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GEOMETRIC INEQUALITIES ON LOCALLY CONFORMALLY FLAT MANIFOLDS

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Abstract. Through the study of some elliptic and parabolic fully nonlinear PDEs, we establish conformal versions of the quermassintegral inequality, the Sobolev inequality and the Moser-Trudinger inequality for the geometric quantities associated to the Schouten tensor on locally conformally flat manifolds.

1. Introduction

In this paper, we are interested in certain global geometric quantities associated to the Schouten tensor and their relationship to each other in conformal geometry. For an oriented compact Riemannian manifold (M, g) of dimension $n > 2$, there is a sequence of geometric functionals arising naturally in conformal geometry, which generalize the Yamabe functional, introduced by Viaclovsky in [35] as curvature integrals of the Schouten tensor. If we let Ric_g and R_g denote the Ricci tensor and the scalar curvature of g respectively, the Schouten tensor can be written as

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)}\cdot g\right).$$

Let σ_k be the kth elementary symmetric function, and let

$$\sigma_k(g) := \sigma_k(g^{-1}\cdot S_g)$$

be the σ_k*-scalar curvature* of g. We are interested in the following functionals defined by

$$\mathcal{F}_k(g) = vol(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g)\, dg, \quad k = 0, 1, ..., n, \tag{1}$$

where dg is the volume form of g. For convenience, set $\sigma_0(g) = 1$.

We note that $\mathcal{F}_0(g)$ is simply the volume of g and $\mathcal{F}_1(g)$ is the Yamabe functional (up to a constant multiple). Let us fix a background metric g_0 and denote by $[g_0]$ the conformal class of g_0. The standard Sobolev inequality states that

$$\mathcal{F}_0(g) \le C_{[g_0]}(\mathcal{F}_1^{\frac{n}{n-1}}(g)), \quad \text{for } g \in [g_0],$$

for some constant $C_{[g_0]}$ depending only on $[g_0]$. Equality holds if and only if the metric g is a Yamabe minimizer. For $g \in [g_0]$, we can write $g = e^{-2u}g_0$. Let S_{g_0} and S_g be

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the corresponding Schouten tensors of g_0 and g respectively. There is a transformation formula relating them:

$$S_g = \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}. \tag{2}$$

From this point of view, $\mathcal{F}_k(g)$ in (1) resembles the kth quermassintegral in the theory of convex bodies. The classical quermasstintegral inequalities suggest that there may exist similar inequalities between the geometric quantities $\mathcal{F}_k(g)$.

The focus of this paper is the investigation of the precise relationship of the geometric quantities defined in (1). As we will see, there exists a complete system of sharp inequalities for $\mathcal{F}_k$ on locally conformally flat manifolds, and the cases of equality will be characterized by certain "extremal" metrics, just as in the case of the Yamabe problem. The main tool used to obtain these geometric inequalities is the theory of parabolic fully nonlinear equations, and this is one of the special features of this paper.

When (M, g_0) is a locally conformally flat manifold and $k \neq n/2$, it was proved by Viaclovsky in [35] that the critical points of $\mathcal{F}_k$ in $[g_0]$ are the metrics g satisfying the equation

$$\sigma_k(g) = constant. \tag{3}$$

For $k = n/2$, $\mathcal{F}_{\frac{n}{2}}(g)$ is a constant in the conformal class (see [35]). In this case, there is another functional which was discovered in [4]. A similar functional was also found in [8]. The functional in [4] is defined by

$$\mathcal{E}_{n/2}(g) \quad = \quad -\int_0^1 \int_M \sigma_{n/2}(g_t) u dg_t dt, \tag{4}$$

where u is the conformal factor of $g = e^{-2u} g_0$ and $g_t = e^{-2tu} g_0$. Unlike $\mathcal{F}_k$, $\mathcal{E}_{n/2}$ depends on the choice of the background metric g_0. However, its derivative $\nabla \mathcal{E}_{n/2}$ *does not* depend on the choice of g_0. The critical points of $\mathcal{E}_{n/2}$ correspond to the metrics g satisfying (3) for $k = n/2$ on locally conformally flat manifolds. These important variational properties are unknown for general (i.e. not locally conformally flat) manifolds for $k > 2$. For this reason, we will restrict ourselves to locally conformally flat manifolds in our study of $\mathcal{F}_k$ in this paper. Another advantage of locally conformally flat manifolds is the availability of the fundamental result of Schoen-Yau [30] on developing maps, which is crucial for us in establishing the gradient estimates for the conformal flow (13) we consider below.

We will establish three types of inequalities depending on the range of k. More precisely, a *Sobolev type* inequality (5) is established for any $k < \frac{n}{2}$ and a *conformal quermassintegral type inequality* (8) for any $k \geq n/2$. And, for the exceptional case $k = n/2$, we establish a *Moser-Trudinger type inequality* (9) for $\mathcal{E}_{n/2}$.

Before giving precise results, let us first recall some notation and definitions. Let

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0, \forall\, j \leq k\}.$$

Let $\overline{\Gamma}_k^+$ be the closure of Γ_k^+. A C^2 metric g is said to be in Γ_k^+ if the eigenvalues of $g^{-1} \cdot S_g(x) \in \Gamma_k^+$ for any $x \in M$ (see [14]). Such a metric is also called *admissible.*

Similarly, a $C^{1,1}$ metric is said to be in $\overline{\Gamma}_k^+$ if the eigenvalues of $g^{-1}\cdot S_g(x)\in\overline{\Gamma}_k^+$ for any $x\in M$. For convenience, we set $\sigma_0(A)=1$ and $\sigma_0(g)=1$. We denote

$$\mathcal{C}_k([g_0])=\{g\in C^{4,\alpha}[g_0]|g\in\Gamma_k^+\},$$

where $[g_0]$ is the conformal class of g_0. Here $0<\alpha<1$.

Unlike the Yamabe equation ($k=1$ in (3)), equation (3) is fully nonlinear when $k>1$. Moreover, from work in [5], it is necessary to assume $g\in\Gamma_k^+$ (or $\Gamma_k^-=-\Gamma_k^+$) for $k>1$ in order to make use of elliptic theory. This is a very restrictive condition. It is proved in [14] that any locally conformally flat (M,g) with $g\in\Gamma_{\frac{n}{2}}^+$ is a space form. Nevertheless, for $k<\frac{n}{2}$, there are plenty of other locally conformally flat manifolds. For example, $\mathbb{S}^1\times\mathbb{S}^{n-1}$ with the standard product metric is in Γ_k^+ for any $k<\frac{n}{2}$, and $\mathbb{H}^p\times\mathbb{S}^{n-p}$ with the standard product metric is in Γ_k^+ for certain range of $2\le k$ when p is sufficiently small compared to n. Furthermore, if (M_1,g_1) and (M_2,g_2) are two n-dimensional locally conformally flat manifolds with $g_i\in\Gamma_k^+$ $(i=1,2)$ for some $1\le k<\frac{n}{2}$, then one can construct a locally conformally flat metric $g\in\Gamma_k^+$ on the connected sum of them (see also [13]). This type of construction yields a wide class of locally conformally flat manifolds with metrics in Γ_k^+ for each $k<\frac{n}{2}$.

We now state our main results. Since any metric $g\in\Gamma_{n/2}^+$ is conformally equivalent to a metric of constant sectional curvature (see [14]), we will assume, whenever we consider (4) in this paper, that g_0 has constant sectional curvature.

Theorem 1. *Suppose that (M,g_0) is a compact, oriented and connected locally conformally flat manifold with $g_0\in\Gamma_k^+$ smooth and $g\in\mathcal{C}_k$. Let $0\le l<k\le n$.*

(A). Sobolev type inequality: *If $0\le l<k<\frac{n}{2}$, then there is a positive constant $C_S=C_S([g_0],n,k,l)$ depending only on n, k, l and the conformal class $[g_0]$ such that*

$$(\mathcal{F}_k(g))^{\frac{1}{n-2k}}\ge C_S\left(\mathcal{F}_l(g)\right)^{\frac{1}{n-2l}}. \tag{5}$$

If we normalize $\int_M\sigma_l(g)dg=1$, then equality holds in (5) if and only if

$$\frac{\sigma_k(g)}{\sigma_l(g)}=C_S^{n-2k}. \tag{6}$$

There exists a metric $g_E\in\mathcal{C}_k$ attaining equality in (5). Furthermore,

$$C_S\le C_{S,k,l}(\mathbb{S}^n)=\binom{n}{k}^{\frac{1}{n-2k}}\binom{n}{l}^{\frac{-1}{n-2l}}\left(\frac{\omega_n^2}{2^n}\right)^{\frac{k-l}{(n-2k)(n-2l)}}, \tag{7}$$

where ω_n is the volume of the standard sphere $\mathbb{S}^n$.

(B). Conformal quermassintegral type inequality: *If $n/2\le k\le n$, $1\le l<k$, then*

$$(\mathcal{F}_k(g))^{\frac{1}{k}}\le\binom{n}{k}^{\frac{1}{k}}\binom{n}{l}^{-\frac{1}{l}}(\mathcal{F}_l(g))^{\frac{1}{l}}. \tag{8}$$

Equality in (8) holds if and only if (M,g) is a spherical space form.

(C). Moser-Trudinger type inequality: *If $k = n/2$, then*

$$(n-2l)\mathcal{E}_{n/2}(g) \geq C_{MT}\left\{\log \int_M \sigma_l(g)dg - \log \int_M \sigma_l(g_0)dg_0\right\}, \tag{9}$$

where

$$C_{MT} = \int_M \sigma_{n/2}(g_0)dg_0 = \frac{\omega_n}{2^{\frac{n}{2}}}\binom{n}{\frac{n}{2}}.$$

Equality holds in (9) if and only if (M, g) is a space form. The above inequality is also true for $l > k = n/2$, provided $g \in \mathcal{C}_l$.

When $l = 0$ and $k = 1$, inequality (5) is the standard Sobolev inequality (e.g., see [2]). For $l = 0$ and $1 \leq k < \frac{n}{2}$, inequality (5) provides the optimal control of the L^n-norm of e^{-u} in terms of $\mathcal{F}_k$. For these reasons, we call (5) a Sobolev type inequality. We were informed by Professor Alice Chang that Maria Del Mar Gonzalez [11] obtained inequality (7) independently for $l = 0$. We suspect that (7) should be true on general compact manifolds. Inequality (5) plays a key role in the proof of a main result in [12].

Inequality (8) is reminiscent of the classical quermassintegral inequality (e.g., see [16] for a discussion), which provides one of the motivations for this paper. In the case $n = 4, k = 2$ and $l = 1$, inequality (8) was proved earlier by Gursky in [17] for general 4-dimensional manifolds. Some cases of the inequality were also verified in [16] and [14] for locally conformally flat manifolds.

Inequality (9) is similar to the Moser-Trudinger inequality on compact Riemannian surfaces (see [32], [26], [27] and [21]). When $l = 0$, (9) was proved by Brendle-Viaclovsky and Chang-Yang in [4] and [8] using a result in [16] on a fully nonlinear conformal flow, and was referred to by them as the Moser-Trudinger inequality. We also refer to [3] for a different form of the Moser-Trudinger inequality in higher dimensions.

