means that, if $(q_0, p_0) \in \operatorname{Im}\alpha$, then there is $\lambda_0 \in \Lambda$ such that $\alpha_{\lambda_0}(q_0) = p_0$, and the integral curve of X_{λ_0} through q_0, lifted by α_{λ_0} to T^*Q, gives the integral curve of Z_H through (q_0, p_0). This justifies the name of "complete solution".

Furthermore, different transversal foliations of (T^*Q, ω), with Z_H tangent to the leaves, are different ways to collect integral curves of Z_H smoothly and such that they project onto Q in a coherent way: integral curves of Z_H in $\operatorname{Im}\alpha_\lambda$ project onto integral curves of the associated vector field X_λ).

The relation between Z_H and complete solutions is the same as in the Lagrangian case, using first integrals of Z_H and the vector fields X_λ associated to α_λ, for $\lambda \in \Lambda$.

Finally, if the foliation is Lagrangian in (T^*Q, ω), then we have a complete solution of the Hamiltonian Hamilton–Jacobi problem. In this case the above family of first integrals are in involution.

3.3 Equivalence between the Lagrangian and Hamiltonian formulations

This section is devoted to the equivalence between the Lagrangian and Hamiltonian Hamilton–Jacobi theory. We have the following:

Theorem 3 *Let (TQ, ω_L, E_L) be a hyper-regular Lagrangian system, and (T^*Q, ω, H) its associated Hamiltonian system. Then there exists a bijection between the set of solutions of the (generalized) Lagrangian Hamilton–Jacobi problem and the set of solutions of the (generalized) Hamiltonian Hamilton–Jacobi problem. This bijection is given by composition with the Legendre map: $X \mapsto \alpha = \mathcal{F}L \circ X$.*

Proof Suppose $X \in \mathfrak{X}(Q)$ satisfies $TX \circ X = \Gamma_L \circ X$, and $T\mathcal{F}L \circ \Gamma_L = Z_H \circ \mathcal{F}L$. Let $\alpha = \mathcal{F}L \circ X$, then

$$T\alpha \circ X = T(\mathcal{F}L \circ X) = T\mathcal{F}L \circ TX \circ X = T\mathcal{F}L \circ \Gamma_L \circ X = Z_H \circ \mathcal{F}L \circ X = Z_H \circ \alpha \ ,$$

hence α is a solution of the Hamiltonian problem.

Furthermore, if $E_L \circ X = const.$, and $\alpha = \mathcal{F}L \circ X$, then $E_L \circ \mathcal{F}H \circ \alpha = const.$, that is, $H \circ \alpha = const.$

Conversely, suppose $\alpha \in \Omega^1(Q)$ satisfies $T\alpha \circ X = Z_H \circ \alpha$, and $\Gamma_L \circ \mathcal{F}H = T\mathcal{F}H \circ Z_H$, where $\mathcal{F}H$ denotes the fiber derivative of the Hamiltonian, which satisfies $\mathcal{F}H = (\mathcal{F}L)^{-1}$, because the system is hyper-regular. Let $X = \mathcal{F}H \circ \alpha$, then

$$TX \circ X = T(\mathcal{F}H \circ \alpha) \circ X = T\mathcal{F}H \circ T\alpha \circ X = T\mathcal{F}H \circ Z_H \circ \alpha = \Gamma_L \circ \mathcal{F}H \circ \alpha = \Gamma_L \circ X \ ,$$

hence X is a solution of the corresponding Lagrangian problem.

In addition, if $H \circ \alpha = const.$, and $X = \mathcal{F}H \circ \alpha$, then $H \circ \mathcal{F}L \circ X = const.$, that is, $E_L \circ X = const.$ ∎

This result can be extended to complete solutions in a natural way.

As is obvious, for regular but non hyper-regular Lagrangians, all this holds only in the local open sets where $\mathcal{F}L$ is a diffeomorphism.

3.4 Alternative Lagrangian descriptions

Let us consider now the case of a regular system admitting alternative Lagrangian descriptions; that is, suppose there are regular Lagrangians functions $L, L' \in C^\infty(TQ)$, $L \neq L'$, giving rise to the same dynamical vector field $\Gamma_L = \Gamma_{L'} \in \mathfrak{X}(TQ)$ solution of the equations

$$i(\Gamma)\omega_L = dE_L \,, \quad i(\Gamma)\omega_{L'} = dE_{L'} \,.$$

So, we have the same dynamics, but two different symplectic structures.

- If X is a solution of the generalized Hamilton–Jacobi problem for one of the Lagrangians, then it is also a solution of the generalized problem for the second Lagrangian. A similar result does not hold for the non generalized problem, that is, a solution X of the Hamilton–Jacobi problem for one of the Lagrangians will not be (in general) a solution of the Hamilton–Jacobi problem for the other Lagrangian, that is, $X^*\omega_L = 0$, but $X^*\omega_{L'} \neq 0$.

- It is natural to compare with the situation in the Hamiltonian formalism, that is, in T^*Q. Instead of ω_L and $\omega_{L'}$, we have a symplectic structure $\omega \in \Omega^2(T^*Q)$, but different Hamiltonian functions $H, H' \in C^\infty(T^*Q)$. Thus the same solution X of the Lagrangian Hamilton–Jacobi problem leads to two solutions α and α' of the Hamiltonian Hamilton–Jacobi problem corresponding to the Hamiltonians H and H' respectively. Nevertheless, notice that
$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}(q,p)\Big|_{p=\frac{\partial W}{\partial q}} = \frac{\partial H'}{\partial p_i}(q,p)\Big|_{p=\frac{\partial W'}{\partial q}}$$
where $\alpha = dW$ and $\alpha' = dW'$ locally. These are the equations for the integral curves of X. However, the corresponding dynamical vector fields on T^*Q related to X are different. In other words, the difference between $Z_H \circ \alpha$ and $Z_{H'} \circ \alpha'$ is a vertical vector field, which in general does not vanish.

- The case of dynamical systems described by two alternative equivalent Lagrangians motivates the use of two different symplectic structures in T^*Q in the following way. Let L and L' be equivalent hyper-regular Lagrangians, and ω_L and $\omega_{L'}$ their corresponding Lagrangian 2-forms. If $\omega_0 \in \Omega^2(T^*Q)$ is the canonical 2-form in T^*Q, we have that $\mathcal{F}L^*\omega_0 = \omega_L$. Then, let $\omega_1 \in \Omega^2(T^*Q)$ be another symplectic structure in T^*Q such that $\mathcal{F}L^*\omega_1 = \omega_{L'}$. Hence, $\mathcal{F}L' \circ \mathcal{F}L^{-1}$ is a base-preserving transformation from (T^*Q, ω_0) to (T^*Q, ω_1). These transformations have been called fouling transformations [28].

As an example, it is not difficult to show this construction for the two-dimensional harmonic oscillator described both by $L = \frac{1}{2}((v^1)^2 + (v^2)^2 - (q^1)^2 - (q^2)^2)$, and $L' = v^1 v^2 - q^1 q^2$ (see example 2 in Section 2.2).

These considerations show that, in order to incorporate alternative Lagrangian or Hamiltonian descriptions in the Hamilton–Jacobi setting, we must introduce generalized solutions.

4   The Hamilton–Jacobi problem for unconstrained singular Lagrangian systems

In this section we are going to show how our procedure can be extended to singular Lagrangians without secondary constraints.

4.1   Lagrangian formulation

Now we consider a singular Lagrangian $L \in C^\infty(TQ)$. We recall that the Euler–Lagrange equation for a SODE Γ is the equation

$$i(\Gamma)\omega_L = dE_L. \tag{19}$$

A curve $\gamma\colon \mathbb{R} \to Q$ is a solution of the Euler–Lagrange equation if $\xi = \dot\gamma$ satisfies

$$i(\dot\xi)\omega_L = dE_L \circ \xi. \tag{20}$$

We shall only consider the case of singular Lagrangians for which the following assumption holds:

Assumption 1 *The Lagrangian dynamical equation (19) has a* SODE *solution* $\Gamma \in \mathfrak{X}(TQ)$ *everywhere defined in TQ and the rank of $T\mathfrak{F}L$ is constant.*

The constancy of the rank of $T\mathfrak{F}L$ is equivalent to saying that ω_L has also constant rank, hence ω_L is a presymplectic form.

Under this assumption, the set of SODE solution vector fields is the set of sections of an affine bundle $\mathcal{A} \to TQ$, modeled on the vector bundle $\mathrm{Ker}\, T\mathfrak{F}L \to TQ$. More precisely, the fiber of $\mathcal{A}$ at $v \in TQ$ is

$$\mathcal{A}_v = \left\{\, V \in T_v(TQ) \mid T\tau_Q(V) = v \quad \text{and} \quad i(V)\omega_L|_v = dE_L|_v \,\right\}.$$

A curve $\gamma\colon \mathbb{R} \to Q$ is a solution of the Euler–Lagrange equations if, and only, the curve $\xi = \dot\gamma$ satisfies $\dot\xi(t) \in \mathcal{A}_{\xi(t)}$, for every $t \in \mathbb{R}$. Observe that if a vector $V \in T_v(TQ)$ satisfies $T\tau_Q(V) = v$ then the linear 1-form $i(V)\omega_L|_v - dE_L|_v$ is semibasic.

