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GEOMETRIC MEANING OF SASAKIAN SPACE FORMS FROM THE VIEWPOINT OF SUBMANIFOLD THEORY

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To the memory of Professor Shûkichi Tanno

Abstract

We show that M^{2n-1} is a real hypersurface all of whose geodesics orthogonal to the characteristic vector ξ are mapped to circles of the same curvature 1 in an n-dimensional nonflat complex space form $\tilde{M}_n(c)(= \mathbf{C}P^n(c)$ or $\mathbf{C}H^n(c))$ if and only if M is a Sasakian manifold with respect to the almost contact metric structure from the ambient space $\tilde{M}_n(c)$. Moreover, this Sasakian manifold M is a Sasakian space form of constant ϕ-sectional curvature $c+1$ for each $c(\neq 0)$.

1. Introduction

We denote by $\tilde{M}_n(c)$, $n \geqq 2$ a complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature $c(\neq 0)$. That is, $\tilde{M}_n(c)$ is holomorphically isometric to either an n-dimensional complex projective space $\mathbf{C}P^n(c)$ of constant holomorphic sectional curvature c or an n-dimensional complex hyperbolic space $\mathbf{C}H^n(c)$ of constant holomorphic sectional curvature c according as c is positive or negative. As is well-known, every real hypersurface M^{2n-1} of $\tilde{M}_n(c)$ admits an almost contact metric structure (ϕ, ξ, η, g) induced from this ambient space. Making use of such a structure, many geometers have studied actively real hypersurfaces in nonflat complex space forms (c.f. [9]).

On the other hand, the theory of contact geometry has been developed also by many geometers (for details, see [5]). Sasakian manifolds and Sasakian space forms are analogues to Kähler manifolds and complex space forms, respectively. J. Berndt ([4]) showed that a Sasakian space form of constant ϕ-sectional curvature $d(\neq 1)$ is realized as a Hopf hypersurface with two distinct constant principal curvatures in a nonflat complex space form (see Lemma 2). In this context we

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study Sasakian space forms from submanifold theoretical point of view. It is well-known that a $(2n-1)$-dimensional Sasakian space form of constant ϕ-sectional curvature 1 is a unit sphere $S^{2n-1}(1)$. This sphere is totally umbilic in $\mathbf{C}^n$ and is the only real hypersurface all of whose geodesics are mapped to circles of the same curvature 1 in the ambient space $\mathbf{C}^n$. It is hence natural to come to consider real hypersurfaces all of whose geodesics are mapped to circles of the same positive curvature in a complex n-dimensional nonflat complex space form $\tilde{M}_n(c)$. However, unfortunately there exist no such real hypersurfaces in this space $\tilde{M}_n(c)$ because this space admits no totally umbilic real hypersurfaces.

Motivated by this fact, we are interested in investigating real hypersurfaces all of whose geodesics orthogonal to the characteristic vector ξ are mapped to circles of the same positive curvature in a nonflat complex space form. The main purpose of this paper is to characterize Sasakian space forms by such a condition among real hypersurfaces in a nonflat complex space form $\tilde{M}_n(c)$ (Theorem 1 and Remark 2). This, together with results in [3], gives us many nice geometric properties of geodesics on Sasakian space forms (Proposition 2). For example, when $c > 8$, the Sasakian space form of constant ϕ-sectional curvature $c+1$ is a Berger sphere.

In the last section, we show directly that in the case of $c > 0$ the metric of our realization of a Sasakian space form of constant ϕ-sectional curvature $c+1$ coincides with the metric of the standard example of a Sasakian space form given in [5].

2. Preliminaries

We first review fundamental notion in contact geometry (see [5]). Let M^{2n-1} $(n \geqq 2)$ be a differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g). That is, this structure satisfies the following identities:

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all vectors X, Y on M. We say M to be a *Sasakian manifold* if the structure tensor ϕ satisfies the differential equation

$$(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y)X \tag{2.1}$$

for all tangent vectors $X, Y \in TM$, where ∇ denotes the Riemannian connection of the metric g of M. A Sasakian manifold is called a *Sasakian space form* of constant ϕ-sectional curvature c if the sectional curvature $K(u, \phi u) := g(R(u, \phi u)\phi u, u)$ satisfies $K(u, \phi u) = c$ for each unit vector u orthogonal to ξ. The following is a Sasakian analogue of Schur's Theorem.

THEOREM A. *If the ϕ-sectional curvature at each point of a Sasakian manifold M of dimension $\geqq 5$ does not depend on the choice of ϕ-section at that point, then it is constant on M. The curvature tensor is given by*

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ & + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ & + g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z\}, \end{aligned}$$

where c is the constant ϕ-sectional curvature of M.

In this paper, we denote by $M^{2n-1}(c)$ a $(2n-1)$-dimensional Sasakian space form of constant ϕ-sectional curvature c. For the standard construction of Sasakian space forms, see pp. 114–115 in [5]. The following is the unique existence theorem of Sasakian space forms.

