Geometric Invariants of the Quantum Hall Effect

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Abstract. We study the two-dimensional Hall effect with a random potential. The Hall conductivity is identified as a geometric invariant associated with an algebra of observables. Using the pairing between K-theory and cyclic cohomology theory, we identify this geometric invariant with a topological index, thereby giving the Hall conductivity a new interpretation.

Introduction

Since its experimental discovery [21], the quantum Hall effect [23] has attracted the attention of many researchers. The general approach in the theoretical work is that one considers a one-particle theory in a two-dimensional sample of infinite size. This leads to the study of the Hamiltonian

$$H_\omega = [K_x^2 + K_y^2]/2 + Q_\omega(x, y), \quad \omega \in \Omega, \tag{0.1}$$

with the commutation relation

$$[K_x, K_y] = i\beta.$$

Here, β accounts for the presence of a magnetic field and Q is a real function on the state space Ω which admits a flow of translation by $\mathbb{R}^2$. D. Thouless et al. [28] considered periodic potentials Q, i.e. the case where Ω is the two dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the natural flow induced by $\mathbb{R}^2$. They showed that with such a potential and a magnetic field of rational flux, the conductance of any filled, isolated band in the spectrum of the Hamiltonian is a topological invariant which is an integral multiple of $e^2/\hbar$. Since then substantial progress has been made on this subject through the contributions of many authors [1, 2, 3, 5-8, 18, 22]. Among these references, J. Bellissard's work has a close relation with this paper.

It was first observed in [5] that the two-dimensional Hall conductivity can be identified as the Chern character of the spectral projection of H_ω corresponding to $(-\infty, E_F)$, where E_F is the Fermi level. This led to the introduction of A. Connes' geometry-operator algebra techniques to the study of the quantum Hall effect

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[6, 7, 8]. In order to use these techniques, Bellissard constructed an algebra of observables $\mathscr{A}_\omega$ which contains sufficient information about the Hamiltonian H_ω. It was noted in [5] that a computation of the current yields that the Hall conductivity σ_H equals $(e^2/2\pi\hbar)\tau(P_F[\delta_1(P_F), \delta_2(P_F)])/2\pi i$, where τ and δ_1, δ_2 are, respectively, a trace and two derivations on $\mathscr{A}_\omega$, and P_F is the Fermi projection. (Also see [3, 22].) According to Connes [14], quantities of form $\tau(P[\delta_1(P), \delta_2(P)])$ are in fact a geometric invariant. This invariant can also be interpreted as the pairing of $[P] \in K_0(\mathscr{A}_\omega)$ with a cyclic 2-cocycle ψ, in the sense of [15], on the algebra of observables. The cocycle gives rise to an element $[\psi]$ in the even cyclic cohomology group of a smooth subalgebra $\mathscr{A}_\omega^\infty$ which contains all the spectral projection of H_ω corresponding to gaps. Thus the proof of the quantization of the conductivity becomes the computation of all possible pairings $\langle [\psi], [P] \rangle$. We will follow Bellissard's formalism to the point of reducing the computation of the Hall conductivity to that of these pairings. But this is also the point at which [5–8] and our work take on divergent courses. Using a Fredholm module $(\mathscr{H}, G, F)$, Bellissard identified the Hall conductivity as an analytical index,

$$\tau(P[\delta_1(P), \delta_2(P)])/2\pi i = \operatorname{Ind} F_P,$$

where F_P is a Fredholm operator. The importance of this formula is that it holds true for projections in the Sobolev space of $\mathscr{A}_\omega$ [6, 7].

The purpose of this paper is to identify $\langle [\psi], [P] \rangle$ and, therefore, the Hall conductivity as a topological index which is obviously an integer. The technique required to achieve this is quite different from Bellissard's; we use the Thom isomorphisms for C^*-crossed products and the cyclic cohomology theory introduced by Connes. We write the Hamiltonian in the form of

$$H_\omega = [-\partial^2/\partial x^2 + (-i\partial/\partial y - \beta x)^2]/2 + Q_\omega(x, y).$$

This particular form of the Hamiltonian makes apparent that the corresponding algebra of observables $C_\omega^*(\Omega, \mathbb{R}^2, \beta)$ is an iterative crossed product of $C(\Omega)$ by two copies of $\mathbb{R}$. We use this fact to prove a labelling theorem for the gaps of the spectrum of H_ω. Composing the two Thom isomorphisms involved here, one obtains a natural isomorphism $\phi \colon K_0(C(\Omega)) \to K_0(C_\omega^*(\Omega, \mathbb{R}^2, \beta))$. Let μ be the ergodic measure that induces the trace τ. It turns out that the pull back (in the sense of the pairing between cyclic cohomology theory and K-theory) of the cyclic cohomology class $[\psi]$ by the transpose of ϕ is the class $-[\mu]$. Since each element of $K_0(C(\Omega))$ is naturally identified with the difference of two classes of vector bundles, we can state our main result as follows.

Theorem. *Assume that β is a nonzero real number.*

(i) *We have $\langle [\psi], [P] \rangle = - \langle [\mu], \phi^{-1}[P] \rangle$ for smooth $[P] \in K_0(C^*(\Omega, \mathbb{R}^2, \beta))$.*

(ii) *Given a real potential $Q \in C(\Omega)$, if the Fermi energy E_F belongs to a gap of the spectrum of H_ω, then there are vector bundles V^+ and V^- on Ω such that $\phi^{-1}[P_F] = [V^+] - [V^-]$ and such that the Hall conductivity equals*

$$\sigma_H = (e^2/2\pi\hbar)\langle [\psi], [P_F] \rangle = (\dim V^- - \dim V^+)(e^2/2\pi\hbar).$$

The rest of the paper is organized as follows. In the first section, we discuss

the algebra of observables associated with the random Hamiltonian and the cyclic cocycle which will later be used to express the conductivity. In Sect. 2, we prove the gap labelling theorem for the Hamiltonian H_ω. This requires the identification of $C_\omega^*(\Omega, \mathbb{R}^2, \beta)$ with the iterative crossed product of $C(\Omega)$ by two copies of $\mathbb{R}$. As it turns out, this identification is crucial to the proof of our main result. We identify the conductivity with the geometric invariant $\langle [\psi], [P_F] \rangle$ in Sect. 3. Section 4 is proof of our main theorem.

1. The Random Hamiltonian and the Algebra of Observables

Throughout the paper, Ω will denote a separable, *connected*, compact Hausdorff space. It will be assumed that there is a group of homeomorphisms $\{\varphi_{x,y}:(x,y)\in\mathbb{R}^2\}$ acting on Ω such that for any $f\in C(\Omega)$, $(x,y)\mapsto f\circ\varphi_{x,y}$ is a continuous map from $\mathbb{R}^2$ into $C(\Omega)$. For the sake of simplicity, $\varphi_{x,y}(\omega)$ will be denoted in the form of flow translation $\omega + (x,y)$. For any $f\in C(\Omega)$, we define $f_\omega(x,y) = f(\omega + (x,y))$, $\omega\in\Omega$, $(x,y)\in\mathbb{R}$. Let $d\mu$ denote a translation invariant, ergodic, probability measure of the flow $(\Omega, \varphi, \mathbb{R}^2)$. Given a real function $Q\in C(\Omega)$, one has a family of Hamiltonians

$$H_\omega = [-\partial^2/\partial x^2 + (-i\partial/\partial y - \beta x)^2]/2 + Q_\omega(x,y) \text{ on } L^2(\mathbb{R}^2), \quad \omega\in\Omega.$$

Here, β is a real number which arises from the physical system. In the case of the two-dimensional Hall effect with a uniform magnetic field **B** perpendicular to the xy-plane, the constant β equals $e|\mathbf{B}|/\hbar c$. We will assume for the rest of the paper that β is a *nonzero* real number. Under this assumption, the free Hamiltonian

$$H_0 = [-\partial^2/\partial x^2 + (-i\partial/\partial y - \beta x)^2]/2$$

is in essence a linear oscillator. This can be seen in the following way. Let $K_1 = -i\partial/\partial x$ and $K_2 = -i\partial/\partial y - \beta x$. Let $\mathscr{F}: L^2(\mathbb{R}^2) \to L^2(\hat{\mathbb{R}}^2)$ be the Fourier transform $(\mathscr{F}f)(u,v) = \int_{\mathbb{R}^2} \exp[-2\pi i(ux+vy)]f(x,y)dxdy$. Then on $L^2(\hat{\mathbb{R}}^2)$, we have $\hat{K}_1 = \mathscr{F}K_1\mathscr{F}^* = 2\pi u$, $\hat{K}_2 = \mathscr{F}K_2\mathscr{F}^* = 2\pi v - (\beta/2\pi)i\partial/\partial u$, and, therefore, $\hat{H}_0 = \mathscr{F}H_0\mathscr{F}^* = [(2\pi u)^2 + (2\pi v - (\beta/2\pi)i\partial/\partial u)^2]/2$. Let U_β be the unitary operator of multiplication by $\exp(i(2\pi)^2 uv/\beta)$ on $L^2(\hat{\mathbb{R}}^2)$. Then

$$\tilde{K}_1 = U_\beta\hat{K}_1 U_\beta^* = U_\beta\mathscr{F}K_1\mathscr{F}^* U_\beta^* = 2\pi u = k_1\otimes 1,$$
$$\tilde{K}_2 = U_\beta\hat{K}_2 U_\beta^* = U_\beta\mathscr{F}K_2\mathscr{F}^* U_\beta^* = -(\beta/2\pi)i\partial/\partial u = k_2\otimes 1, \qquad (1.1)$$

where k_1 is the multiplication by $2\pi u$ on $L^2(\mathbb{R})$ and $k_2 = -(\beta/2\pi)id/du$ on the same space. Hence H_0 is unitarily equivalent to

$$\tilde{H}_0 = U_\beta\hat{H}_0 U_\beta^* = U_\beta\mathscr{F}H_0\mathscr{F}^* U_\beta^* = [-(\beta/2\pi)^2\partial^2/\partial u^2 + (2\pi u)^2]/2$$
$$= 2^{-1}[-(\beta/2\pi)^2 d^2/du^2 + (2\pi u)^2]\otimes 1. \qquad (1.2)$$

In this paper, we use a function on a given Ω to describe the disorder associated with the physical system. We take this approach because of its mathematical convenience. On the other hand, if a potential Q on $\mathbb{R}^2$ is given, one can think of Ω as the hull of Q and the flow on Ω as that induced by translating Q [5]. Oftentimes, one can assume that the flow $(\Omega, \varphi, \mathbb{R}^2)$ is minimal and uniquely ergodic. This is the case, for example, when the potential Q is periodic, as was considered in [28].

For such a potential, Ω is the two-dimensional torus $T^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ and the flow is that induced by the group structure of Ω.

Let $C_c(\Omega \times \mathbb{R}^2)$ denote the collection of continuous functions on $\Omega \times \mathbb{R}^2$ with compact support. For any $a \in C_c(\Omega \times \mathbb{R}^2)$ and $\omega \in \Omega$, let

$$(\pi_\omega(a)f)(x,y) = \int_{\mathbb{R}^2} a(\omega + (x,y),\xi,\eta)\exp(-i\beta x\eta)f(x+\xi,y+\eta)d\xi d\eta, \quad f \in L^2(\mathbb{R}^2).$$
(1.3)

For a fixed ω, the C^*-algebra generated by all $\pi_\omega(a)$, $a \in C_c(\Omega \times \mathbb{R}^2)$, will be denoted by $C_\omega^*(\Omega, \mathbb{R}^2, \beta)$. This is the *algebra of observables*. If β were 0, this algebra would be a representation of the C^*-crossed product $C(\Omega) \times_\varphi \mathbb{R}^2$. We will later identify $C_\omega^*(\Omega, \mathbb{R}^2, \beta)$ as an iterative crossed product of $C(\Omega)$ by two copies of $\mathbb{R}$. This identification is absolutely essential to our work. Although the algebra of observables constructed here has a different appearance from the algebra introduced by Bellissard in [5, 6, 7], they are actually unitarily equivalent. We choose this particular representation of the algebra because of our need to identify it as a double crossed product.