The Yamabe problem is the problem of finding a metric in the conformal class $[g_0]$ which minimizes the functional $\mathcal{F}_1(g)$. The final solution of the Yamabe problem by Aubin [2] and Schoen [28] is one of the triumphs of geometric analysis. There have been several important generalizations of Yamabe type problems in other settings such as the CR Yamabe problem studied by Jerison-Lee [22]. Viaclovsky [35] considered another type of Yamabe problem: for $1 \leq k \leq n$, find a metric $g \in [g_0]$ satisfying equation (3), or equivalently the following partial differential equation for the function u in local orthonormal frames with respect to g_0:

$$\sigma_k\left(u_{ij} + u_i u_j - \frac{|\nabla u|^2}{2}\delta_{ij} + S_{ij}\right) = ce^{-2ku}, \tag{10}$$

where S_{ij} are the entries of S_{g_0}, u_i and u_{ij} are the first and second order covariant derivatives of u with respect to the local orthonormal frames. When $k > 1$, equation (10) is fully nonlinear. There has been much recent activity surrounding equation (10). Viaclovsky [35] obtained some existence results for $k = n$ under certain geometric conditions. Equation (3) was solved by Chang-Gursky-Yang [6, 7] in the case $n = 4, k = 2$. Gursky-Viaclovsky studied the equation for the negative cone case in [19], and more recently for the lower dimensional cases $n = 3, 4$ in [18]. For the locally conformally flat manifolds, equation (3) was completely solved by Guan-Wang [16] and Li-Li [24], and for more general conformally invariant equations by Guan-Lin-Wang [12] and Li-Li [25]. We remark that for $k \geq \frac{n}{2}$, (M, g_0) is conformally equivalent to a space form by [14], so any solution of (3) is a maximizer of $\mathcal{F}_k$. But, for $k < \frac{n}{2}$, the previous solutions to equation (3) obtained in [16, 24] are not necessarily the Yamabe type solutions. That is, they are not necessarily minimizers of the functional $\mathcal{F}_k$. Similarly, neither are the solutions of (6) on a locally conformally flat manifold obtained in [12] and [25] necessarily minimizers of a corresponding functional.

One immediate consequence of Theorem 1 is the existence of a minimizer (or maximizer) solution of the general conformal quotient equation (6). For simplicity of notation, we will denote $\frac{\sigma_k(A)}{\sigma_l(A)}$ by $\frac{\sigma_k}{\sigma_l}(A)$. By the transformation formula (2), equation (6) can be expressed in the fully nonlinear form:

$$\frac{\sigma_k}{\sigma_l}\left(u_{ij} + u_i u_j - \frac{|\nabla u|^2}{2}\delta_{ij} + S_{ij}\right) = ce^{-2(k-l)u}. \tag{11}$$

Corollary 1. *For any $0 \leq l < k$, there is an extremal metric $g_E \in \mathcal{C}_k$ satisfying equation (11) which is a global minimizer (maximizer) of $\mathcal{F}_k$ in $\mathcal{C}_k$ with $\int_M \sigma_l(g_E)dg_E = 1$ ($\mathcal{E}_{n/2}(g_E) = 1$ if $l = n/2$) when $k < n/2$ ($k > n/2$); when $k = n/2$, there is a global minimizer of $\mathcal{E}_{n/2}$.*

We will prove Theorem 1 by studying an associated flow equation. The approach of using flows to study geometric inequalities was previously considered in different contexts by various authors in [1, 9, 38, 34]. In the conformal context, a simpler flow was introduced in [16]

$$\begin{cases} \dfrac{d}{dt}g &= -(\log \sigma_k(g) - \log r_k(g)) \cdot g, \\ g(0) &= g_0, \end{cases} \tag{12}$$

where $r_k(g)$ is given by

$$r_k(g) = \exp\left(\frac{1}{vol(g)}\int_M \log \sigma_k(g)\, dg\right).$$

The global existence, regularity and convergence for the flow (12) was proved there. For the purpose of this paper, we need to deal with the following general fully nonlinear flow:

$$\begin{cases} \dfrac{d}{dt}g &= -\left(\log \dfrac{\sigma_k(g)}{\sigma_l(g)} - \log r_{k,l}\right) \cdot g, \\ g(0) &= g_0, \end{cases} \tag{13}$$

where

$$r_{k,l} = \exp\left(\frac{\int \sigma_l(g) \log(\sigma_k(g)\sigma_l(g)^{-1})dg}{\int \sigma_l(g)dg}\right)$$

is defined so that the flow (13) preserves $\int \sigma_l(g)dg$ when $l \neq n/2$ and $\mathcal{E}_{n/2}$ when $l = n/2$. We have the following result for the flow (13).

Theorem 2. *For any $C^{4,\alpha}$ initial metric $g_0 \in \Gamma_k^+$, the flow (13) has a global solution $g(t) \in \mathcal{C}_k$ for any $t > 0$. Moreover, there is $h \in \mathcal{C}_k$ satisfying equation (6) such that,*

$$\lim_{t\to\infty} \|g(t) - h\|_{C^{4,\alpha}(M)} = 0.$$

Flows (12) and (13) are fully nonlinear conformal equations. The first step is to obtain a Harnack inequality (i.e., the boundedness of $|\nabla g(t)|$, see (39) below). This can be done using a fundamental result of Schoen-Yau on developing maps and the method of moving planes as in [39]. This is where the assumption of locally conformally flat is used crucially. For the flow (12), since the volume of the evolved metric is preserved, C^0 boundedness is a direct consequence of the Harnack inequality (see [16] and [39]). However the flow (13) may not preserve the volume. This is the major difference between the flows (12) and (13). To deal with this problem we first use the Harnack inequality (39) to bound $|\nabla^2 g(t)|$. From there we obtain a C^0 bound using our previous results [16] and the local estimates in [15]. The preliminaries for this are developed in Section 2.

We will also need local estimates for equation (6) in the case $0 \le l < k < \frac{n}{2}$. These estimates are needed to prove Theorem 1.A. The existence of such local estimates is a very special feature of this type of conformal fully nonlinear equation and is crucial for the study of the elliptic equation (6).

This paper is organized as follows. In Section 2, we list some basic facts regarding $\frac{\sigma_k}{\sigma_l}$ and the flow (13) and prove Theorem 1 for the case $l = 0$. In Section 3, we establish some local estimates for equation (6), which will be used in a crucial way to obtain a positive lower bound for the geometric functionals. In Section 4, we study the flow (13) and prove Theorem 2. Theorem 1 for the case $l > 0$ will be proved in Section 5. In Section 6, we discuss the best constant, which appears in part (**A**) of Theorem 1.

Notation: In the rest of the paper, $u_i, u_{ij}, \cdots$ denote the covariant derivatives of the function u with respect to some local orthonormal frames of the background metric g_0, unless it is otherwise indicated.

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2. Some basic facts

Let $\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$. The kth elementary symmetric function is defined as

$$\sigma_k(\lambda_1, \dots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

A real symmetric $n \times n$ matrix A is said to lie in Γ_k^+ if its eigenvalues lie in Γ_k^+. Let A_{ij} be the $\{i, j\}$-entry of an $n \times n$ matrix. Then for $0 \le k \le n$, the kth *Newton transformation* associated with A is defined to be

$$T_k(A) = \sigma_k(A)I - \sigma_{k-1}(A)A + \cdots + (-1)^k A^k.$$

We have

$$T_k(A)^i_j = \frac{1}{k!}\delta^{i_1\ldots i_k i}_{j_1\ldots j_k j} A_{i_1j_1}\cdots A_{i_kj_k},$$

where $\delta^{i_1\ldots i_k i}_{j_1\ldots j_k j}$ is the generalized Kronecker delta symbol. Here we use the summation convention. By definition,

$$\sigma_k(A) = \frac{1}{k!}\delta^{i_1\ldots i_k}_{j_1\ldots j_k} A_{i_1j_1}\cdots A_{i_kj_k}, \quad T_{k-1}(A)^i_j = \frac{\partial\sigma_k(A)}{\partial A_{ij}}.$$

For $0 < l < k \le n$, let

$$\tilde{T}_{k-1,l-1}(A) = \frac{T_{k-1}(A)}{\sigma_k(A)} - \frac{T_{l-1}(A)}{\sigma_l(A)}.$$

It is important to note that if $A \in \Gamma_k^+$, then $\tilde{T}_{k-1,l-1}(A)$ is positive definite. This follows the fact that the function $\frac{\sigma_k}{\sigma_l}(A)$ is monotonic in Γ_k^+, that is if $B - A$ is semi-positive, then $\frac{\sigma_k}{\sigma_l}(A) \le \frac{\sigma_k}{\sigma_l}(B)$. Another important fact is that $(\frac{\sigma_k}{\sigma_l}(A))^{\frac{1}{k-l}}$ is concave in Γ_k^+ (e.g., see [33]). Let $\Lambda_i = (\lambda_1, \lambda_2, \cdots, \lambda_{i-1}, \lambda_{i+1}, \cdots, \lambda_n)$.

Lemma 1 (Newton-MacLaurin Inequality [20])**.**

$$l(n-k+1)\sigma_l(\Lambda)\sigma_{k-1}(\Lambda) \ge k(n-l+1)\sigma_k(\Lambda)\sigma_{l-1}(\Lambda). \tag{14}$$

Lemma 2 (Garding's Inequality)**.** *Let* $\Lambda = (\lambda_1, \cdots, \lambda_n), \Lambda_0 = (\mu_1, \cdots, \mu_n) \in \Gamma_k^+$,

$$F(\Lambda) = \left(\frac{\sigma_k}{\sigma_l}(\Lambda)\right)^{\frac{1}{k-l}}.$$

Then,

$$\sum_i \{\frac{\sigma_{k-1}(\Lambda_i)}{\sigma_k(\Lambda)} - \frac{\sigma_{l-1}(\Lambda_i)}{\sigma_l(\Lambda)}\}\mu_i \ge (k-l)\frac{F(\Lambda_0)}{F(\Lambda)}. \tag{15}$$

Proof. The main argument of the proof follows from [5]. From the concavity of F in Γ_k^+, for $\Lambda, \Lambda_0 = (\mu_1, \cdots, \mu_n) \in \Gamma_k^+$ we have

$$\begin{aligned} F(\Lambda_0) &\le F(\Lambda) + \sum_l (\mu_i - \lambda_i)\frac{\partial F(\Lambda)}{\partial\lambda_i} \\ &= F(\Lambda) + \frac{1}{k-l}F(\Lambda)\sum_i \{\frac{\sigma_{k-1}(\Lambda_i)}{\sigma_k(\Lambda)} - \frac{\sigma_{l-1}(\Lambda_i)}{\sigma_l(\Lambda)}\}(\mu_i - \lambda_i) \\ &= \frac{1}{k-l}F(\Lambda)\sum_i \{\frac{\sigma_{k-1}(\Lambda_i)}{\sigma_k(\Lambda)} - \frac{\sigma_{l-1}(\Lambda_i)}{\sigma_l(\Lambda)}\}\mu_i. \end{aligned}$$

In the last equality, we have used the fact that F is homogeneous of degree one. Then (15) follows. ∎

Lemma 3. *A conformal class of metric* $[g]$ *with* $[g] \cap \Gamma_k^+ \ne \emptyset$ *does not have a* $C^{1,1}$ *metric* $g_1 \in \overline{\Gamma}_k^+$ *with* $\sigma_k(g_1) = 0$*, where* $\overline{\Gamma}_k^+$ *is the closure of* Γ_k^+*.*

Proof. By the assumption, there is a smooth admissible metric g_0 with $\sigma_k(g) > 0$. Assume by contradiction that there is a $C^{1,1}$ metric g_1 with $\sigma_k(g_1) = 0$. Write $g_1 = e^{-2u} g_0$, so u satisfies

$$\sigma_k\left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) = 0. \tag{16}$$

Let

$$W = \left(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\right) \quad \text{and} \quad a_{ij}(W) = \frac{\partial \sigma_k(W)}{\partial w_{ij}}.$$