THEOREM B. *For any two simply connected complete Sasakian manifolds of constant ϕ-sectional curvature c, there exists an isomorphism between them which preserves their almost contact metric structures.*

We next review the fundamental theory of real hypersurfaces. Let M^{2n-1} be a real hypersurface with a unit normal local vector field $\mathcal{N}$ of an n-dimensional Kähler manifold $(\tilde{M}, g, J)$ with Riemannian metric g and Kähler structure J. The Riemannian connections $\tilde{\nabla}$ of $\tilde{M}$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX,Y)\mathcal{N}, \tag{2.2}$$

$$\tilde{\nabla}_X \mathcal{N} = -AX \tag{2.3}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric of M induced from the ambient space $\tilde{M}$ and A is the shape operator of M in $\tilde{M}$. An eigenvector X of the shape operator A is called a *principal curvature vector* of M in $\tilde{M}$ and an eigenvalue λ of A is called a *principal curvature* of M in $\tilde{M}$. We denote by V_λ the eigenspace associated to the principal curvature λ, namely we set $V_\lambda = \{v \in TM \mid Av = \lambda v\}$.

It is well-known that M has an almost contact metric structure induced from the Kähler structure of the ambient space $\tilde{M}$. That is, we have a quartet (ϕ, ξ, η, g) defined by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

It follows from (2.2), (2.3) and $\tilde{\nabla}J = 0$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi AX. \tag{2.5}$$

We here clarify the meaning of the condition that a real hypersurface M is a Sasakian manifold with respect to the almost contact metric structure of the ambient Kähler manifold $\tilde{M}$. On an orientable connected real hypersurface M in a Kähler manifold $\tilde{M}$, we have an almost contact metric structure (ϕ, ξ, η, g) associated with a unit normal vector $\mathcal{N}$ of M in $\tilde{M}$. Clearly the quartet $(\phi, -\xi, -\eta, g)$ is also an almost contact metric structure on M which is associated with a unit normal $-\mathcal{N}$. We call a real hypersurface M *Sasakian* if M satisfies either (2.1) or

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X$$

for all vectors $X, Y \in TM$.

In the following, we consider real hypersurfaces in an n-dimensional nonflat complex space form $\tilde{M}_n(c)(= \mathbf{C}P^n(c)$ or $\mathbf{C}H^n(c))$. Denoting the curvature tensor of M by R, we have the equation of Gauss given by

$$\begin{aligned}(2.6)\quad & g(R(X, Y)Z, W)\\ &= (c/4)\{g(Y,Z)g(X, W) - g(X,Z)g(Y, W) + g(\phi Y, Z)g(\phi X, W)\\ &\qquad - g(\phi X, Z)g(\phi Y, W) - 2g(\phi X, Y)g(\phi Z, W)\}\\ &\quad + g(AY, Z)g(AX, W) - g(AX, Z)g(AY, W).\end{aligned}$$

We usually call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M. For a Hopf hypersurface M^{2n-1} $(n \geqq 2)$ in a nonflat complex space form $\tilde{M}_n(c)$, the principal curvature δ corresponding to the characteristic vector field ξ is locally constant on M. Furthermore, every tube of sufficiently small constant radius around each Kähler submanifold of a nonflat complex space form $\tilde{M}_n(c)$ is a Hopf hypersurface. This fact tells us that the notion of Hopf hypersurfaces is natural in the theory of real hypersurfaces in a nonflat complex space form (see [9]).

In $\mathbf{C}P^n(c)$ $(n \geqq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following:

- (A_1) A geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;
- (A_2) A tube of radius r around totally geodesic $\mathbf{C}P^{\ell}(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \pi/\sqrt{c}$;
- (B) A tube of radius r around complex hyperquadric $\mathbf{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c})$;
- (C) A tube of radius r around $\mathbf{C}P^1(c) \times \mathbf{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and $n(\geqq 5)$ is odd;
- (D) A tube of radius r around complex Grassmann $\mathbf{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 9$;
- (E) A tube of radius r around Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Summing up real hypersurfaces of types (A_1) and (A_2), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. The principal curvatures of these real hypersurfaces in $\mathbf{C}P^n(c)$ are given as follows (see [9]):

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r-\frac{\pi}{4}\right)$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\right)$	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r+\frac{\pi}{4}\right)$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot\left(\frac{\sqrt{c}}{2}r\right)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan\left(\frac{\sqrt{c}}{2}r\right)$
δ	$\sqrt{c}\cot(\sqrt{c}r)$	$\sqrt{c}\cot(\sqrt{c}r)$	$\sqrt{c}\cot(\sqrt{c}r)$	$\sqrt{c}\cot(\sqrt{c}r)$

One should notice that in $\mathbf{C}P^n(c)$ a tube of radius r $(0 < r < \pi/\sqrt{c})$ around totally geodesic $\mathbf{C}P^\ell(c)$ $(0 \leqq \ell \leqq n-1)$ is congruent to a tube of radius $((\pi/\sqrt{c}) - r)$ around totally geodesic $\mathbf{C}P^{n-\ell-1}(c)$.