For any $z \in \mathbb{C} \setminus \mathbb{R}$, the resolvent $(z - H_\omega)^{-1}$ belongs to $C_\omega^*(\Omega, \mathbb{R}^2, \beta)$. But this is not an obvious fact and a little explanation is in order. Using the usual power series expansion, one sees that it suffices to show that the resolvent of the free Hamiltonian, $(z - H_0)^{-1}$, belongs to $C_\omega^*(\Omega, \mathbb{R}^2, \beta)$. By (1.3), the C^*-algebra $\mathscr{K}$ generated by all

$$\int_{\mathbb{R}^2} b(\xi,\eta)\exp(i\xi K_1)\exp(i\eta K_2)d\xi d\eta, \quad b \in C_c(\mathbb{R}^2)$$
(1.4)

is contained in $C_\omega^*(\Omega, \mathbb{R}^2, \beta)$. On the other hand, the C^*-algebra generated by

$$\int_{\mathbb{R}^2} b(\xi,\eta)\exp(i\xi k_1)\exp(i\eta k_2)d\xi d\eta, \quad b \in C_c(\mathbb{R}^2)$$
(1.5)

is $\mathscr{K}(L^2(\mathbb{R}))$, the collection of all the compact operators on $L^2(\mathbb{R})$. By (1.1), the map that takes (1.4) to (1.5) extends to an isomorphism $\mathscr{J}_\beta$ from $\mathscr{K}$ onto $\mathscr{K}(L^2(\mathbb{R}))$. It is a well known fact that if z is a nonreal number, $(z - (k_1^2 + k_2^2)/2)^{-1} \in \mathscr{K}(L^2(\mathbb{R}))$. Therefore $(z - \tilde{H}_0)^{-1} \in \mathscr{K}(L^2(\mathbb{R})) \otimes \mathbb{C}$. One deduces from this, (1.1) and (1.2) that $(z - H_0)^{-1} \in \mathscr{K} \subset C_\omega^*(\Omega, \mathbb{R}^2, \beta)$.

On the space $L^2(\Omega \times \mathbb{R}^2)$, where the measure is $d\mu \times dxdy$, let $C^*(\Omega, \mathbb{R}^2, \beta)$ be the C^*-algebra generated by twisted convolution operators

$$(C_a f)(\omega, x, y) = \int_{\mathbb{R}^2} a(\omega, \xi, \eta)\exp(-i\beta x\eta)f(\omega + (\xi,\eta), x+\xi, y+\eta)d\xi d\eta,$$

$$a \in C_c(\Omega \times \mathbb{R}^2), \quad f \in L^2(\Omega \times \mathbb{R}^2).$$
(1.6)

We call the function a the kernal of C_a. It is easy to verify that the kernel of $(C_a)^*$ is

$$a^*(\omega, x, y) = \exp(-i\beta xy)\overline{a(\omega + (x,y), -x, -y)}.$$
(1.7)

For any $a, b \in C_c(\Omega \times \mathbb{R}^2)$, $C_a C_b$ has a kernel

$$a * b(\omega, x, y) = \int_{\mathbb{R}^2} a(\omega, \xi, \eta)\exp(-i\beta\xi(y-\eta))b(\omega + (\xi,\eta), x-\xi, y-\eta)d\xi d\eta.$$
(1.8)

These formulas show that

$$\tau(C_a) = \int_\Omega a(\omega, 0, 0) d\mu(\omega), \quad a \in C_c(\Omega \times \mathbb{R}^2) \tag{1.9}$$

defines a trace on $C^*(\Omega, \mathbb{R}^2, \beta)$. On $L^2(\Omega \times \mathbb{R}^2)$, define $(D_1 f)(\omega, x, y) = 2\pi i x f(\omega, x, y)$ and $(D_2 f)(\omega, x, y) = 2\pi i y f(\omega, x, y)$. These operators induce two derivations on $C^*(\Omega, \mathbb{R}^2, \beta)$ in the following way. For $A \in C^*(\Omega, \mathbb{R}^2, \beta)$, define

$$\delta_1(A) = [A, D_1] = AD_1 - D_1 A \quad \text{and} \quad \delta_2(A) = [A, D_2] = AD_2 - D_2 A$$

if these operators belong to $C^*(\Omega, \mathbb{R}^2, \beta)$. Obviously for every $a \in C_c(\Omega \times \mathbb{R}^2)$, C_a belongs to the domain of δ_1 and δ_2. The operators $\delta_1(C_a)$ and $\delta_2(C_a)$ have kernels $2\pi i x a(\omega, x, y)$ and $2\pi i y a(\omega, x, y)$ respectively. Let $C^\infty(\Omega, \mathbb{R}^2, \beta)$ be the collection of elements A such that $\delta_1^n(\delta_2^m(A)) \in C^*(\Omega, \mathbb{R}^2, \beta)$, and such that $\tau(\delta_1^n(\delta_2^m(A))^* \delta_1^n(\delta_2^m(A))) < \infty$ for any nonnegative integers n and m. The algebra $C^\infty(\Omega, \mathbb{R}^2, \beta)$ is a smooth subalgebra of $C^*(\Omega, \mathbb{R}^2, \beta)$ closed under analytic functional calculus [13]. According to [13, Page 52], we have

Proposition 1.1. *The inclusion map* $\iota : C^\infty(\Omega, \mathbb{R}^2, \beta) \to C^*(\Omega, \mathbb{R}^2, \beta)$ *induces an isomorphism*

$$\iota_* : K_0(C^\infty(\Omega, \mathbb{R}^2, \beta)) \to K_0(C^*(\Omega, \mathbb{R}^2, \beta))$$

between the K-groups.

Let $\mathscr{W} = W^*(\Omega, \mathbb{R}^2, \beta)$ be the weak*-closure of $C^*(\Omega, \mathbb{R}^2, \beta)$ in $\mathscr{L}(L^2(\Omega \times \mathbb{R}^2))$. It is known that the von Neumann algebra $\mathscr{W}$ is a factor if the action of $\mathbb{R}^2$ on Ω is free and it is of type-II_∞ if the orbits of the flow are not closed [14]. The trace τ naturally extends to a normal trace, which we denote by the same letter, on $\mathscr{W}$. One can easily deduce from formulas (1.6–1.9) that the GNS representation of $\mathscr{W}$ induced by τ is unitarily equivalent to the identity representation of $\mathscr{W}$ on $L^2(\Omega \times \mathbb{R}^2)$. Hence τ is always a faithful trace on $\mathscr{W}$. If $A \in \mathscr{W}$ is such that $\tau(A^* A) < \infty$, then A has a square-integrable kernel a on $\Omega \times \mathbb{R}^2$ in the sense that for any $f \in L^2(\Omega \times \mathbb{R}^2)$, Af is given by the right-hand side of (1.6) [20].

Let $A_1, A_2 \in C^*(\Omega, \mathbb{R}^2, \beta)$ be two elements in the domain of δ_1 and δ_2 such that $\tau(A_j^* A_j) < \infty$ and $\tau(\delta_k(A_j)^* \delta_k(A_j)) < \infty$, $j, k = 1, 2$. Then $\delta_k(A_1 A_2) = \delta_k(A_1)A_2 + A_1 \delta_k(A_2)$ is of τ-trace class, $k = 1, 2$. Since $\delta_k(A_j)$'s have square-integrable kernels, a simple computation using formulas (1.6), (1.8) and (1.9) yields $\tau(\delta_k(A_1)A_2) = -\tau(A_1 \delta_k(A_2))$. Hence for A_1 and A_2 satisfying the above condition, we have

$$\tau(\delta_k(A_1 A_2)) = 0, \quad k = 1, 2. \tag{1.10}$$

In other words these derivations annihilate the trace. Because δ_1 and δ_2 commute,

$$\psi(a_0, a_1, a_2) = \tau(a_0 [\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2)]) \tag{1.11}$$

defines a cyclic 2-cocycle on $C^\infty(\Omega, \mathbb{R}^2, \beta)$ [15].

Let V be the unitary operator $(Vf)(\omega, x, y) = f(\omega + (x, y), x, y)$ on $L^2(\Omega \times \mathbb{R}^2)$. It is straightforward to verify that for any $a \in C_c(\Omega \times \mathbb{R})$,

$$VC_a = \int_\Omega \oplus \pi_\omega(a) d\mu(\omega) V,$$

where the direct integral corresponds to the space decomposition

$$L^2(\Omega \times \mathbb{R}) = \int_\Omega \oplus L^2(\mathbb{R}) d\mu(\omega).$$

If the flow $(\Omega, \varphi, \mathbb{R}^2)$ is minimal, then $\|\pi_\omega(a)\|$ is independent of ω [20]. In this case, for any $\omega \in \Omega$, $C_a \mapsto \pi_\omega(a)$ extends to an isomorphism from $C^*(\Omega, \mathbb{R}^2, \beta)$ onto $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$. If the flow is not minimal, one can still show, by using the ergodicity of μ, that there is a Borel set $N \subset \Omega$ of measure 0 such that π_ω is an isomorphism for every $\omega \in \Omega \backslash N$. Let

$$\mathbf{H} = V^* \int_\Omega \oplus H_\omega d\mu(\omega) V.$$

Then

$$(\mathbf{H}f)(\omega, x, y) = [-\partial^2/\partial s^2 + (-i\partial/\partial t - \beta x)^2] f(\omega + (s, t), x + s, y + t)/2|_{t=s=0} + Q(\omega) f(\omega, x, y). \tag{1.12}$$

Thus we have

Proposition 1.2. (i) *The map π_ω is an isomorphism from $C^*(\Omega, \mathbb{R}^2, \beta)$ onto $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$ for a.e. $\omega \in \Omega$. If the flow is minimal, then every π_ω is an isomorphism.*

(ii) *Given a real potential $Q \in C(\Omega)$, for a.e. $\omega \in \Omega$, the spectrum of H_ω is the same as that of $\mathbf{H}$ defined by (1.12). If the flow is minimal, then every H_ω has the same spectrum as $\mathbf{H}$.*

Because of the isomorphism π_ω, for a.e. ω, τ will be regarded as a trace on $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$. Similarly, δ_1 and δ_2 are regarded as derivations on $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$ and ψ defined by (1.11) as a cyclic cocycle on $C^\infty_\omega(\Omega, \mathbb{R}^2, \beta) = \pi_\omega(C^\infty(\Omega, \mathbb{R}^2, \beta))$.

Lemma 1.3. *Let G be the resolvent set of H_ω and let $R_\omega(z) = (z - H_\omega)^{-1}$. Then $K_j R_\omega(z), j = 1, 2$, are $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$-valued analytic functions on G and for any $z \in G$, $\tau(R_\omega(z)^* R_\omega(z)) < \infty$.*

Proof. Let $R_0(z) = (z - H_0)^{-1}$, then for nonreal z, $R_\omega(z) = R_0(z) + R_0(z) Q_\omega R_\omega(z)$ and, therefore, $K_j R_\omega(z) = K_j R_0(\lambda) + K_j R_0(z) Q_\omega R_\omega(z)$. Let $S = \{(n + 1/2)|\beta| : n = 0, 1, 2, \ldots\}$, the spectrum of H_0 [25]. We first show that, off the set S, $K_j R_0(z)$ belongs to $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$. According to (1.1) and (1.2), this will follow if we can show that for $z \in \mathbb{C} \backslash S$, $k_j(z - (k_1^2 + k_2^2)/2)^{-1}$ is a compact operator on $L^2(\mathbb{R})$. Let $p = k_1 + ik_2$. Then, by the commutation relation $[k_1, k_2] = i\beta$, we have $p^* p = k_1^2 + k_2^2 - \beta$. Since $(\bar{z} - (k_1^2 + k_2^2)/2)^{-1} p^* p(z - (k_1^2 + k_2^2)/2)^{-1}$ is a compact operator, by polar decomposition, $p(z - (k_1^2 + k_2^2)/2)^{-1}$ is compact. A similar argument shows that $p^*(z - (k_1^2 + k_2^2)/2)^{-1}$ is also compact. Hence $k_j(z - (k_1^2 + k_2^2)/2)^{-1}, j = 1, 2$, are compact operators. Once this has been established, the analyticity of the map $z \mapsto K_j R_0(z)$ is obvious. Hence $K_j R_\omega(z)$ is a $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$-valued analytic function on $G \backslash S$. Since S has no points of accumulation and since $\langle K_j R_\omega(z) f, g \rangle = \langle R_\omega(z) f, K_j g \rangle$ for any $f \in L^2(\mathbb{R}^2)$ and g in the domain of K_j, all the singularities of $K_j R_\omega(z)$ in G are removable.