(Here, the notation $a_{ij}(W) = \frac{\partial \sigma_k(W)}{\partial w_{ij}}$ is usually used by analysts. More intrinsically, $a_{ij} = T_{k-1}(W)^i_j$, where $T_{k-1}(W)$ is the $(k-1)$th Newton transformation defined above and W is viewed as the matrix $g_0^{-1} \cdot W$.) We may assume $u \le 1$ and $u(x_0) = 1$ for some $x_0 \in M$, since $u + c$ also satisfies (16) for any constant c. Let $v = e^{-u} - e^{-1}$, $h_t = te^{-u} + (1-t)e^{-1}$, $u_t = -\log h_t$ and $W_t = \nabla^2 u_t + du_t \otimes du_t - \frac{|\nabla u_t|^2}{2} g_0 + S_{g_0}$. As in [37], one can check that (a) $W_t \in \Gamma_k^+$ and (b) $(a_{ij}(W_t))$ positive definite (nonnegative definite, resp.) for all $0 \le t < 1$ ($0 \le t \le 1$ resp.). We have the following

$$0 > \sigma_k(W_1) - \sigma_k(W_0) = \sum_{ij} \left(\int_0^1 \frac{a_{ij}(W_t)}{h_t^2} dt\right) \nabla^2_{ij} v + \sum_l b^l(t,x) \nabla_l v + dv, \tag{17}$$

for some bounded functions d and b^l, $l = 1, ..., n$. Since $v \ge 0$ and $v(x_0) = 0$, this is a contradiction to the strong maximum principle. ■

For any $0 < k \le n$, let

$$\tilde{\mathcal{F}}_k(g) = \begin{cases} \dfrac{1}{n-2k} \displaystyle\int_M \sigma_k(g) dg, & k \ne n/2, \\ \mathcal{E}_{n/2}(g), & k = n/2. \end{cases} \tag{18}$$

Lemma 4. *Flow (13) preserves $\tilde{\mathcal{F}}_l$, it also decreases the functional $\tilde{\mathcal{F}}_k$. In fact, we have the following evolution identities. The evolution equations for $\log \frac{\sigma_k}{\sigma_l}$ and $\tilde{\mathcal{F}}_k$ are*

$$\frac{d}{dt} \log \frac{\sigma_k}{\sigma_l}(g) = \frac{1}{2} \mathrm{tr}\{\tilde{T}_{k-1,l-1}(S_g) \nabla_g^2 \log \frac{\sigma_k}{\sigma_l}(g)\} + (k-l)(\log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}) \tag{19}$$

and

$$\frac{d}{dt} \tilde{\mathcal{F}}_k(g) = -\frac{1}{2} \int_M \left(\frac{\sigma_k}{\sigma_l}(g) - r_{k,l}\right) \left(\log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}\right) \sigma_l(g) dg. \tag{20}$$

Proof. It is easy to verify $\frac{d}{dt} \tilde{\mathcal{F}}_l(g) = 0$ from the equation and the definition of $r_{k,l}$. The first equality follows the same argument in [16]. We verify the second equality. When $k \ne \frac{n}{2}$,

$$\begin{aligned}\frac{d}{dt}\tilde{\mathcal{F}}_k(g) &= \frac{1}{2}\int_M \sigma_k(g) g^{-1}\frac{d}{dt} g\, dg \\ &= -\frac{1}{2}\int_M \left(\frac{\sigma_k}{\sigma_l}(g) - r_{k,l}\right)\left(\log\frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}\right)\sigma_l(g)\, dg.\end{aligned}$$

The first equality was proven in [35], where the assumption of locally conformally flat is used. By [4], the above also holds for $k = \frac{n}{2}$. ■

In the rest of this section we use results obtained in [16] for flow (12) to establish Theorem 1 in the case $l = 0$.

Proposition 1. *Let (M, g_0) be a locally conformally flat manifold with $g_0 \in \Gamma_k^+$. We have*

(a). *When $k > n/2$, there is a constant $C_Q = C_Q(n, k) > 0$ depending only on n and k such that for any metric $g \in \mathcal{C}_k$.*

$$\int_M \sigma_k(g) vol(g) \le C_Q vol(g)^{\frac{n-2k}{n}}.$$

(b). *When $k < n/2$, there is a constant $C_S = C_S([g_0], n) > 0$ such that for any metric $g \in \mathcal{C}_k$.*

$$\int_M \sigma_k(g) vol(g) \ge C_S vol(g)^{\frac{n-2k}{n}}.$$

(c). *If $k = n/2$ and g_0 is a metric of constant sectional curvature, then for any $g \in \mathcal{C}_k$*

$$\mathcal{E}_{n/2}(g) \ge \frac{1}{n} C_{MT}(\log vol(g) - \log vol(g_0)),$$

where $C_{MT} = \int_M \sigma_{n/2}(g_0) dg_0$.

Moreover, in cases (a) and (c) the equality holds if and only if g is a metric of constant sectional curvature.

Proof. (c) was already proved in [4].

When $k > n/2$, from [14] we know that (M, g_0) is conformally equivalent to a spherical space form. In this case, it was proved in [35] that any solution of (6) for $l = 0$ is of constant sectional curvature. By Theorem 1 in [16] ($k > \frac{n}{2}$), for any $g \in \mathcal{C}_k$ there is a metric $g_e \in \mathcal{C}_k$ of constant sectional curvature with $vol(g) = vol(g_e)$ and

$$\tilde{\mathcal{F}}_k(g) \ge \tilde{\mathcal{F}}_k(g_e). \tag{21}$$

When $k > n/2$, (21) implies that

$$vol(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) dg \le vol(g_e)^{-\frac{n-2k}{n}} \int_M \sigma_k(g_e) dg_e,$$

and the equality holds if and only if (M, g) is a space form. It is clear that

$$vol(g_e)^{-\frac{n-2k}{n}} \int_M \sigma_k(g_e) dg_e$$

is a constant depending only on n, k. This proves (a).

It remains to prove (b). For this case, we only need to prove that

$$\inf_{\mathcal{C}_k\cap\{vol(g)=1\}} \mathcal{F}_k(g) =: \beta_0 > 0.$$

Assume by contradiction that $\beta_0 = 0$. By Theorem 1 in [16], we can find a sequence of solutions $g_i = e^{-2u_i} g_0 \in \mathcal{C}_k$ of (3) with $vol(g_i) = 1$ and $\sigma_k(g_i) = \beta_i$ such that $\lim_{i\to\infty} \beta_i = 0$. $\sigma_k(g_l) = \beta_i$ means

$$\sigma_k(\nabla^2 u_i + du_i \otimes du_i - \frac{|\nabla u_i|^2}{2} g_0 + S_{g_0}) = \beta_i e^{-2ku_i}. \tag{22}$$

Consider the scaled metric $\tilde g_i = e^{-2\tilde u_i} g_0$ with $\tilde u_i = u_i - \frac{1}{2k}\log\beta_i$, which satisfies clearly that

$$\sigma_k\left(\nabla^2 \tilde u_i + d\tilde u_i \otimes d\tilde u_i - \frac{|\nabla \tilde u_i|^2}{2} g_0 + S_{g_0}\right) = e^{-2k\tilde u_i} \tag{23}$$

and

$$vol(\tilde g_i) = \beta_i^{\frac{n}{2k}} \to 0 \quad \text{as } i \to \infty.$$

By Corollary 1.2 in [15], we conclude that

$$\tilde u_i \to +\infty \text{ uniformly as } i \to \infty.$$

Hence $m_i := \inf_M \tilde u_i \to +\infty$ as $i \to \infty$. Now at the minimum point x_i of $\tilde u_i$, by (23),

$$\sigma_k(S_{g_0}) \le \sigma_k\left(\nabla^2 \tilde u_i + d\tilde u_i \otimes d\tilde u_i - \frac{|\nabla \tilde u_i|^2}{2} g_0 + S_{g_0}\right) = e^{-2km_i} \to 0.$$

This is a contradiction to the fact $g_0 \in \Gamma_k^+$. ■

3. Local estimates for conformal quotient equations

In order to get a positive lower bound for the constant in (**A**) of Theorem 1, as in the proof of Proposition 1 in the previous section, we prove some estimates for solutions of

$$\frac{\sigma_k}{\sigma_l}(g) = f, \tag{24}$$

defined locally in any open subset of M with a nonnegative C^2 function f. When $l = 0$ local estimates were established in [15]. The same estimates hold for equation (24) when $0 \le l < k \le n$. Since the proof for the special case $(n-k+1)(n-l+1) > 2(n+1)$ is much simpler and it is suffice for the purpose of this paper, we only treat this case here. The local estimates for equation (24) in general case $0 \le l < k \le n$ will appear elsewhere. We emphasize that, in this section, we *do not* assume (M, g_0) is locally conformally flat .

Theorem 3. *Let (M, g_0) be a smooth compact, n-dimensional manifold and let $g = e^{-2u} g_0$ be a C^4 solution of (24) in B_r with*

$$(n-k+1)(n-l+1) > 2(n+1). \tag{25}$$

Then there exist two positive constants c_1 *(depending only on* $\|g_0\|_{C^3(B_r)}$*,* n,k,l *and* $\|f\|_{C^1(B_r)}$*)) and* c_2 *(depending only on* $\|g_0\|_{C^4(B_r)}$*,* n,k,l *and* $\|f\|_{C^2(B_r)}$*) such that* $\forall x \in B_{\frac{r}{2}}$*,*

$$|\nabla u(x)|^2 \leq c_1(1 + e^{-2\inf_{B_r} u}) \tag{26}$$

and

$$|\nabla^2 u(x)| \leq c_2(1 + e^{-2\inf_{B_r} u}). \tag{27}$$

In particular, if $k < \frac{n}{2}$*, condition (25) holds, and hence inequalities (26) and (27) are valid in this case.*

Theorem 3 has the following consequence.

Corollary 2. *Let* k,l *as in Theorem 3. There exists a constant* $\varepsilon_0 > 0$ *such that for any sequence of* $C^{4,\alpha}$ *solutions* u_i *of (24) in* B_r *with*

$$\int_{B_1} e^{-nu} dvol(g_0) \leq \varepsilon_0,$$

either

(1) *There is a subsequence* u_{i_l} *uniformly converging to* $+\infty$ *in any compact subset in* B_r*, or*
(2) *There is a subsequence* u_{i_l} *converges strongly in* $C^{4,\alpha}_{loc}(B_r)$*. If* f *is smooth and strictly positive in* B_r *and* u_i *is smooth, then* u_{i_l} *converges strongly in* $C^m_{loc}(B_r)$*,* $\forall m$*.*

The proof of Corollary 2 follows from the same lines in the proof of Corollary 1.2 in [15], the same argument can be traced back to Schoen [29], we will not repeat it here.

Proof of Theorem 3. The proof in [15] works for (27) as $(\frac{\sigma_k}{\sigma_l})^{\frac{1}{k-l}}$ is elliptic and concave in Γ_k^+. We only need to prove (26).

We follow the same lines of proof in [15] to prove the Theorem, together with Lemma 5 proven at the end of this section. For convenince of the reader, we will use the same notations as in [15].