In $\mathbf{C}H^n(c)$ $(n \geqq 2)$, a Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following (cf. [9]):

- (A_0) A horosphere in $\mathbf{C}H^n(c)$;
- $(A_{1,0})$ A geodesic sphere of radius r $(0 < r < \infty)$;
- $(A_{1,1})$ A tube of radius r around totally geodesic $\mathbf{C}H^{n-1}(c)$, where $0 < r < \infty$;
- (A_2) A tube of radius r around totally geodesic $\mathbf{C}H^\ell(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;
- (B) A tube of radius r around totally real totally geodesic $\mathbf{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of types (A_0), (A_1), (A_1), (A_2) and (B). Here, type (A_1) means either type $(A_{1,0})$ or type $(A_{1,1})$. Summing up real hypersurfaces of types (A_0), (A_1) and (A_2), we call them hypersurfaces of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$ has two distinct constant principal curvatures. Except this real hypersurface of type (B) with radius $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3})$, the numbers of distinct principal curvatures of these real hypersurfaces are $2, 2, 2, 3, 3$, respectively. The principal curvatures of these real hypersurfaces in $\mathbf{C}H^n(c)$ are given as follows (cf. [9]):

	(A_0)	$(A_{1,0})$	$(A_{1,1})$	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh\left(\frac{\sqrt{\lvert c\rvert}}{2}r\right)$
δ	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}r)$

A real hypersurface M of a nonflat complex space form $\tilde{M}_n(c)$, $n \geqq 2$ is called *totally η-umbilic* if its shape operator A is of the form $A = \alpha I + \beta\eta \otimes \xi$ for some smooth functions α and β on M. This definition is equivalent to saying that $Au = \alpha u$ for each vector u on M which is orthogonal to the characteristic vector ξ of M, where α is a smooth function on M. It is known that every totally η-umbilic hypersurface is a member of Hopf hypersurfaces with constant principal curvatures. The following classification theorem of totally η-umbilic hypersurfaces M shows that these two functions α and β are automatically constant on M (see [9]):

Theorem C. *Let M^{2n-1}, $n \geqq 2$ be a totally η-umbilic hypersurface of a nonflat complex space form $\tilde{M}_n(c)$ with shape operator $A = \alpha I + \beta\eta \otimes \xi$. Then M is locally congruent to one of the following*:

(P) *A geodesic sphere of radius r $(0 < r < \pi/\sqrt{c})$ in $\mathbf{C}P^n(c)$, where $\alpha = (\sqrt{c}/2)\cot(\sqrt{c}r/2)$ and $\beta = -(\sqrt{c}/2)\tan(\sqrt{c}r/2)$;*

($\mathrm{H_i}$) *A horosphere in $\mathbf{C}H^n(c)$, where $\alpha = \beta = \sqrt{|c|}/2$;*

($\mathrm{H_{ii}}$) *A geodesic sphere of radius r $(0 < r < \infty)$ in $\mathbf{C}H^n(c)$, where $\alpha = (\sqrt{|c|}/2)\coth(\sqrt{|c|}r/2)$ and $\beta = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}r/2)$;*

($\mathrm{H_{iii}}$) *A tube of radius r $(0 < r < \infty)$ around totally geodesic complex hyperplane $\mathbf{C}H^{n-1}(c)$ in $\mathbf{C}H^n(c)$, where $\alpha = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}r/2)$ and $\beta = (\sqrt{|c|}/2)\coth(\sqrt{|c|}r/2)$.*

Every totally η-umbilic hypersurface has two distinct constant principal curvatures α and $\alpha + \beta (= \delta)$. For the later use we prepare the following lemma (see [9]).

Lemma 1. *For a real hypersurface M in a nonflat complex space form $\tilde{M}_n(c)$ $(n \geqq 2)$, the following conditions are mutually equivalent.*

(1) *M is of type* (A).

(2) *$\phi A = A\phi$.*

(3) *$g((\nabla_X A)Y, Z) = (c/4)(-\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y))$ for arbitrary vectors X, Y and Z on M.*

In this paper, real hypersurfaces of types (A), (B), (C), (D) and (E) in $\tilde{M}_n(c)$ are said to be *standard real hypersurfaces*. It is well-known that every standard real hypersurface M is a homogeneous real hypersurface of $\tilde{M}_n(c)$, namely M is an orbit of some subgroup of the full isometry group $\mathrm{I}(\tilde{M}_n(c))$ of $\tilde{M}_n(c)$ (see [9]).

At the end of this section we review the definition of circles in Riemannian geometry. A real smooth curve $\gamma = \gamma(s)$ parameterized by its arclength s in a Riemannian manifold M with Riemannian connection ∇ is called a *circle* of curvature k if it satisfies the ordinary differential equations $\nabla_{\dot\gamma}\dot\gamma = kY_s$, $\nabla_{\dot\gamma}Y_s = -k\dot\gamma$ with a field Y_s of unit vectors along γ. Here $k(\geqq 0)$ is constant and Y_s is called the unit principal normal vector of γ. A circle of null curvature is nothing but a geodesic. The definition of a circle is equivalent to saying that it is a curve

$\gamma = \gamma(s)$ on M with Riemannian metric g satisfying the ordinary differential equation

$$\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) + g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)\dot\gamma = 0. \tag{2.7}$$

3. Geodesics on Sasakian hypersurface in $\tilde{M}_n(c)$

We shall start by classifying all Sasakian real hypersurfaces in a nonflat complex space form $\tilde{M}_n(c)$. The following lemma is essentially due to Berndt [4]. We here give its complete proof for the sake of readers' convenience.