The generalized Lagrangian Hamilton–Jacobi problem for these kinds of Lagrangians can be stated as follows:

Generalized Lagrangian Hamilton–Jacobi problem for unconstrained singular Lagrangians   *To find a vector field $X\colon Q \to TQ$ such that, if $\gamma\colon \mathbb{R} \to Q$ is an integral curve of X, then $\xi = X \circ \gamma\colon I \to TQ$ is a solution of the Euler–Lagrange equation (20).*

In a similar way to the first part of Section 2.1, we have the following result:

Theorem 4 *The following conditions for a vector field $X \in \mathfrak{X}(Q)$ are equivalent:*

1. *X is a solution of the generalized Lagrangian Hamilton–Jacobi problem.*

2. *X satisfies the condition $\mathrm{Im}\,(TX \circ X) \subset \mathcal{A}|_{\mathrm{Im}\,X}$.*

3. *X satisfies the equation $i(X)(X^*\omega_L) = d(X^*E_L)$.*

4. *For every $v \in \mathrm{Im}\,X$ there exists $w \in \mathcal{A}_v$ such that w is tangent to $\mathrm{Im}\,X$.*

5. *The submanifold $\mathrm{Im}\,X$ is such that for every initial condition $v \in \mathrm{Im}\,X$ there is a solution of the Euler–Lagrange equations which is entirely contained in the submanifold $\mathrm{Im}\,X$.*

Proof

[(1) $\Rightarrow$ (2)] Let $q \in Q$ be arbitrary, and consider the integral curve γ of X such that $\gamma(0) = q$. Denote by $\xi = \dot{\gamma}$ the tangent lift of γ. Since condition (1) is satisfied and γ is an integral curve of X, we have that $\dot{\xi}(t) \in \mathcal{A}_{\xi(t)}$. But since $\xi = X \circ \gamma$ we have $\dot{\xi} = TX \circ \dot{\gamma} = TX \circ X \circ \gamma$, which at $t = 0$ gives $TX(X(q)) = \dot{\xi}(0) \in \mathcal{A}_{\xi(0)} = \mathcal{A}_{X(q)}$.

[(2) $\Rightarrow$ (3)] Assume that $\mathrm{Im}\,(TX \circ X) \subset \mathcal{A}|_{\mathrm{Im}\,X}$, so that for every $q \in Q$ we have that $T_qX(X(q)) \in \mathcal{A}_{X(q)}$. Then, from the definition of $\mathcal{A}$ we have that $\omega_L(T_qX(X(q)), W) = \langle dE_L|_{X(q)}, W \rangle$ for every $W \in T_{X(q)}(TQ)$. In particular, if we take $W = T_qX(w)$ for arbitrary $w \in T_qQ$, we have that $\omega_L(T_qX(X(q)), T_qX(w)) = \langle dE_L|_{X(q)}, T_qX(w) \rangle$, or in other words $(X^*\omega_L)_q(X(q), w) = \langle X^*(dE_L)|_q, w \rangle$. Since this equality holds for every $w \in T_qQ$ and every $q \in Q$, we deduce that $i(X)(X^*\omega_L) = X^*dE_L$.

[(3) $\Rightarrow$ (4)] Let X be a vector field satisfying condition (3) and $v \in \mathrm{Im}\,X$, so that $v = X(q)$ for $q = \tau_Q(v)$. The vector $w = TX(v)$ satisfies the required properties. Indeed, on one hand it is clear that w is tangent to the image of X, and on the other we have that $T\tau_Q(w) = T\tau_Q(TX(v)) = T(\tau_Q \circ X)(v) = v$, so that we have to prove that that linear 1-form $i(w)\omega_L|_v - dE_L|_v$ vanishes. Since such 1-form is semibasic, we just need to prove that it vanishes when applied to elements of the form $TX(u)$ for $u \in T_qQ$:

$$
\begin{aligned}
(i(w)\omega_L|_v - dE_L|_v)(TX(u)) &= \omega_L|_{X(q)}(TX(v), TX(u)) - dE_L|_{X(q)}(TX(u)) \\
&= (X^*\omega_L)_q(v, u) - d(X^*E_L)_q(u) \\
&= (X^*\omega_L)_q(X(q), u) - d(X^*E_L)_q(u),
\end{aligned}
$$

which vanishes in view of the condition $i(X)X^*\omega_L - d(X^*E_L) = 0$.

[(4) $\Rightarrow$ (1)] Assume that for every element $v \in \mathrm{Im}\,X$ there exists $w \in \mathcal{A}_v$, which is tangent to $\mathrm{Im}\,X$. In other words, for every $q \in Q$ (and hence $v = X(q)$) there exists $w \in \mathcal{A}_{X(q)}$ such that $w = T_qX(z)$ for some $z \in T_qQ$. But the first condition for the element w to be in $\mathcal{A}$ is $T\tau_Q(w) = \tau_{TQ}(w)$, which for $w = T_qX(z)$ is just $z = X(q)$. Therefore, the vector w is $w = T_qX(X(q))$ and it is $\mathcal{A}_{X(q)}$. Since this is true for every $q \in Q$, we have proved that $\mathrm{Im}\,(TX \circ X) \subset \mathcal{A}|_{\mathrm{Im}\,X}$, which was shown to be equivalent to condition (1).

Finally, (4) and (5) are clearly equivalent, and both are equivalent to the integrability of the restriction of $\mathcal{A}$ to $\mathrm{Im}\,X$ (see the remark below this proof). ∎

Remark: Let us recall a few facts from the theory of implicit differential systems, in particular, when the implicit system is just an affine subbundle of the tangent bundle. Let $\mathcal{A} \to M$ be an affine subbundle of TM and let N be a submanifold of M. Consider the restriction $\mathcal{A}|_N$ of the subbundle $\mathcal{A}$ to N. The following properties are equivalent:

1. The restriction of $\mathcal{A}$ to N satisfies the integrability condition for implicit differential equations [21, 32].

2. For every initial condition $m \in N$ there exists a curve solution of the system which is entirely contained in N.

3. For every $m \in M$ there exists $w \in \mathcal{A}_m$ such that w is tangent to N.

Roughly speaking, the proofs of these facts are as follows: (1) and (2) are equivalent by definition of an integrable implicit differential system. $[(2){\Rightarrow}(3)]$ is obvious: given $m \in M$ take the solution $\gamma(t)$ passing through m and contained in M, and then $w = \dot{\gamma}(0)$ is tangent to M. $[(3){\Rightarrow}(2)]$ Take a local section of $\mathcal{A} \cap TN$, and an integral curve of such a section is a curve contained in M.

As in the regular case, we can state the following particular problem:

Lagrangian Hamilton–Jacobi problem for unconstrained singular Lagrangians   *To find solutions X to the generalized Lagrangian Hamilton–Jacobi problem for unconstrained singular Lagrangians satisfying $X^*\omega_L = 0$.*

The main results for this situation are summarized in the following:

Proposition 9 *The following assertions for a vector field $X \in \mathfrak{X}(Q)$ are equivalent:*

1. *X is a solution of the Lagrangian Hamilton–Jacobi problem.*

2. *$\operatorname{Im} X$ is an isotropic submanifold of (TQ, ω_L) and $\operatorname{Im}(TX \circ X) \subset \mathcal{A}|_{\operatorname{Im} X}$.*

3. *$d(X^*\theta_L) = 0$ and $d(X^*E_L) = 0$.*

4. *$\operatorname{Im} X$ is an isotropic submanifold of (TQ, ω_L) and for every $v \in \operatorname{Im} X$ there exists $w \in \mathcal{A}_v$ such that w is tangent to $\operatorname{Im} X$.*

5. *$\operatorname{Im} X$ is an isotropic submanifold of (TQ, ω_L), and for every initial condition in $\operatorname{Im} X$ there exists a solution of the Euler–Lagrange equations entirely contained in $\operatorname{Im} X$.*

Proof   They are consequences of the last theorem, taking into account that $\operatorname{Im} X$ is isotropic if, and only if, $X^*\omega_L = 0$, and this is equivalent to $d(X^*\theta_L) = 0$.   ∎

4.2 Hamiltonian formulation

When the Lagrangian is singular, in general, there is no satisfactory Hamiltonian formalism unless certain regularity conditions hold. We will assume in what follows that:

Assumption 2 *The Lagrangian L is* almost-regular, *that is: $P = \mathcal{F}L(TQ)$ is a closed submanifold of T^*Q, $\mathcal{F}L$ is a submersion onto its image P, and the fibers $\mathcal{F}L^{-1}(\mathcal{F}L(p))$, for every $p \in TQ$, are connected submanifolds of TQ.*

The natural embedding of P into T^*Q will be denoted $\jmath_0 \colon P \hookrightarrow T^*Q$. Denote by $\mathcal{F}L^0$ the map $\mathcal{F}L^0 \colon TQ \to P$ defined by the relation $\jmath_0 \circ \mathcal{F}L^0 = \mathcal{F}L$.

For an almost-regular Lagrangian system (TQ, L) there exists a Hamiltonian formalism. The associated Hamiltonian system is (P, ω_0, H_0), where $\omega_0 = \jmath_0^* \omega$ is a presymplectic form, and $H_0 \in C^\infty(P)$ is the Hamiltonian function, defined by the equation $\mathcal{F}L^{0*} H_0 = E_L$.

