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Geometric Properties and Coincidence Theorems with Applications to Generalized Vector Equilibrium Problems[1]

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Abstract. The present paper is divided into two parts. In the first part, we derive a Fan-KKM type theorem and establish some Fan type geometric properties of convex spaces. By applying our results, we also obtain some coincidence theorems and fixed-point theorems in the setting of convex spaces. The second part deals with the applications of our coincidence theorem to establish some existence results for a solution to the generalized vector equilibrium problems.

Key Words. Fan-KKM type theorem, coincidence theorems, fixed-point theorems, generalized vector equilibrium problems, maximal pseudomonotone maps.

1. Introduction

The present paper is divided into two parts. In the first part, we derive a Fan–Knaster–Kuratowski–Mazurkiewicz (Fan-KKM) type theorem and establish some Fan type geometric properties of convex spaces. By applying our results, we also obtain some coincidence theorems and fixed-point theorems in the setting of convex spaces. The second part deals with the applications of our coincidence theorem to establish some existence results

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for a solution of generalized vector equilibrium problems with or without maximal pseudomonotonicity and diagonality conditions.

2. Geometric Properties and Coincidence Theorems on Convex Spaces

In 1961, Fan (Ref. 1) established the following geometric result.

Theorem 2.1. Let X be a nonempty compact convex subset of a Hausdorff topological vector space E, and let $A \subseteq X \times X$ be a set such that:

(i) for each $x \in X$, $(x, x) \in A$;
(ii) for each $y \in X$, the set $\{x \in X: (x, y) \in A\}$ is closed in X;
(iii) for each $x \in X$, the set $\{y \in X: (x, y) \notin \in A\}$ is convex or empty.

Then, there exists a point $x_0 \in X$ such that $\{x_0\} \times X \subseteq A$.

Since then, numerous generalizations with their applications have been studied in the literature; see for example Refs. 2–6 and references therein. In 1980, Ha (Ref. 2) generalized Theorem 2.1 by relaxing the compactness assumption on X. In this section, we derive a Fan-KKM type theorem and extend Theorem 2.1 in various directions. By applying our results, we establish also some coincidence theorems and fixed-point theorems in the setting of convex spaces.

We mention some preliminaries which will be used in the sequel. We denote by 2^K the family of all subsets of the set K and by $\langle K \rangle$ the class of all finite subsets of K. Let K be a nonempty subset of a topological space E; then, we shall denote by $\bar{K}$ or cl K the closure of K and by int K the interior of K. If K is a nonempty subset of $D \subseteq E$, then $\mathrm{int}_D K$ denotes the interior of K in D. For K a nonempty subset of a vector space, we denote by co$K$ the convex hull of K.

Let $P: X \to 2^Y$ be a multivalued map from a space X to another space Y. The graph of P, denoted by $\mathscr{G}(P)$, is

$$\mathscr{G}(P) = \{(x, y) \in X \times Y: x \in X, y \in P(x)\}.$$

The inverse P^{-1} of P is also a multivalued map from the range of P to X defined by

$$x \in P^{-1}(y), \qquad \text{if and only if } y \in P(x).$$

Let X and Y be topological spaces. A multivalued map $P: X \to 2^Y$ is called:

(i) closed if $\mathscr{G}(P)$ is a closed subset of $X \times Y$;
(ii) compact if $\overline{P(X)}$ is a compact subset of Y;

(iii) upper semicontinuous if $P^{-1}(A) = \{x \in X: P(x) \cap A \neq \emptyset\}$ is closed in X for each closed subset A of Y;
(iv) u-hemicontinuous if, for any $x, y \in K$ and $t \in [0, 1]$, the mapping $t \to P(y + t(x - y))$ is upper semicontinuous at 0^+;
(v) transfer closed (Ref. 7) if, for any $x \in X$ and $y \notin P(x)$, there exists $\tilde{x} \in X$ such that $y \notin \overline{P(\tilde{x})}$;
(vi) transfer open (Ref. 7) if, for any $x \in X$ and $y \in P(x)$, there exists $\tilde{x} \in X$ such that $y \in \operatorname{int} P(\tilde{x})$.

A subset B of Y is said to be compactly open (respectively, compactly closed) if, for each compact subset D of Y, the set $B \cap D$ is open (respectively, closed) in D.