By the resolvent identity we used in the previous paragraph and the Cauchy integral formula, in order to show $\tau(R_\omega(z)^* R_\omega(z)) < \infty$ on G, it suffices to show $\tau(R_0(z)^* R_0(z)) < \infty$ on $\mathbb{C} \backslash S$. Since the multiplicity of $(k_1^2 + k_2^2)/2$ at every point of

S is 1 [16], $(z - (k_1^2 + k_2^2)/2)^{-1} = \mathscr{J}_\beta(R_0(z))$ is a Hilbert–Schmidt operator on $L^2(\mathbb{R})$ whenever $z \in \mathbb{C} \backslash S$. By (1.3),(1.4),(1.8) and a simple computation, one finds that an element $A \in \mathscr{K}$ is of τ-trace class if and only if $\mathscr{J}_\beta(A)$ is a trace class operator on $L^2(\mathbb{R})$ and that

$$\tau(A) = (|\beta|/2\pi)\operatorname{tr}(\mathscr{J}_\beta(A)). \tag{1.13}$$

Hence $\tau(R_0(z)^* R_0(z)) < \infty$. $\square$

Remark. The key in the above proof is that the singularities of the analytic function $K_j R_0(z)$ are isolated points in $\mathbb{C}$ if $\beta \neq 0$. If $\beta = 0$, however, $K_j R_0(z)$ does not have analytic continuation on $[0, \infty)$ and the proof of the above lemma will fail. Consequently, the following proposition may not be true. This is the why we must assume $\beta \neq 0$.

Proposition 1.4. *Let λ be a real number not in the spectrum of H_ω and let P be the spectral projection of H_ω corresponding to the interval $(-\infty, \lambda)$. Then*

$$P \in C_\omega^\infty(\Omega, \mathbb{R}^2, \beta).$$

Proof. Since the spectrum of H_ω in $(-\infty, \lambda)$ is contained in some finite interval (b, λ), we can choose a counter-clockwise oriented contour $\Gamma \subset G$ which encircles (b, λ) once. Thus $P = \int_\Gamma R_\omega(z)dz/2\pi i$. Because $\delta_j(R_\omega(z)) = 2\pi R_\omega(z)K_j R_\omega(z)$, it follows from Lemma 1.3 that $\delta_j(P) = -i\int_\Gamma R_\omega(z)K_j R_\omega(z)dz \in C_\omega^*(\Omega, \mathbb{R}^2, \beta)$. Formally, we have $\delta_j(K_j) = 2\pi$ and $\delta_j(K_k) = 0$ for $j \neq k$. Hence that $\delta_j^n(\delta_k^m(P))$ belongs to $C_\omega^*(\Omega, \mathbb{R}^2, \beta)$ can be proved by mathematical induction and repeated use of Lemma 1.3. $\square$

2. The Integrated Density of States and the Fermi Energy

Let $\mathbf{H} = \int_{\mathbb{R}} \lambda d\mathbf{P}_\lambda$ be the spectral decomposition of the operator (1.12). Since the resolvent $(z - \mathbf{H})^{-1}$ belongs to the ideal generated by the τ-trace class operators in $\mathscr{W}$, every spectral projection $\mathbf{P}(-\infty, \lambda)$, $\lambda < \infty$, is finite in $\mathscr{W}$. As usual, the increasing function $N_Q(\lambda) = \tau(\mathbf{P}(-\infty, \lambda))$ is called the *integrated density of states* (IDS) for the potential Q. Since τ is a faithful trace, the spectrum of $\mathbf{H}$ and, for a.e. ω, that of H_ω, is the complement in $\mathbb{R}$ of the open intervals on which N_Q is a constant. If the action of $\mathbb{R}^2$ action on Ω is uniquely ergodic, then $N_Q(\lambda)$ can be recovered from the limit distribution of the eigenvalues of H_ω restricted to finite domains of $\mathbb{R}^2$ (hence the term IDS). In the case $\beta = 0$, this was done by M. Shubin in [27]. One can trace the steps of [27] to reproduce such a result for the case $\beta \neq 0$. But since this is not essential to our investigation, we omit the details.

Like the Hamiltonians for many other disordered systems, the function N_Q provides a labelling for the gaps of the spectrum of H_ω [4]. A commonly used method for determining all the possible labels is to compute the image under the trace map of the K_0-group of the algebra of observables. For this purpose we will express $C^*(\Omega, \mathbb{R}^2, \beta)$ as an iterative C^*-crossed product of $C(\Omega)$ by two copies of $\mathbb{R}$. This idea comes from a careful inspection of (1.8). For the validity of the following

constructions of C^*-crossed product algebras, the reader is referred to [26, Sect. 7.7].

From the group $\{\varphi_{x,y}:(x,y)\in\mathbb{R}^2\}$, one obtains two one-parameter groups of homeomorphisms on Ω:

$$\theta_t(\omega) = \varphi_{0,t}(\omega), \quad \text{and} \quad \alpha_t(\omega) = \varphi_{t,0}(\omega), \quad \omega\in\Omega, \quad t\in\mathbb{R}.$$

The C^*-crossed product $C(\Omega) \times_\theta \mathbb{R}$ of $C(\Omega)$ by the one-parameter group θ can be represented on $L^2(\Omega\times\mathbb{R})$ as follows: For $a\in C_c(\Omega\times\mathbb{R})$, and $f\in L^2(\Omega\times\mathbb{R})$, let

$$(a*f)(\omega,y) = \int_\mathbb{R} a(\omega,t)f(\omega+(0,t),y+t)dt. \tag{2.1}$$

Then for any $a,b\in C_c(\Omega\times\mathbb{R})$, we have

$$(a*b)(\omega,y) = \int_\mathbb{R} a(\omega,t)b(\omega+(0,t),y-t)dt = \int_\mathbb{R} a(\omega,t)b(\theta_t(\omega),y-t)dt. \tag{2.2}$$

The algebra $C(\Omega) \times_\theta \mathbb{R}$ is the C^*-algebra generated by all $a\in C_c(\Omega\times\mathbb{R})$ with the algebraic structure described above. The invariant measure $d\mu$ gives rise to a trace τ_1 on $C(\Omega) \times_\theta \mathbb{R}$:

$$\tau_1(a) = \int_\Omega a(\omega,0)d\mu(\omega). \tag{2.3}$$

Let us now define a one-parameter group of automorphisms $\{\gamma_s:s\in\mathbb{R}\}$ on $C(\Omega) \times_\theta \mathbb{R}$. For an $a\in C_c(\Omega\times\mathbb{R})$, let

$$\gamma_s(a)(\omega,y) = \exp(-i\beta sy)a(\alpha_s(\omega),y) = \exp(-i\beta sy)a(\omega+(s,0),y). \tag{2.4}$$

It is straightforward to verify that each γ_s is an automorphism. Indeed $\gamma_s(a) = U_s^* a U_s$, where $(U_sf)(\omega,y) = \exp(-i\beta sy)f(\omega+(s,0),y)$ on $L^2(\Omega\times\mathbb{R})$. The C^*-crossed product $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$ on $L^2((\Omega\times\mathbb{R})\times\mathbb{R})$ is the norm-closure of $C_c(\mathbb{R}, C(\Omega) \times_\theta \mathbb{R})$, the collection of compactly supported, $C(\Omega) \times_\theta \mathbb{R}$-valued continuous functions on $\mathbb{R}$. If we view $C_c(\Omega\times\mathbb{R})$ as a dense subalgebra of $C(\Omega) \times_\theta \mathbb{R}$ in the sense of (2.1), then $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$ is generated by

$$(\widetilde{C}_a f)(\omega,y,x) = \int_\mathbb{R} \exp(-i\beta sy)\left(\int_\mathbb{R} a(s,\omega,t)f(\omega+(s,t),y+t,x+s)dt\right)ds, \tag{2.5}$$

where $a\in C_c(\mathbb{R}, C_c(\Omega\times\mathbb{R})) = C_c(\mathbb{R}\times\Omega\times\mathbb{R})$ and $f\in L^2((\Omega\times\mathbb{R}_y)\times\mathbb{R}_x)$. Furthermore, for any $a,b\in C_c(\mathbb{R}, C_c(\Omega\times\mathbb{R}))$, $\widetilde{C}_a\widetilde{C}_b$ has a kernel

$$(a*b)(x,\omega,t) = \int_\mathbb{R} \exp(-i\beta s(y-t))\left(\int_\mathbb{R} a(s,\omega,t)b(x-s,\omega+(s,t),y-t)dt\right)ds. \tag{2.6}$$

Naturally, the trace on $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$ is

$$\tau(\widetilde{C}_a) = \int_\Omega a(0,\omega,0)d\mu(\omega). \tag{2.7}$$

For $a\in C_c(\Omega\times\mathbb{R}^2)$, let $\tilde{a}(x,\omega,y) = a(\omega,x,y)$. A comparison of (1.6) and (1.8) with (2.5) and (2.6) shows that $C_a\mapsto\widetilde{C}_{\tilde{a}}$ extends to an isomorphism from $C^*(\Omega,\mathbb{R}^2,\beta)$ onto $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$. Hence we have proved

Proposition 2.1. *There is a natural isomorphism $\mathscr{I}:C^*(\Omega,\mathbb{R}^2,\beta)\to[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$ which respects the traces.*

For the rest of the paper, we will identify these algebras. Thus δ_1 and δ_2 will be regarded as derivations on $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$ and ψ as a cyclic cocycle on this algebra.

Theorem 2.2.
$$\tau_*(K_0(C^*(\Omega, \mathbb{R}^2, \beta))) = \tau_*(K_0(C(\Omega) \times_\varphi \mathbb{R}^2)) + (\beta/2\pi)\mathbb{Z}.$$

Proof. By (1.13), we have $(\beta/2\pi)\mathbb{Z} = \tau_*(K_0(\mathcal{K})) \subset \tau_*(K_0(C^*(\Omega, \mathbb{R}^2, \beta)))$. For the rest of the proof, we need Connes' formula for the image of $K_0([C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R})$ under the trace map. According to [13, Corollary 1], $\tau_*(K_0([C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}))$ consists of numbers

$$(\tau_1 \otimes \text{tr}_n)(\delta_\gamma(u)u^{-1})/2\pi i,$$

where $\delta_\gamma(a) = \lim_{s \to 0} (\gamma_s(a) - a)/s$ on $C(\Omega) \times_\theta \mathbb{R}$ and u is any smooth invertible element in $M_n \otimes C(\Omega) \times_\theta \mathbb{R}^\sim$. By the definition (2.4) of γ_s, we have $\delta_\gamma = \delta_\alpha + \delta_\beta$, where, for an $a \in C_c(\Omega \times \mathbb{R})$ smooth along the orbits of $\mathbb{R}$, $\delta_\alpha(a)(\omega, y) = da(\alpha_s(\omega), y)/ds|_{s=0}$ and $\delta_\beta(a)(\omega, y) = -i\beta ya(\omega, y)$. Because $[C(\Omega) \times_\theta \mathbb{R}] \times_\alpha \mathbb{R} = C(\Omega) \times_\varphi \mathbb{R}$, by [13], $(\tau_1 \otimes \text{tr}_n)(\delta_\alpha(u)u^{-1})/2\pi i \in \tau_*(K_0([C(\Omega) \times_\theta \mathbb{R}] \times_\alpha \mathbb{R})) = \tau_*(K_0(C(\Omega) \times_\varphi \mathbb{R}^2))$. On the other hand, it is obvious that $\delta_\beta = (-\beta/2\pi)\delta_t$, where δ_t is the derivation induced by the one-parameter group of automorphisms $\iota_s(a)(\omega, y) = \exp(2\pi i s y)a(\omega, y)$. Since $(\tau_1 \otimes \text{tr}_n) (\delta_\iota(u)u^{-1})/2\pi i \in \tau_*(K_0([C(\Omega) \times_\theta \mathbb{R}] \times_\iota \mathbb{R}))$, which is known to be $\mathbb{Z}$ [13], we have $(\tau_1 \otimes \text{tr}_n)(\delta_\beta(u)u^{-1})/2\pi i \in (\beta/2\pi)\mathbb{Z}$. Hence $\tau_*(K_0([C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R})) \subset \tau_*(K_0(C(\Omega) \times_\varphi \mathbb{R}^2)) + (\beta/2\pi)\mathbb{Z}$. The inclusion the other way follows from the fact that for a smooth, invertible u, $(\tau_1 \otimes \text{tr}_n)(\delta_\alpha(u)u^{-1})/2\pi i \in (\tau_1 \otimes \text{tr}_n)(\delta_\gamma(u)u^{-1})/2\pi i + (\beta/2\pi)\mathbb{Z} \subset \tau_*(K_0([C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}))$. $\square$

Corollary 2.3. *If λ belongs to a gap of the spectrum of H_ω for a.e. ω, then*

$$N_Q(\lambda) \in \tau_*(K_0(C(\Omega) \times_\varphi \mathbb{R}^2)) + (\beta/2\pi)\mathbb{Z}.$$

The computation of $\tau_*(K_0(C(\Omega) \times_\varphi \mathbb{R}^2))$ is contained in Connes' work [14]. He showed, for example, that if Ω is a smooth manifold and if the action φ is generated by two smooth vector fields X and Y, then $\tau_*(K_0(C(\Omega) \times_\varphi \mathbb{R}^2))$ consists of numbers $\langle C, \text{ch}[e] \rangle$, where $\text{ch}: K_0(C(\Omega)) \to H^2(\Omega, \mathbb{R})$ is the Chern character and C is the Ruelle–Sullivan current of the foliation of Ω by the orbits of $\mathbb{R}^2$. Hence there is a large class of Ω and β for which $\tau_*(K_0(C^*(\Omega, \mathbb{R}^2, \beta)))$ is a dense subset of $\mathbb{R}$. Because of this density, it is not unreasonable to speculate that the gaps of H_ω may not always correspond to Landau levels and that the Hall conductivity may not always be identified with Landau band index. We will discuss this point further in the next section.