The Hamilton equation is the presymplectic equation

$$i(Z)\omega_0 = dH_0, \tag{21}$$

for a vector field $Z \in \mathfrak{X}(P)$. Under our assumptions this equation has solution everywhere in P, although it is not unique [6, 8, 20]. The set of solutions is the set of sections of an affine subbundle $\mathcal{B} \to P$ of $T(T^*Q)$, modeled on the vector subbundle $\mathrm{Ker}(\omega_0) \to P$. The fiber over a point $\alpha \in P$ is

$$\mathcal{B}_\alpha = \{V \in T_\alpha(T^*Q) \mid i(V)\omega_{0|\alpha} = dH_{0_\alpha}\}.$$

A curve $\mu \colon \mathbb{R} \to T^*Q$ is a solution of the Hamilton equations if it satisfies

$$i(\dot{\mu})\,\omega_0 = dH_0 \circ \mu. \tag{22}$$

Hence, the curve μ is a solution of the Hamilton equation if, and only if, $\dot{\mu}(t) \in \mathcal{B}_{\mu(t)}$.

Bearing in mind the above comments and the results for the Lagrangian and the Hamiltonian regular cases, the generalized version of the Hamiltonian Hamilton–Jacobi problem for these kinds of singular systems can be stated in the following way, which is not exactly as in the regular case:

Generalized Hamiltonian Hamilton–Jacobi problem for unconstrained singular Lagrangians *To find vector fields $X \colon Q \to TQ$ such that, if $\gamma \colon \mathbb{R} \to Q$ is an integral curve of X then $\mu = \mathcal{F}L^0 \circ X \circ \gamma$ is a curve solution of the Hamilton equation (22).*

Observe that $\alpha = \mathcal{F}L^0 \circ X \colon Q \to P$ is a section of the projection $\pi_Q^0 = \pi_Q \circ \jmath_0 \colon P \to Q$. We will say that α is the 1-form associated with the particular chosen solution X.

In this way, all the definitions, results and comments stated in Section 3 hold for the manifold P instead of T^*Q. In particular:

Theorem 5 *The following conditions for a vector field $X \in \mathfrak{X}(Q)$ are equivalent*

1. *X is a solution of the generalized Hamiltonian Hamilton–Jacobi problem for the unconstrained singular Lagrangian L, with associated 1-form α.*

2. *X satisfies the condition* $\mathrm{Im}\,(T\alpha \circ X) \subset \mathcal{B}|_{\mathrm{Im}\,\alpha}$

3. *X satisfies the equation* $i(X)(\alpha^*\omega_0) = d(\alpha^*H_0)$.

4. *For every* $\lambda \in \mathrm{Im}\,\alpha$ *there exists* $w \in \mathcal{B}_\lambda$ *such that* w *is tangent to* $\mathrm{Im}\,\alpha$.

5. *The submanifold* $\mathrm{Im}\,\alpha$ *is such that, for every initial condition in* $\mathrm{Im}\,\alpha$ *there is a curve solution of the Hamilton equations which is entirely contained in the submanifold* $\mathrm{Im}\,\alpha$. ∎

As above, we can state the particular case:

Hamiltonian Hamilton–Jacobi problem for unconstrained singular Lagrangians   *To find solutions α of the generalized singular Hamiltonian Hamilton–Jacobi problem for unconstrained singular Lagrangians satisfying $\alpha^*\omega_0 = 0$.*

And we obtain:

Proposition 10 *The following assertions for a 1-form $\alpha\colon Q \to P \subset T^*Q$ are equivalent:*

1. *α is a solution of the Hamiltonian Hamilton–Jacobi problem for the unconstrained singular Lagrangian L.*

2. *$\mathrm{Im}\,\alpha$ is an isotropic submanifold of $P, \omega_0)$ and $\mathrm{Im}\,(T\alpha \circ X) \subset \mathcal{B}|_{\mathrm{Im}\,\alpha}$.*

3. *$d(j_0 \circ \alpha) = 0$ and $d(\alpha^*H) = 0$.*

4. *$\mathrm{Im}\,\alpha$ is an isotropic submanifold of (P, ω_0) and, for every $\lambda \in \mathrm{Im}\,\alpha$, there exists $w \in \mathcal{B}_\lambda$ such that w is tangent to $\mathrm{Im}\,\alpha$.*

5. *$\mathrm{Im}\,\alpha$ is an isotropic submanifold of (P, ω_0), and for every initial condition in $\mathrm{Im}\,\alpha$ there exists a curve solution of the Hamilton equations entirely contained in $\mathrm{Im}\,\alpha$.* ∎

As a final remark, the equivalence between the Lagrangian and Hamiltonian Hamilton–Jacobi problem in the unconstrained singular case is straightforward, taking into account how the problem has been stated in the Hamiltonian formalism.

5   The Hamilton–Jacobi problem for time-dependent regular systems

5.1   The extended homogeneous Lagrangian formalism

The geometric formalism for non-autonomous Lagrangian and Hamiltonian systems exhibits some differences with respect to the autonomous formalism (see [14, 15] for the details). In the

non-relativistic Lagrangian formalism of time-dependent dynamical systems, the configuration space is a bundle $\pi\colon E \to \mathbb{R}$, known as the configuration bundle, and the velocity-phase space is the first-order jet bundle $\pi^1\colon J^1\pi \to E$. Given fibered coordinates (t, q^i) on E, we get fibered coordinates (t, q^i, v^i) on $J^1\pi$.

A non-autonomous Lagrangian is a function $L \in C^\infty(J^1\pi)$. In this case, the associated Cartan 1-form Θ_L is the 1-form on $J^1\pi$ whose coordinate expression is

$$\Theta_L = \frac{\partial L}{\partial v^i}(dq^i - v^i dt) + L dt.$$

The Lagrangian is regular if the dimension of the kernel of the Cartan 2-form $\Omega_L = -d\Theta_L$ is 1. In this case a unique vector field Γ, the dynamical vector field, is determined by the dynamical equation $i(\Gamma)\Omega_L = 0$, together with the normalization condition $i(\Gamma)dt = 1$, which ensures that integral curves of Γ are parametrized by the time coordinate t.

It follows from the above description that our Hamilton–Jacobi theory does not apply directly to time-dependent systems. A way to solve this problem is to describe non-autonomous systems by the so-called homogeneous formalism (see [22], for a friendly introduction see Section 2.3.1 in [17]), as we are about to explain. Instead of the first jet bundle $J^1\pi$, we consider the tangent bundle $\tau_E\colon TE \to E$, which is called the extended Lagrangian phase space, and we will define a new Lagrangian in this extended space whose solutions are related to the solutions of the original system. Natural coordinates on TE will be denoted by (x^0, x^i, w^0, w^i).

The manifold $J^1\pi$ can be canonically embedded into TE by means of the map $i\colon J^1\pi \to TE$ given by $i(j^1_{t_0}\sigma) = \dot{\sigma}(t_0)$. In fact, the image of i is included into the open submanifold $\widehat{TE} \subset TE$ of vectors which are not vertical over $\mathbb{R}$. Conversely, we can define a map $p\colon \widehat{TE} \to J^1\pi$, which is a left inverse of i, defined as follows: for $w = \dot{\gamma}(0) \in \widehat{TE}$, we consider the function $\varphi = \pi \circ \gamma\colon \mathbb{R} \to \mathbb{R}$; this function is locally invertible in a neighborhood of $s = 0$ since $\dot{\varphi}(0) \neq 0$, thus we can consider $\sigma = \gamma \circ \varphi^{-1}$, which is a local section of π. The 1-jet of σ at the point $t_0 = \varphi(0)$ is well defined (it does not depend on the choice of the curve γ that represents w), and we define $p(w) = j^1_{t_0}\sigma$. In coordinates, the expression of i and p are

$$i(t, q^i, v^i) = (t, q^i, 1, v^i) \qquad \text{and} \qquad p(x^0, x^i, w^0, w^i) = \left(x^0, x^i, \frac{w^i}{w^0}\right).$$

From the above expression it is clear that $p \circ i = \mathrm{id}$ (but $i \circ p \neq \mathrm{id}$), and it is easy to see that the kernel of Tp is generated by the Liouville vector field Δ on TE restricted to $\widehat{TE}$.

Next we define a Lagrangian function $\hat{L} \in C^\infty(\widehat{TE})$ in such a way that the action defined by a curve in the jet formalism and the action defined by the corresponding curve in the extended formalism coincide, that is, with the same notation as above we look for a function $\hat{L}$ such that

$$\int_{t_0}^{t_1} (j^1\sigma)^* L \, dt = \int_{s_0}^{s_1} \dot{\gamma}^* \hat{L} \, ds$$

under the change of variable $t = \varphi(s)$. It follows that the Lagrangian $\hat{L}$ is defined by

$$\hat{L}(\dot{\gamma}(0)) = L(j^1_{t_0}\sigma)\dot{\varphi}(0).$$

In other words $\hat{L} = w^0(p^*L)$, which in coordinates reads

$$\hat{L}(x^0, x^i, w^0, w^i) = L(x^0, x^i, w^i/w^0)w^0$$

which is homogeneous of degree 1.

Proposition 11 *The following relations hold:*

1. $\theta_{\hat{L}} = p^*\Theta_L$ *and* $\Theta_L = i^*\theta_{\hat{L}}$.

2. $\omega_{\hat{L}} = p^*\Omega_L$ *and* $\Omega_L = i^*\omega_{\hat{L}}$.

3. $E_{\hat{L}} = 0$.