The following characterization lemma for a transfer-open multivalued map is given in Ref. 8.

Lemma 2.1. Let X and Y be two topological spaces, and let $P: X \to 2^Y$ be a multivalued map. Then, the following two statements are equivalent.

(i) P^{-1} is transfer open and $P(x)$ is nonempty for all $x \in X$.
(ii) $X = \cup\{\operatorname{int} P^{-1}(y): y \in Y\}$.

Remark 2.1. Let X and Y be two topological spaces, and let $P: X \to 2^Y$ be a multivalued map such that $P^{-1}(y)$ is open for all $y \in Y$; then, P^{-1} is transfer open. But the converse is not true. Wu and Shen (Ref. 9) gave an example which shows that the transfer-open property is more general than the open-fiber property, that is, $P^{-1}(y)$ is open for all $y \in Y$.

A convex space (Ref. 10) X is a nonempty convex set in a vector space with any topology that induces the Euclidean topology on the convex hulls of its finite subsets.

Let X be a convex space, and let Y be a Hausdorff topological space. If $S, T: X \to 2^Y$ are multivalued maps such that

$$T(\mathrm{co}N) \subseteq S(N), \qquad \text{for each } N \in \langle X \rangle,$$

then S is said to be generalized KKM mapping w.r.t. T (Ref. 11). The multivalued map $T: X \to 2^Y$ is said to have the KKM property (Ref. 11) if $S: X \to 2^Y$ is a generalized KKM mapping w.r.t. T such that the family $\{\overline{S(x)}: x \in X\}$ has the finite intersection property.

We denote by KKM(X, Y) the family of all multivalued maps having the KKM property.

Recall that a nonempty space is acyclic if all of its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus any nonempty convex or star-shaped set is acyclic.

We denote by $V(X, Y)$ the family of all upper semicontinuous multivalued maps with compact acyclic values. Then, $V(X, Y) \subseteq \mathrm{KKM}(X, Y)$; see for example Ref. 11.

The following lemmas will be used in the sequel.

Lemma 2.2. See Ref. 12. Let X be a convex space, and let Y be a Hausdorff topological space. Let $T \in \mathrm{KKM}(X, Y)$ be compact, and let $G: X \to 2^Y$ be a multivalued map. Assume that, for each $x \in X$, $G(x)$ is compactly closed in Y and, for any $N \in \langle X \rangle$, $T(\mathrm{co}N) \subseteq G(N)$. Then, $\overline{T(X)} \cap \bigcap \{G(x): x \in X\} \neq \emptyset$.

Lemma 2.3. See Ref. 12. Let X be a convex subset of a topological vector space, Y a topological space, A a nonempty convex subset of X, and $T \in \mathrm{KKM}(X, Y)$. Then, $T|_A \in \mathrm{KKM}(A, Y)$.

Lemma 2.4. See Ref. 13. Let X and Y be topological spaces, and let $P: X \to 2^Y$ be a multivalued map. If Y is compact and P is closed, then P is upper semicontinuous.

Lemma 2.5. Let X and Y be topological spaces, and let $P: X \to 2^Y$ be a multivalued map. Then:

(i) P is transfer closed if and only if $\bigcap_{x \in X} P(x) = \bigcap_{x \in X} \overline{P(x)}$; see Ref. 14.

(ii) P is transfer open if and only if $G: X \to 2^Y$, defined by $G(x) = Y \setminus P(x)$ for all $x \in X$, is transfer closed; see Ref. 7.

Now, we present the main results of this section. In the rest of the paper, all topological spaces are assumed to be Hausdorff. The following result follows immediately from Lemma 2.2.

Theorem 2.2. Let X be a convex space, let Y be a topological space, and let $T \in \mathrm{KKM}(X, Y)$ be compact. Assume that $G: X \to 2^Y$ is transfer closed and, for any $N \in \langle X \rangle$, $T(\mathrm{co}N) \subseteq G(N)$. Then, $\overline{T(X)} \cap \bigcap \{G(x): x \in X\} \neq \emptyset$.