Without loss of generality, we may assume $r = 1$. Let

$$F = \left(\frac{\sigma_k}{\sigma_l}(W)\right)^{\frac{1}{k-l}} \quad \text{and} \quad F^{ij} = \frac{\partial F}{\partial w_{ij}},$$

where $W = (\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0})$. More intrinsically,

$$F^{ij} = \frac{1}{k-l}\left(\frac{\sigma_k}{\sigma_l}(W)\right)^{\frac{1}{k-l}-1} \frac{1}{\sigma_l^2(W)} \left\{\sigma_l(W) T_{k-1}(W)^i_j - \sigma_k(W) T_{l-1}(W)^i_j\right\},$$

where T_j is the jth Newton transformation. Let ρ be a test function $\rho \in C_0^\infty(B_1)$ such that

$$\begin{aligned} &\rho \geq 0, \text{ in } B_1 \quad \text{ and } \quad \rho = 1, \text{ in } B_{1/2}, \\ &|\nabla \rho(x)| \leq 2b_0 \rho^{1/2}(x) \quad \text{ and } \quad |\nabla^2 \rho| \leq b_0, \text{ in } B_1. \end{aligned} \tag{28}$$

Here $b_0 > 1$ is a constant depending only on the background metric g_0. Set $H = \rho|\nabla u|^2$. It suffices to bound $\max_{B_1} H$. Let $x_0 \in B_1$ be a maximum point of H. We may assume that W is diagonal at the point x_0 by choosing a suitable normal coordinates around x_0. Set $\lambda_i = w_{ii}$ and $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$. In what follows, all computations are given at the point x_0. We denote Λ_i the vector with ith component deleted from Λ and

$$F^* = \frac{1}{k-l}\left(\frac{\sigma_k}{\sigma_l}(\Lambda)\right)^{\frac{1}{k-l}-1}\frac{1}{\sigma_l^2(\Lambda)}.$$

At the point x_0, (F^{ij}) is diagonal and

$$F^{ii} = F^*\{\sigma_l(\Lambda)\sigma_{k-1}(\Lambda_i) - \sigma_k(\Lambda)\sigma_{l-1}(\Lambda_i)\}. \tag{29}$$

Since W is diagonal at x_0, we have at x_0

$$w_{ii} = u_{ii} + u_i^2 - \frac{1}{2}|\nabla u|^2 + S_{ii}, \quad u_{ij} = -u_i u_j - S_{ij}, \quad \forall i \neq j, \tag{30}$$

where S_{ij} are entries of S_{g_0}.

We may assume that

$$H(x_0) \geq b_0^2 A_0^2,$$

for some constant A_0 will be fixed later. We may also assume that

$$|S_{g_0}|(x_0) \leq A_0^{-1}|\nabla u|^2(x_0). \tag{31}$$

Otherwise, we are done. Since x_0 is the maximum point of H, we can get, for any i,

$$\left|\sum_{l=1}^n u_{il}u_l\right|(x_0) \leq \frac{|\nabla u|^3}{A_0}(x_0). \tag{32}$$

By applying maximum principle and following the same deduction in [15] (formulas (2.18) and (2.20) there), we have

(33)
$$\begin{aligned} 0 \;\geq\;& F^{ij}H_{ij} = F^{ij}\left\{\left(-2\frac{\rho_i\rho_j}{\rho} + \rho_{ij}\right)|\nabla u|^2 + 2\rho u_{lij}u_l + 2\rho u_{il}u_{jl}\right\} \\ \geq\;& 2\sum_{i,j,l} F^{ij}\rho u_{il}u_{jl} + \sum_{j\geq 1} F^{jj}\left\{-10nb_0^2|\nabla u|^2 - Ce^{-\inf_{B_1} u}|\nabla u|^2 - \frac{(n+2)^2}{A_0}\rho|\nabla u|^4\right\} \end{aligned}$$

where C is a constant depending only on n, k, l, $\|f\|_{C^1(B_1)}$ and $\|g\|_{C^3(B_1)}$. Set $\tilde{u}_{ij} = u_{ij} + S_{ij}$. By (31), we have

$$\sum_{i,j,l} F^{ij}u_{il}u_{jl} \geq \frac{1}{2}\sum_{i,l} F^{ii}\tilde{u}_{il}^2 - \frac{n}{A_0^2}|\nabla u|^4 \sum_i F^{ii}.$$

This, together with (33), implies

$$\sum_{i,l} F^{ii}\tilde{u}_{il}^2 \le \sum_{j\ge 1} F^{jj}\left\{10nb_0^2|\nabla u|^2 + Ce^{-\inf_{B_1} u}|\nabla u|^2 + \frac{2(n+2)^2}{A_0}\rho|\nabla u|^4\right\}.$$

Hence, the needed bound $H(x_0) \le c_1(1+e^{-2\inf_{B_1} u})$ can be reduced to the verification of the following:

Claim: There is a constant A_0 (depending only on k and n) such that

$$\sum_{i,j} F^{ii}\tilde{u}_{ij}^2 \ge A_0^{-\frac{5}{8}} \sum_i F^{ii}|\nabla u|^4. \tag{34}$$

This claim is just Claim 2.5 in [15]. As in [15], the verification of the Claim is a crucial step to prove the Theorem.

Set $\delta_0 = A_0^{-1/4} < 0.1$. We divide the set $I = \{1, 2, \cdots, n\}$ as in [15] into two parts:

$$I_1 = \{i \in I \mid u_i^2 \ge \delta_0|\nabla u|^2\} \quad \text{and} \quad I_2 = \{i \in I \mid u_i^2 < \delta_0|\nabla u|^2\}.$$

Clearly, I_1 is non-empty.

Case 1. There is j_0 satisfying

$$\tilde{u}_{jj}^2 \le \delta_0^2|\nabla u|^4 \quad \text{and} \quad u_j^2 < \delta_0|\nabla u|^2. \tag{35}$$

We may assume that $j_0 = n$. We have $|w_{nn} + \frac{|\nabla u|^2}{2}| = |\tilde{u}_{nn} + u_n^2| < 2\delta_0|\nabla u|^2$ by (35). From (32) and (30), we have for any $i \in I$

$$\left|u_i(u_{ii} - (|\nabla u|^2 - u_i^2)) - \sum_l S_{il}u_l\right| = \left|\sum_l u_{il}u_l\right| \le \delta_0^4|\nabla u|^3.$$

This, together with (31), implies

$$|u_i(u_{ii} - (|\nabla u|^2 - u_i^2))| \le 2\delta_0^4|\nabla u|^3, \tag{36}$$

which, in turn, implies

$$\left|w_{ii} - \tfrac{|\nabla u|^2}{2}\right| \quad = \quad \left|u_{ii} + u_i^2 - |\nabla u|^2 + S_{ii}\right| \le 3\delta_0^2|\nabla u|^2,$$

for any $i \in I_1$. Using these inequalities, we can repeat the derivation of equation (2.38) in [15] to obtain

$$\sum_{i,l} F^{ii}\tilde{u}_{il}^2 \ge \tilde{F}^1\frac{|\nabla u|^4}{4} - \tilde{F}^1|\nabla u|^4 + F^{nn}\frac{|\nabla u|^4}{4} + (1 - 32\delta_0^2)\frac{|\nabla u|^4}{4}\sum_i F^{ii}, \tag{37}$$

where $\tilde{F}^1 = \max_{i\in I_1} F^{ii}$.

Recall that I_1 is necessarily non-empty. We may assume that $i_0 \in I_1$ with $\tilde{F}^1 = F^{i_0i_0}$. Without loss of generality, we assume that $i_0 = 1$. From above, we know $w_{11} \ge w_{nn}$. By

Lemma 5 below, we have $F^{nn} \geq \tilde{F}^{11}$ and $\tilde{F}^{11} \leq \frac{1}{2+c(n,k,l)} \sum_l F^{ll}$, because $w_{11} > 0$. Hence, (37) implies that

$$\sum_{i,l} F^{ii} \tilde{u}_{il}^2 \geq \{\frac{1}{4} - \frac{1}{2(2+c(n,k,l))} - 32\delta_0^2\} \sum_i F^{ii} |\nabla u|^4.$$

The inequality (34) then follows for this case by choosing $A_0 \geq \left(\frac{128(2+c(n,k,l))}{c(n,k,l)}\right)^2$.

Case 2. There is no $j \in I$ satisfying (35).

We may assume that there is i_0 such that $\tilde{u}_{i_0 i_0}^2 \leq \delta_0^2 |\nabla u|^4$, otherwise the claim is automatically true. Assume $i_0 = 1$. As in Case 4 in [15], we have $u_1^2 \geq (1-2\delta_0)|\nabla u|^2$ and $\tilde{u}_{jj}^2 + u_j^2(|\nabla u|^2 - u_j^2) \geq \delta_0^2 |\nabla u|^4$ for $j > 1$. From equation (2.50) in [15] and Lemma 5, we have

$$\begin{aligned}
\sum_{i,l} F^{ii} \tilde{u}_{il}^2 &= \sum_i F^{ii} \left\{\tilde{u}_{ii}^2 + u_i^2(|\nabla u|^2 - u_i^2)\right\} \\
&\geq \sum_{i \geq 2} F^{ii} (\tilde{u}_{ii}^2 + u_i^2(|\nabla u|^2 - u_i^2)) \\
&\geq \delta_0^2 |\nabla u|^4 \sum_{i\geq 2} F^{ii} \geq \frac{1}{2} \delta_0^2 |\nabla u|^4 \sum_{i \geq 1} F^{ii}
\end{aligned}$$

The **Claim** is verified, so the local gradient estimate (26) is proved. ■

Lemma 5. *If $\lambda_i \leq \lambda_j$, then $F^{ii} \geq F^{jj}$. If $(n-k+1)(n-l+1) > 2(n+1)$ and $\lambda_1 > 0$, then there is a positive constant $c(n,k,l)$ such that*

$$\sum_i F^{ii} \geq (2 + c(n,k,l)) F^{11}.$$

Proof. The first statement follows from the monotonicity of σ_{l-1} and $\frac{\sigma_{k-1}}{\sigma_{l-1}}$, since

$$F^{ii} = F^* \sigma_{l-1}(\Lambda_i) \{\sigma_l(\Lambda) \frac{\sigma_{k-1}}{\sigma_{l-1}}(\Lambda_i) - \sigma_k(\Lambda)\}.$$

Similarly, we have

$$\begin{aligned}
F^{11} &= F^* \sigma_{l-1}(\Lambda_1) \{\sigma_l(\Lambda) \frac{\sigma_{k-1}}{\sigma_{l-1}}(\Lambda_1) - \sigma_k(\Lambda)\} \\
&\leq F^* \sigma_{l-1}(\Lambda) \{\sigma_l(\Lambda) \frac{\sigma_{k-1}}{\sigma_{l-1}}(\Lambda) - \sigma_k(\Lambda)\}.
\end{aligned}$$

From the Newton-MacLaurin inequality (14),

$$q_0 := (n-k+1-\alpha_0)\sigma_l(\Lambda)\sigma_{k-1}(\Lambda) - (n-l+1-\alpha_0)\sigma_k(\Lambda)\sigma_{l-1}(\Lambda) \geq 0,$$

where $\alpha_0 = \frac{(n-k+1)(n-l+1)}{n+1}$. Hence, we have

$$\begin{aligned}\sum_i F^{ii} &= F^*\{(n-k+1)\sigma_l(\Lambda)\sigma_{k-1}(\Lambda) - (n-l+1)\sigma_k(\Lambda)\sigma_{l-1}(\Lambda)\\ &= F^*\{q_0 + \alpha_0[s_l(\Lambda)\sigma_{k-1}(\Lambda) - \sigma_k(\Lambda)\sigma_{l-1}(\Lambda)]\}\\ &\geq \alpha_0 F^{11}.\end{aligned}$$

The Lemma follows with $c(n,k,l) = 2 - \alpha_0 > 0$. ■

4. The global convergence of flow (13)

We treat flow (13) in this section. If $g = e^{-2u}\cdot g_0$, one may compute that

$$\sigma_k(g) = e^{2ku}\sigma_k\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right).$$

Equation (13) can be written in the following form

$$\left\{\begin{aligned} 2\frac{du}{dt} &= \log\frac{\sigma_k}{\sigma_l}\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right) + 2(k-l)u - \log r_{k,l}(g)\\ u(0) &= u_0.\end{aligned}\right. \tag{38}$$

In these equations and below, the norms and covariant derivatives are with respect to the background metric g_0.