Proof A simple calculation in coordinates shows that

$$\theta_{\hat{L}} = \frac{\partial\hat{L}}{\partial w^0}dx^0 + \frac{\partial\hat{L}}{\partial w^i}dx^i = -p^*E_L dx^0 + p^*\left(\frac{\partial L}{\partial v^i}\right)dx^i = p^*\left(-E_L dt + \frac{\partial L}{\partial v^i}dq^i\right) = p^*\Theta_L.$$

and

$$E_{\hat{L}} = (\Delta w^0)p^*L - w^0 p^*L = w^0 p^*L - w^0\Delta(p^*L) - w^0 p^*L = 0.$$

The other properties follow easily form the first one by using $p \circ i = \mathrm{id}$. ∎

When the Lagrangian L is regular, it follows from the preceding proposition that the kernel of $\omega_{\hat{L}}$ is 2-dimensional and that Δ is in the kernel. Thus $\hat{L}$ is a singular Lagrangian, but it is easy to see that its dynamical equation has solutions defined everywhere and furthermore, among them, there are SODE solutions, since L is a type II-Lagrangian (see [5] for the details).

Locally, if $\Gamma = \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial q^i} + f^i(t, q^j, v^j)\frac{\partial}{\partial v^i}$ is the solution of the dynamics in the time-dependent formalism, then the solution of the dynamics in the homogeneous extended formalism is

$$\hat{\Gamma} = w^0\frac{\partial}{\partial x^0} + w^i\frac{\partial}{\partial x^i} + (w^0)^2 f(x^0, x^j, w^j/w^0)\frac{\partial}{\partial w^i} + \lambda\Delta,$$

for λ an arbitrary function on $\widehat{TE}$.

Once we have transformed our time-dependent problem into an autonomous one, and taking into account that the Lagrangian is singular but does not generate constraints, we can apply the theory that we have developed in the previous sections. We look for a vector field Y on E such that its integral curves are also integral curves of the dynamical vector fields $\hat{\Gamma}$. Since we are interested in integral curves parametrized by time, we must chose such vector field Y in the image of the map i, that is, we will take a jet field $X\colon E \to J^1\pi$ and the vector field $Y = i \circ X$. Then we have that

$$Y^*\theta_{\hat{L}} = Y^*p^*\Theta_L = X^*i^*p^*\Theta_L = X^*\Theta_L,$$

so that the Hamilton–Jacobi equation amounts to $d(X^*\Theta_L) = 0$. Locally, the form $X^*\Theta_L$ will be exact, $X^*\Theta_L = dS$, i.e.

$$X^*\left(-E_L dt + \frac{\partial L}{\partial v^i}dq^i\right) = \frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial q^i}dq^i.$$

Thus the Hamilton–Jacobi equation reads in coordinates

$$\frac{\partial S}{\partial t} = -E_L(t, q^i, X^i)$$
$$\frac{\partial S}{\partial q^i} = \frac{\partial L}{\partial v^i}(t, q^i, X^i),$$

which are the expected expressions of the Hamilton–Jacobi theory for time-dependent Lagrangian systems.

5.2 The Hamiltonian formalism

In time-dependent non-relativistic Hamiltonian Mechanics, the Hamiltonian is not a function but a section h of a certain bundle. Given a bundle $\pi\colon E \to \mathbb{R}$ we consider the affine-dual bundle $\mathrm{Aff}(J^1\pi, \mathbb{R})$, which is canonically isomorphic to T^*E, and the vector bundle $\nu\colon J^{1*}\pi \equiv \mathrm{Ver}(\pi)^* \to E$ dual to the vertical bundle. We have an affine bundle fibration $\mu\colon T^*E \to J^{1*}\pi$ and a Hamiltonian is a section h of the projection μ.

Given a Hamiltonian section $h\colon J^{1*}\pi \to T^*E$, the pullback by h of the canonical symplectic form $\omega = -d\theta$ on T^*E defines a 2-form $\Omega_h = h^*\omega$ on $J^{1*}\pi$. The associated Hamiltonian vector fields are the solutions Γ_h to the equations

$$i(\Gamma_h)\Omega_h = 0 \qquad \text{and} \qquad i(\Gamma_h)dt = 1. \qquad (23)$$

It is clear that $\Omega_h = -d\Theta_h$ where $\Theta_h = h^*\theta$.

The relation with the Lagrangian formalism is as follows (see [7] for details). From the Lagrangian L we can define two maps, usually called the Legendre transformation $\mathcal{F}L\colon J^1\pi \to J^{1*}\pi$ and the extended Legendre transformations $\hat{\mathcal{F}}_L\colon J^1\pi \to T^*E$, related by $\mu \circ \hat{\mathcal{F}}_L = \mathcal{F}L$. When the Lagrangian is hyper-regular we have that $\mathcal{F}L$ is invertible and a unique section h of μ is determined by the equation $\hat{\mathcal{F}}_L = h \circ \mathcal{F}L$. When L is regular we must restrict the study to the image of $\mathcal{F}L$. For simplicity, we will assume that the Lagrangian L is hyper-regular.

Let us consider the homogeneous Lagrangian $\hat{L} \in C^\infty(\widehat{TE})$ and the Legendre transformation $\mathcal{F}_{\hat{L}}\colon \widehat{TE} \to T^*E$ defined by $\hat{L}$. Then the relation between the Legendre transformation $\mathcal{F}_{\hat{L}}$ and $\hat{\mathcal{F}}_L$ is given by $\hat{\mathcal{F}}_L = i \circ \mathcal{F}_{\hat{L}}$. In coordinates (x^0, x^i, w^0, w^i) in $\widehat{TE}$ and (x^0, x^i, u, p_i) on T^*E, the expression of the Legendre transformation is

$$\mathcal{F}_{\hat{L}}(x^0, x^i, w^0, w^i) = \left(x^0, x^i, -p^*E_L, p^*\left(\frac{\partial L}{\partial v^i}\right) \right),$$

the composition $\hat{\mathcal{F}}_L = \mathcal{F}_{\hat{L}} \circ i\colon J^1\pi \to T^*E$ is given by

$$\hat{\mathcal{F}}_L(t, x^i, v^i) = \left(t, x^i, -E_L, \frac{\partial L}{\partial v^i} \right),$$

and composing with the projection $\mu\colon T^*E \to J^{1*}\pi$ we get the map $\mathcal{F}L\colon J^1\pi \to J^{1*}\pi$, which in coordinates reads

$$\mathcal{F}L(t, x^i, v^i) = \left(t, x^i, \frac{\partial L}{\partial v^i} \right).$$

Since we are assuming that the Lagrangian L is hyper-regular, it follows that the Lagrangian $\hat{L}$ is almost-regular, and we can construct the Hamiltonian formulation. The kernel of $T\mathcal{F}_{\hat{L}}$ is spanned by the Liouville vector field Δ on $\widehat{TE}$, and moreover we have $\mathcal{F}_{\hat{L}}(\lambda w) = \mathcal{F}_{\hat{L}}(w)$ for every $\lambda \neq 0$, so that the image of $\mathcal{F}_{\hat{L}}$ coincides with the image of $\hat{\mathcal{F}}_L$. Since $\mathcal{F}L$ is invertible, we can identify the image of $\mathcal{F}_{\hat{L}}$ with $J^{1*}\pi$, or better, with the image of $J^{1*}\pi$ by a unique section $h\colon J^{1*}\pi \to T^*E$ of μ given explicitly by $h = \hat{\mathcal{F}}_L \circ \mathcal{F}L^{-1}$.

Thus, with the same notation as in the general case, we have that $P = J^{1*}\pi$ and $j_0 = h$. If we denote by $\omega = -d\theta$ the canonical symplectic form on T^*E, then the 2-form $\omega_0 = j_0^*\omega$ is $\Omega_h = h^*\omega$, that is the differential of $\Theta_h = h^*\theta$. Following our general theory for unconstrained singular systems, we must look for a section α such that $\alpha^*\Theta_h$ is locally an exact form dS. In coordinates $h(t, x^i, p_i) = (t, x^i, -H(t, x^i, p_i), p_i)$ and hence

$$\alpha^*\Theta_h = \alpha_i dq^i - (H \circ \alpha)\, dt = \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial x^i} dx^i,$$

from where we get

$$\alpha_i = \frac{\partial S}{\partial x^i} \qquad \text{and} \qquad \frac{\partial S}{\partial t} + H(t, x^i, \alpha_i) = 0,$$

or equivalently

$$\frac{\partial S}{\partial t} + H\left(t, x^i, \frac{\partial S}{\partial x^i}\right) = 0,$$

which is the classical time dependent Hamilton–Jacobi equation.

6 Distance on a Riemann manifold: the free relativistic particle

6.1 General features

We consider a Riemannian or semi-Riemannian manifold (Q, g) and the Lagrangian $L(v) = \sqrt{g(v, v)}$. In the semi-Riemannian case we restrict v to be time-like, i.e., $g(v, v) > 0$. In particular, if g is the Lorentz metric, this Lagrangian models a free relativistic particle on the manifold Q.

Lagrangian dynamics   The Lagrangian L is singular. In fact, it is homogeneous of degree one, hence the energy function vanishes identically $E_L = 0$. The Cartan 1-form is given by

$$\theta_L(U) = \frac{g(v, w)}{\sqrt{g(v, v)}}$$

for all $U \in T(TQ)$, where $v = \tau_{TQ}(U)$ and $w = T_v\tau_Q(U)$. The kernel of the Cartan 2-form ω_L is generated by the geodesic spray Γ and the Liouville vector field Δ. There exists underdetermined global second-order dynamics given by $\Gamma + \lambda\Delta$, for any function $\lambda \in C^\infty(TQ)$. See [6, 19].