LEMMA 2. *Let M^{2n-1} $(n \geqq 2)$ be a connected Sasakian real hypersurface of a nonflat complex space form $\tilde{M}_n(c)$. Then M is locally congruent to one of the following homogeneous real hypersurfaces of the ambient space $\tilde{M}_n(c)$:*

i) *A geodesic sphere $G(r)$ of radius r with $\tan(\sqrt{c}r/2) = \sqrt{c}/2$ $(0 < r < \pi/\sqrt{c})$ in $\mathbf{C}P^n(c)$;*

ii) *A horosphere in $\mathbf{C}H^n(-4)$;*

iii) *A geodesic sphere $G(r)$ of radius r with $\tanh(\sqrt{|c|}r/2) = \sqrt{|c|}/2$ $(0 < r < \infty)$ in $\mathbf{C}H^n(c)$ $(-4 < c < 0)$;*

iv) *A tube of radius r around totally geodesic $\mathbf{C}H^{n-1}(c)$ with $\tanh(\sqrt{|c|}r/2) = 2/\sqrt{|c|}$ $(0 < r < \infty)$ in $\mathbf{C}H^n(c)$ $(c < -4)$.*

In these cases, M has constant ϕ-sectional curvature $c + 1$.

Proof. Suppose that our real hypersurface M is a Sasakian manifold. Then it follows from (2.1) and (2.4) that

$$g(X, Y)\xi - \eta(Y)X = \eta(Y)AX - g(AX, Y)\xi \tag{3.1}$$

for all vectors $X, Y \in TM$. Setting $X = Y = \xi$ in Equation (3.1), we see that ξ is principal. Hence we can choose a principal curvature vector u orthogonal to ξ. Then, setting $Y = \xi$ in Equation (3.1), we find that $Au = -u$, so that the tangent bundle TM of M is decomposed as $TM = \{\xi\}_{\mathbf{R}} \oplus V_{-1}$, where $V_{-1} = \{X \in TM \mid AX = -X\}$. This, together with Theorem C, shows that our real hypersurface M is a totally η-umbilic hypersurface with coefficients $\alpha = -1$ and $\beta = c/4$ in $\tilde{M}_n(c)$. Here, we change the unit normal vector $\mathcal{N}$ into $-\mathcal{N}$ for each hypersurface in Theorem C. Then we know that M is locally congruent to one of i), ii), iii) and iv) in Lemma 2. Next, for each unit vector u perpendicular to ξ, we compute the ϕ-sectional curvature $K(u, \phi u)$ of our real hypersurface M. It follows from Equation (2.6) and the equality $A = -I + (c/4)\eta \otimes \xi$ that $K(u, \phi u) = c + 1$.

Conversely, we suppose that our real hypersurface M is locally congruent to one of i), ii), iii) and iv) in Lemma 2. We then see that the shape operator A of our real hypersurface M is of the form $A = -I + (c/4)\eta \otimes \xi$ by changing $\mathcal{N}$ into $-\mathcal{N}$ for each hypersurface in Theorem C. This, combined with (2.4), yields

(2.1), so that M is a Sasakian manifold. Thus we can obtain the conclusion of our Theorem. □

Remark 1. (1) Each real hypersurface i), ii), iii) and iv) in Lemma 2 is complete and simply connected. In particular, when $c > 0$ or $-4 < c < 0$, it is compact.
(2) The proof of Lemma 2 shows that M is a Sasakian manifold if and only if M is totally η-umbilic in $\tilde{M}_n(c)$ with shape operator $A = -I + (c/4)\eta \otimes \xi$.

The following theorem is closely related to Lemma 2 in the case of $c > 0$ (for details, see [6]).

Theorem D. *Let M be a real hypersurface in $\mathbf{C}P^n(c)$, $n \geqq 3$ on which the ϕ-sectional curvature H is constant. Then M is locally congruent to one of the following*:
(1) *A geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c})$ with $H = c + (c/4)\cot^2(\sqrt{c}r/2) > c$*;
(2) *A ruled real hypersurface with $H = c$*;
(3) *A real hypersurface on which there is an integrable distribution of codimension two such that its each leaf lies on some totally geodesic $\mathbf{C}P^{n-1}(c)$ as a ruled real hypersurface with $H = c$.*

Note that a real hypersurface M in Theorem D is Sasakian if and only if M is of Case i) in Lemma 2.

We are now in a position to prove the following which gives a geometric meaning of all Sasakian real hypersurfaces M in a nonflat complex space form $\tilde{M}_n(c)$ in terms of the extrinsic shape of some geodesics on M.

Theorem 1. *Let M be a connected real hypersurface in a nonflat complex space form $\tilde{M}_n(c)$. Then the following three conditions are mutually equivalent*:
(1) *M is a Sasakian manifold*;
(2) *M is a Sasakian space form of constant ϕ-sectional curvature $c + 1$*;
(3) *There exist orthonormal vectors $v_1, v_2, \dots, v_{2n-2}$ orthogonal to ξ at each point p of M satisfying the following two conditions*:
   i) *All geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ are mapped to circles of the same curvature* 1 *in $\tilde{M}_n(c)$*;
   ii) *All geodesics $\gamma_{ij} = \gamma_{ij}(s)$ on M with $\gamma_{ij}(0) = p$ and $\dot{\gamma}_{ij}(0) = (v_i + v_j)/\sqrt{2}$ $(1 \leqq i < j \leqq 2n-2)$ are mapped to circles of the same curvature* 1 *in $\tilde{M}_n(c)$.*

Proof. By virtue of the discussion in the proof of Lemma 2, we only need to verify that Conditions (1) and (3) are equivalent.