Because $g_0 \in \Gamma_k^+$, the highest order term on the right hand side is uniformly elliptic. Consequently, the short time existence of flow (13) follows from the standard parabolic theory. We want to prove the long time existence and convergence. Let

$$T^* = \sup\{T_0 > 0\,|\,(13) \text{ exists in } [0,T_0] \text{ and } g(t)\in\mathcal{C}_k \text{ for } t\in[0.T_0]\}.$$

For any $T < T^*$, we will establish a C^2 bound for the conformal factor u, which is independent of T. As in [16], we have a Harnack inequality

$$|\nabla u| \leq c, \tag{39}$$

for some positive constant c independent of T, see a complete proof in [39]. But, as mentioned above, we can not deduce C^0 boundedness as in [16] because the flow (13) may not preserve the volume of the evolved metric $g(t)$. In order to to get C^1 bound we first use (39) to bound $|\nabla u|^2$.

Proposition 2. *There is a constant $C > 0$ independent of T such that*

$$|\nabla^2 u| \leq C.$$

Proof. Set

$$F = \log\frac{\sigma_k}{\sigma_l}\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right).$$

By equation (38), $F = 2u_t - 2(k-l)u - \log r_{k,l}$. We only need to consider the case $k > 1$. From the property of Γ_2,

$$|\lambda_i| \le \max\{1, (n-2)\} \sum_{j=1}^n \lambda_j.$$

Therefore we only need to give a upper bound of Δu which dominates all other second order derivatives.

Consider $G = \Delta u + m|\nabla u|^2$ on $M \times [0,T]$, where m is a large constant which will be fixed later. Note that Δu is analyst's Laplacian. Without loss of generality, we may assume that the maximum of G on $M \times [0,T]$ achieves at a point $(x_0, t_0) \in M \times (0,T]$ and $G(x_0,t_0) \ge 1$. We may assume that at (x_0,t_0)

$$2\sigma_1(W) \ge G \ge \frac{1}{2}\sigma_1(W), \tag{40}$$

where $W = \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}$. Consider everything in a small neighborhood near x_0. We may consider W as a matrix with entry $w_{ij} = u_{ij} + u_i u_j - \frac{1}{2}|\nabla u|^2 \delta_{ij} + S(g_0)_{ij}$. In the rest of the proof, c denotes a positive constant independent of T, which may vary from line to line.

Since G achieves its maximum at (x_0,t_0), we have at this point

$$G_t = \sum_l (u_{llt} + 2m u_{lt} u_l) \ge 0, \tag{41}$$

and

$$G_i = \sum_l (u_{lli} + 2m u_{li} u_l) = 0, \quad \forall\ i. \tag{42}$$

(42) and (39) imply that at (x_0,t_0)

$$|\sum_l u_{lli}| \le cG. \tag{43}$$

By the Harnack inequality (39) and the fact $|u_{ij}| \le G$, we may assume that

$$|u_{lij} - u_{ijl}| < c \quad \text{and} \quad |u_{ijkl} - u_{ijlk}| < cG, \tag{44}$$

where c is a constant depending only on the background metric g_0. We may assume by choosing appropriate local orthonormal frames that the matrix (w_{ij}) at (x_0,t_0) is diagonal. At the maximum point, G_{ij} is non-positive definite. Set $F^{ij} = \frac{\partial F}{\partial w_{ij}}$. Since $g(t) = e^{-2u(t)} g_0 \in \Gamma_k^+$, we know that the matrix (F^{ij}) is positive. Hence in view of

(41)-(44) and the concavity of F we have

$$
\begin{aligned}
0 &\geq \sum_{i,j} F^{ij}G_{ij} = \sum F^{ij}(u_{llij} + 2mu_{li}u_{lj} + 2mu_{lij}u_l) \\
&\geq \sum_{i,j,l} F^{ij}(u_{ijll} + 2mu_{li}u_{lj} + 2mu_{ijl}u_l) - c\sum_i F^{ii}G \\
&= -c\sum_i F^{ii}G + \sum_{i,j,l} F^{ij}\{w_{ijll} - (u_iu_j - \frac{1}{2}|\nabla u|^2\delta_{ij} + S(g_0)_{ij})_{ll} \\
&\quad +2mu_{li}u_{lj} + 2mw_{ijl}u_l - 2mu_l(u_iu_j - \frac{1}{2}|\nabla u|^2\delta_{ij} + S(g_0)_{ij})_l\} \\
&\geq \Delta F + 2m\sum_l F_lu_l + \sum_i F^{ii}u_{jl}^2 + 2(m-1)\sum_{i,l} F^{ii}u_{li}^2 - c\sum_i F^{ii}G \\
&\geq \Delta F + 2m\sum_l F_lu_l + \frac{1}{n}G^2\sum_i F^{ii} + 2(m-1)\sum_{i,l} F^{ii}u_{li}^2 - c\sum_i F^{ii}G.
\end{aligned}
\tag{45}
$$

From equation (38), $F = 2u_t - 2(k-l)u - \log r_{k,l}(g)$. In view of (41) and (42), (45) yields

$$
\begin{aligned}
0 &\geq -2(k-l)G + \frac{1}{n}\sum_i F^{ii}G^2 + 2(m-1)\sum_i F^{ii}u_{ii} - c\sum_i F^{ii}G \\
&\geq -2(k-l)\Delta u + \sum_i F^{ii}G^2 + 2(m-1)\sum F^{ii}u_{ii}^2 - c\sum F^{ii}G \\
&\geq \{-2(k-l)G + 2(m-1)\sum F^{ii}u_{ii}^2\} + \frac{1}{n}\sum_i F^{ii}(G^2 - cG).
\end{aligned}
\tag{46}
$$

The Proposition follows from (39), (46) and Lemma 6 below. ■

Lemma 6. *There is a large constant $m > 0$ such that*

$$
\frac{1}{2n}G^2\sum_i F^{ii} + 2(m-1)\sum_i F^{ii}w_{ii}^2 \geq 2(k-l)G. \tag{47}
$$

Proof. It is easy to check, from the Newton-MacLaurin inequality (14), that

$$
\begin{aligned}
\sum F^{ii}w_{ii}^2 &= \frac{\sigma_1(W)\sigma_k(W) - (k+1)\sigma_{k+1}(W)}{\sigma_k(W)} - \frac{\sigma_1(W)\sigma_l(W) - (l+1)\sigma_{l+1}(W)}{\sigma_l(W)} \\
&= (l+1)\frac{\sigma_{l+1}}{\sigma_l}(W) - (k+1)\frac{\sigma_{k+1}}{\sigma_k}(W) \geq c_{n,k,l}\frac{\sigma_{l+1}}{\sigma_l}(W),
\end{aligned}
\tag{48}
$$

and

$$
\sum_i F^{ii} = (n-k+1)\frac{\sigma_{k-1}}{\sigma_k}(W) - (n-l+1)\frac{\sigma_{l-1}}{\sigma_l}(W) \geq \frac{\tilde{c}_{n,k,l}}{\sigma_1(W)}, \tag{49}
$$

where $c_{n,k,l}$ and $\tilde{c}_{n,k,l}$ are two positive constant depending only on n, k and l. From these two facts, we can prove the claim as follows. First, if

$$\frac{\tilde{c}_{n,k,l}}{4n}\frac{\sigma_1(W)\sigma_{k-1}(W)}{\sigma_k(W)} \geq 4(k-l),$$

then the claim follows from (49) and (40). Hence we may assume that

$$\frac{\sigma_1(W)\sigma_{k-1}(W)}{\sigma_k(W)} \leq c^*_{n,k,l},$$

for some positive constant $c^*_{n,k,l}$ depending only on n, k and l. Together with the Newton-MacLaurin inequality, it implies

$$\frac{\sigma_{l+1}(W)}{\sigma_l(W)} \geq \hat{c}_{n,k,l}\frac{\sigma_k(W)}{\sigma_{k-1}(W)} \geq \hat{c}_{n,k,l}c^*_{n,k,l}\sigma_1(w),$$

which, in turn, together with (48) implies

$$\sum_i F^{ii}w_{ii}^2 \geq c_{n,k,l}\frac{\sigma_{l+1}(W)}{\sigma_l(W)} \geq c^1_{n,k,l}\sigma_1(W) \geq \frac{1}{2}c^1_{n,k,l}G.$$

Hence, if we choose m large, then the lemma is true. ■

Now we can prove the C^0 boundedness (and hence C^2 boundedness).

Proposition 3. *Let $g = e^{-2u}g_0$ be a solution of flow (13) with $\sigma_k(g(t)) \in \Gamma_k^+$ on $M \times [0, T^*)$. Then there is a constant $c > 0$ depending only on v_0, g_0, k and n (independent of T^*) such that*

$$\|u(t)\|_{C^2} \leq c, \quad \forall\, t \in [0, T^*). \tag{50}$$

Proof. We only need to show the boundedness of $|u|$. First we consider the case $l \neq n/2$. By Proposition 2 and the preservation of $\int \sigma_l(g)dg$, we have

$$\begin{aligned} c_l &= \int_M e^{(2l-n)u}\sigma_l\left(\nabla^2 u + du\otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}\right)dg_0 \\ &\leq c_1\textstyle\int_M e^{(2l-n)u}dg_0. \end{aligned} \tag{51}$$

If $l < n/2$, then (51), together with (39), implies that $u < c$ for some constant $c > 0$. On the other hand, in this case Proposition 1 gives

$$vol(g) \leq C\left(\int_M \sigma_l(g)dg\right)^{\frac{n}{n-2l}} = c_0 C,$$

which, together with (39) implies $u > c_1$, hence $|u| \leq C$ in this case.

If $l > n/2$, (51) gives a lower bound of u. Suppose there is no upper bound, we have a sequence of u, with ∇u and $\nabla^2 u$ bounded, but $\sup u$ goes to infinity (so does $\inf u$). Set $v = u - \inf u$, so v is bounded and so is the C^2 norm of v. But, for $\tilde{g} = e^{-2v}g_0$, we get $\tilde{F}_l(\tilde{g})$ tends to 0. Take a subsequence, we get $\sigma_l(e^{-2v^*}g_0) = 0$ with v^* in $C^{1,1}\cap\overline{\Gamma}_k^+$. This is a contradiction to Lemma 3.