Recall that the *Fermi energy* of the physical system can be computed from N_Q [27]. Let M_Q be the "inverse" of N_Q, i.e.

$$M_Q(t) = \sup\{\lambda : N_Q(\lambda) < t\}.$$

If $\rho > 0$ is the electron density, then we can express the Fermi energy as the average value of M_Q on the interval $[0, \rho]$, that is

$$E_F = E_F(\rho) = \rho^{-1}\int_0^\rho M_Q(t)dt \quad [27].$$

This formula tells us that if the Fermi energy belongs to a gap of the spectrum, it will remain so if the electron density varies a small amount. If the spectrum of H_ω is a Cantor set, a small variation of the electron density could move the Fermi energy into a gap.

3. The Conductivity and Cyclic Cohomology

Assume that the Fermi energy E_F for H_ω lies in a gap of its spectrum. We will denote by P_F the spectral projection of H_ω corresponding to the interval $(-\infty, E_F)$. As was explained in [5, 6], the Hall conductivity σ_H can be measured in the following way. Assuming that there is an electric field **E** in the direction of the Hall potential difference at time $t = 0$, one calculates the evolution of the current as the electric field is being turned off. This involves taking the limits $|\mathbf{E}| \to 0$ and $t \to \infty$. A recent result of J. Avron, R. Seiler and L. Yaffe [3] allows the interchange of the order of these limits. Therefore the Kubo–Greenwood formula yields that at temperature 0, the Hall conductivity equals

$$\sigma_H = \lim_{\varepsilon \downarrow 0} \frac{e^2}{i\hbar\varepsilon} \int_0^\infty e^{-\varepsilon t} \tau(P_F[K_1, \exp(-itH_\omega)K_2 \exp(itH_\omega)]P_F)dt. \qquad (3.1)$$

The same formula was used by H. Kunz [22].

Note that $\exp(itH_\omega)$ is a multiplier of $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$. Indeed it is fairly obvious that for any $z \in \mathbb{C} \backslash \mathbb{R}$ and $A \in C^*_\omega(\Omega, \mathbb{R}^2, \beta)$, $\exp(itH_\omega)R_\omega(z)A \in C^*_\omega(\Omega, \mathbb{R}^2, \beta)$. Hence that $\exp(itH_\omega)A$ belongs to $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$ follows from the resolvent identity $R_\omega(z) = R_0(z) + R_0(z)Q_\omega R_\omega(z)$ and the fact $\lim_{\lambda \to \infty} \| \lambda i R_0(i\lambda)A - A \| = 0$, which can be easily derived from the structure of $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$. Lemma 1.3 say that $K_j P_F$ and $P_F K_j$, $j = i, 2$, are of τ-trace class in $C^*_\omega(\Omega, \mathbb{R}^2, \beta)$. Thus the integrand on the right-hand side of (3.1) is well defined. We will now rewrite σ_H in terms of the cyclic cocycle (1.11).

Lemma 3.1. *Let P be the spectral projection of H_ω corresponding to $(-\infty, \lambda)$, where λ belongs to a gap of the spectrum of H_ω. Then*

$$\lim_{\varepsilon \downarrow 0} \int_0^\infty e^{-\varepsilon t} \tau(P[K_1, \exp(-itH_\omega)K_2 \exp(itH_\omega)]P)dt/\varepsilon = \tau(P[\delta_1(P), \delta_2(P)])/4\pi^2.$$
$$(3.2)$$

Proof. Let

$$A_\varepsilon = \int_0^\infty e^{-\varepsilon t} \exp(-itH_\omega)K_2 \exp(itH_\omega)dt.$$

Since the trace τ has the property $\tau(B^*) = \overline{\tau(B)}$, we have $\mathrm{Im}\, \tau(PK_1 PA_\varepsilon P) = -i[\tau(PK_1 PA_\varepsilon P) - \tau(PA_\varepsilon PK_1 P)]/2 = 0$. Therefore the left-hand side of (3.2) equals

$$\lim_{\varepsilon \downarrow 0} 2i\, \mathrm{Im}\, \tau(PK_1 A_\varepsilon P)/\varepsilon = \lim_{\varepsilon \downarrow 0} 2i\, \mathrm{Im}\, \tau(PK_1[A_\varepsilon, P]P)/\varepsilon. \qquad (3.3)$$

Let $H_\omega = \int_\mathbb{R} \xi dP_\xi$ be the spectral decomposition. Then $P = P_\lambda$ and

$$[A_\varepsilon, P] = \iint_{\mathbb{R}\,\mathbb{R}} \left(\int_0^\infty e^{-\varepsilon t} e^{it(\eta-\xi)} [\chi_{(-\infty,\lambda)}(\eta) - \chi_{(-\infty,\lambda)}(\xi)]\, dt \right) dP_\xi K_2 dP_\eta$$

$$= \iint_{\mathbb{R}\,\mathbb{R}} \frac{\chi_{(-\infty,\lambda)}(\eta) - \chi_{(-\infty,\lambda)}(\xi)}{\varepsilon + i(\xi - \eta)}\, dP_\xi K_2 dP_\eta.$$

Suppose that the spectrum of H_ω is contained in $(a+1, \infty)$. Let Γ be a counter-clockwise oriented contour in $\mathbb{C}$ that encircles the interval $(a+1, \lambda)$ once and that crosses the real axis at a and λ. If $\xi, \eta \in (a, +\infty) \backslash \{\lambda\}$, we have

$$\frac{\chi_{(-\infty,\lambda)}(\eta) - \chi_{(-\infty,\lambda)}(\xi)}{\varepsilon + i(\xi - \eta)} = \frac{1}{2\pi} \int_\Gamma \frac{dz}{(z-\xi)(z-\eta-i\varepsilon)}.$$

Using the identity $R_\omega(z - i\varepsilon) = R_\omega(z) + i\varepsilon R_\omega(z) R_\omega(z - i\varepsilon)$ and the identity above, we have

$$[A_\varepsilon, P] = \int_\Gamma R_\omega(z) K_2 R_\omega(z - i\varepsilon)\, dz/2\pi$$

$$= i\delta_2(P)/2\pi - \varepsilon \int_\Gamma R_\omega(z) K_2 R_\omega(z) R_\omega(z - i\varepsilon)\, dz/2\pi i.$$

Since $K_j \delta_k(P) \in C^*(\Omega, \mathbb{R}^2, \beta)$, $\delta_j(P)$ is self-adjoint, and $P = P^2$, by (1.10), we have $\operatorname{Im} \tau(-iPK_1\delta_2(P)/2\pi) = \operatorname{Re} \tau(PK_1\delta_2(P)/2\pi) = \tau(PK_1\delta_2(P) + \delta_2(P)K_1 P)/2\pi = \tau(\delta_2(PK_1 P))/2\pi = 0$. Hence

$$\text{LHS of (3.2)} = -\lim_{\varepsilon \downarrow 0} 2i \operatorname{Im}\left[\int_\Gamma \tau(PK_1 R_\omega(z) K_2 R_\omega(z) R_\omega(z - i\varepsilon)P)\, dz/2\pi i \right]$$

$$= -2i \operatorname{Im}\left[\int_\Gamma \tau(PK_1 R_\omega(z) K_2 R^2(z) P)\, dz/2\pi i \right]$$

$$= -2i \operatorname{Im}\left[\int_\Gamma \tau(PR_\omega(z) K_1 R_\omega(z) K_2 R_\omega(z) P)\, dz/2\pi i \right]. \qquad (3.4)$$

Let ζ be a contour which is slightly larger than Γ and which still crosses the real axis through the gap containing λ. Then

$$\delta_1(P)\delta_2(P) = \left(-i \int_\Gamma R_\omega(z) K_1 R_\omega(z)\, dz \right)\left(-i \int_\zeta R_\omega(u) K_2 R_\omega(u)\, du \right).$$

For $z \in \Gamma$ and $u \in \zeta$, we have $R_\omega(z) R_\omega(u) = (R_\omega(z) - R_\omega(u))/(u - z)$. Hence

$$P\delta_1(P)\delta_2(P)P = -\iint_{\Gamma\zeta} PR_\omega(z) K_1 \frac{R_\omega(z) - R_\omega(u)}{u - z} K_2 R_\omega(u) P\, dz\, du. \qquad (3.5)$$

Using the spectral decomposition of H_ω and the standard contour integral tricks, one finds $\int_\Gamma R_\omega(z)(u - z)^{-1}\, dz = 2\pi i P R_\omega(u)$ and $\int_\zeta R_\omega(u)(u - z)^{-1}\, dz = 2\pi i R_\omega(z)(1 - P)$. Thus (3.5) yields

$$P\delta_1(P)\delta_2(P)P = 2\pi i \int_\zeta PR_\omega(u) K_1 R_\omega(u) K_2 R_\omega(u) P\, du.$$

Comparing this with (3.4), we obtain

LHS of (3.2) $= 2i \operatorname{Im} \tau(P\delta_1(P)\delta_2(P)/(2\pi)^2)$

$$= \tau(P\delta_1(P)\delta_2(P) - \delta_2(P)\delta_1(P)P)/4\pi^2 = \tau(P[\delta_1(P),\delta_2(P)])/4\pi^2,$$

as was to be proved. $\square$

Corollary 3.2. *The Hall conductivity can be expressed as*

$$\sigma_H = (e^2/2\pi\hbar)\tau(P_F[\delta_1(P_F),\delta_2(P_F)])/2\pi i.$$

Quantities of form $\tau(P[\delta_1(P),\delta_2(P)])/2\pi i$ were first studied and recognized as a geometric invariant in the irrational rotation algebra by A. Connes [14]. Here, as it was the case in [14], we can identify this invariant as the "integral" of the curvature of a connection associated with the algebra of observables. Given a projection $P \in C_\omega^\infty(\Omega, \mathbb{R}^2, \beta)$, $\mathscr{E}_P = PC_\omega^\infty(\Omega, \mathbb{R}^2, \beta)$ is a finite projective modular over $C_\omega^\infty(\Omega, \mathbb{R}^2, \beta)$. Given the derivations δ_1 and δ_2, a compatible connection ∇ on $\mathscr{E}_P$ is a pair of linear maps $\nabla_1, \nabla_2 : \mathscr{E}_P \to \mathscr{E}_P$ such that $\nabla_j(\xi u) = \nabla_j(\xi)u + \xi\delta_j(u)$ for $\xi \in \mathscr{E}_P$ and $u \in C_\omega^\infty(\Omega, \mathbb{R}^2, \beta)$. We can simply choose these maps to be $\nabla_j(\xi) = P\delta_j(\xi)$, $j = 1, 2$. A simple calculation shows $(\nabla_1\nabla_2 - \nabla_2\nabla_1)\xi = [PD_1P, PD_2P]\xi$. Therefore the curvature $\mathscr{C} = \nabla_1\nabla_2 - \nabla_2\nabla_1$ is an endomorphism on $\mathscr{E}_P$, i.e. $\mathscr{C}(\xi u) = \mathscr{C}(\xi)u$ for $\xi \in \mathscr{E}_P$ and $u \in C_\omega^\infty(\Omega, \mathbb{R}^2, \beta)$. On the other hand, $P[\delta_1(P),\delta_2(P)]P = [PD_1P, PD_2P] = \nabla_1\nabla_2 - \nabla_2\nabla_1 = \mathscr{C}$. Hence $\tau(P[\delta_1(P),\delta_2(P)])/2\pi i = \tau(\mathscr{C})/2\pi i$ if one extends, or, perhaps more appropriately, restricts τ to End $\mathscr{E}_P$. The main purpose of this paper is to identify the geometric invariant $\tau(P_F[\delta_1(P_F),\delta_2(P_F)])/2\pi i$ with a topological invariant which is obviously an integer. To accomplish this, we need the pairing between the K-theory and the cyclic cohomology theory of the algebra of observables, and the Thom isomorphisms for K-groups and those for cyclic cohomology groups.