Suppose Condition (1). The shape operator A of M is expressed as $A = -I + (c/4)\eta \otimes \xi$. We take a unit vector v orthogonal to ξ at an arbitrary fixed point p of M. It satisfies $Av = -v$. Let $\gamma = \gamma(s)$ be the geodesic $\gamma = \gamma(s)$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$ on M. We then have

$$\begin{aligned}\nabla_{\dot{\gamma}} g(\dot{\gamma}, \xi) &= g(\dot{\gamma}, \nabla_{\dot{\gamma}}\xi) = g(\dot{\gamma}, \phi A\dot{\gamma}) \quad (\text{from } (2.5))\\ &= g(\dot{\gamma}, A\phi\dot{\gamma}) \quad (\text{from Lemma 1})\\ &= -g(\phi A\dot{\gamma}, \dot{\gamma}) = 0.\end{aligned}$$

This, together with $g(\dot{\gamma}(0), \xi) = g(v, \xi) = 0$, shows that the tangent vector $\dot{\gamma}(s)$ is perpendicular to $\xi_{\gamma(s)}$ for each s. Hence, again by using Lemma 1 we see that

$$\nabla_{\dot{\gamma}}\|A\dot{\gamma} + \dot{\gamma}\|^2 = 2g((\nabla_{\dot{\gamma}}A)\dot{\gamma}, A\dot{\gamma} + \dot{\gamma}) = 2g((\nabla_{\dot{\gamma}}A)\dot{\gamma}, A\dot{\gamma}) + 2g((\nabla_{\dot{\gamma}}A)\dot{\gamma}, \dot{\gamma}) = 0.$$

This, combined with $A\dot{\gamma}(0) + \dot{\gamma}(0) = Av + v = 0$, yields that $A\dot{\gamma}(s) + \dot{\gamma}(s) = 0$ for each s. Therefore, from (2.2) and (2.3) we have

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = g(A\dot{\gamma}, \dot{\gamma})\mathcal{N} = -\mathcal{N}$$

and

$$\tilde{\nabla}_{\dot{\gamma}}(-\mathcal{N}) = -\dot{\gamma}.$$

Thus we can see that the geodesic γ on a Sasakian real hypersurface M is a mapped to a circle of the same curvature 1 in the ambient space $\tilde{M}_n(c)$, and get Condition (3).

Next, we take orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ at a point p of a real hypersurface M satisfying Condition (3). Then, from (2.7) they satisfy

$$\tilde{\nabla}_{\dot{\gamma}_i}\tilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i = -\dot{\gamma}_i. \tag{3.2}$$

On the other hand, from (2.2) and (2.3) we have

$$\tilde{\nabla}_{\dot{\gamma}_i}\tilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i = g((\nabla_{\dot{\gamma}_i}A)\dot{\gamma}_i, \dot{\gamma}_i)\mathcal{N} - g(A\dot{\gamma}_i, \dot{\gamma}_i)A\dot{\gamma}_i. \tag{3.3}$$

Comparing the tangential components of (3.2) and (3.3), we see that

$$g(A\dot{\gamma}_i, \dot{\gamma}_i)A\dot{\gamma}_i = \dot{\gamma}_i,$$

so that at $s = 0$ we get

$$g(Av_i, v_i)Av_i = v_i \quad \text{for } 1 \leqq i \leqq 2n-2,$$

which yields that

$$Av_i = v_i \quad \text{or} \quad Av_i = -v_i \quad \text{for } 1 \leqq i \leqq 2n-2. \tag{3.4}$$

This implies that ξ is a principal curvature vector, because $\langle A\xi, v_i\rangle = \langle \xi, Av_i\rangle = 0$ for $1 \leqq i \leqq 2n-2$. Therefore M is a Hopf hypersurface with at most three distinct principal curvatures $1, -1$ and $\delta = g(A\xi, \xi)$ at its each point. On the other hand, applying the same discussion as above to Condition ii) of (3) in Theorem 1, we get the following corresponding to Equation (3.4):

$$A((v_i + v_j)/\sqrt{2}) = (v_i + v_j)/\sqrt{2} \quad \text{or} \quad A((v_i + v_j)/\sqrt{2}) = -(v_i + v_j)/\sqrt{2} \tag{3.5}$$

for $1 \leqq i < j \leqq 2n-2$. Thus, from (3.4) and (3.5) we can see that either $Av_i = v_i$ $(1 \leqq i \leqq 2n-2)$ or $Av_i = -v_i$ $(1 \leqq i \leqq 2n-2)$ holds. This implies that our real hypersurface M is totally η-umbilic with coefficient $\alpha = \pm 1$ in the ambient space $\tilde{M}_n(c)$. We hence get Condition (1). □

Remark 2. As an immediate consequence of the proof of Theorem 1, we know that on each Sasakian real hypersurface M^{2n-1} $(n \geqq 2)$ every geodesic $\gamma = \gamma(s)$ whose initial vector $\dot{\gamma}(0)$ is orthogonal to $\xi_{\gamma(0)}$ is mapped to a circle of the same curvature 1 in a nonflat complex space form $\tilde{M}_n(c)$.

In consideration of Lemma 1 and the proof of Theorem 1 we find the following:

Proposition 1. *Let M be a connected real hypersurface of a nonflat complex space form $\tilde{M}_n(c)$. Then M is of type either* (A_0) *or type* (A_1) *if and only if there exist orthonormal vectors $v_1, v_2, \dots, v_{2n-2}$ orthogonal to ξ at each point p of M satisfying the following two conditions*:

i$'$) *All geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ are mapped to circles of positive curvature in $\tilde{M}_n(c)$;*

ii$'$) *All geodesics $\gamma_{ij} = \gamma_{ij}(s)$ on M with $\gamma_{ij}(0) = p$ and $\dot{\gamma}_{ij}(0) = (v_i + v_j)/\sqrt{2}$ $(1 \leqq i < j \leqq 2n-2)$ are mapped to circles of positive curvature in $\tilde{M}_n(c)$.*

The following theorem tells us that Theorem 1 is no longer true if we remove Condition ii) of (3) in the assumption of Theorem 1.