Then we consider the case $l = n/2$. In this case, $\mathcal{E}_{n/2}(g)$ is constant. First it is easy to check that $g_t = e^{-2tu}g_0 \in \Gamma^+_{\frac{n}{2}}$ when $0 \le t \le 1$ (using the fact $(1, \cdots, 1, -1) \in \overline{\Gamma}^+_{\frac{n}{2}}$ when n even). In particular, $\sigma_{\frac{n}{2}}(g_t) > 0$ for $t > 0$. From the expression of $\mathcal{E}_{n/2}(g)$,

$$-\sup(u)\int_M \sigma_{\frac{n}{2}}(g)dg \le \mathcal{E}_{n/2}(g) \le -\inf(u)\int_M \sigma_{\frac{n}{2}}(g)dg.$$

Since

$$\int_M \sigma_{\frac{n}{2}}(g)dg = \int_M \sigma_{\frac{n}{2}}(g_0)dg_0,$$

So we have

$$-\sup(u)\int_M \sigma_{\frac{n}{2}}(g_0)dg_0 \le \mathcal{E}_{n/2}(g) \le -\inf(u)\int_M \sigma_{\frac{n}{2}}(g_0)dg_0.$$

Thus, $\inf(u)$ is bounded from above and $\sup(u)$ is bounded from below. By (39) again, u is bounded from above and away from 0. Now we have proved the boundedness of $|u|$ in all cases. Combining that with (39) and Proposition 2 gives a C^2 bound u independent of T. ■

Proposition 4. *There is a constant $C_0 > 0$ independent of T such that*

$$\frac{\sigma_k}{\sigma_l}(g)(t) \ge C_0, \qquad \text{for } t \in [0, \infty).$$

Proof. We follow the similar argument in [16]. Here we will make use of Lemma 2. We consider $H = \log \frac{\sigma_k}{\sigma_l}(g) - e^{-u}$ on $M \times [0, T]$ for any $T < T^*$. From (13) and (19) we have

$$\begin{aligned}\frac{dH}{dt} &= \frac{1}{2}\mathrm{tr}\{\tilde{T}_{k-1,l-1}(S_g)\nabla_g^2 \log \frac{\sigma_k}{\sigma_l}(g)\} + (k - l + \frac{1}{2}e^{-u})\left(\log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g)\right) \\ &= \frac{1}{2}\mathrm{tr}\{\tilde{T}_{k-1,l-1}(S_g)\nabla_g^2 (H + e^{-u})\} + (k - l + \frac{1}{2}e^{-u})\left(\log \frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g)\right).\end{aligned}$$

Without loss of generality, we may assume that the minimum of H in $M \times [0, T]$ achieves at $(x_0, t_0) \in M \times (0, T]$. Let H_j and H_{ij} are the first and second derivatives with respect to the background metric g_0. At this point, we have $\frac{dH}{dt} \le 0$, $0 = H_l = \sum_{ij} F^{ij} w_{ijl} + e^{-u}u_l$ for all l, and (H_{ij}) is non-negative definite. Also we have (F^{ij}) is positive definite and

$$\sum_{i,j} F^{ij} w_{ij} = \frac{1}{\sigma_k(g)}\frac{\partial \sigma_k(g)}{\partial w_{ij}} w_{ij} - \frac{1}{\sigma_l(g)}\frac{\partial \sigma_l(g)}{\partial w_{ij}} w_{ij} = k - l.$$

Recall that in local coordinates $\tilde{T}^{ij}_{k-1,l-1}(S_g) = F^{ij}$ and

$$\sum_{i,j} F^{ij}(\nabla_g^2)_{ij} H = \sum_{i,j} F^{ij}(H_{ij} + u_i H_j + u_j H_j - \sum_l u_l H_l \delta_{ij}).$$

It follows that at the point,
(52)

$$
\begin{aligned}
0 &\geq H_t - \frac{1}{2}\sum_{i,j} F^{ij} H_{ij} \\
&= \frac{1}{2}\mathrm{tr}\{\tilde{T}_{k-1,l-1}(S_g)\nabla_g^2 e^{-u}\} + (k-l+\frac{1}{2}e^{-u})\left(\log\frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g)\right) \\
&= \frac{1}{2}\sum_{i,j} F^{ij}\{(e^{-u})_{ij} + u_i(e^{-u})_j + u_j(e^{-u})_i - u_l(e^{-u})_l\delta_{ij}\} \\
&\quad +(k-l+\frac{1}{2}e^{-u})\left(\log\frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g)\right) \\
&= \frac{e^{-u}}{2}\sum_{i,j} F^{ij}\{-u_{ij} - u_iu_j + |\nabla u|^2\delta_{ij}\} + (k-l+\frac{1}{2}e^{-u})\left(\log\frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g)\right) \\
&= \frac{e^{-u}}{2}\sum_{i,j} F^{ij}\{-w_{ij} + S_{ij} + \frac{1}{2}|\nabla u|^2\delta_{ij}\} + (k-l+\frac{e^{-u}}{2})\left(\log\frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g)\right) \\
&\geq \frac{e^{-u}}{2}\sum_{i,j} F^{ij}\{-w_{ij} + S_{ij}\} + (k-l+\frac{1}{2}e^{-u})\left(\log\frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g)\right) \\
&= \frac{e^{-u}}{2}\sum_{i,j} F^{ij}S_{ij} + (k-l+\frac{1}{2}e^{-u})\left(\log\frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g)\right) - \frac{k-l}{2}e^{-u},
\end{aligned}
$$

where S_{ij} are the entries of $S(g_0)$. Since $S(g_0) \in \Gamma_k^+$, by Lemma 2,

$$
(53)\quad F^{ij}S_{ij} = \{\frac{1}{\sigma_k(g)}\frac{\partial\sigma_k(g)}{\partial w_{ij}} - \frac{1}{\sigma_l(g)}\frac{\partial\sigma_l(g)}{\partial w_{ij}}\}S_{ij} \geq (k-l)e^{2u}\left(\frac{\sigma_k}{\sigma_l}(g_0)\right)^{\frac{1}{k-l}}\left(\frac{\sigma_k}{\sigma_l}(g)\right)^{-\frac{1}{k-l}}.
$$

By C^2 estimates, $\log r_{k,l}(g)$ is bounded from above, we have

$$
\begin{aligned}
0 &\geq \frac{(k-l)e^u}{2}\left(\frac{\sigma_k}{\sigma_l}(g_0)\right)^{\frac{1}{k-l}}\left(\frac{\sigma_k}{\sigma_l}(g)\right)^{-\frac{1}{k-l}} \\
&\quad +(k-l+\frac{1}{2}e^{-u})\left(\log\frac{\sigma_k}{\sigma_l}(g) - \log r_{k,l}(g)\right) - \frac{k-l}{2}e^{-u} \\
&\geq c_1\left(\frac{\sigma_k}{\sigma_l}(g)\right)^{-\frac{1}{k-l}} + c_2\log\frac{\sigma_k}{\sigma_l}(g) - c_3
\end{aligned}
$$

for positive constants c_1, c_2 and c_3 independent of T. It follows that there is a positive constant c_4 independent of T such that

$$
\frac{\sigma_k}{\sigma_l}(g) \geq c_4,
$$

at point (x_0, t_0). Then the Proposition follows, as $|u|$ is bounded by Proposition 3. ■

Proof of Theorem 2. Now we can prove Theorem 2. First, from Propositions 3 and 4 we have $T^* = \infty$. Then, by Krylov's Theorem [23], the flow has $C^{2,\alpha}$ estimates. (20) implies that

$$\int_0^\infty \int_M (\sigma_k(g) - r_{k,l}\sigma_l(g))^2 dgdt < \infty,$$

which, in turn, implies that there is a sequence $\{t_l\}$ such that

$$\int_M (\sigma_k(g) - r_{k,l}\sigma_l(g))^2(t_l)dg \to 0$$

as $t_l \to \infty$. The above estimates imply that $g(t_l)$ converges in $C^{2,\alpha}$ to a conformal metric h, which is a solution of (6).

Now we want to use Simon's argument [31] to prove that h is the unique limit of flow (13) as in [16] (see also [1]). Since the arguments are essentially the same, here we only give a sketch. First, with the $C^{2,\alpha}$ regularity estimates (and higher regularity estimates follows from the standard parabolic theory) established for flow (13), one can show that,

$$\lim_{t\to\infty} \|\frac{\sigma_k}{\sigma_l}(g(t)) - \beta\|_{C^{4,\alpha}(M)} = 0,$$

for some positive constant β. It is clear that $\frac{\sigma_k}{\sigma_l}(h) = \beta$. By Proposition 4 and the Newton-MacLaurin inequality, there is a constant $c > 1$ such that $c^{-1} \le \sigma_l(g(t)) \le c$. We want to show that flow (13) is a pseudo-gradient flow, though it is not a gradient flow. The crucial step is to establish the "angle estimate" (54) for the L^2 gradient of some proper functionals. The estimate (54) enables one to conclude that the flow will converge to a unique stationary solution, we refer to [16] for the detailed argument.

Now we may switch the background metric to h and all derivatives and norms are taken with respect to the metric h (since we have all the a priori estimates for h).

Here we only give a proof for $l < k < n/2$. The proof for the other cases is similar after taking consideration of the corresponding functionals.

Consider a functional defined by

$$\mathcal{F}_{k,l}(g) = \left(\int \sigma_l(g)dg\right)^{-\frac{n-2k}{n-2l}} \int_M \sigma_k(g)dg.$$

Its L^2-gradient is

$$\nabla \mathcal{F}_{k,l} = -c_0((\sigma_k(g) - \tilde{r}_{k,l}(g)\sigma_l(g))e^{-nu},$$

where c_0 is a non-zero constant and $\tilde{r}_{k,l}(g)$ is given by

$$\tilde{r}_{k,l}(g) := \frac{\int_M \sigma_k(g)dg}{\int_M \sigma_l(g)dg},$$

which is different from $r_{k,l}$. But it is easy to check that $r_{k,l}(t)-\tilde r_{k,l}(t)\to 0$ as $t\to\infty$. Since $\frac{\sigma_k}{\sigma_l}(g(t))$ is very close to a constant for large t, from (20) we have

(54)

$$\begin{aligned}
\tfrac{d}{dt}\mathcal{F}_{k,l}(g) &\le -c\int_M (\frac{\sigma_k}{\sigma_l}(g)-r_{k,l})\left(\log\frac{\sigma_k}{\sigma_l}(g)-\log r_{k,l}\right)\sigma_l(g)dg\\
&\le -c\left(\int_M\left|\frac{\sigma_k}{\sigma_l}(g)-r_{k,l}\right|^2\sigma_l(g)dg\int_M\left|\log\frac{\sigma_k}{\sigma_l}(g)-\log r_{k,l}\right|^2\sigma_l(g)dg\right)^{1/2}\\
&\le -c\left(\int_M|\frac{\sigma_k}{\sigma_l}(g)-r_{k,l}|^2\sigma_l(g)dg\right)^{1/2}\left(\int_M|\frac{dg}{dt}|^2\sigma_l(g)dg\right)^{1/2}\\
&\le -c\left(\int_M|\frac{\sigma_k}{\sigma_l}(g)-\tilde r_{k,l}|^2\sigma_l(g)dg\right)^{1/2}\left(\int_M|\frac{dg}{dt}|^2\sigma_l(g)dg\right)^{1/2}\\
&\le -c\left(\int_M|\sigma_k(g)-\tilde r_{k,l}\sigma_l(g)|^2\sigma_l(g)dg\right)^{1/2}\left(\int_M|\frac{dg}{dt}|^2\sigma_l(g)dg\right)^{1/2}\\
&\le -c\left(\int_M|\nabla\mathcal{F}_{k,l}|^2dh\right)^{1/2}\left(\int_M|\frac{dg}{dt}|^2dh\right)^{1/2},
\end{aligned}$$

where $c>0$ is a constant varying from line to line. The angle estimate (54) means that flow (13) is a pseudo-gradient flow. Finally from (54) we can apply the argument given in [31] to show the uniqueness of the limit as in [16]. ■

5. Proof of Theorem 1

From the flow approach we developed, we have

Proposition 5. *Let (M,g_0) be a compact, connected and oriented locally conformally flat manifold with $g_0\in\Gamma_k^+$ and $0\le l<k\le n$. Let $\tilde{\mathcal{F}}_k$ defined as in (18), then there is $g_E\in\mathcal{C}_k$ satisfying equation (6) such that*

$$0<\tilde{\mathcal{F}}_k(g_E)\le\tilde{\mathcal{F}}_k(g),$$

for any $g\in\mathcal{C}_k$ with $\tilde{\mathcal{F}}_l(g_E)=\tilde{\mathcal{F}}_l(g)$. Moreover, if (M,g_0) is conformally equivalent to a space form, then (M,g_E) is also a space form.

Proof. The case $l=0$ has been treated in Proposition 1. We may assume $l\ge 1$ in the rest of proof. When (M,g_0) is conformally equivalent to a space form, then any solutions of (6) are metrics of constant sectional curvature (this can be proved using the method of moving plane of [10] as in [36], see Corollary 1.1 in [24] for more general statement), and hence have the same $\tilde{\mathcal{F}}_k$ if they have been the same $\tilde{\mathcal{F}}_l$. Hence the Proposition follows from Theorem 2.