Hamilton–Jacobi equation   Let X be a nowhere vanishing vector field on Q, and everywhere time-like in the semi-Riemannian case. From the expression of θ_L above we immediately have that

$$X^*\theta_L = \frac{1}{\sqrt{g(X, X)}} X^\flat,$$

where we have denoted by $X^\flat$ the 1-form on Q such that $\langle X^\flat, Y \rangle = g(X, Y)$ for all vector fields Y on Q. If we define $\hat{X}$ as the unitary vector field in the direction of X, that is

$$\hat{X} = \frac{1}{\sqrt{g(X, X)}} X,$$

then we have that $X^*\theta_L = \hat{X}^\flat$.

Since the energy function vanishes identically, the Hamilton–Jacobi equation reduces to $d(X^*\theta_L) = 0$. Let us find an alternative expression for this condition in terms of the Levi-Civita connection associated with the metric.

Proposition 12 *A time-like vector field $X \in \mathfrak{X}(Q)$ is a solution of the Hamilton–Jacobi equation of the Lagrangian $L(v) = \sqrt{g(v,v)}$ if, and only if,*

1. *the distribution $X^\perp$ is integrable, and*

2. *$\nabla_X X = \lambda X$ for some function $\lambda \in C^\infty(Q)$.*

Proof   If X is a solution of the Hamilton–Jacobi equation, i.e. $d(X^*\theta_L) = 0$, we have that $d\hat{X}^\flat = 0$, hence $X^\perp = (\hat{X}^\flat)^\circ$ is an integrable distribution.

Observe that, for every vector field $Z \in \mathfrak{X}(Q)$, using the Levi–Civita connection associated with the metric g, since this connection is torsion-free, the exterior differential can be calculated by skew-symmetrization of the covariant differential, and thus

$$dX^\flat(Y,Z) = \nabla_Y X^\flat(Z) - \nabla_Z X^\flat(Y) = g(\nabla_Y X, Z) - g(\nabla_Z X, Y). \tag{24}$$

Then we have

$$0 = (d\hat{X}^\flat)(\hat{X}, Z) = g(\nabla_{\hat{X}}\hat{X}, Z) - g(\nabla_Z \hat{X}, \hat{X})$$
$$= g(\nabla_{\hat{X}}\hat{X}, Z) - \frac{1}{2}\nabla_Z(g(\hat{X}, \hat{X})) = g(\nabla_{\hat{X}}\hat{X}, Z)$$

then $\nabla_{\hat{X}}\hat{X} = 0$, and hence $\nabla_X X = \lambda X$ with $\lambda = \nabla_X(\ln\sqrt{g(X,X)})$. This proves the direct statement.

Conversely, assume that X satisfies conditions (1) and (2). We will prove that $d\hat{X}^\flat = 0$, so that $d(X^*\theta_L) = 0$. Taking the derivative ∇_X of $g(X,X)$ we find that the function λ is given by the relation $\nabla_X\sqrt{g(X,X)} = \lambda\sqrt{g(X,X)}$, from where $\nabla_{\hat{X}}\hat{X} = 0$ follows.

On the other hand, if the distribution $X^\perp$ is integrable, there exists locally a nowhere vanishing function φ such that $d(\varphi\hat{X}^\flat) = 0$. First we will prove that $d\varphi = (\nabla_{\hat{X}}\varphi)\hat{X}^\flat$. Indeed, for every $Z \in \mathfrak{X}(Q)$,

$$0 = d(\varphi\hat{X}^\flat)(\hat{X}, Z) = \nabla_{\hat{X}}(\varphi\, g(\hat{X}, Z)) - \nabla_Z(\varphi\, g(\hat{X}, \hat{X})) - \varphi\, g(\hat{X}, [\hat{X}, Z])$$
$$= (\nabla_{\hat{X}}\varphi)g(\hat{X}, Z) - \nabla_Z\varphi$$

where we have used that $g(\hat{X}, \hat{X}) = 1$ and

$$g(\hat{X}, [\hat{X}, Z]) = g(\hat{X}, \nabla_{\hat{X}}Z) - g(\hat{X}, \nabla_Z\hat{X}) = g(\hat{X}, \nabla_{\hat{X}}Z) - \frac{1}{2}\nabla_Z(g(\hat{X}, \hat{X})) = g(\hat{X}, \nabla_{\hat{X}}Z)$$

Therefore $\nabla_Z\varphi = (\nabla_{\hat{X}}\varphi)g(\hat{X}, Z)$, for every $Z \in \mathfrak{X}(Q)$, which proves that $d\varphi = (\nabla_{\hat{X}}\varphi)\hat{X}^\flat$. But we have

$$0 = d(\varphi\hat{X}^\flat) = d\varphi \wedge \hat{X}^\flat + \varphi\, d\hat{X}^\flat = \varphi\, d\hat{X}^\flat,$$

that is $d\hat{X}^\flat = 0$, and hence $d(X^*\theta_L) = 0$.   ∎

6.2 Alternative Lagrangian description

It is well known [4] that $L(v) = \frac{1}{2}g(v,v)$ provides a Lagrangian description of a free motion in a Riemannian manifold, that is, the geodetic spray Γ, which is, moreover, regular. It is interesting to compare the solutions of the Hamilton–Jacobi equation for this Lagrangian with the above one.

The Lagrangian L is homogeneous of degree two. Therefore, we have that $E_L = L = \frac{1}{2}g(v,v)$. The Cartan 1-form is given by $\theta_L(W) = g(v,w)$ for $W \in T_v(TQ)$, where $w = T_v\tau_Q(W)$, so that $X^*\theta_L = X^\flat$. Then the Hamilton–Jacobi equation is

$$dX^\flat = 0 \qquad \text{and} \qquad g(X,X) = c,$$

for some constant $c \geq 0$. For $c = 0$ we have the trivial solution $X = 0$, and for $c > 0$ we can rescale X to $X/\sqrt{c}$, so that we can consider only the case $c = 1$, which means that we can restrict our study to the case that X is a unit vector field.

Proposition 13 *A unit vector field X is a solution of the Hamilton–Jacobi equation for the Lagrangian $L(v) = \frac{1}{2}g(v,v)$ if, and only if, $X^\perp$ is integrable and $\nabla_X X = 0$.*

Proof By (24), we have $dX^\flat(Y,Z) = g(\nabla_Y X, Z) - g(\nabla_Z X, Y)$, so that the Hamilton–Jacobi equation is equivalent to

$$g(\nabla_Y X, Z) = g(\nabla_Z X, Y).$$

If we take $Y = Z = X$ the condition is identically satisfied. If we take $Y = X$ and $Z \in X^\perp$, we have $g(\nabla_X X, Z) = 0$, thus the vector field X satisfies that $\nabla_X X = \lambda X$ for some function $\lambda \in C^\infty(Q)$. But from the normalization condition $g(X,X) = 1$ we have that $\lambda = g(\nabla_X X, X) = \frac{1}{2}\nabla_X[g(X,X)] = 0$, so that $\nabla_X X = 0$. Finally, for $Y, Z \in X^\perp$ we have $g(\nabla_Y X, Z) = g(\nabla_Z X, Y)$, which, as above, is equivalent to $g(X, [Y,Z]) = 0$. ∎

Remarks

1. The condition $\nabla_X X = 0$ gives the generalized solution of the Hamilton–Jacobi problem, and together with the integrability of $X^\perp$ give the classical Hamilton–Jacobi solution.

2. Recall that a vector field X satisfying that $\nabla_X X = 0$ is called a *geodetic vector field*, and its integral curves are geodesics parametrized by arc length. If we reparametrize the curves we have the vector field $\bar{X} = fX$ for some function f nowhere vanishing. Therefore $\nabla_{\bar{X}}\bar{X} = f(Xf)X$, so that $\nabla_{\bar{X}}\bar{X} = \lambda\bar{X}$ with $\lambda = f(Xf)$. Notice the relation between the unit length parametrization in the regular case with the projective theory in the singular case. In the regular case, the vector field X must be unitary in order to have integral curves parametrized by arc-length.

3. The interpretation of the above results is (in both cases) as follows: the vector X points in the direction of propagation of the rays, and the orthogonal distribution to X is the tangent to the wavefront. Wavefronts are manifolds, so the orthogonal distribution to X is integrable. Furthermore the rays are the geodesics of the metric, and therefore the vector field X must be a geodetic vector field.

7   Free motion on Lie groups, rigid bodies and the electron monopole system

In this section we wish to show that the notion of generalized solution is the only one available in generic situations, because solutions in terms of characteristic functions are not available globally either for topological reasons or because of invariance requirements. We are going to present a simplified approach to dynamics on Lie groups (see [11]), since we wish to isolate the main conceptual aspects of the Hamilton–Jacobi problem on spaces with nontrivial topology, although parallelizable.

7.1   Free motion on Lie groups

By free motions on a Lie group G we mean motions associated with equations of motion analogous to the equation $d^2x/dt^2 = 0$, which are written in some affine space. Thus for simplicity we consider our group realized as a group of matrices $g \in GL(n, \mathbb{R})$. The equations of free motion will be written as

$$\frac{d}{dt}\left(g^{-1}(t)\,\dot{g}(t)\right) = 0\,.$$

These differential equations admit a Lagrangian description in terms of a Lagrangian function

$$L(g, \dot{g}) = \frac{1}{2}\mathrm{Tr}\,\left[(g^{-1}\,\dot{g})^2\right]\,.$$

The geometrical objects associated with L are simply written

$$\theta_L(g, \dot{g}) = \mathrm{Tr}\,\left[(g^{-1}\,\dot{g})\,(g^{-1}\,dg)\right]\,,\quad \omega_L = -d\theta_L,\quad E_L = L\,.$$

We will show that every left-invariant vector field X provides us with a solution of the generalized Hamilton–Jacobi problem. So, let us consider $X \in \mathfrak{X}_{()}LG$. Denote by ξ the value of X at the identity e in the group G, that is $\xi = X(e) \in \mathfrak{g}$. In this way, we have that $X(g) = g\xi$, or $g^{-1}X(g) = \xi$.