Theorem 2. *For a connected real hypersurface M^{2n-1} $(n \geqq 2)$ in a nonflat complex space form $\tilde{M}_n(c)$, the following two conditions are mutually equivalent*:

(1) *M^{2n-1} is a Sasakian space form of constant ϕ-sectional curvature $c+1$ or it is locally congruent to a tube of radius $\pi/4$ around totally geodesic $\mathbf{C}P^{\ell}(4)$ $(1 \leqq \ell \leqq n-2)$ in $\mathbf{C}P^n(4)$;*

(2) *There exist orthonormal vectors $v_1, v_2, \dots, v_{2n-2}$ orthogonal to ξ at each point p of M satisfying that all geodesics $\gamma_i = \gamma_i(s)$ on M with $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ are mapped to circles of the same curvature 1 in $\tilde{M}_n(c)$.*

Proof. We suppose Condition (2). The discussion in the proof of Theorem 1 shows that M has constant ϕ-sectional curvature $c+1$ or is a Hopf hypersurface with at most three distinct (constant) principal curvatures $1, -1$ and $\delta = g(A\xi, \xi)$ at its each point. In the following, we shall consider the latter case. The classification theorems of Hopf hypersurfaces with constant principal curvatures imply that this non-Sasakian Hopf hypersurface M is one of the hypersurfaces of types (A_0), (A_1), (A_2) and (B). This, together with the tables of principal curvatures in Section 2, implies that the real hypersurface M is locally congruent to a tube of radius $\pi/4$ around totally geodesic $\mathbf{C}P^{\ell}(4)$ $(1 \leqq \ell \leqq n-2)$ in $\mathbf{C}P^n(4)$. Hence we obtain Condition (1).

Condition (1) follows from Condition (2) by using Lemma 2 and the proof of Theorem 1. □

Remark 3. (1) Theorems 1 and 2 are local statements. If we add the condition that M is complete and simply connected to the assumptions, then these theorems are global results.

(2) Every circle of curvature 1 in Theorems 1 and 2 lies on a totally real totally geodesic surface $\mathbf{R}P^2(c/4)$ (resp. $\mathbf{R}H^2(c/4)$) of constant sectional curvature $c/4$ in $\mathbf{C}P^n(c)$ (resp. $\mathbf{C}H^n(c)$). It is a simple curve, but is not necessarily closed. A circle of curvature 1 in Theorems 1 and 2 is closed if and only if $c > -4$. In particular, when $c = -4$, every circle of curvature 1 in Theorems 1 and 2 is a horocycle on $\mathbf{C}H^n(-4)$ (for details, see [1, 2]).

4. The length spectrum of a Sasakian space form

In this section, let M denote an arbitrary totally η-umbilic hypersurface of a nonflat complex space form $\tilde{M}_n(c)$ with $n \geqq 2$. In order to investigate geometric properties of geodesics on a complete and simply connected Sasakian space form $M^{2n-1}(c+1)$ of constant ϕ-sectional curvature $c+1$, we study those of geodesics on this hypersurface M. The following discussion is indebted to [3].

Our hypersurface M is a *nice* Riemannian homogeneous manifold. Such a manifold M is known as an example of a naturally reductive Riemannian homogeneous manifold (see [7, 8]). This fact implies that each geodesic γ on the hypersurface M is a homogeneous curve, namely the curve γ is an orbit of some one-parameter subgroup of the isometry group $\mathrm{I}(M)$ of M, so that it is a simple curve.

We recall the congruence theorem for geodesics on the hypersurface M. To do this, for a geodesic γ on M, we define its *structure torsion* ρ_γ by $\rho_\gamma = g(\dot{\gamma}, \xi_\gamma)$. Clearly, it satisfies $-1 \leqq \rho_\gamma \leqq 1$. Moreover, the computation in the proof of Theorem 1 shows that ρ_γ is constant along γ.

For geodesics on the hypersurface M, we can classify them by means of their structure torsions as follows. We say that two smooth curves γ_1, γ_2 on M are congruent to each other if there exists an isometry φ of M satisfying $\gamma_2(s) = (\varphi \circ \gamma_1)(s)$ for all s. When these curves γ_1, γ_2 are geodesics on the hypersurface M, it is known that they are congruent to each other with respect to the isometry group $\mathrm{I}(M)$ of M if and only if their structure torsions ρ_{γ_1} and ρ_{γ_2} satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$.

As a matter of course, for each $c(<0)$ every horosphere of $\mathbf{C}H^n(c)$ has no closed geodesics, because each geodesic on every horosphere is a horocycle in the ambient space $\mathbf{C}H^n(c)$. But for other totally η-umbilic hypersurfaces M, using the above congruence theorem, we can investigate the number of congruence classes of closed geodesics on M. We emphasize that each totally η-umbilic hypersurface M, which is not congruent to a horosphere of $\mathbf{C}H^n(c)$, has countably infinite congruence classes of closed geodesics.