Proof. We define a multivalued map $S: X \to 2^Y$ by $S(x) = \overline{G(x)}$. Then, all the conditions of Lemma 2.2 are satisfied and hence,

$$\overline{T(X)} \cap \bigcap \{S(x): x \in X\} \neq \emptyset.$$

Since G is a transfer-closed multivalued map, it follows from Lemma 2.5 (i) that

$$\bigcap \{G(x): x \in X\} = \bigcap \{\overline{G(x)}: x \in X\} = \bigcap \{S(x): x \in X\}.$$

Therefore,

$$\overline{T(X)} \cap \bigcap \{G(x): x \in X\} \neq \emptyset. \qquad \square$$

By using Theorem 2.2, we establish the following geometric properties of convex spaces.

Theorem 2.3. Let Y be a convex space, let X be a topological space, and let $T \in \mathrm{KKM}(Y, X)$ be compact. Let A and $B \subseteq X \times Y$ satisfy the following conditions:

(i) for all $y \in Y$ and $x \in T(y)$, $(x, y) \in B$;
(ii) the multivalued map $P^{-1}: Y \to 2^X$ is transfer open, where $P: X \to 2^Y$ is defined by $P(x) = \{y \in Y: (x, y) \notin A\}$ for all $x \in X$;
(iii) for all $x \in X$, the set $\{y \in Y: (x, y) \notin A\}$ is convex;
(iv) $B \subseteq A$.

Then, there exists a point $\bar{x} \in \overline{T(Y)}$ such that $(\bar{x}, y) \in A$ for all $y \in Y$.

Proof. Define a multivalued map $G: X \to 2^Y$ by

$$G(x) = \{y \in Y: (x, y) \in A\}, \qquad \text{for all } x \in X.$$

Since P^{-1} is transfer open, by Lemma 2.5 (ii), G^{-1} is transfer closed. We want to show that, for each $N = \{y_1, \ldots, y_n\} \in \langle Y \rangle$,

$$T(\mathrm{co}N) \subseteq G^{-1}(N) = \bigcup_{i=1}^{n} G^{-1}(y_i).$$

Suppose to the contrary that there exist a finite set $N = \{y_1, \ldots, y_n\} \in \langle Y \rangle$, a point $\hat{y} \in \mathrm{co}N$, and $\hat{x} \in T(\hat{y})$ such that $\hat{x} \notin \bigcup_{i=1}^{n} G^{-1}(y_i)$. Then,

$$\hat{x} \notin G^{-1}(y_i), \qquad \text{for all } i = 1, \ldots, n.$$

This implies that

$$(\hat{x}, y_i) \notin A, \qquad \text{for all } i = 1, \ldots, n.$$

Since $P(x)$ is convex for all $x \in X$,

$$(\hat{x}, v) \notin A, \qquad \text{for all } v \in \mathrm{co}N.$$

Since $B \subseteq A$,

$$(\hat{x}, v) \notin B \qquad \text{for all } v \in \mathrm{co}N.$$

This is a contradiction of condition (i). Hence, by Theorem 2.2,

$$\overline{T(Y)} \cap \bigcap \{G^{-1}(y): y \in Y\} \neq \emptyset,$$

that is, there exists a point $\bar{x} \in \overline{T(Y)}$ such that $(\bar{x}, y) \in A$ for all $y \in Y$. $\square$

Remark 2.2. If $X = Y$, $A = B$, $T(y) = \{y\}$ for all $y \in Y$, and $P^{-1}(y)$ is open for all $y \in Y$, then Theorem 2.3 reduces to Theorem 2.1.

By using Theorem 2.3, we have the following result.

Theorem 2.4. Let Y be a convex space, and let X be a topological vector space. Let $A \subseteq X \times Y$ satisfy the following conditions:

(i) the multivalued map $P^{-1}: Y \to 2^X$ is transfer open, where $P: X \to 2^Y$ is defined by $P(x) = \{y \in Y: (x, y) \notin A\}$ for all $x \in X$;
(ii) for all $x \in X$, the set $\{y \in Y: (x, y) \notin A\}$ is convex;
(iii) there exist a compact set $D \subseteq X$ and a closed set $B \subseteq A$ such that, for each $y \in Y$, $T(y) := \{x \in D: (x, y) \in B\}$ is a nonempty acyclic subset of D.

Then, there exists a point $\bar{x} \in \overline{T(Y)}$ such that $(\bar{x}, y) \in A$ for all $y \in Y$.