On the one hand, it is clear that the pullback of the energy is constant:

$$(X^*E_L)(g) = (X^*L)(g) = \frac{1}{2}\,\mathrm{Tr}[(g^{-1}X(g))^2] = \frac{1}{2}\,\mathrm{Tr}(\xi^2).$$

On the other hand, the pullback of the symplectic form does not vanish. Indeed, we calculate $\langle X^*\theta_L, Y\rangle$ for a vector field $Y \in \mathfrak{X}(G)$, which we may take to be left-invariant, $Y(g) = g\zeta$, for some $\zeta \in \mathfrak{g}$:

$$\langle X^*\theta_L, Y\rangle(g) = \frac{d}{dt}L(X(g) + tY(g))\Big|_{t=0} = \frac{1}{2}\frac{d}{dt}\,\mathrm{Tr}[(\xi + t\zeta)^2]\Big|_{t=0} = \mathrm{Tr}(\xi\zeta).$$

Thus the differential evaluated on two left-invariant vector fields Y_1, Y_2, $Y_1(g) = g\zeta_1$ and $Y_2(g) = g\zeta_2$, is

$$d(X^*\theta_L)(g)(Y_1, Y_2) = Y_1(g)(\mathrm{Tr}(\xi\zeta_2)) - Y_2(g)(\mathrm{Tr}(\xi\zeta_1)) - \mathrm{Tr}(\xi\,g^{-1}[Y_1, Y_2](g)))$$
$$= 0 - 0 - \mathrm{Tr}(\xi\,[\zeta_1, \zeta_2]),$$

so that $(X^*\omega_L)(Y_1, Y_2) = \mathrm{Tr}(\xi\,[\zeta_1, \zeta_2])$.

It follows that X is not a solution of the Hamilton–Jacobi problem in the standard sense (except for Abelian groups). Nevertheless X is a solution of the generalized Hamilton–Jacobi problem, since $i_X(X^*\omega_L) = 0$:

$$i_X(X^*\omega_L)(Y) = \mathrm{Tr}(\xi\,[\xi, \zeta]) = \mathrm{Tr}(\xi^2\zeta - \xi\zeta\xi) = 0,$$

where use has been made of the properties of the trace.

Therefore, the left trivialization provides a diffeomorphism $\Phi\colon G \times \mathfrak{g} \to TG$, given by $\Phi(g, \xi) = T_eL_g(\xi)$, such that for every $\xi \in \mathfrak{g}$ we get a solution of the generalized problem. In other words we have a complete solution of the generalized Hamilton–Jacobi problem, where the parameter space Λ is the Lie algebra, $\Lambda = \mathfrak{g}$.

The associated constant of the motion $F = \mathrm{pr}_2 \circ \Phi^{-1}\colon TG \to \mathfrak{g}$ is explicitly given by $F(g, \dot{g}) = g^{-1}\dot{g}$, or in other words $F(g, g\xi) = \xi$. We can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the trace operation, that is we identify $\nu \in \mathfrak{g}^*$ with $\xi \in \mathfrak{g}$ if $\langle \nu, \zeta \rangle = \mathrm{Tr}(\xi\zeta)$ for every $\zeta \in \mathfrak{g}$. Under this identification, we get a map $\mu\colon TG \to \mathfrak{g}^*$ given by

$$\langle \mu(g, \dot{g}), \zeta \rangle = \mathrm{Tr}(g^{-1}\dot{g}\zeta),$$

which is the momentum map for the left action of G on TG.

Notice that we have exploited the left-invariance of the Lagrangian function. Furthermore L is also right invariant, since we can write it in the form $L(g, \dot{g}) = \frac{1}{2}\mathrm{Tr}[(\dot{g}g^{-1})^2]$. Therefore we can also define a second foliation by taking the right-invariant vector fields.

Finally, it should be remarked that whenever G is a compact Lie group we cannot have functions on G whose differentials are never vanishing, therefore any invariant foliation (solutions of the generalized Hamilton–Jacobi problem) could never be associated with some dW, for some function $W : G \times \Lambda \to \mathbb{R}$.

7.2 Rigid bodies

Consider a (generalized) rigid body defined on a configuration Lie group G with symmetric inertia tensor $I\colon \mathfrak{g} \to \mathfrak{g}^*$. We will analyze it in the Hamiltonian formalism on the cotangent bundle T^*G. The Hamiltonian function $H \in C^\infty(T^*G)$ is

$$H(\lambda_g) = \frac{1}{2}\langle T_e^*L_g(\lambda_g), I^{-1}T_e^*L_g(\lambda_g)\rangle,$$

where $\langle\ ,\ \rangle$ is the standard pairing. We will show that every right-invariant 1-form $\alpha \in \Omega^1(G)$ is a solution of the generalized Hamilton–Jacobi problem. If $\mu \in \mathfrak{g}^*$ is the value of α at the identity $e \in G$, then we have that $\alpha(g) = T_g^*R_{g^{-1}}(\mu)$. The value of α on a right-invariant vector field $Y = T_eR_g(\zeta)$ is constant,

$$\langle \alpha, Y \rangle(g) = \langle T_g^*R_{g^{-1}}\mu, T_eR_g(\zeta)\rangle = \langle \mu, \zeta \rangle,$$

and hence the differential of α over two right-invariant vector fields Y_1 and Y_2 is

$$
\begin{aligned}
d\alpha(Y_1, Y_2)(g) &= Y_1(g)\langle \alpha, Y_2 \rangle - Y_2(g)\langle \alpha, Y_1 \rangle - \langle \alpha, [Y_1, Y_2]\rangle(g) \\
&= Y_1(g)\langle \mu, \zeta_2 \rangle - Y_2(g)\langle \mu, \zeta_1 \rangle - \langle \mu, -[\zeta_1, \zeta_2]\rangle \\
&= 0 - 0 + \langle \mu, [\zeta_1, \zeta_2]\rangle.
\end{aligned}
$$

Therefore, for every $\zeta_1, \zeta_2 \in \mathfrak{g}$ we have

$$d\alpha(g)(T_e R_g(\zeta_1), T_e R_g(Z\zeta_2)) = \langle \mu, [\zeta_1, \zeta_2] \rangle.$$

The fiber derivative of the Hamiltonian is given by $\mathcal{F}H(\lambda_g) = T_e L_g I^{-1} T_e^* L_g(\lambda_g)$, for every $\lambda - g \in T_g^* G$. Indeed

$$\langle \lambda_g', FH(\lambda_g) \rangle = \frac{d}{dt} H(\lambda_g + t\lambda_g') \Big|_{t=0} = \langle T_e^* L_g(\lambda_g'), I^{-1} T_e^* L_g(\lambda_g) \rangle = \langle \lambda_g', T_e L_g I^{-1} T_e^* L_g(\lambda_g) \rangle.$$

Therefore, the vector field $X \in \mathfrak{X}(G)$ associated to α is

$$X(g) = \mathcal{F}H(\alpha(g)) = \mathcal{F}H(T_g^* R_{g^{-1}}(\mu)) = T_e L_g I^{-1}(Ad_g^*(\mu)),$$

The contraction of $d\alpha$ with X is given by

$$\begin{aligned}
(i_X d\alpha)(g)(TR_g(\zeta)) &= d\alpha(g)(X(g), TR_g(\zeta)) \\
&= d\alpha(g)(T_e R_g(T_g R_{g^{-1}} X(g)), TR_g(\zeta)) \\
&= \langle \mu, [Ad_g I^{-1} Ad_g^* \mu, \zeta] \rangle,
\end{aligned}$$

where we have used that

$$T_g^* R_{g^{-1}}(X(g)) = T_g^* R_{g^{-1}} T_e L_g I^{-1} Ad_g^* \mu = Ad_g I^{-1} Ad_g^* \mu.$$

Furthermore, the pullback of the Hamiltonian by α is

$$(\alpha^* H)(g) = \frac{1}{2} = \langle T_e^* L_g(T_g^* R_{g^{-1}}(\mu)), I^{-1} T_e^* L_g(T_g^* R_{g^{-1}}(\mu)) \rangle = \frac{1}{2} \langle Ad_g^*(\mu), I^{-1} Ad_g^*(\mu) \rangle.$$

To calculate its differential evaluated at $T_e R_g(\zeta)$, we consider its integral curve $\gamma(t) = \exp(t\zeta)g$ through the point g and hence

$$\begin{aligned}
\langle d(\alpha^* H)(g), T_e R_g(\zeta) \rangle &= \frac{d}{dt}(\alpha^* H)(\gamma(t)) \Big|_{t=0} \\
&= \langle \frac{d}{dt} Ad_{\exp(t\zeta)g}^* \mu \Big|_{t=0}, I^{-1} Ad_g^* \mu \rangle \\
&= \langle Ad_g^* ad_\zeta^* \mu, I^{-1} Ad_g^* \mu \rangle \\
&= \langle ad_\zeta^* \mu, Ad_g I^{-1} Ad_g^* \mu \rangle \\
&= \langle \mu, ad_\zeta Ad_g I^{-1} Ad_g^* \mu \rangle
\end{aligned}$$

and finally adding both terms we get

$$\big(i(X)d\alpha + d(\alpha^* H)\big)(T_e R_g(\zeta)) = \langle \mu, [Ad_g I^{-1} Ad_g^* \mu, \zeta] \rangle + \langle \mu, ad_\zeta Ad_g I^{-1} Ad_g^* \mu \rangle = 0.$$