We here pay particular attention to totally η-umbilic hypersurfaces of $\mathbf{C}P^n(c)$. Every geodesic sphere $G(r)$ $(0 < r < \pi/\sqrt{c})$ of $\mathbf{C}P^n(c)$ is a Riemannian homogeneous space, which is not isometric but diffeomorphic to a standard sphere, with maximal sectional curvature $K = c + (c/4)\cot^2(\sqrt{c}r/2)$ and minimal sectional curvature $(c/4)\cot^2(\sqrt{c}r/2)$. Moreover, when the radius r satisfies $\tan^2(\sqrt{c}r/2) > 2$, this geodesic sphere $G(r)$ in $\mathbf{C}P^n(c)$ is so-called a Berger sphere. That is, the sectional curvatures of $G(r)$ with $\tan^2(\sqrt{c}r/2) > 2$ lie in the interval $[\delta K, K]$ with some $\delta \in (0, 1/9)$ but it has closed geodesics of length shorter than $2\pi/\sqrt{K}$ (see [10]). Indeed, if we take an arbitrary integral curve of the characteristic vector field ξ on $G(r)$, then it is a closed geodesic whose length is $2\pi\sin(\sqrt{c}r)/\sqrt{c}$. This length is shorter than $2\pi/\sqrt{K}$ if $\tan^2(\sqrt{c}r/2) > 2$. Furthermore, this closed geodesic is the only closed geodesic on $G(r)$ with respect to the isometry group $\mathrm{I}(G(r))$ of $G(r)$ of length less than $2\pi/\sqrt{K}$.

We here clarify some fundamental properties of geodesics on a complete and simply connected Sasakian space form $M^{2n-1}(c+1)$ of constant ϕ-sectional curvature $c+1$ for each $c(\neq 0)$. By virtue of the above discussion and Lemma 2 we have the following:

Proposition 2. (1) *Every geodesic γ on $M^{2n-1}(c+1)$ is a homogeneous curve, that is the curve γ is an orbit of some one-parameter subgroup of the isometry group* $\mathrm{I}(M^{2n-1}(c+1))$;

(2) *Two geodesics γ_1, γ_2 on $M^{2n-1}(c+1)$ are congruent to each other with respect to the isometry group* $\mathrm{I}(M^{2n-1}(c+1))$ *of $M^{2n-1}(c+1)$ if and only if their structure torsions ρ_{γ_1} and ρ_{γ_2} satisfy* $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$;

(3) *$M^{2n-1}(-3)$ has no closed geodesics and every geodesic on this manifold can be considered as a horocycle in* $\mathbf{C}H^n(-4)$;

(4) *$M^{2n-1}(c+1)$, $c \neq -4$ has countably infinite congruence classes of closed geodesics*;

(5) *When $c > 8$, $M^{2n-1}(c+1)$ is a Berger sphere*.

For other detailed information on the length spectrum, see [3].

5. The metric of a certain Sasakian space form $M^{2n-1}(c+1)$

In this section, we pay particular attention to the metric of a complete and simply connected Sasakian space form with constant ϕ-sectional curvature $c+1$ for each $c > 0$.

We investigate the metric g of i) in Lemma 2. To do this, we must clarify the relation between the metrics of $\mathbf{C}P^n(c)$ and $S^{2n+1}(c/4)$ of constant sectional curvature $c/4$ through the Hopf fibration $\pi : S^{2n+1}(c/4) \to \mathbf{C}P^n(c)$. We denote by $S^m[R]$ an m-dimensional standard sphere of radius R. So we can set

$$S^{2n+1}[R] = \{z = (z_0, \dots, z_n) \in \mathbf{C}^{n+1} \mid |z_0|^2 + \cdots + |z_n|^2 = R^2\} = S^{2n+1}\left(\frac{1}{R^2}\right).$$

As is well-known, the standard inner product $\langle\ ,\ \rangle$ of $\mathbf{C}^{n+1}$ is given by $\langle X, Y\rangle = \mathfrak{Re}(\sum_{i=0}^{n} X^i \overline{Y^i})$. Note that the horizontal part $\mathscr{H}_{\hat{z}}$ of $\pi : S^{2n+1}[R] \to \mathbf{C}P^n(4/R^2)$ at $\hat{z} \in S^{2n+1}[R]$ is expressed as

$$\mathscr{H}_{\hat{z}} = \{(\hat{z}, \hat{X}) \in \{\hat{z}\} \times \mathbf{C}^{n+1} \mid \langle \hat{z}, \hat{X}\rangle = \langle i\hat{z}, \hat{X}\rangle = 0\}.$$

Then the meric g defines the metric $\hat{g}$ of $S^{2n+1}[R]$ which degenerates along the vertical vector $i\hat{z}$ as follows. For any $(\hat{z}, \hat{X}), (\hat{z}, \hat{Y}) \in T_{\hat{z}}S^{2n+1}[R] = \{\hat{X} \in \mathbf{C}^{n+1} \mid \langle \hat{z}, \hat{X}\rangle = 0\}$, we take two horizontal vectors

$$\hat{X} - \left\langle \frac{i\hat{z}}{R}, \hat{X}\right\rangle \frac{i\hat{z}}{R} = \hat{X} - \frac{1}{R^2}\langle i\hat{z}, \hat{X}\rangle i\hat{z}, \quad \hat{Y} - \left\langle \frac{i\hat{z}}{R}, \hat{Y}\right\rangle \frac{i\hat{z}}{R} = \hat{Y} - \frac{1}{R^2}\langle i\hat{z}, \hat{Y}\rangle i\hat{z} \in \mathscr{H}_{\hat{z}}.$$