Proof. Let $(y, x) \in \overline{\mathscr{G}(T)}$. Then, there exists a net $\{(y_\alpha, x_\alpha)\}$ in $\mathscr{G}(T)$ such that $(y_\alpha, x_\alpha) \to (y, x)$. Since $(y_\alpha, x_\alpha) \in \mathscr{G}(T)$, we have $x_\alpha \in D$ and $(x_\alpha, y_\alpha) \in B$. Since D is compact and B is closed, $x \in D$ and $(x, y) \in B$. This implies that $x \in T(y)$ and $(y, x) \in \mathscr{G}(T)$. Therefore, T is closed. It is easy to show that, for each $y \in Y$, $T(y)$ is a closed set. Thus, for each $y \in Y$, $T(y) \subseteq D$ is compact. By Lemma 2.4 and condition (iii), T is an upper semicontinuous multivalued map with compact acyclic values. Then, $T \in V(Y, X) \subseteq \text{KKM}(Y, X)$ is compact and the conclusion follows from Theorem 2.3. □

From Theorem 2.4, we derive the following result on the sets with convex sections due to Ha (Ref.2).

Corollary 2.1. Let X and Y be nonempty convex subsets of topological vector spaces E and W, respectively, and let $A \subseteq X \times Y$ be a set such that:

(i) for each $y \in Y$, the set $\{x \in X: (x, y) \in A\}$ is closed in X;
(ii) for each $x \in X$, the set $\{y \in Y: (x, y) \notin A\}$ is convex or empty;
(iii) there exist a compact convex subset D of X and a closed set $B \subseteq A$ such that, for each $y \in Y$, the set $\{x \in D: (x, y) \in B\}$ is nonempty and convex.

Then, there exists a point $\bar{x} \in X$ such that $\{\bar{x}\} \times Y \subseteq A$.

Proof. By (i), $P^{-1}(y) := \{x \in X: (x, y) \notin A\}$ is open in X and therefore, P^{-1} is transfer open. By (iii), for each $y \in Y$, $T(y) := \{x \in D: (x, y) \in B\}$ is nonempty convex, and so nonempty acyclic. The conclusion follows from Theorem 2.4. □

Remark 2.3. Theorems 2.3 and 2.4 generalize Theorem 3 in Ref. 2.

By using Theorem 2.3 with $A = B$, we derive the following coincidence theorems.

Theorem 2.5. Let Y be a convex space, let X be a topological space, and let $T \in \mathrm{KKM}(Y, X)$ be compact. Let $P: X \to 2^Y$ be a multivalued map such that, for all $x \in X$, $P(x)$ is convex and $X = \bigcup \{\mathrm{int}\, P^{-1}(y): y \in Y\}$. Then, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in P(\bar{x})$.

Proof. We define

$$A = \{(x, y) \in X \times Y: (x, y) \notin \mathscr{G}(P)\}.$$

Suppose to the contrary that, for all $x \in X$, $P(x) \cap T^{-1}(x) = \varnothing$. Then, for all $y \in Y$, $x \in T(y)$ implies $y \notin P(x)$, and therefore $(x, y) \in A$. From the definition of A, it follows that

$$P(x) = \{y \in Y: (x, y) \notin A\}.$$

Since

$$X = \bigcup \{\mathrm{int}\, P^{-1}(y): y \in Y\}$$

and by Lemma 2.1, $P^{-1}: Y \to 2^X$ is transfer open and for all $x \in X$, $P(x)$ is nonempty. Since for all $x \in X$, $P(x)$ is convex, we have that the set $\{y \in Y: (x, y) \notin A\}$ is convex for all $x \in X$.

By Theorem 2.3 (with $A = B$), there exists $\hat{x} \in \overline{T(Y)}$ such that $(\hat{x}, y) \in A$ for all $y \in Y$. Therefore, $P(\hat{x}) = \varnothing$. This contradicts the fact that $P(x) \neq \varnothing$ for all $x \in X$. Hence, there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in P(\bar{x})$. □

Remark 2.4. Following the arguments of Theorem 1 in Ref. 15, it is easy to derive the fixed-point theorem of Chang and Yen (Ref. 11) by using Theorem 2.5.

When T is not necessarily compact, we have the following result.