Thus we have a complete solution of the generalized Hamilton–Jacobi problem, $\Phi \colon G \times \mathfrak{g}^* \to T^* G$ explicitly given by the inverse of the right trivialization map, $\Phi(g, \mu) = T_g^* R_{g^{-1}}(\mu)$. The associated constant of the motion is $F = \mathrm{pr}_2 \circ \Phi^{-1} \colon T^* G \to \mathfrak{g}^*$, which is the momentum map $F = J_L$ associated to the left action of G on $T^* G$, that is,

$$F(\lambda_g) = T_e^* R_g(\lambda_g).$$

As the theory predicts, if $g(t)$ is an integral curve of the vector field X, i.e. it satisfies $\dot{g}(t) = T_e L_{g(t)} I^{-1}(Ad_{g(t)}^* \mu)$, then $\Omega = g^{-1} \dot{g}$ is given by $I\Omega = Ad_g^* \mu$ (with $\mu \in \mathfrak{g}^*$ constant) and hence it satisfies the differential equation $I\dot{\Omega} = -ad_\Omega^* Ad_g^* \mu = -ad_\Omega^*(I\Omega)$, which is the Euler equation.

7.3 The electron monopole system

The equations of motion for a charged particle with electric charge e moving in the external magnetic field of a monopole with magnetic charge g are described by the following second order vector field in $Q = \mathbb{R}^3 - \{0\}$

$$\Gamma = v^j \frac{\partial}{\partial q^j} + \frac{n}{r^3} \epsilon_{ijk} x^j v^k \frac{\partial}{\partial v^i} \, ,$$

where ϵ_{ijk} is the completely skew-symmetric Levi Civita tensor, i.e. such that $\epsilon_{123} = 1$ and with

$$n = \frac{e\,g}{4\,\pi\,m} \, .$$

The vector field Γ admits a symplectic description with the symplectic structure (see e.g [29])

$$\omega = dx^i \wedge dv^i - \frac{n}{2\,r^3} \epsilon_{ijk} x^i \, dx^j \wedge dx^k \, ,$$

and Hamiltonian function

$$H = \frac{1}{2} v^j \, v^j \, .$$

Because ω is closed but not exact, ω cannot be written as a Lagrangian 2-form ω_L. It is however possible to write it as a Lagrangian 2-form locally by using a local Lagrangian.

In addition to the Hamiltonian function, the dynamical system admits other constants of the motion associated with the rotational symmetry group; they are

$$l_i = \epsilon_{ijk} x^j \, v^k + \frac{n\,x^i}{r} \, .$$

They are made up of the expected components of the orbital angular momentum plus the "helicity term" $n\,x^j/r$.

It is possible to find local solutions of the standard Hamilton–Jacobi equation by using constants of the motion H, $\mathbf{l}^2$ and l_3, for instance. We may solve for the velocities, and by replacing them in Γ we find a 3-parameter family of vector fields defined on some open submanifold of $\mathbb{R}^3 - \{0\}$.

It should be noticed, however, that it is not possible to find globally defined vector valued solutions, because if we denote the sought solution by $Y = Y^j \frac{\partial}{\partial x^j}$, we would have

$$Y^*(dx^i \wedge dv^i) = \frac{n}{2\,r^3} \epsilon_{jki} x^j \, dx^k \wedge dx^i \, ,$$

which is not possible because the left hand side is exact while the right hand side is not.

Nevertheless, it is possible to describe the electron monopole system as a reduction of a globally defined Lagrangian system with a singular Lagrangian but without secondary constraints. To this end we replace the configuration space $Q = \mathbb{R}^3 - \{0\} \approx S^2 \times \mathbb{R}_+$ with a covering by replacing S^2 with S^3 in the product of manifolds. The new configuration space will be $SU(2,\mathbb{C}) \times \mathbb{R}_+$.

The covering map $\pi : SU(2,\mathbb{C}) \to S^2$ is given by the following construction. Let (x^1, x^2, x^3) be the coordinates in $\mathbb{R}^3 - \{0\}$ and let $\widehat{x}^j = x^j/r \in S^2$, so that they satisfy $\widehat{x}^j \, \widehat{x}^j = 1$. Now we

describe $\mathbb{R}^3$ in terms of the 2×2 traceless Hermitian matrices using as a basis Pauli matrices, we have

$$M = \vec{x} \cdot \vec{\sigma} = \begin{bmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{bmatrix}.$$

Now we describe our covering map by introducing the following matrices to describe $\mathbb{R}^4$

$$\mathbf{s} = y^0 I + i\vec{y} \cdot \vec{\sigma} = \begin{bmatrix} y^0 + iy^3 & y^2 + iy^1 \\ -y^2 + iy^1 & y^0 - iy^3 \end{bmatrix},$$

and setting $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ by means of

$$\pi(\mathbf{s}) = \mathbf{s}\,\sigma_3\,\mathbf{s}^\dagger = \vec{x} \cdot \vec{\sigma}.$$

This map is also known as the Kustaanheimo-Stiefel map (for a classical and quantum version of this map see the recent papers [2, 3]).

This relation makes sense because both sides are traceless Hermitian matrices and Pauli matrices are a basis for the real linear space of Hermitian matrices with zero trace. We notice that $\mathbf{s}$ represent elements of $SU(2, \mathbb{C})$ when the constraint

$$\mathbf{s}\,\mathbf{s}^\dagger = ((y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2)\,I = \det s\,I = I$$

is imposed.

To spell out the way (x^1, x^2, x^3) depends on (y^0, y^1, y^2, y^3), i.e. the pull-back of coordinate functions from $\mathbb{R}^3 - \{0\}$ to $\mathbb{R}^4 - \{0\}$, we notice that

$$\mathbf{s}^\dagger = \begin{bmatrix} y^0 - iy^3 & -y^2 - iy^1 \\ y^2 - iy^1 & y^0 + iy^3 \end{bmatrix}$$

so that

$$\mathbf{s}\,\sigma_3\,\mathbf{s}^\dagger = \begin{bmatrix} y^0 + iy^3 & y^2 + iy^1 \\ -y^2 + iy^1 & y^0 - iy^3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y^0 - iy^3 & -y^2 - iy^1 \\ y^2 - iy^1 & y^0 + iy^3 \end{bmatrix}$$

which is given by

$$\mathbf{s}\,\sigma_3\,\mathbf{s}^\dagger = \begin{bmatrix} (y^0)^2 + (y^3)^2 - (y^1)^2 - (y^2)^2 & 2(y^0\,y^1 + y^2\,y^3) - 2i(y^0\,y^2 - y^1\,y^3) \\ 2(y^0\,y^1 + y^2\,y^3) + 2i(y^0\,y^2 - y^1\,y^3) & -(y^0)^2 - (y^3)^2 + (y^1)^2 + (y^2)^2 \end{bmatrix}$$

provides us with

$$\begin{aligned} x^1 &= 2(y^0\,y^1 + y^2\,y^3) \\ x^2 &= 2(y^0\,y^2 - y^1\,y^3) \\ x^3 &= (y^0)^2 + (y^3)^2 - (y^1)^2 - (y^2)^2. \end{aligned}$$

Now we find the tangent map of the covering map

$$T\pi : T(SU(2, \mathbb{C}) \times \mathbb{R}_+) \to T(S^2 \times \mathbb{R}_+),$$

more explicitly

$$v^1 = 2(y^0 u^1 + u^0 y^1 + u^2 y^3 + y^2 u^3)$$

$$\begin{aligned}
v^2 &= 2(y^0 u^2 + u^0 y^2 - u^3 y^1 + y^3 u^1) \\
v^3 &= 2(y^0 u^0 + u^3 y^3 - y^1 u^1 - y^2 u^2)
\end{aligned}$$

and the pull-back of the 2-form

$$\omega = dx^i \wedge dv^i - \frac{n}{2\,r^3}\,\epsilon_{ijk}\,x^i\,dx^j \wedge dx^k\ ,$$

namely, $(T\pi)^*\omega$, will be exact and moreover Lagrangian

$$(T\pi)^*\omega = \omega_L,$$

with Lagrangian L on $T(SU(2,\mathbb{C}) \times \mathbb{R}_+)$ given by

$$L = \frac{1}{2}T\pi^*(v^j v^j) + i\,n\,\mathrm{Tr}\,\sigma_3\,\mathbf{s}^{-1}\,\dot{\mathbf{s}}\,.$$

The fibering map $\pi : SU(2,\mathbb{C}) \times \mathbb{R}_+ \to S^2 \times \mathbb{R}_+$ is actually a principal bundle projection with group $U(1)$ given by

$$U(1) = \{\exp(i\,t\,\sigma_3) \mid t \in \mathbb{R}\}\,,$$

and acting on $SU(2,\mathbb{C})$ on the right. The tangent bundle group $TU(1) = U(1) \otimes \mathbb{R}$ will now be the structure group of the tangent bundle $T(SU(2,\mathbb{C}) \times \mathbb{R}_+) \to T(S^2 \times \mathbb{R}_+)$.