By direct computation we have

$$\begin{aligned}\hat{g}((\hat{z}, \hat{X}), (\hat{z}, \hat{Y})) &= \left\langle \hat{X} - \frac{1}{R^2}\langle i\hat{z}, \hat{X}\rangle i\hat{z}, \hat{Y} - \frac{1}{R^2}\langle i\hat{z}, \hat{Y}\rangle i\hat{z}\right\rangle \\ &= \langle \hat{X}, \hat{Y}\rangle - \frac{2}{R^2}\langle i\hat{z}, \hat{X}\rangle\langle i\hat{z}, \hat{Y}\rangle + \frac{1}{R^4}\langle i\hat{z}, \hat{X}\rangle\langle i\hat{z}, \hat{Y}\rangle \|i\hat{z}\|^2 \\ &= \langle \hat{X}, \hat{Y}\rangle - \frac{1}{R^2}\langle i\hat{z}, \hat{X}\rangle\langle i\hat{z}, \hat{Y}\rangle,\end{aligned}$$

so that

$$\hat{g}((\hat{z}, \hat{X}), (\hat{z}, \hat{Y})) = \langle \hat{X}, \hat{Y}\rangle - \frac{1}{R^2}\langle i\hat{z}, \hat{X}\rangle\langle i\hat{z}, \hat{Y}\rangle \tag{5.1}$$

for each $(\hat{z}, \hat{X}), (\hat{z}, \hat{Y}) \in T_{\hat{z}}S^{2n+1}[R]$.

On the other hand, we denote by $G(r) = G(r; 4/R^2)$ a geodesic sphere of radius r $(0 < r < R\pi/2)$ in $\mathbf{C}P^n(4/R^2)$. The inverse image $\pi^{-1}(G(r))$ is expressed as

$$\pi^{-1}(G(r)) = S^1[R_1] \times S^{2n-1}[R_2], \quad R_1^2 + R_2^2 = R^2.$$

We here remark that $R_1 = R\cos(r/R)$ and $R_2 = R\sin(r/R)$. Indeed, we can set $R_1 = R\cos\theta$, $R_2 = R\sin\theta$. It is known that the hypersurface $\pi^{-1}(G(r))$ of $S^{2n+1}[R]$ has two constant principal curvatures $-R_2/(RR_1)$, $R_1/(RR_2)$. This shows that the principal curvatures λ_1, δ of the geodesic sphere $G(r)$ are expressed as $\lambda_1 = R_1/(RR_2)$, $\delta = (R_1/(RR_2)) - (R_2/(RR_1))$. On the other hand, λ_1 can be expressed as $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}r/2) = (1/R)\cot(r/R)$. We hence find that $\theta = r/R$.

Next, in order to define a diffeomorphism $\varphi : S^{2n-1}(1)(= S^{2n-1}[1]) \to G(r)$, we consider a diffeomorphism $\tilde{\varphi} : S^{2n-1}(1) \to \tilde{\varphi}(S^{2n-1}(1))$ $(\subset S^1[R_1] \times S^{2n-1}[R_2])$ given by $\tilde{\varphi}(z) = (R_1, R_2 z)$. Then we get a desirable mapping $\varphi : S^{2n-1}(1) \to G(r)$ as $\varphi = \pi \circ \tilde{\varphi}$. We here note that

$$(d\tilde{\varphi})_z(z, X) = (0, R_2 X). \tag{5.2}$$

We shall compute the metric $g_\#$ on $S^{2n-1}(1)$ defined by $g_\# = \varphi^* g$, which is the pullback of g by φ. It follows from (5.1) and (5.2) that

$$\begin{aligned} g_\#((z,X),(z,Y)) &= \hat{g}(((R_1,R_2z),(0,R_2X)),((R_1,R_2z),(0,R_2Y))) \\ &= \langle (0,R_2X),(0,R_2Y)\rangle \\ &\quad - \frac{1}{R^2}\langle i(R_1,R_2z),(0,R_2X)\rangle\langle i(R_1,R_2z),(0,R_2Y)\rangle \\ &= R_2^2\langle X,Y\rangle - \frac{R_2^4}{R^2}\langle iz,X\rangle\langle iz,Y\rangle \\ &= R_2^2(\langle X,Y\rangle - \langle iz,X\rangle\langle iz,Y\rangle) + \frac{R_1^2R_2^2}{R^2}\langle iz,X\rangle\langle iz,Y\rangle. \end{aligned}$$

On the other hand, Remark 1(2) gives us equations $R_1 = RR_2$, $R_2/(RR_1) = c/4$ and $R_1^2 + R_2^2 = R^2$. Thus we see $R = 2/\sqrt{c}$, $R_1 = 4/\sqrt{c(c+4)}$, $R_2 = 2/\sqrt{c+4}$. We hence know that the metric g of the geodesic sphere $G(r)$ is realized as the following metric $g_\#$ which is nothing but the deformation of the standard metric g_0 of $S^{2n-1}(1)$:

$$g_\# = \frac{4}{c+4}(g_0 - \eta\otimes\eta) + \left(\frac{4}{c+4}\right)^2 \eta\otimes\eta.$$

Therefore we conclude that the metric g of i) in Lemma 2 coincides with the well-known metric on a standard example of a complete and simply connected Sasakian space form with constant ϕ-sectional curvature $c+1$ for each $c>0$ (see [5]).

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