Recall that there is an operator S for the cyclic cohomology theory such that for any algebra $\mathscr{A}$, $S : H_\lambda^n(\mathscr{A}) \to H_\lambda^{n+2}(\mathscr{A})$ is a group homomorphism [15]. Using this operator, one obtains two groups of inductive limits $H_\lambda^{\mathrm{ev}}(\mathscr{A}) = \varinjlim_S H^{2n}(\mathscr{A})$ and $H_\lambda^{\mathrm{odd}}(\mathscr{A}) = \varinjlim_S H^{2n+1}(\mathscr{A})$. There are pairings between $K_0(\mathscr{A})$ and $H_\lambda^{\mathrm{ev}}(\mathscr{A})$ and between $K_1(\mathscr{A})$ and $H_\lambda^{\mathrm{odd}}(\mathscr{A})$. That is, there are bilinear maps $\langle .,.\rangle : H_\lambda^{\mathrm{ev}}(\mathscr{A}) \times K_0(\mathscr{A}) \to \mathbb{R}$ and $\langle .,.\rangle : H_\lambda^{\mathrm{odd}}(\mathscr{A}) \times K_1(\mathscr{A}) \to \mathbb{R}$. For the exact definition of these pairings, the reader is referred to [15]. As we have mentioned, the tri-linear functional ψ defined by (1.11) is a cyclic 2-cocycle. Hence ψ gives rise to a class $[\psi] \in H_\lambda^{\mathrm{ev}}(C^\infty(\Omega, \mathbb{R}^2, \beta))$. Thus $\langle[\psi], .\rangle : K_0(C^\infty(\Omega, \mathbb{R}^2, \beta)) \to \mathbb{R}$ is a group homomorphism. We will also consider $\langle[\psi], .\rangle$ as a homomorphism from $K_0(C_\omega^\infty(\Omega, \mathbb{R}^2, \beta))$ into $\mathbb{R}$. We will show that the range of this map is $\mathbb{Z}$. For a projection $P \in C_\omega^\infty(\Omega, \mathbb{R}^2, \beta)$, $\langle[\psi],[P]\rangle = \tau(P[\delta_1(P),\delta_2(P)])/2\pi i$ [15, 108]. In particular, the Hall conductivity can be expressed as

$$\sigma_H = (e^2/2\pi\hbar)\langle[\psi],[P_F]\rangle. \tag{3.6}$$

In [22], Kunz considered the Hall effect where the physical units are normalized so that $\beta = 1$, and where the potential is relatively small, i.e. $\|Q\|_\infty < 1/2$. It follows from general principles of analysis that, in this case, the spectrum of H_ω is *contained* in $B = \bigcup_{n=0}^\infty B_n$, where $B_n = [n + (1/2) - \|Q\|_\infty, n + (1/2) + \|Q\|_\infty]$. He showed that

if $\lambda \notin B$, that is, if λ belongs to a "natural gap" of the spectrum of the Hamiltonian, then the conductivity corresponding to $(-\infty, \lambda)$ is quantized and is given by the corresponding Landau band index. Using our notations, this amounts to computing $\langle [\psi], [P_\lambda] \rangle$ if P_λ is the spectral projection of H_ω corresponding to $(-\infty, \lambda)$. He achieved this by approximating Q by periodic potentials. But from the viewpoint of K-theory and cyclic cohomology theory, there is a much simpler approach. For $t \in [0, 1]$, define $H_\omega^t = H_0 + t Q_\omega$. The spectrum of every H_ω^t is also contained in B. The resolvent $R_\omega^t(z) = (z - H_\omega^t)^{-1}$ is a continuous function of t. Therefore, if P_λ^t is the spectral projection of H_ω^t for the interval $(-\infty, \lambda)$, then $t \mapsto P_\lambda^t$ is a continuous function from $[0, 1]$ onto $C_\omega^\infty(\Omega, \mathbb{R}^2, 1)$. Hence in $K_0(C_\omega^\infty(\Omega, \mathbb{R}^2, 1))$, $[P_\lambda] = [P_\lambda^1] = [P_\lambda^0]$. Thus $\langle [\psi], [P_\lambda] \rangle = \langle [\psi], [P_\lambda^0] \rangle$. Since $H_\omega^0 = H_0$, $P_\lambda^0 \in \mathcal{K} \cap C_\omega^\infty(\Omega, \mathbb{R}^2, 1) = \mathcal{K}_\infty = C^\infty(\text{point}, \mathbb{R}^2, 1)$, the class $[P_\lambda]$ in $K_0(C_\omega^\infty(\Omega, \mathbb{R}^2, 1))$ belongs to the image of the inclusion map $\iota_*: K_0(\mathcal{K}_\infty) \to K_0(C^\infty(\Omega, \mathbb{R}^2, 1))$. It is known that the pairing of $[\psi]$ with any element in $K_0(\mathcal{K}_\infty)$ is an integer. The main contribution of Kunz's work is, of course, the treatment of the case where the Fermi energy lies in the edge of one of the bands B_n, where the spectral projection is sufficiently localized.

Given a C^*-dynamical system $(A, \mathbb{R}, \alpha)$, there are two Thom isomorphisms $\phi_{\alpha, j}: K_j(A) \to K_{1-j}(A \times_\alpha \mathbb{R})$, $j = 0, 1$ [13]. For the algebra $C^*(\Omega, \mathbb{R}^2, \beta) = [C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$, the composition of the Thom isomorphisms $\phi_{\theta, 0}: K_0(C(\Omega)) \to K_1(C(\Omega) \times_\theta \mathbb{R})$ and $\phi_{\gamma, 1}: K_1(C(\Omega) \times_\theta \mathbb{R}) \to K_0([C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R})$ provides an isomorphism

$$\phi = \phi_{\gamma,1} \circ \phi_{\theta,0}: K_0(C(\Omega)) \to K_0([C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}) = K_0(C^*(\Omega, \mathbb{R}^2, \beta)). \quad (3.7)$$

Because of Proposition 1.1 and because of the isomorphism π_ω, for a.e. $\omega \in \Omega$, ϕ^{-1} can be considered as an isomorphism from $K_0(C_\omega^\infty(\Omega, \mathbb{R}^2, \beta))$ onto $K_0(C(\Omega))$. The map $f \mapsto \int_\Omega f(\omega) d\mu(\omega)$ defines a 0-cocycle on $C(\Omega)$; its class in $H_\lambda^{ev}(C(\Omega))$ will be denoted by $[\mu]$. It is trivial that $\langle [\mu], [e] \rangle \in \mathbb{Z}$ for any $[e] \in K_0(C(\Omega))$. The following is the main result of the paper.

Theorem 3.3. *For any* $[P] \in K_0(C^\infty(\Omega, \mathbb{R}^2, \beta))$, *we have*

$$\langle [\psi], [P] \rangle = -\langle [\mu], \phi^{-1}[P] \rangle \in \mathbb{Z}.$$

The proof of this theorem will be presented in the next section. Since the elements in $K_0(C(\Omega))$ can be identified as $[V_1] - [V_2]$, where V_1 and V_2 are vector bundles on Ω, we can interpret the Hall conductivity as follows.

Corollary 3.4. *If the Fermi energy E_F belongs to a gap of the spectrum of H_ω, then there are vector bundles V^+ and V^- on Ω such that* $\phi^{-1}[P_F] = [V^+] - [V^-]$ *and*

$$\sigma_H = (e^2/2\pi\hbar)(\dim V^- - \dim V^+).$$

At this point, it is necessary to compare our result with Bellissard's work on the same subject. In [5, 6, 7], he considered the symmetric Hamiltonian

$$H(\omega, \beta) = [(-i\partial/\partial x + \beta y/2)^2 + (-i\partial/\partial y - \beta x/2)^2]/2 + Q_\omega(x, y) = W_\beta H_\omega W_\beta^*,$$

where $(W_\beta f)(x, y) = \exp(-i\beta xy/2) f(x, y)$ for $f \in L^2(\mathbb{R}^2)$. Accordingly, the algebra of observables constructed there has the form $\mathcal{A}_\omega = W_\beta C_\omega^*(\Omega, \mathbb{R}^2, \beta) W_\beta^*$. That is, a

function $a \in C_c(\Omega \times \mathbb{R}^2)$ represents an element

$$(\tilde{\pi}_\omega(a)f)(x,y) = \int\limits_{\mathbb{R}^2} a(\omega + (x,y), \xi, \eta) \exp(i\beta(y\xi - x\eta)/2)f(x + \xi, y + \eta)d\xi d\eta$$

in $\mathscr{A}_\omega$. To compute the Hall conductivity, he constructed a Fredholm module $(\mathscr{H}, G, F)$ with $\mathscr{H} = L^2(\mathbb{R}^2)_+ \oplus L^2(\mathbb{R}^2)_-$, $G = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$F = \begin{bmatrix} 0 & (x+iy)/|x+iy| \\ (x-iy)/|x+iy| & 0 \end{bmatrix}, \quad \text{and } \mathbf{A} = \begin{bmatrix} \pi_\omega(A) & 0 \\ 0 & \pi_\omega(A) \end{bmatrix}$$

for $A \in \mathscr{A}$. He then derived the formula

$$\psi(A_0, A_1, A_2)/2\pi i = \int\limits_\Omega \mathrm{Tr}_s(\mathbf{A}_0[F, \mathbf{A}_1][F, \mathbf{A}_2])d\mu$$

where $\mathrm{Tr}_s(T) = \mathrm{Tr}(GF[F, T])/2$. Thus for a projection $P \in \mathscr{A}$,

$$\tau(P[\delta_1(P), \delta_2(P)])/2\pi i = \int\limits_\Omega \mathrm{Tr}_s(\mathbf{P}[F, \mathbf{P}][F, \mathbf{P}])d\mu. \tag{3.7}$$

From this and a formula of Connes [15], it follows

$$\tau(P[\delta_1(P), \delta_2(P)])/2\pi i = \mathrm{Index}(F_P),$$

where F_P is the restriction of $\{\mathbf{P}F\mathbf{P}\}_{+,-}$ to $\mathbf{P}\mathscr{H}$, which is a Fredholm operator. Thus Bellissard identified the Hall conductivity as an *analytical* index $\mathrm{Index}(F_P)$. The importance of the formula (3.7) is that it holds true for projections in the Sobolev space $H_1(\mathscr{A})$, i.e. for projections $P \in \mathscr{W}$ with finite localization length

$$\tau([\delta_1(P)]^2 + [\delta_2(P)]^2) < +\infty.$$

Therefore he was able to identify the Hall conductivity as an analytical index when the Fermi energy lies in a gap of extended states in the spectrum of the Hamiltonian.