Now we remain to consider the case $k<n/2$ and (M,g_0) is not conformally equivalent to a space form. We will follow the same argument in the proof of Proposition 1. Here we need the local estimates in Theorem 3 for the quotient equation (6).

First we want to show

$$\inf_{g\in\mathcal{C}_k,\tilde{\mathcal{F}}_l(g)=1}\tilde{\mathcal{F}}_k(g)=:\beta_0>0. \tag{55}$$

Suppose $\beta_0=0$. By the result for flow (13), there is a sequence $g_i=e^{-2u_i}g_0\in\mathcal{C}_k$ with $\tilde{\mathcal{F}}_l(g_i)=1$ and

$$\frac{\sigma_k}{\sigma_l}(g_i)=\beta_i,\quad \lim_{i\to\infty}\beta_i=0,$$

The scaled metric $\tilde{g}_i=e^{-2\tilde{u}_i}g_0$ with $\tilde{u}_i=u_i-\frac{1}{2(k-l)}\log\beta_i$ satisfies

$$\frac{\sigma_k}{\sigma_l}(\nabla^2\tilde{u}_i+d\tilde{u}_i\otimes d\tilde{u}_i-\frac{|\nabla\tilde{u}_i|^2}{2}g_0+S_{g_0})=e^{-2(k-l)\tilde{u}_i}. \tag{56}$$

By Proposition 1,

$$Cvol(\tilde{g}_i)^{\frac{n-2l}{n}}\le\tilde{\mathcal{F}}_l(\tilde{g}_i)=\beta_i^{\frac{n-2l}{2(k-l)}}\to 0\qquad\text{as } i\to\infty,$$

Since $vol(\tilde{g}_i)$ is tending to 0, by Corollary 2,

$$m_i:=\inf_M\tilde{u}_i\to+\infty\quad\text{as}\quad i\to\infty.$$

Now at the minimum point x_i of $\tilde{u}_i$, by equation (56),

$$\frac{\sigma_k}{\sigma_l}(S_{g_0})\le\frac{\sigma_k}{\sigma_l}(\nabla^2\tilde{u}_i+d\tilde{u}_i\otimes d\tilde{u}_i-\frac{|\nabla\tilde{u}_i|^2}{2}g_0+S_{g_0})=e^{-2(k-l)m_i}\to 0.$$

This is a contradiction to the fact that $g_0\in\Gamma_k^+$, proving (55).

Finally we prove the existence of an extremal metric in this case. From above argument, there is a minimization sequence $g_i\in\mathcal{C}_k$, with $\tilde{\mathcal{F}}_l(g)=1$, and $\frac{\sigma_k(g_i)}{\sigma_l(g_i)}=\beta_i$, with β_i decreasing and bound below by a positive constant. As (M,g_0) is not conformally equivalent to $\mathbb{S}^n$ by assumption, it follows from Theorem 1.3 in [12] (see also [25]) that the metrics converge (by taking a subsequence) to some g_E which attains the infimum C_S. ■

Proof of (**B**) *of Theorem 1.* The cases $l=n/2$ and $k=n/2$ were considered in [16] and [14]. Hence we assume that $k\neq n/2$ and $l\neq n/2$. Let us consider

$$\mathcal{F}_{k,l}(g)=\left(\int\sigma_l(g)dg\right)^{-\frac{n-2k}{n-2l}}\int_M\sigma_k(g)dg.$$

Since $\mathcal{F}_{k,l}$ is invariant under the transformation g to $e^{-2a}g$ for any constant a, Proposition 5 implies that for any $g\in\mathcal{C}_k$

$$\mathcal{F}_{k,l}(g)\le\mathcal{F}_{k,l}(g_E)=:C(n,k,l).$$

It is clear that $C(n,k,l)$ depends only on n,k,l.

Let $c_0=\mathcal{F}_{k,l}(g_E)$. From Proposition 5, we have

$$\begin{aligned}\int_M\sigma_k(g)dg &\le C(n,k,l)\left(\int_M\sigma_l(g)dg\right)^{\frac{n-2k}{n-2l}}\\ &= C(n,k,l)\left(\int_M\sigma_l(g)dg\right)^{\gamma k}\left(\int_M\sigma_l(g)dg\right)^{\frac{k}{l}},\end{aligned} \tag{57}$$

where $\gamma=\frac{n-2k}{k(n-2l)}-\frac{1}{l}$. It is clear that $\gamma>0$ when $l>n/2$ and $\gamma<0$ when $l<n/2$.

We first consider the case $l > n/2$. In this case, by Proposition 1 we have

$$\int_M \sigma_l(g)dg \le c_1 vol(g)^{\frac{n-2l}{n}},$$

where $c_1 = \mathcal{F}_l(g_e)$. It follows that

$$\left(\int_M \sigma_l(g)dg\right)^\gamma \le c_0^\gamma vol(g)^{\frac{l-k}{kl}}. \tag{58}$$

Hence

$$\begin{aligned}(\mathcal{F}_k(g))^{1/k} &= \left(vol(g)^{-\frac{n-2k}{n}}\int_M \sigma_k(g)dg\right)^{\frac{1}{k}}\\ &\le c_0^{\frac{1}{k}}\left(vol(g)^{-\frac{n-2l}{n}}\int_M \sigma_l(g)dg\right)^{\frac{1}{l}}\\ &= c_0^{\frac{1}{k}}(\mathcal{F}_l(g))^{1/l}.\end{aligned}$$

The equality holds if and only if g is a metric of constant sectional curvature.

Consider the case $l < n/2$. In this case, by Proposition 1 again we have

$$\int_M \sigma_l(g)dg \ge c_1 vol(g)^{\frac{n-2l}{n}},$$

where $c_1 = \mathcal{F}_l(g_e)$. Since $\gamma < 0$, we have (58). The same argument given in the previous case gives the same conclusion.

Finally, since $k \ge n/2$, (M, g_0) is conformally equivalent to a space form ([14]). The existence of the extremal metric which attains the equality case follows from Proposition 5. And the constant $C(n, k, l)$ is easy to calculate. ∎

Proof of (**C**) *of Theorem 1.* Let us first consider the case $l < n/2$. Let $g \in \mathcal{C}_{n/2}$. Choose a such that $\int_M \sigma_l(e^{-2a}g)dvol(e^{-2a}g) = \int_M \sigma_l(g_0)dg_0$. It is easy to see that

$$a = \frac{1}{n-2l}\{\log\int_M \sigma_l(g)dg - \log\int_M \sigma_l(g_0)dg_0\}.$$

By Proposition 5, we have

$$\begin{aligned}\mathcal{E}_{n/2}(g) &= \mathcal{E}_{n/2}(e^{-2a}g) + a\int_M \sigma_{n/2}(g)dg\\ &\ge a\int_M \sigma_{n/2}(g_0)dg_0\\ &= \frac{1}{n-2}\int_M \sigma_{n/2}(g_0)dg_0\left\{\log\int_M \sigma_l(g)dg - \log\int_M \sigma_l(g_0)dg_0\right\}.\end{aligned}$$

This proves the Theorem for the case $l < n/2$.

Now we consider the case $l > n/2$. 5. For any $g \in \mathcal{C}_l$ we choose

$$a = (\int_M \sigma_{n/2}(g)dg)^{-1}\mathcal{E}_{n/2}(g)$$

such that $\mathcal{E}_{n/2}(e^{-2a}g) = \mathcal{E}_{n/2}(g_0)$. Recall that $\tilde{\mathcal{F}}_{n/2} = \mathcal{E}_{n/2}$. By Proposition 5 again, we have

$$\begin{aligned}\tilde{\mathcal{F}}_l(g) &= \frac{1}{n-2l}\int_M \sigma_l(g)dg \\ &= \frac{1}{n-2l}e^{-(2l-n)a}\int_M \sigma_l(e^{-2a}g)dvol(e^{-2a}g) \\ &\geq \frac{1}{n-2l}e^{-(2l-n)a}\int_M \sigma_l(g_0)dg_0 \\ &= \frac{1}{n-2l}\exp\left\{(n-2l)\left(\int_M \sigma_{n/2}(g)dg\right)^{-1}\mathcal{E}_{n/2}(g)\right\}\int_M \sigma_l(g_0)dg_0.\end{aligned}$$

Since (M, g_0) is conformally equivalent to a space form in this case, the existence of the extremal metric can be proved along the same line as in part (**B**) of the Theorem. Note that since n is even, (M, g_0) is the standard sphere. The computation of C_{MT} is straightforward. ∎

Proof of (**A**) *of Theorem 1.* Inequality (5) follows from (55) in the proof of Proposition 5. The existence of the extremal metric has also proved there. As for inequality (7), since its proof is of different spirit and inequality itself is of independent interest, we will devote it in the next section (Theorem 4). The constant $C_{S,k,l}(\mathbb{S}^n)$ in (7) can be computed easily. ∎

6. The best constant

In this section, we address the question of the best constant in part (**A**) of Theorem 1. As in the Yamabe problem (i.e., $k = 1$ and $l = 0$), for $0 \leq l < k < n/2$ we define

$$Y_{k,l}(M,[g_0]) = \inf_{g\in\mathcal{C}_k}(\mathcal{F}_l(g))^{-\frac{n-2k}{n-2l}}\mathcal{F}_k(g) = \inf_{g\in\mathcal{C}_k}(\int_M \sigma_l(g)dg)^{-\frac{n-2k}{n-2l}}\int_M \sigma_k(g)dg.$$

It is clear that $Y_{k,l}(M,[g_0]) = C_s^{n-2k}$. In this section we prove

Theorem 4. *For any compact, oriented locally conformally flat manifold (M, g_0), we have*

$$Y_{k,l}(M,[g_0]) \leq Y_{k,l}(\mathbb{S}^n, g_{\mathbb{S}^n}), \tag{59}$$

where $g_{\mathbb{S}^n}$ is the standard metric of the unit sphere.

When $k = 1$ and $l = 0$, this was proven by Aubin (e.g., see [2]) for general compact manifolds. To prove Theorem 4 we need to construct a sequence of "blow-up" functions which belong to $\mathcal{C}_k$. This is a delicate part of the problem.

We need two Lemmas.

Lemma 7. *Let D be the unit disk in $\mathbb{R}^n$ and ds^2 the standard Euclidean metric. Let $g_0 = e^{-2u_0}ds^2$ be a metric on D and $g_0 \in \Gamma_k$ with $k < n/2$. Then there is a conformal metric $g = e^{-2u}ds^2$ on $D\backslash\{0\}$ of positive Γ_k-curvature with the following properties:*

1). *$\sigma_k(g) > 0$ in $D\backslash\{0\}$.*
2). *$u(x) = u_0(x)$ for $r = |x| \in (r_0, 1]$.*

3). $u(x) = a + \log r$ *for* $r = |x| \in (0, r_3)$ *and some constant* a.
for some constants r_0 *and* r_3 *with* $0 < r_3 < r_0 < 1$.