Within the notation we have used, the left-invariant vector field along σ_3 generator of the $U(1)$ action is

$$X^3 = y^0 \frac{\partial}{\partial y^3} - y^3 \frac{\partial}{\partial y^0} - \epsilon_{3ij}\,y^i \frac{\partial}{\partial y^j}.$$

while the infinitesimal generator of $TU(1)$ will be the tangent lift

$$(X^3)^T = X^3 + \dot{y}^0 \frac{\partial}{\partial \dot{y}^3} - \dot{y}^3 \frac{\partial}{\partial \dot{y}^0} - \epsilon_{3ij}\,\dot{y}^i \frac{\partial}{\partial \dot{y}^j}$$

and the vertical lift

$$(X^3)^v = y^0 \frac{\partial}{\partial \dot{y}^3} - y^3 \frac{\partial}{\partial \dot{y}^0} - \epsilon_{3ij}\,y^i \frac{\partial}{\partial \dot{y}^j}\,.$$

By using the pull-back of constants of the motion from $T(\mathbb{R}^3 - \{0\})$ to $T(\mathbb{R}^4 - \{0\})$, we will find generalized extended space. On the 8-dimensional carrier space they will define submanifolds of codimension three. If the constants of the motion used are pairwise in involution, the invariant submanifold will be isotropic, otherwise we will give rise to solutions of the generalized Hamilton–Jacobi problem.

8 Conclusions and outlook

In this paper we show that to deal with bi-Hamiltonian systems in the Hamilton–Jacobi setting it is convenient to introduce generalized solutions, i.e. invariant submanifolds (or foliations) with dimension equal to the dimension of the configuration space without the requirement of Lagrangianity. Thus the associated PDE will have solutions given by vector valued functions. When the "Lagrangian" requirement is made, these functions will be the coefficients of an exact

1-form, and we recover the standard PDE for the principal function W or the characteristic function S. The link via the Feynman approach to quantum mechanics between these solutions and the phase of the wave function seems to suggest that only invariant foliations with Lagrangian leaves with respect to the admissible alternative symplectic structure should be accepted.

According to von Neumann's representation theory we would have to accept as Hilbert spaces the space of square integrable functions defined on some invariant Lagrangian submanifold (according to the chosen symplectic structure). This raises the problem of the selection of the appropriate "Lebesgue measure" and how to compare the descriptions on these alternative Hilbert spaces. These aspects will be taken up elsewhere.

Formulation of the Hamilton–Jacobi theory on the tangent bundle in terms of the Lagrangian formalism, prepares us ready to consider the problem of Hamilton–Jacob theory connected with degenerate Lagrangians (gauge theories) in full generality. Extension of the ideas in this paper for Lagrangian and Hamiltonian systems on Lie algebroids [31, 25] is also worthy of study. These aspects should be addressed in the future.

Acknowledgments

References

[1] R. Abraham, J.E. Marsden, *Foundations of Mechanics*, 2nd edition, Benjamin–Cumming, Reading, 1978.

[2] A. D'Avanzo, G. Marmo, "Reduction and unfolding : The Kepler problem", *Int. J. Geom. Methods Mod. Phys.* **2** (2005) 83–109.

[3] A. D'Avanzo, G. Marmo, A. Valentino, "Reduction and unfolding for quantum systems: The Hydrogen atom", *Int. J. Geom. Methods Mod. Phys.* **2** (2005) 1043–1062.

[4] P. G. Bergmann, *Introduction to the Theory of Relativity*, Prentice-Hall, New York, 1942.

[5] F. Cantrijn, J.F. Cariñena, M. Crampin, L.A. Ibort, "Reduction of degenerate Lagrangian systems", *J. Geom. Phys.* **3** (1986) 353-400.

[6] J.F. Cariñena, "Theory of singular Lagrangians", *Fortschr. Phys.* **38** (1990) 641-679.

[7] J. F. Cariñena, M. Crampin, L. A. Ibort, "On the multisymplectic formalism for first order field theories", *Diff. Geom. Appl. 1 (1991) 345–374.*

[8] J. F. Cariñena, C. López, "The time-evolution operator for singular Lagrangians", *Lett. Math. Phys. 14 (1987) 203–210.*

[9] J.F. Cariñena, C. López, N. Román-Roy, "Geometric study of the connection between the Lagrangian and Hamiltonian Constraints", *J. Geom. Phys.* **4** (1987) 315–334.

[10] J.F. CARIÑENA, M.A. DEL OLMO, P. WINTERNITZ, "On the relation between weak and strong invariance of differential equations", *Lett. Math. Phys.* **29** (1994) 151-63.

[11] C. C. CHIANG, S. C. LEE, G. MARMO, "Lagrangian dynamics on higher-dimensional spaces with application to Kaluza-Klein theories", *J. Math. Phys.* **26** (1985) 1083–1092.

[12] M. CRAMPIN, "On the differential geometry of the Euler–Lagrange equations and the inverse problem of Lagrangian dynamics", *J. Phys. A: Math. Gen.* **14** (1981) 2567–2575.

[13] M. CRAMPIN, "Tangent bundle geometry for Lagrangian dynamics", *J. Phys. A: Math. Gen.* **16** (1983) 3755–3772.

[14] M. CRAMPIN, G.E. PRINCE, G. THOMPSON, " A geometrical version of the Helmholtz conditions in time-dependent Lagrangian dynamics", *J. Phys. A: Math. Gen.* **17** (1984)1437–1447.

[15] M. CRAMPIN, E. MARTÍNEZ, W. SARLET, "Linear connections for systems of second-order ordinary differential equations", *Ann. Inst. H. Poincaré* **65** (1996) 223–249.

[16] B. DUBROVIN, G. MARMO, A. SIMONI, "Alternative Hamiltonian description for quantum systems", *Mod. Phys. Lett. A* **5** (1990) 1229–1234.

[17] G. ESPOSITO, G. MARMO, G. SUDARSHAN, *From classical to quantum mechanics*, Cambridge University Press, 2004.

[18] Z. GE, J. E. MARSDEN, "Lie-Poisson Hamilton–Jacobi theory and Lie-Poisson integrators", *Phys. Lett. A* **133**(3) (1988) 134–139.

[19] X. GRÀCIA, "Fibre derivatives: some applications to singular lagrangians", *Rep. Math. Phys.* **45** (2000) 67-84.

[20] X. GRÀCIA, J. M. PONS, "On an evolution operator connecting Lagrangian and Hamiltonian formalisms", *Lett. Math. Phys.* **17** *(1989) 175–180.*

[21] X. GRÀCIA, J.M. PONS, "A generalized geometric framework for constrained systems", *Diff. Geom. Appl.* **2** (1992) 223-247.

[22] J. KLEIN, "Espaces variationnels et mécanique" *Ann. Inst. Fourier (Grenoble)* **12** *(1962) 1–124*

[23] J. KLEIN "Opérateurs différentielles sur les varietés presque tangentes" *C. R. Acad Sci. Paris* **257** *(1963) 2392–2394*

[24] M. DE LEÓN, J. MARÍN-SOLANO, J.C. MARRERO, M.C. MUÑOZ-LECANDA, N. ROMÁN-ROY, "Singular Lagrangian Systems on Jet bundles", *Fortschr. Phys.* **50** (2002) 105-169.

[25] M. DE LEÓN, J. C. MARRERO, E. MARTÍNEZ, "Lagrangian submanifolds and dynamics on Lie algebroids", *J. Phys. A.: Math. Gen. (Topical Review)* **38** (2005) R241–R308.

[26] P. LIBERMANN, C.M. MARLE, *Symplectic geometry and analytical dynamics*, D. Reidel Publishing Company, Dordrecht, 1987.

[27] G. MARMO, G. MORANDI, N. MUKUNDA, "A geometrical approach to the Hamilton–Jacobi form of the dynamics and its generalizations", *Riv. Nuovo Cim.* **13** (1990) 1–74.

[28] G. MARMO, N. MUKUNDA, J. SAMUEL, "Dynamics and symmetry for constrained systems: a geometrical analysis", *Riv. Nuovo Cim.* **6** (1983) 1–52.

[29] G. MARMO, C. RUBANO, *Particle Dynamics on Fibre bundles.* Bibliopolis, Napoli, 1988.

[30] G. Marmo, E.J. Saletan, A. Simoni, B. Vitale, *Dynamical Systems: A Differential Geometric Approach to Symmetry and Reduction*. Wiley, Chichester, 1985.

[31] E. Martínez, "Lagrangian Mechanics on Lie algebroids", *Acta Appl. Math.* **67** (2001) 295–320.

[32] G. Mendella, G. Marmo, W.M. Tulczyjew, "Integrability of implicit differential equations", *J. Phys. A: Math. Gen.* **28** (1995) 149–163.

[33] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo, C. Rubano, "The inverse problem in the calculus of variations and the geometry of the tangent bundle", *Phys. Rep.* **188** (1990) 147–284.

[34] G.E. Prince, J.E. Aldridge, G.B. Byrnes, "A universal Hamilton–Jacobi equation for second-order ODEs", *J. Phys. A: Math. Gen.* **32** (1999) 827–844.

[35] J. Samuel, "A phase space approach to genmeralized Hamilton–Jacobi theory", *J. Math. Phys.* **21** (1980) 55–59.

[36] W.M. Tulczyjew, "Les sous-varietés lagrangiennes et la dinamique hamiltonienne", *C.R. Acad. Sci. Paris* **283A** (1976) 15–18.

[37] A.M. Vinogradov, I.S. Krasil'schik. "What is Hamiltonian formalism?", *Russian Math. Surveys* **30**(1) (1975) 177–202.