By contrast, our formula $\langle [\psi], [P] \rangle = -\langle [\mu], \phi^{-1}[P] \rangle$ identifies the Hall conductivity as a *topological* index $-\langle [\mu], \phi^{-1}[P] \rangle$. This formula not only provides an alternate interpretation of the Hall conductivity but also has the following advantages. First, as it is the case for pseudo-differential operators, topological index is generally considered to be more readily calculable. (When $Q = 0$, for example, it follow immediately from the construction of the Thom isomorphisms that $-\langle [\mu], \phi^{-1}[P_F] \rangle$ coincides with the Landau band index.) Second, it suggests that the K-theory class $\phi^{-1}[P_F]$ may contain more physical information of the quantum Hall effect than the conductivity σ_H. One sees from our formula, for example, that the Hall conductivity should not be interpreted as Landau band index unless $\phi^{-1}[P_F]$ lies in the subgroup $\mathbb{Z}[1]$ of $K_0(C(\Omega))$. It would be extremely interesting if one could come up with an example where $\phi^{-1}[P_F] \in K_0(C(\Omega))\backslash\mathbb{Z}[1]$, or equivalently, $[P_F] \in K_0(C^*_\omega(\Omega, \mathbb{R}^2, \beta))\backslash K_0(\mathscr{K})$. We would consider such a class as a nontrivial topological invariant of the quantum Hall effect. Given the fact that $K_0(C(\Omega))$ is generally quite larger for Ω with sufficiently complicated topology, this is not entirely inconceivable. But while our approach to the quantization of the Hall conductivity has its mathematical beauty, it also has the shortcoming that the Fermi energy is required to be in a gap of the

spectrum of the Hamiltonian. With Bellissard's results for Sobolev projections in mind, one certainly would like to extend Theorem 3.4 to this class of projections. But that is the task of another paper. In the meantime, combining Bellissard's work and ours, we obtain the following analogue of the classical index theorem (analytical index) = − (topological index):

$$\text{Index}(F_P) = - \langle [\mu], \phi^{-1}[P] \rangle \quad P \in \text{Proj } C_\omega^\infty(\Omega, \mathbb{R}^2, \beta).$$

4. The Thom Isomorphisms and the Pairing

As a first step in the proof of Theorem 3.3, let us examine how the two derivations δ_1 and δ_2 are constructed on $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$ when one uses this iterative crossed product to represent $C^*(\Omega, \mathbb{R}^2, \beta)$. On the space $L^2(\Omega \times \mathbb{R}_y)$, where $C(\Omega) \times_\theta \mathbb{R}$ lives, one defines the operator $(\tilde{D}_2 f)(\omega, y) = 2\pi i y f(\omega, y)$. Then $A \mapsto [A, \tilde{D}_2] = \tilde{\delta}_2(A)$ extends to a closed derivation on $C(\Omega) \times_\theta \mathbb{R}$. Since this derivation commutes with the automorphism group $\{\gamma_s : s \in \mathbb{R}\}$, it can be elevated to a derivation on $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$ in the following way. For an element $a \in C_c(\mathbb{R}, C(\Omega) \times_\theta \mathbb{R}))$, which is the collection of compactly supported, $C(\Omega) \times_\theta \mathbb{R}$-valued continuous functions, such that $a(x)$ belongs to the domain of $\tilde{\delta}_2$ for every x and $x \mapsto \tilde{\delta}_2(a(x))$ is continuous, simply define $\delta_2(a)(x) = \tilde{\delta}_2(a(x))$. One can easily check that δ_2 extends to a closed derivation on $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$ and that it is precisely the derivation denoted by the same symbol on $C^*(\Omega, \mathbb{R}^2, \beta)$ when these algebras are identified with each other. Similarly, $a \mapsto 2\pi i x a$ $(a \in C_c(\mathbb{R}, C(\Omega) \times_\theta \mathbb{R}))$ extends to the derivation δ_1 on $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$. The trace τ on $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$ can be realized as follows. For a compactly supported, $C(\Omega) \times_\theta \mathbb{R}$-valued continuous function $a(x)$, which represents an element a in $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}$,

$$\tau(a) = \tau_1(a(0)).$$

In order to prove Theorem 3.3, we need the Thom isomorphism for cyclic cohomology introduced recently by G. Elliott, T. Natsume and R. Nest [17]. Let us briefly recall the construction of this map. For a locally convex topological algebra $\mathscr{A}$, there is a universal differential graded algebra $(\Omega(\mathscr{A}), d)$ with $\Omega(\mathscr{A}) = \bigoplus_{n \geq 0} \Omega_n(\mathscr{A})$ [15, 17]. (In this paper, we will only consider those $\mathscr{A}$'s which are norm dense subalgebras of C^*-algebras.) Let $\mathscr{S}^*(\mathbb{R})$ be the collection of smooth, rapidly decreasing function on $\mathbb{R}$ with convolution as its product. The differential graded algebra $\Omega(\mathscr{S}^*(\mathbb{R}))/\bigoplus \{\Omega_n(\mathscr{S}^*(\mathbb{R})): n \geq 2\}$ will be denote by $\mathbf{E}$. Suppose that $\mathscr{A}$ is a Fréchet algebra with a smooth action v of $\mathbb{R}$. Let $(\mathscr{A} \times_v \mathbb{R})^\infty$ denote the set of $\mathscr{A}$-valued smooth, rapidly decreasing function on $\mathbb{R}$ with the product structure $a * b(t) = \int_\mathbb{R} a(s) v_s(b(t-s)) ds$. (We will use the superscript ∞ to distinguish this smooth crossed product from the ordinary crossed product.) The space $\Omega(\mathscr{A}) \otimes_v \mathbf{E}$ will be given a structure of locally convex differential graded algebra as follows [17].

(1) For $w \in \Omega(\mathscr{A})$ and $f \in \mathbf{E}$,

$$d(w \otimes f) = dw \otimes f + (-1)^{\deg w} w \otimes df.$$

(2) Define a left $\mathbf{E}_0$-module structure on $\Omega(\mathscr{A}) \otimes \mathbf{E}_0$ as the one induced from the product structure of $(\Omega(\mathscr{A})^\sim \times_v \mathbb{R})^\infty$.

(3) Define a left **E**-module structure on $\Omega(\mathscr{A}) \otimes \mathbf{E}$ by the formulas

$$f(w \otimes gdh) = (fw \otimes g)dh, \quad df(w \otimes g) = d(fw \otimes g) - fd(w \otimes g), \quad df(w \otimes gdh) = 0.$$

(4) Define the product on $\Omega(\mathscr{A}) \otimes \mathbf{E}$ by

$$(w \otimes f)(w_1 \otimes f_1) = w(f(w_1 \otimes f_1)).$$

Given a closed continuous n-trace T on $\Omega(\mathscr{A})$, define an $n+1$-trace $\#_v T$ on $\Omega(\mathscr{A}) \otimes {}_v \mathbf{E}$ as follows. For $f \in \Omega_n(\mathscr{A}) \otimes \mathscr{S}^*(\mathbb{R}) \otimes \mathscr{S}^*(\mathbb{R}) \subset \Omega_n(\mathscr{A}) \otimes \mathbf{E}_1$,

$$\#_v T(f) = 2\pi i \int_{\mathbb{R}} \left(\int_0^t T(v_\lambda(f(-t, t))) d\lambda \right) dt. \tag{4.1}$$

For other f, $\#_v T(f) = 0$. It was shown in [9] that $\#_v$ induces isomorphisms

$$H_\lambda^{\mathrm{ev}}(\mathscr{A}) \to H_\lambda^{\mathrm{odd}}((\mathscr{A} \times {}_v \mathbb{R})^\infty),$$

$$H_\lambda^{\mathrm{odd}}(\mathscr{A}) \to H^{\mathrm{ev}}((\mathscr{A} \times {}_v \mathbb{R})^\infty).$$

Given a smooth action v on a Fréchet algebra $\mathscr{A}$, there is a dual action $\hat{v}$ on $(\mathscr{A} \times {}_v \mathbb{R})^\infty$. Indeed if $f \in (\mathscr{A} \times {}_v \mathbb{R})^\infty$ is regarded as an $\mathscr{A}$-valued function, then $(\hat{v}_s f)(t) = e^{2\pi i s} f(t)$ [15, 17]. It was shown in [17] that there is a natural isomorphism $T_v : \mathscr{A} \otimes \mathscr{K}^\infty \to [(\mathscr{A} \times {}_v \mathbb{R})^\infty \times {}_{\hat{v}} \mathbb{R}]^\infty$, where $\mathscr{K}^\infty$ is the collection of Hilbert–Schmidt operators on $L^2(\mathbb{R})$ with kernels in $\mathscr{S}(\mathbb{R}^2)$. There is a map $r : \mathscr{A} \to \mathscr{A} \otimes \mathscr{K}^\infty$ such that $r(a) = a \otimes e_0$, where e_0 is the projection corresponding to the 0th Hermite function [17]. These maps induce an isomorphism $T_{v*} : K_1(\mathscr{A} \otimes \mathscr{K}^\infty) \to K_1([(\mathscr{A} \times {}_v \mathbb{R})^\infty \times {}_{\hat{v}} \mathbb{R}]^\infty)$ and a homomorphism $r_* : K_1(\mathscr{A}) \to K_1(\mathscr{A} \otimes \mathscr{K}^\infty)$. Hence the map $T_v r : \mathscr{A} \to [(\mathscr{A} \times {}_v \mathbb{R})^\infty \times {}_{\hat{v}} \mathbb{R}]^\infty$ induces an homomorphism $(T_v r)_* = T_{v*} r_* : K_1(\mathscr{A}) \to K_1([(\mathscr{A} \times {}_v \mathbb{R})^\infty \times {}_{\hat{v}} \mathbb{R}]^\infty)$. These maps also induce homomorphisms between cyclic cohomologies: $T_v^* : H_\lambda^*([(\mathscr{A} \times {}_v \mathbb{R})^\infty \times {}_{\hat{v}} \mathbb{R}]^\infty) \to H_\lambda^*(\mathscr{A} \otimes \mathscr{K}^\infty)$ and $r^* : H_\lambda^*(\mathscr{A} \otimes \mathscr{K}^\infty) \to H_\lambda^*(\mathscr{A})$.

Lemma 4.1. *Let Λ be an odd cyclic cocycle on a Fréchet algebra $\mathscr{A}$ which admits a smooth action v of $\mathbb{R}$ and let $[\Lambda]$ be the class of Λ in $H_\lambda^{\mathrm{odd}}(\mathscr{A})$. Then for any $v \in K_1(\mathscr{A})$,*

$$\langle \#_{\hat{v}} \#_v [\Lambda], T_{v*} r_* v \rangle = \langle [\Lambda], v \rangle.$$

Proof. By the definition of the pairing between H_λ^{odd} and K_1, we have $\langle \#_{\hat{v}} \#_v [\Lambda], T_{v*} r_* v \rangle = \langle r^* T_v^* \#_{\hat{v}} \#_v [\Lambda], v \rangle$. According to the proofs of Theorems 4.2 and 5.2 of [17], $r^* T_v^* \#_{\hat{v}} \#_v [\Lambda] = [S\Lambda] = [\Lambda]$ in H_λ^{odd}. Therefore $\langle \#_{\hat{v}} \#_v [\Lambda], T_{v*} r_* v \rangle = \langle [\Lambda], v \rangle$. $\square$

Let $C^\infty(\Omega)$ be the collection of functions $f \in C(\Omega)$ such that $(x, y) \mapsto f \circ \varphi_{x,y}$ is a $C(\Omega)$-valued C^∞ function on $\mathbb{R}^2$. Then the group $\theta_t = \varphi_{0,t}$ acts on $C^\infty(\Omega)$ smoothly. Hence we can form the smooth crossed product $(C^\infty(\Omega) \times {}_\theta \mathbb{R})^\infty$. If μ denotes the 0-cocycle on $C^\infty(\Omega)$ of integration against the ergodic measure $d\mu$, then $\#_\theta \mu$ is a 1-cocycle on $(C^\infty(\Omega) \times {}_\theta \mathbb{R})^\infty$. The automorphism group $\{\gamma_s : s \in \mathbb{R}\}$ of $C(\Omega) \times {}_\theta \mathbb{R}$ defined in Sect. 2 acts smoothly on the subalgebra $(C^\infty(\Omega) \times {}_\theta \mathbb{R})^\infty$. Thus we have a 2-cocycle $\#_\hat{v} \#_\theta \mu$ on the smooth crossed product $[(C^\infty(\Omega) \times {}_\theta \mathbb{R})^\infty \times {}_v \mathbb{R}]^\infty$. When Ω is a single point, this cocycle was computed in [17]. We will do the same computation for nontrivial Ω's. But as was the case in [17], this computation is no more than a chasing of the definition of the operator $\#$, the constructions of

$(C^\infty(\Omega) \times_\theta \mathbb{R})^\infty$ and $[(C^\infty(\Omega) \times_\theta \mathbb{R})^\infty \times_\gamma \mathbb{R}]^\infty$, and the traces and derivations thereon.

Lemma 4.2. (i) *For any $b_0, b_1 \in (C^\infty(\Omega) \times_\theta \mathbb{R})^\infty$,*

$$\#_\theta \mu(b_0, b_1) = \tau_1(b_0 \tilde{\delta}_2(b_1)),$$

where τ_1 is the trace defined by (2.3) and $\tilde{\delta}_2$ the derivation defined in the first paragraph of this section.