Proof. Let v be a function on D and $\tilde g = e^{-2v} g_0$. By the transformation formula of the Schouten tensor, we have

$$\begin{aligned} S(\tilde g)_{ij} &= \nabla^2_{ij}(v+u_0) + \nabla_i(v+u_0)\nabla_j(v+u_0) - \frac{1}{2}|\nabla(v+u_0)|^2\delta_{ij} \\ &= \nabla^2_{ij} v + \nabla_i v \nabla_j v + \nabla_i v \nabla_j u_0 + \nabla_j v \nabla_i u_0 \\ &\quad + (\frac{1}{2}|\nabla v|^2 + \nabla v \nabla u_0)\delta_{ij} + S_{g_0} \end{aligned} \tag{60}$$

Here ∇ and ∇^2 are the first and the second derivatives with respect to the standard metric ds^2. Let $r = |x|$. We want to find a function $v = v(r)$ with $\tilde g \in \Gamma_k^+$ and

$$v' = \frac{\alpha(r)}{r},$$

where $\alpha = 1$ near 0 and $\alpha = 0$ near 1. From (60) we have

$$S(\tilde g)_{ij} = \frac{2\alpha - \alpha^2}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2} + S(g_0)_{ij} + O(|\nabla u_0|)\frac{\alpha}{r}, \tag{61}$$

where $O(|\nabla u_0|)$ is a term bounded by a constant C_1 depending only on $\max|\nabla u_0|$. Let $A(r)$ be an $n \times n$ matrix with entry $a_{ij} = S(\tilde g)_{ij} - S(g_0)_{ij}$. Hence

$$\sigma_k(\tilde g) = e^{-2k(v+u_0)}\sigma_k\left(A + S(g_0)\right).$$

To our aim, we need to find α such that $A + S(g_0) \in \Gamma_k^+$. Let $\varepsilon \in (0, 1/2)$ and $r_0 = \min\{\frac{1}{2}, C_1\varepsilon\}$. We will choose α such that

$$\alpha(r) \in [0,1] \text{ and } \alpha(r) = 0, \text{ for } r \in [r_0, 1]. \tag{62}$$

Since $\sigma_k(\tilde g) = e^{2k(v+u_0)}\sigma_k(A(r) + S(g_0))$, we want to find α such that $\sigma_k(A(r) + S(g_0)) > 0$. It is clear to see that for $r \in [0, r_0]$

$$A(r) \geq \left(\frac{2\alpha - \alpha^2 - \varepsilon\alpha}{2r^2}\delta_{ij} + \left(\frac{\alpha'}{r} + \frac{\alpha^2 - 2\alpha}{r^2}\right)\frac{x_i x_j}{r^2}\right),$$

as a matrix. This implies that

$$\sigma_k(A(r)) \geq \frac{(n-1)!}{k!(n-k)!}\left(\frac{2\alpha - \alpha^2 - \varepsilon\alpha}{2r^2}\right)^k \left(n - 2k + 2\frac{r\alpha' - \varepsilon\alpha}{2\alpha - \alpha^2 - \varepsilon\alpha}\right). \tag{63}$$

One can easily check that for any small $\delta > 0$,

$$\alpha(r) = \frac{2(1-\varepsilon)\delta}{\delta + r^{\frac{1-\varepsilon}{2}}} \tag{64}$$

is a solution of

$$(2 - \varepsilon)\alpha - \alpha^2 = -4(r\alpha' - \varepsilon\alpha).$$

Now we can finish our construction of α. Since $S(g_0) \in \Gamma_k^+$, by the openness of Γ_k^+ we can choose $r_1 \in (0, r_0)$ and an non-increasing function $\alpha : [r_1, r_0] \subset [0, 1)$ such that

$\sigma_k(\tilde{g}) > 0$ and $\alpha(r_1) > 0$. Now we choose a suitable $\delta > 0$ and α in the form (64).Then find $r_2 \in (0, r_1)$ with $\alpha(r_2) = 1$. It is clear that $\sigma_k(A(r)) > 0$ on $[r_2, r_1]$. Define $\alpha(r) = 1$ on $[0, r_2]$. We may smooth α such that the new resulted conformal metric g satisfying all conditions in Lemma 7. ■

Remark 1. *From Lemma 7, one can prove that the connected sum of two locally conformally flat manifolds* (M_1, g_1) *and* (M_2, g_2) *with* $g_1, g_2 \in \Gamma_k^+$ $(k < n/2)$ *admits a metric in* Γ_k^+. *This is also true for general manifolds. Namely, the connected sum of two compact manifolds* (M_1, g_1) *and* (M_2, g_2) *with* $g_1, g_2 \in \Gamma_k^+$ $(k < n/2)$ *admits a metric in* Γ_k^+. *The proof is given in* [13].

Lemma 8. *For any small constants* $\delta > 0$ *and* $\varepsilon > 0$, *there exists a function* $u : \mathbb{R}^n \backslash \{0\} \to 0$ *satisfying:*

1. *The metric* $g = e^{-2u} dx^2$ *has positive* Γ_k*-curvature.*
2. $u = \log(1 + |x|^2) + b_0$ *for* $|x| \geq \delta$, *i.e.,* $(\{x \in \mathbb{R}^n \,|\, |x| \geq \delta\}, g)$ *is a part of a sphere.*
3. $u = \log|x|$ *for* $|x| \leq \delta_1$, *i.e.,* $(\{x \in \mathbb{R}^n \,|\, 0 < |x| \leq \delta_1\}, g)$ *is a cylinder.*
4. $vol(B_\delta \backslash B_{\delta_1}, g) \leq C\delta^{-\frac{2n}{1-\varepsilon_0}}$.
5. $\int_{B_\delta \backslash B_{\delta_1}} \sigma_k(g) dvol(g) \leq C\delta^{-\frac{2(n-2k)}{1-\varepsilon_0}}$, *for any* $k < n/2$,

where C *is a constant independent of* δ, $\delta_1 = \delta^{\frac{3-\varepsilon_0}{1-\varepsilon_0}}$ *and* $b_0 \sim \frac{3-\varepsilon_0}{1-\varepsilon_0} \log \delta$.

Proof. Let $\delta \in (0, 1)$ be any small constant. For any small constant $\varepsilon_0 > 0$, we define u by

$$u(r) = \begin{cases} \log(1 + r^2) + b_0, & r \geq \delta \\ -\dfrac{2}{1-\varepsilon_0} \log \dfrac{1 + \delta^{3-\varepsilon_0} r^{-(1-\varepsilon_0)}}{2} + \dfrac{3-\varepsilon_0}{1-\varepsilon_0} \log \delta & r \in (\delta_1, \delta) \\ \log r, & r \leq \delta_1, \end{cases},$$

where $\delta_1 = \delta^{\frac{3-\varepsilon_0}{1-\varepsilon_0}}$ and

$$b_0 = -\log(1 + \delta^2) - \frac{2}{1-\varepsilon_0} \log \frac{1+\delta^2}{2} + \frac{3-\varepsilon_0}{1-\varepsilon_0} \log \delta.$$

As in the proof of Lemma 5, we write $u'(r) = \frac{\alpha(r)}{r}$. It is easy to see that $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\alpha(r) = \begin{cases} \dfrac{2r^2}{1+r^2}, & r \geq \delta, \\ \dfrac{2\delta^{3-\varepsilon_0}}{\delta^{3-\varepsilon_0} + r^{1-\varepsilon_0}}, & r \in (\delta_1, \delta), \\ 1, & r \leq \delta_1. \end{cases}$$

One can check all conditions in the Lemma, except the smoothness of u, which is $C^{1,1}$. We first check (1). By a direct computation, see for example (13), we have

$$\sigma_k(e^{-2u}|dx|^2) = e^{2ku(r)} \frac{(n-1)!}{k!(n-k)!} \left(\frac{2\alpha - \alpha^2}{2r^2}\right)^k \left(n - 2k + 2\frac{r\alpha'}{2\alpha - \alpha^2}\right).$$

In the interval (δ_1, δ), $\alpha \in (0,2)$ satisfies
$$\frac{2r\alpha'}{2\alpha-\alpha^2} = -(1-\varepsilon_0).$$
Since $k < n/2$, we have $\sigma_k(e^{-2u}|dx|^2) > 0$. One can also directly to check (4) and (5). Here we only check (5). A direct computation gives
$$\begin{aligned}\int_{B_\delta\backslash B_{\delta_1}} \sigma_k(g)dvol(g) &\le c\int_{\delta_1}^{\delta} e^{-(n-2k)u(r)}r^{-2k}r^{n-1}dr\\ &\le c\delta^{-(n-2k)\frac{3-\varepsilon_0}{1-\varepsilon_0}}\int_{\delta_1}^{\delta} r^{n-2k-1}dr\\ &\le c\delta^{-\frac{2(n-2k)}{1-\varepsilon_0}}.\end{aligned}$$
From our construction, we only have $u \in C^{1,1}$. But, for $\delta > 0$ fixed, we can smooth α so that $u \in C^\infty$ satisfies all conditions (1)-(5). ∎

Proof of Theorem 4. Let $p \in M$ and U a neighborhood of p such that (U,g) is conformally flat, namely $(U,g) = (D, e^{-2u_0}|dx|^2)$. Applying Lemma 7, we obtain a conformal metric u satisfying conditions 1)-3) in Lemma 7 with constants r_0, r_3 and a. By adding a constant we may assume $a = 0$. Now applying Lemma 8 for any small constant $\delta > 0$ we have a conformal metric $g_\delta = e^{-2u_\delta}|dx|^2$ on $\mathbb{R}^n\backslash\{0\}$. Consider the rescaled function
$$\tilde u_\delta = u_\delta(\frac{\delta_1}{r_3}x) - \log\frac{\delta_1}{r_3}.$$
Now u and $\tilde u_\delta$ are the same in $\{0 < |x| < r_3\}$. Consider the following conformal transformation
$$f(x) = \frac{r_3^2}{2}\frac{x}{|x|^2},$$
which maps $\{r_3/2 \le |x| \le r_3\}$ into itself and maps one of boundary components to another with opposite orientations. Now we define a new function on M by
$$w_\delta(x) = \begin{cases} 0, & |x| \ge r_0,\\ u - u_0, & r_3/2 \le |x| \le r_0,\\ \tilde u_\delta(f(x)) + 2\log|x| - \log\frac{r_3^2}{2} - u_0, & |x| \le r_3/2.\end{cases}$$
Since u and $\tilde u_\delta$ are the same in $\{0 < |x| < r_3\}$, it clear that $w_\delta(x)$ is smooth on M. Consider the conformal metric $g_\delta = e^{-2w_\delta}g$ and compute, using Lemmas 7 and 8
$$\begin{aligned}\int_M \sigma_k(g_\delta)dvol(g_\delta) &= \int_{\{|x|\le r_3/2\}} \sigma_k(g_\delta)dg_\delta + O(1)\\ &= e^{-(n-2k)b_0}\int_{\mathbb{R}^n\backslash\{|x|\le\delta\}} \sigma_k(g_{\mathbb{S}^n})dvol(g_{\mathbb{S}^n}) + O(1)\delta^{-\frac{2(n-2k)}{1-\varepsilon_0}}\\ &= \delta^{-\frac{3-\varepsilon_0}{1-\varepsilon_0}(n-2k)}\int_{\mathbb{R}^n\backslash\{|x|\le\delta\}} \sigma_k(g_{\mathbb{S}^n})dvol(g_{\mathbb{S}^n}) + o(\delta^{-\frac{3-\varepsilon_0}{1-\varepsilon_0}(n-2k)})\end{aligned}$$

and

$$\int_M \sigma_l(g_\delta)dvol(g_\delta) = \delta^{-\frac{3-\varepsilon_0}{1-\varepsilon_0}(n-2l)} vol(\mathbb{R}^n \backslash \{|x| \le \delta\}, g_{\mathbb{S}^n}) + o(\delta^{-\frac{3-\varepsilon_0}{1-\varepsilon_0}(n-2l)}),$$

where $g_{\mathbb{S}^n} = \frac{1}{(1+|x|^2)^2}|dx|^2$ is the standard metric of the sphere and $O(1)$ is a term bounded by a constant independent of δ. Now it is readily to see

$$Y_{k,l}(M) \le \lim_{\delta \to 0} Y_{k,l}(g_\delta) \to Y_{k,l}(\mathbb{S}^n),$$

as $\delta \to 0$. ■

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