Theorem 2.6. Let Y be a convex space, let X be a topological space, and let $T \in \mathrm{KKM}(Y, X)$. Let $P: X \to 2^Y$ be a multivalued map such that, for each $x \in X$, $P(x)$ is convex, $X = \bigcup \{\text{int}\, P^{-1}(y) \colon y \in Y\}$ and, for each compact subset A of X, $\overline{T(A)}$ is compact. Assume that there exist a nonempty compact subset D of X and, for each $N \in \langle Y \rangle$, a compact convex subset L_N of Y containing N such that $T(L_N) \backslash D \subseteq \bigcup \{\text{int}\, P^{-1}(y) \colon y \in L_N\}$. Then, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in P(\bar{x})$.

Proof. Since

$$D \subseteq X = \bigcup \{\text{int}\, P^{-1}(y) \colon y \in Y\},$$

there exists a finite set $N = \{y_1, \ldots, y_n\} \in \langle Y \rangle$ such that

$$D \subseteq \bigcup_{i=1}^{n} \text{int}\, P^{-1}(y_i). \tag{1}$$

By hypothesis, there exists a compact convex subset L_N of Y containing N such that

$$T(L_N) \backslash D \subseteq \bigcup \{\text{int}\, P^{-1}(y) \colon y \in L_N\}. \tag{2}$$

By (1), we get

$$T(L_N) \cap D \subseteq \bigcup \{\text{int}\, P^{-1}(y) \colon y \in N\} \subseteq \bigcup \{\text{int}\, P^{-1}(y) \colon y \in L_N\}. \tag{3}$$

From (2) and (3), we have

$$T(L_N) \subseteq \bigcup \{\text{int}\, P^{-1}(y) \colon y \in L_N\}.$$

Therefore,

$$T(L_N) = \bigcup \{\text{int}_{T(L_N)} P^{-1}(y) \colon y \in L_N\}.$$

Since Y is a convex space and L_N is a compact convex subset of Y, L_N is a convex space. By hypothesis, $\overline{T(L_N)}$ is compact and thus $T|_{L_N}$ is compact. Since $T \in \text{KKM}(Y, X)$ and L_N is a nonempty convex subset of Y, it follows from Lemma 2.3 that $T \in \text{KKM}(L_N, X)$. By Theorem 2.5, there exist $\bar{x} \in T(L_N) \subseteq X$ and $\bar{y} \in L_N \subseteq Y$ such that $\bar{x} \in T|_{L_N}(\bar{y}) = T(\bar{y})$ and $\bar{y} \in P(\bar{x})$. □

As a simple consequence of Theorem 2.6, we have the following fixed-point result.

Corollary 2.2. See Ref. 8. Let X be a convex space, and let $P \colon X \to 2^X$ be a multivalued map such that, for each $x \in X$, $P(x)$ is convex and $X = \bigcup \{\text{int}\, P^{-1}(y) \colon y \in Y\}$. Assume that there exist a nonempty compact subset D of X and, for each $N \in \langle X \rangle$, a compact convex subset L_N of X containing N such that $L_N \backslash D \subseteq \bigcup \{\text{int}\, P^{-1}(y) \colon y \in L_N\}$. Then, there exists a point $\bar{x} \in X$ such that $\bar{x} \in P(\bar{x})$.

Proof. The conclusion follows from Theorem 2.6 by letting $T(x) = \{x\}$ for all $x \in X$. □

Theorem 2.7. Let Y be a convex space, and let X be a topological space. Let $P: X \to 2^Y$ be a multivalued map such that, for all $x \in X$, $P(x)$ is convex and $X = \bigcup \{\text{int}\, P^{-1}(y): y \in Y\}$. Assume that there exist a compact set $D \subseteq Y$ and a closed set $B \subseteq X \times Y$ such that, for each $y \in Y$, the set $T(y) := \{x \in D: (x, y) \in B\}$ is a nonempty acyclic subset of X. Then, there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in P(\bar{x})$.

Proof. Using the same argument as in the proof of Theorem 2.4, we see that

$$T \in V(Y, X) \subseteq \text{KKM}(Y, X)$$

is compact and the conclusion follows from Theorem 2.5. □

Now, by using coincidence Theorem 2.6, we derive the following geometric property theorem for convex spaces.