(ii) *For any $a_0, a_1, a_2 \in [(C^\infty(\Omega) \times_\theta \mathbb{R})^\infty \times_\gamma \mathbb{R}]^\infty$,*

$$\#_\gamma \#_\theta \mu(a_0, a_1, a_2) = \tau(a_0[\delta_2(a_1)\delta_1(a_2) - \delta_1(a_1)\delta_2(a_2)]) = -\psi(a_0, a_1, a_2).$$

Proof: (i) We start with the graded differential algebra $(\Omega(C^\infty(\Omega)) \otimes_\theta \mathbf{E}, d)$. Because of the linearity and continuity of $\#_\theta \mu$ on this algebra, it suffices to evaluate the 1-cocycle at elements $b_0 db_1$, where $b_j = (u_j \otimes f_j)$ with $u_j \in C^\infty(\Omega)$ and $f_j \in \mathscr{S}^*(\mathbb{R})$, $j = 0, 1$. ($u_j \otimes f_j$ represents the element $u_j(\omega)f_j(y)$ in $(C^\infty(\Omega) \times_\theta \mathbb{R})^\infty$.) By the definitions (1),...,(4), we have $b_0 db_1 = (u_0 \otimes f_0)d(u_1 \otimes f_1) = (u_0 \otimes f_0)(du_1 \otimes f_1 + u_1 \otimes df_1) = u_0(f_0(du_1 \otimes f_1)) + u_0(f_0(u_1 \otimes df_1))$. Since $u_0(f_0(du_1 \otimes f_1)) \in \Omega_1(C^\infty(\Omega)) \otimes \mathbf{E}_0$ and, as an $\Omega_0(C^\infty(\Omega))$-valued function on $\mathbb{R}^2$, $u_0(f_0(u_1 \otimes df_1))(s, t) = u_0 \theta_s(u_1)f_0(s)f_1(t)$, by (4.1),

$$\begin{aligned}
\#_\theta \mu(b_0, b_1) &= 2\pi i \int_\mathbb{R} \int_0^t \int_\Omega \theta_\lambda(u_0 \theta_{-t}(u_1) f_0(-t) f_1(t)) d\mu d\lambda dt \\
&= 2\pi i \int_\Omega \int_\mathbb{R} t u_0(\omega) \theta_{-t}(u_1(\omega)) f_0(-t) f_1(t) dt d\mu(\omega) \\
&= -2\pi i \int_\Omega \int_\mathbb{R} t u_0(\omega) f_0(t) \theta_t(u_1(\omega)) f_1(-t) dt d\mu(\omega) \\
&= -\tau_1(\tilde{\delta}_2(b_0) b_1) = \tau_1(b_0 \tilde{\delta}_2(b_1)).
\end{aligned}$$

(ii) Once again, it suffices to evaluate $\#_\gamma \#_\theta \mu$ at $a_0 da_1 da_2$, where $a_j = v_j \otimes g_j$ with $v_j \in (C^\infty(\Omega) \times_\theta \mathbb{R})^\infty$ and $g_j \in \mathscr{S}^*(\mathbb{R})$, $j = 0, 1, 2$. By the definition of the product on $\Omega((C^\infty(\Omega) \times_\theta \mathbb{R})^\infty) \otimes_\gamma \mathbf{E}$, we have

$$\begin{aligned}
(v_0 \otimes g_0)d(v_1 \otimes g_1)d(v_2 \otimes g_2) &= \int_\mathbb{R} v_0 d(\gamma_r(v_1)\gamma_s(v_2)) g_0(r)g_1(s - r)dr g_2(t) \\
&\quad - \int_\mathbb{R} v_0 \gamma_s(v_1 d(\gamma_r(v_2))) g_0(s)g_1(r)g_2(t - r)dr + A,
\end{aligned}$$

where the first two terms are functions in $\Omega_1((C^\infty(\Omega) \times_\theta \mathbb{R})^\infty) \otimes \mathscr{S}^*(\mathbb{R}) \otimes \mathscr{S}^*(\mathbb{R})$ and A does not belong to this space. Applying (4.1) to the first two terms and using the definitions of $\tau_1, \tilde{\delta}_2, \tau, \delta_1, \delta_2$, we have

$$\begin{aligned}
\#_\gamma \#_\theta \mu(a_0, a_1, a_2) &= -\tau(\delta_1(a_0 \delta_2(a_1))a_2) - \tau(\delta_1(a_0 a_1)\delta_2(a_2)) + \tau(\delta_1(a_0)a_1 \delta_2(a_2)) \\
&= \tau(a_0 \delta_2(a_1)\delta_1(a_2)) - \tau(a_0 \delta_1(a_1)\delta_2(a_2)) = -\psi(a_0, a_1, a_2). \quad \square
\end{aligned}$$

For a nonunital algebra $\mathscr{A}$, let $\tilde{\mathscr{A}}$ denote the algebra obtained from $\mathscr{A}$ by adjoining an identity. For a complex algebra $\mathscr{A}$, in the pairing $\langle ., . \rangle$: $H^{\text{odd}}(\mathscr{A}) \times K_1(\mathscr{A}) \to \mathbb{R}$ defined in [15], $K_1(\mathscr{A})$ is the algebraic K_1-group of $\mathscr{A}$. The algebraic $K_1(\mathscr{A})$ is defined as follows [14, 15]. Let $GL_n(\tilde{\mathscr{A}})$ denote the collection of invertible elements in $\tilde{\mathscr{A}} \otimes M_n$, where M_n is the collection of $n \times n$ matrices. Let $GL_\infty(\tilde{\mathscr{A}})$ denote the inductive limit of $GL_n(\tilde{\mathscr{A}})$ in the obvious way. If one

denotes the subgroup of $GL_\infty(\tilde{\mathscr{A}})$ generated by commutators by $GL_\infty^c(\tilde{\mathscr{A}})$, then the algebraic $K_1(\mathscr{A}) = K_1^a(\mathscr{A})$ is defined to be $GL_\infty(\tilde{\mathscr{A}})/GL_\infty^c(\tilde{\mathscr{A}})$. If $\mathscr{A}$ is a Fréchet algebra and Λ is an odd cyclic cocycle on $\mathscr{A}$ which is continuous with respect to the Fréchet topology, then it is possible to pair the class of Λ in $H^{\mathrm{odd}}(\mathscr{A})$ with a certain topological K_1-group of $\mathscr{A}$. We will make this statement precise for certain subalgebras of C^*-algebras.

Let B be a C^*-algebra satisfying the following conditions:

(i) B has a trace τ_B with a dense domain and the property that $|\tau_B(ab)| \leq \|a\| \tau_B((b^*b)^{1/2})$ for any $a \in B$ and any τ_B-traces class element $b \in B$.

(ii) B is equipped with closed, densely defined derivations $\partial_1, \ldots, \partial_p$ which commute with each other.

Let $\mathscr{B}$ be the collection of $a \in B$ such that for any nonnegative integers $n_1, \ldots, n_p$,

(iii) $a_{n_1, \ldots, n_p} = \partial_1^{n_1} \cdots \partial_p^{n_p}(a) \in B$ and $\tau_B(a_{n_1, \ldots, n_p}^* a_{n_1, \ldots, n_p}) < \infty$.

Define

(iv) $\|a\|_{n_1, \ldots, n_p} = \|a_{n_1, \ldots, n_p}\| + (\tau_B(a_{n_1, \ldots, n_p}^* a_{n_1, \ldots, n_p}))^{1/2}$ for $a \in \mathscr{B}$.

With these norms, $\mathscr{B}$ becomes a Fréchet algebra. The algebra $\mathscr{B}$ is closed under holomorphic functional calculus [13]. Thus each $GL_n(\tilde{\mathscr{B}})$ contains a subgroup $\exp_n(\tilde{\mathscr{B}})$ consisting of $\exp(a_1) \cdots \exp(a_m)$ with $a_1, \ldots, a_m \in \tilde{\mathscr{B}} \otimes M_n$. Let $\exp_\infty(\tilde{\mathscr{B}})$ be the inductive limit of $\exp_n(\tilde{\mathscr{B}})$. For an algebra $\mathscr{B}$ satisfying (i), (ii) and (iii), we use the symbol $K_1(\mathscr{B})$ to denote $GL_\infty(\tilde{\mathscr{B}})/\exp_\infty(\tilde{\mathscr{B}})$ rather than the usual algebraic K_1-group $GL_\infty(\tilde{\mathscr{B}})/GL_\infty^c(\tilde{\mathscr{B}})$, which will now be denoted by $K_1^a(\mathscr{B})$. (For Fréchet algebras $\mathscr{A}$ which does not satisfy (i), (ii) and (iii) and which is not a C^*-algebra, $K_1(\mathscr{A})$ will still denote the algebraic K_1-group.) Since $\exp_\infty(\tilde{\mathscr{B}}) \supset GL_\infty^c(\tilde{\mathscr{B}})$, there is a natural surjective homomorphism $\mathscr{I}: K_1^a(\mathscr{B}) \to K_1(\mathscr{B})$. If $\mathscr{B}$ is a dense subalgebra of B, then the inclusion map $\iota: \mathscr{B} \to B$ induces an isomorphism $\iota_*: K_1(\mathscr{B}) \to K_1(B)$ [15].

Let Λ be a cyclic $(2k-1)$-cocycle on $\mathscr{B}$ which is continuous with respect to the Fréchet topology (iv). We will now show that there is a pairing of the class of Λ in $H_\lambda^{\mathrm{odd}}(\mathscr{B})$ with $K_1(\mathscr{B})$ such that for any $u \in K_1^a(\mathscr{B})$, the pairing of $[\Lambda]$ with $\mathscr{I}u$ $(\in K_1(\mathscr{B}))$ equals $\langle [\Lambda], u \rangle$. Because of the pairing between $H_\lambda^{\mathrm{odd}}(\mathscr{B})$ and $K_1^a(\mathscr{B})$, it suffices to show that the $\langle [\Lambda], v \rangle = 0$ for any $v \in \exp_\infty(\tilde{\mathscr{B}})/GL_\infty^c(\tilde{\mathscr{B}}) = \ker \mathscr{I}$. Furthermore, because of the linearity, it suffices to show that $\langle [\Lambda], [\exp(a)] \rangle = 0$ for any $a \in \tilde{\mathscr{B}} \otimes M_n$. Recall that to evaluate this pairing, one first extends Λ to $\tilde{\mathscr{B}}$ such that $\Lambda(1, a_1, \ldots, a_{2k-1}) = 0$ [15]. It is a routine calculation to verify that Λ is a cyclic cocycle on $\tilde{\mathscr{B}}$. For any $w \in GL_n(\tilde{\mathscr{B}})$, $\langle [\Lambda], [w] \rangle = c_k(\Lambda \# \mathrm{tr}_n) \times (w^{-1} - 1, w - 1, \ldots, w^{-1} - 1, w - 1)$, where c_k is a coefficient depending only on k [15]. Since $t \mapsto \exp(ta)$ is a function differentiable with respect to the topology (iv), and since Λ is continuous with respect to this topology, one can repeat the argument in [15, pp. 109, 110] to show that $\langle [\Lambda], [\exp(a)] \rangle = 0$. But there is a simpler way to do this. By the linearity of the pairing, we have, for any natural number p, $\langle [\Lambda], [\exp(a)] \rangle = p \langle [\Lambda], [\exp(a/p)] \rangle = c_k(\Lambda \# \mathrm{tr}_n)([\exp(-a/p) - 1]/[(1/p)^{1/2k}], \ldots, [\exp(a/p) - 1]/[(1/p)^{1/2k}])$. Since $\lim\limits_{p \to \infty} [\exp(a/p) - 1]/(1/p) = a$ in the topology (iv), we have, for $k \geq 1$, $\lim\limits_{p \to \infty} [\exp(a/p) - 1]/[(1/p)^{1/2k}] = 0$. Thus, by the continuity of Λ, $\langle [\Lambda], [\exp(a)] \rangle = 0$. Hence for any cyclic odd-cocycle Λ on $\mathscr{B}$ which is continuous with respect to the topology (iv), there is a homomorphism

$\langle[\Lambda],.\rangle : K_1(\mathcal{B}) \to \mathbb{R}$ such that $\langle[\Lambda], \mathcal{I}u\rangle = \langle[\Lambda], u\rangle$ for $u \in K_1^a(\mathcal{B})$. In what follows, the pairing between the K_1-group of a holomorphically closed subalgebra of a C^*-algebra equipped with a Fréchet topology (iv) and a continuous cyclic odd-cocycle will be understood in this sense.