Theorem 2.8. Let Y be a convex space, let X be a topological space, and let $T \in \text{KKM}(Y, X)$. Let $A \subseteq X \times Y$ satisfy the following conditions:

(i) for all $y \in Y$, $x \in T(y)$, $(x, y) \in A$;
(ii) $P^{-1}: Y \to 2^X$ is transfer open, where $P: X \to 2^Y$ is defined by $P(x) = \{y \in Y: (x, y) \notin A\}$ for $x \in X$;
(iii) for all $x \in X$, $\{y \in Y: (x, y) \notin A\}$ is convex;
(iv) for each compact subset C of Y, $\overline{T(C)}$ is compact;
(v) there exists a nonempty compact subset D of X and, for each $N \in \langle Y \rangle$, a compact convex subset L_N of Y containing N such that $T(L_N) \cap \bigcap \{\{x \in X: (x, y) \in A\}: y \in L_N\} \subseteq D$.

Then, there exists a point $\bar{x} \in X$ such that $(\bar{x}, y) \in A$ for all $y \in Y$.

Proof. Suppose that, for each $x \in X$, there exists a point $y \in Y$ such that $(x, y) \notin A$. Then, for each $x \in X$, $P(x) \neq \emptyset$. By (ii) and Lemma 2.1,

$$X = \bigcup \{\text{int}\, P^{-1}(y): y \in Y\}.$$

From condition (v), we have

$$T(L_N) \cap \bigcap \{X \backslash \text{int}\, P^{-1}(y): y \in L_N\} \subseteq D.$$

By (iii), for each $x \in X$, $P(x)$ is convex. Therefore, all the conditions of Theorem 2.6 are satisfied; hence, there exist $\bar{x} \in X$ and $\bar{y} \in Y$ such that

$\bar{x} \in T(\bar{y})$ and $\bar{y} \in P(\bar{x})$, that is, $\bar{x} \in T(\bar{y})$ such that $(\bar{x}, \bar{y}) \notin A$, a contradiction of (i). Hence, there exists $\bar{x} \in X$ such that $(\bar{x}, y) \in A$ for all $y \in Y$. □

3. Generalized Vector Equilibrium Problems

Let K be a nonempty convex subset of a topological vector space X. For a given bifunction $f: K \times K \to \mathbb{R}$ such that $f(x, x) \geq 0$ for all $x \in K$, the equilibrium problem (EP) is to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq 0, \qquad \text{for all } y \in K. \tag{4}$$

This problem contains optimization, Nash equilibrium, fixed-point, complementarity, variational inequality, and many other problems as special cases; see for example Refs. 6, 16–22, and references therein.

If we replace $\mathbb{R}$ by a topological vector space Z with ordered cone C, that is, a closed and convex cone with int $C \neq \emptyset$, where int C denotes the interior of C, then one possibility to generalize (4) can be in the following way:

$$f(\bar{x}, y) \notin -\text{int } C, \qquad \text{for all } y \in K. \tag{5}$$

In this case, (EP) is called the vector equilibrium problem (VEP), which includes vector optimization problems, noncooperative vector equilibrium problems, vector complementarity problems, and vector variational inequality problems as special cases. For a more general form of the VEP, we replace the ordered cone C by a moving cone. We consider a multivalued map $C: K \to 2^Z$ such that, for each $x \in K$, $C(x)$ is a closed and convex cone with int $C(x) \neq \emptyset$; then, the VEP can be written so as to find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -\text{int } C(\bar{x}), \qquad \text{for all } y \in K. \tag{6}$$

For further details on the VEP, we refer to Refs. 23–30 and references therein.

Further, if we replace the bifunction f by a multivalued map $F: K \times K \to 2^Z \setminus \{\emptyset\}$, then the VEP is known as the generalized vector equilibrium problem (GVEP), which is to find $\bar{x} \in K$ such that

$$F(\bar{x}, y) \nsubseteq -\text{int } C(\bar{x}), \qquad \text{for all } y \in K. \tag{7}$$

It is considered and studied in Refs. 31–36 and contains generalized implicit vector variational inequality problems and generalized vector variational and variational-like inequality problems as special cases; see for example Ref. 36. For further details on generalized vector variational and variational-like inequality problems, we refer to Ref. 25 and references therein.

A problem closely related to the GVEP (7) is to find $\bar{x} \in K$ such that

$$F(y, \bar{x}) \not\subseteq \text{int}\, C(\bar{x}), \qquad \text{for all } y \in K. \tag{8}$$

It is termed the dual generalized vector equilibrium problem (DGVEP) by Konnov and Yao (Ref. 33). For the different forms of the GVEP, we refer to Refs. 37–39.

If we replace the multivalued map F by another multivalued map $G: K \times K \to 2^Z \setminus \{\emptyset\}$, then the DGVEP (8) becomes the problem of finding $\bar{x} \in K$ such that

$$G(y, \bar{x}) \not\subseteq \text{int}\, C(\bar{x}), \qquad \text{for all } y \in K. \tag{9}$$

Throughout this section, we denote by K^p, K_F^d, and K_G^d the solution sets of the GVEP (7), DGVEP (8), and DGVEP (9), respectively.

The motivation of this section is to provide some applications of our coincidence theorem (Theorem 2.6) to establish some existence results for a solution to the GVEP (7) with or without maximal pseudomonotonicity and diagonality conditions. First, we prove that $K^p = K_G^d$ (in particular, $K^p = K_F^d$ under certain conditions); then, we apply our result to prove that K^p is nonempty.

Let K be a convex space, and let Z be topological vector space. Let $F, G: K \times K \to 2^Z \setminus \{\emptyset\}$ and $C: K \to 2^Z$ be multivalued maps such that, for each $x \in K$, $C(x)$ is a closed and convex cone with $\text{int}\, C(x) \neq \emptyset$. Then, F is called:

(i) G-pseudomonotone if, for all $x, y \in K$,

$$F(x, y) \not\subseteq -\text{int}\, C(x) \text{ implies } G(y, x) \not\subseteq \text{int}\, C(x);$$

(ii) maximal G-pseudomonotone if F is G-pseudomonotone and, for all $x, y \in K$, $G(z, x) \not\subseteq \text{int}\, C(x)$, for all $z \in]x, y]$ implies $F(x, y) \not\subseteq -\text{int}\, C(x)$, where $]x, y] = \{z \in K: z = ty + (1-t)x, t \in (0, 1]\}$ is a line segment in K joining x and y but not containing x.

Remark 3.1. When F and G are single-valued maps, definition (ii) reduces to the definition of maximal G-monotonicity used by Oettli (Ref. 29).

Let K, Z, C be the same as above. A multivalued map $F: K \times K \to 2^Z \setminus \{\emptyset\}$ is called:

(i) C_x-quasiconvex-like (Ref. 31) if, for all $x, y_1, y_2 \in K$ and $t \in [0, 1]$, we have either

$$F(x, ty_1 + (1-t)y_2) \subseteq F(x, y_1) - C(x)$$

or

$$F(x, ty_1 + (1 - t)y_2) \subseteq F(x, y_2) - C(x);$$

(ii) explicitly $\delta(C_x)$-quasiconvex (Ref. 33), if, for all $y_1, y_2 \in K$ and $t \in (0, 1)$, we have either

$$F(y_t, y_1) \subseteq F(y_t, y_t) + C(y_1)$$

or

$$F(y_t, y_2) \subseteq F(y_t, y_t) + C(y_1),$$

and in case $F(y_t, y_1) - F(y_t, y_2) \subseteq \text{int}\, C(y_1)$, for all $t \in (0, 1)$, we have

$$F(y_t, y_1) \subseteq F(y_t, y_t) + \text{int}\, C(y_1),$$

where $y_t = ty_1 + (1 - t)y_2$.

Proposition 3.1. Let K be a convex space, let Z be a topological vector space, and let $C: K \to 2^Z$ be a multivalued map such that, for each $x \in K$, $C(x)$ is a proper, closed, and convex cone with $\text{int}\, C(x) \neq \emptyset$. Let $F, G: K \times K \to 2^Z \setminus \{\emptyset\}$ be multivalued maps such that:

(i) for all $x, y \in K$, $F(y, y) \subseteq C(x)$;
(ii) F is explicitly $\delta(C_x)$-quasiconvex and G-pseudomonotone;
(iii) for all $x, y \in K$, $F(y, x) \subseteq \text{int}\, C(x)$ implies $G(y, x) \subseteq \text{int}\, C(x)$;
(iv) for all $y \in K$, the multivalued map $x \mapsto F(x, y)$ is u-hemicontinuous.