Let $[C(\Omega) \times_\theta \mathbb{R}]_\infty$ be the collection of elements $A \in C(\Omega) \times_\theta \mathbb{R}$ such that for any nonnegative integers m, n, $\tilde{\delta}_2^m(\delta_\gamma^n(A)) \in C(\Omega) \times_\theta \mathbb{R}$ and $\tau_1([\tilde{\delta}_2^m(\delta_\gamma^n(A))]^* \tilde{\delta}_2^m(\delta_\gamma^n(A))) < \infty$. Here, δ_γ is the derivation induced by the automorphism group $\{\gamma_s : s \in \mathbb{R}\}$. Obviously (i)–(iv) are satisfied by $\mathcal{B} = [C(\Omega) \times_\theta \mathbb{R}]_\infty$. Thus we have an isomorphism ι_*: $K_1([C(\Omega) \times_\theta \mathbb{R}]_\infty) \to K_1(C(\Omega) \times_\theta \mathbb{R})$. Therefore the Thom isomorphism $\phi_{\theta,0}$: $K_0(C(\Omega)) \to K_1(C(\Omega) \times_\theta \mathbb{R})$ can be regarded as an isomorphism $\phi_{\theta,0} : K_0(C^\infty(\Omega)) \to K_1([C(\Omega) \times_\theta \mathbb{R}]_\infty)$. On the algebra $[C(\Omega) \times_\theta \mathbb{R}]_\infty$, define a cyclic 1-cocycle

$$\Lambda_\theta(a_0, a_1) = \tau_1(a_0 \tilde{\delta}_2(a_1)), \quad a_0, a_1 \in [C(\Omega) \times_\theta \mathbb{R}]_\infty. \tag{4.2}$$

Obviously, Λ_θ is a continuous extention of the cyclic cocycle $\#_\theta\mu$ on $(C^\infty(\Omega) \times_\theta \mathbb{R})^\infty$ to $[C(\Omega) \times_\theta \mathbb{R}]_\infty$.

Lemma 4.3. *For any* $u \in K_1([C(\Omega) \times_\theta \mathbb{R}]_\infty)$,

$$\langle[\Lambda_\theta], u\rangle = \langle[\mu], \phi_{\theta,0}^{-1} u\rangle.$$

Proof. It was shown in [17, Sect. 6.1] that $u = \phi_{\theta,0}[e]$, where $[e] \in K_0(C^\infty(\Omega))$ and $\phi_{\theta,0}[e]$ is represented by a class in $K_1((C^\infty(\Omega) \times_\theta \mathbb{R})^\infty)$. Therefore $\langle[\Lambda_\theta], u\rangle = \langle[\Lambda_\theta], \phi_{\theta,0}[e]\rangle$ [15, p. 109]. By [17, Theorem 6.2], $\langle\#_\theta[\mu], \phi_{\theta,0}[e]\rangle = \langle[\mu], [e]\rangle = \langle[\mu], \phi_{\theta,0}^{-1} u\rangle$. $\square$

Since the action of $\{\gamma_s : s \in \mathbb{R}\}$ on $[C(\Omega) \times_\theta \mathbb{R}]_\infty$ is smooth, we can define the smooth crossed product

$$([C(\Omega) \times_\theta \mathbb{R}]_\infty \times_\gamma \mathbb{R})^\infty. \tag{4.3}$$

Accordingly, one obtains a cyclic 2-cocycle $\#_\gamma\Lambda_\theta$ on this algebra. By the definition of the $\#$ operator, $\#_\gamma\Lambda_\theta$ is a continuous extension of $\#_\gamma\#_\theta\mu$ to $([C(\Omega) \times_\theta \mathbb{R}]_\infty \times \mathbb{R})^\infty$. Because $-\psi$ is the continuous extension of $\#_\gamma\#_\theta\mu$ to $C^\infty(\Omega, \mathbb{R}^2, \beta)$, which contains $([C(\Omega) \times_\theta \mathbb{R}]_\infty \times_\gamma \mathbb{R})^\infty$, we have

$$\#_\gamma\Lambda_\theta(a_0, a_1, a_2) = -\psi(a_0, a_1, a_2). \tag{4.4}$$

Let $\{\hat{\gamma}_s : s \in \mathbb{R}\}$ be the automorphism group on $[C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R} = C^*(\Omega, \mathbb{R}^2, \beta)$. dual to the action of $\{\gamma_s : s \in \mathbb{R}\}$. The trace τ induces a trace $\hat{\tau}$ the algebra $C^*(\Omega, \mathbb{R}^2, \beta) \times_{\hat{\gamma}} \mathbb{R} = ([C(\Omega) \times_\theta \mathbb{R}] \times_\gamma \mathbb{R}) \times_{\hat{\gamma}} \mathbb{R}$ in the usual way. That is, for an $a \in C_c(\mathbb{R}, C^*(\Omega, \mathbb{R}^2, \beta))$ representing an element in the triple crossed product, $\hat{\tau}(a) = \tau(a(0))$. Since δ_1 and δ_2 on $C^*(\Omega, \mathbb{R}^2, \beta)$ commute with the action of the automorphism group $\{\hat{\gamma}_s : s \in \mathbb{R}\}$, these derivations naturally induce derivations $\hat{\delta}_1$ and $\hat{\delta}_2$ on $C^*(\Omega, \mathbb{R}^2, \beta) \times_{\hat{\gamma}} \mathbb{R}$. Indeed for an $a \in C_c(\mathbb{R}, C^*(\Omega, \mathbb{R}^2, \beta))$, we have $(\hat{\delta}_j(a))(t) = \delta_j(a(t))$, if this function also represents an element in the triple crossed product, $j = 1, 2$. On this algebra, we define a third derivation $(\hat{\delta}_3(a))(t) = 2\pi i t a(t)$ if the $C^*(\Omega, \mathbb{R}^2, \beta)$-valued functions $a(t)$ and $2\pi i t a(t)$ represent elements in $C^*(\Omega, \mathbb{R}^2, \beta) \times_{\hat{\gamma}} \mathbb{R}$. It is obvious that the derivations $\hat{\delta}_1, \hat{\delta}_2$ and $\hat{\delta}_3$ commute. Let $[C^*(\Omega, \mathbb{R}^2, \beta) \times_{\hat{\gamma}} \mathbb{R}]_\infty$ denote the collection of elements A such that for any nonnegative integers m, n, k, $A_{n,m,k} = \hat{\delta}_1^n \hat{\delta}_2^m \hat{\delta}_3^k(A) \in C^*(\Omega, \mathbb{R}^2, \beta) \times_{\hat{\gamma}} \mathbb{R}$ and $\hat{\tau}(A_{n,m,k}^* A_{n,m,k}) < \infty$. This algebra

again satisfies (i)–(iv). Since $[C^*(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R}]_\infty$ is obviously dense in $C^*(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R}$, we have an isomorphism

$$\iota_*: K_1([C^*(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R}]_\infty) \to K_1(C^*(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R}). \tag{4.5}$$

Since $C^\infty(\Omega, \mathbb{R}^2, \beta)$ is smooth under the action of $\{\hat{\gamma}_s : s \in \mathbb{R}\}$, one has the smooth crossed product $(C^\infty(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R})^\infty$. By (4.1) and by calculations similar to those in the proof of Lemma 4.2, the cyclic 3-cocycle $\#_\gamma \psi$ on $(C^\infty(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R})^\infty$ extends continuously to a cyclic 3-cocycle Ψ on $[C^*(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R}]_\infty$. Indeed if the calculation in the proof of Lemma 4.2(b) is carried out one step further, one finds that $\#_\gamma \psi(a_0, a_1, a_2, a_3)$ can be expressed in terms of the $\hat{\tau}$-trace of a finite, linear and multiplicative combination of a_0 and derivations of a_1, a_2 and a_3. Hence we have a closed continuous cyclic 3-cocycle Ψ on $[C^*(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R}]_\infty$ such that

$$\Psi(a_0, a_1, a_2, a_3) = \#_\gamma \psi(a_0, a_1, a_2, a_3) \tag{4.6}$$

for $a_0, \ldots, a_3$ in the subalgebra $(C^\infty(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R})^\infty$. It is obvious that we have an inclusion map

$$\iota: (([C(\Omega) \times_\theta \mathbb{R}]_\infty \times_\gamma \mathbb{R})^\infty \times_\gamma \mathbb{R})^\infty \to [C^*(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R}]_\infty, \tag{4.7}$$

which induces a homomorphism ι_* between the corresponding K-groups.

Lemma 4.4. *The map*

$$\iota_* T_{\gamma*} r_* : K_1([C(\Omega) \times_\theta \mathbb{R}]_\infty) \to K_1([C^*(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R}]_\infty)$$

is a surjective homomorphism.

Proof. Let h_j be j^{th} Hermite function, $j = 0, 1, 2, \ldots$, and let e_{jk} be the rank-one operator $h_j \otimes h_k (\in \mathcal{K}^\infty)$ on $L^2(\mathbb{R})$ [17]. Let A_n be the algebra generated by e_{jk}, $j, k = 0, 1, \ldots, n$. For any n, the map $a \mapsto a \otimes e_{00}$ induces a surjective homomorphism from $K_1([C(\Omega) \times_\theta \mathbb{R}]_\infty)$ onto $K_1([C(\Omega) \times_\theta \mathbb{R}]_\infty \otimes A_n)$. It is easy to see that $\bigcup_{n=0}^\infty T_\gamma([C(\Omega) \times_\theta \mathbb{R}]_\infty \otimes A_n)$ is dense (in the C^*-norm) in $(([C(\Omega) \times_\theta \mathbb{R}]_\infty \times_\gamma \mathbb{R})^\infty \times_\gamma \mathbb{R})^\infty$. Because $[C^*(\Omega, \mathbb{R}^2, \beta) \times_\gamma \mathbb{R}]_\infty$ is closed under holomorphic functional calculus, to complete the proof, it suffices to show that if $B \in ([C(\Omega) \times_\theta \mathbb{R}]_\infty \otimes A_n)^\sim \otimes M_k$ is invertible in $([C(\Omega) \times_\theta \mathbb{R}] \otimes A_n)^\sim \otimes M_k$, then $B^{-1} \in ([C(\Omega) \times_\theta \mathbb{R}]_\infty \otimes A_n)^\sim \otimes M_k$. But this is obvious because, by the definition of $[C(\Omega) \times_\theta \mathbb{R}]_\infty$, for any $b \in [C(\Omega) \times_\theta \mathbb{R}]_\infty \otimes A_n \otimes M_k$ with C^*-norm less than 1, $\sum_{j=1}^\infty b^j \in [C(\Omega) \times_\theta \mathbb{R}]_\infty \otimes A_n \otimes M_k$. $\quad\square$

Proof of Theorem 3.3. By [17, Theorem 6.2], we have, for any $[P] \in K_0(C^\infty(\Omega, \mathbb{R}^2, \beta))$, $\langle [\psi], [P] \rangle = \langle [\#_\gamma \psi], \phi_{\gamma,0}[P] \rangle$, which, according to [15, p. 109] and (4.6), equals $\langle [\Psi], \phi_{\gamma,0}[P] \rangle$. According to Lemma 4.4, $\phi_{\gamma,0}[P]$ can be represented by $T_{\gamma*} r_* v$ with $v \in K_1([C(\Omega) \times_\theta \mathbb{R}]_\infty)$. Because of (4.6) and (4.7), we have $\langle [\psi], [P] \rangle = \langle [\Psi], T_{\gamma*} r_* v \rangle = \langle [\#_\gamma \psi], T_{\gamma*} r_* v \rangle$. Since, by (4.4), the restriction of ψ to $([C(\Omega) \times_\theta \mathbb{R}]_\infty \times_\gamma \mathbb{R})^\infty$ is $-\#_\gamma \Lambda_\theta$, the restriction of $\#_\gamma \psi$ to $(([C(\Omega) \times_\theta \mathbb{R}]_\infty \times_\gamma \mathbb{R})^\infty \times_\gamma \mathbb{R})^\infty$ equals $-\#_\gamma \#_\gamma \Lambda_\theta$. Therefore $\langle [\psi], [P] \rangle = -\langle \#_\gamma \#_\gamma [\Lambda_\theta], T_{\gamma*} r_* v \rangle$ [15, p. 109]. By Lemmas 4.1 and 4.3, we have $\langle [\psi], [P] \rangle = -\langle [\Lambda_\theta], v \rangle = -\langle [\mu], \phi_{\theta,0}^{-1} v \rangle$. To

prove $v = \phi_{\gamma,1}^{-1}[P]$, it suffices to consider the K-groups of the C^*-*algebras* involved here. Since $\phi_{\gamma,0}\phi_{\gamma,1} = T_{\gamma*}r_*$ on $K_1(C(\Omega) \times_\theta \mathbb{R})$ [13], the equation $T_{\gamma*}r_*v = \phi_{\gamma,0}[P]$. implies $\phi_{\gamma,1}v = [P]$. Hence, by (3.7),

$$\langle[\psi],[P]\rangle = -\langle[\psi], \phi_{\theta,0}^{-1}\phi_{\gamma,1}^{-1}[P]\rangle = -\langle[\mu], \phi^{-1}[P]\rangle. \quad \square$$

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