Then, F is maximal G-pseudomonotone.

Proof. It is similar to the proof of Lemma 2.1 in Ref. 33. □

To prove the main results, we define a multivalued map $P: K \to 2^K$ by

$$P(x) = \{y \in K: G(y, x) \subseteq \text{int}\, C(x)\}, \qquad \text{for all } x \in K.$$

Theorem 3.1. Let K be a convex space, and let Z be a topological vector space. Let $T \in \text{KKM}(K, K)$ such that, for each compact subset A of K, $\overline{T(A)}$ is compact. Let $F, G: K \times K \to 2^Z \setminus \{\emptyset\}$, and $C: K \to 2^Z$ be multivalued maps such that, for each $x \in K$, $C(x)$ is a pointed, closed, and convex cone with $\text{int}\, C(x) \neq \emptyset$. Assume that:

(i) for all $y \in K$, $x \in T(y)$, $F(x, y) \nsubseteq -\text{int}\, C(x)$;
(ii) $P^{-1}: K \to 2^K$ is transfer open;
(iii) for each $x \in K$, the set $P(x)$ is convex;

(iv) F is maximal G-pseudomonotone;
(v) there exists a nonempty compact subset D of K and, for each $M \in \langle K \rangle$, a compact convex subset L_M of K containing M such that $T(L_M) \backslash D \subseteq \bigcup \{\text{int } P^{-1}(y): y \in L_M\}$.

Then, the GVEP (7) has a solution $\bar{x} \in K$ and $K^p = K_G^d$.

Proof. Let $\bar{x} \in K_G^d$; then,

$$G(y, \bar{x}) \nsubseteq \text{int } C(\bar{x}), \qquad \text{for all } y \in K.$$

For any $y \in K$, $[\bar{x}, y] \subseteq K$. Therefore,

$$G(z, \bar{x}) \nsubseteq \text{int } C(\bar{x}), \qquad \text{for all } z \in]\bar{x}, y].$$

Since F is maximal G-pseudomonotone,

$$F(\bar{x}, y) \nsubseteq -\text{int } C(\bar{x}).$$

Hence,

$$\bar{x} \in K^p \quad \text{and} \quad K_G^d \subset K^p.$$

So, it is sufficient to show that the DGVEP (9) has a solution.

Suppose to the contrary that the DGVEP (9) does not have any solution. Then, for each $x \in K$,

$$P(x) = \{y \in K: G(y, x) \subseteq \text{int } C(x)\} \neq \emptyset.$$

By (ii) and Lemma 2.1,

$$K = \bigcup \{\text{int } P^{-1}(y): y \in K\}.$$

By Theorem 2.6, there exist $\bar{x} \in K$ and $\bar{y} \in Y$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in P(\bar{x})$. Thus, by (i), $F(\bar{x}, \bar{y}) \nsubseteq -\text{int } C(\bar{x})$ and $G(\bar{y}, \bar{x}) \subseteq \text{int } C(\bar{x})$, because $\bar{y} \in P(\bar{x})$. Since F is maximal G-pseudomonotone, F is G-pseudomonotone; therefore, $F(\bar{x}, \bar{y}) \nsubseteq -\text{int } C(\bar{x})$ implies $G(\bar{y}, \bar{x}) \nsubseteq \text{int } C(\bar{x})$, a contradiction.

Since F is maximal G-pseudomonotone, F is G-pseudomonotone and the relation $K^p \subset K_G^d$ follows from the definition of the G-pseudomonotone property. Therefore, $K^p = K_G^d$. □

Remark 3.2.

(i) If $T: K \to 2^K$ is a multivalued map defined as $T(x) = \{x\}$, for all $x \in K$, then condition (i) of Theorem 3.1 can be replaced by the diagonality condition $F(x, x) \nsubseteq -\text{int } C(x)$, for all $x \in K$.

(ii) If F is C_x-quasiconvex-like, then for each $x \in K$, $P(x)$ is convex; see for example the proof of Theorem 2.1 in Ref. 31.

(iii) For each $y \in K$, if $G(y, \cdot)$ is upper semicontinuous with compact values on K and if the multivalued map $W: K \to 2^K$ defined as $W(x) = Z \backslash \operatorname{int} C(x)$, for all $x \in K$, is upper semicontinuous, then the set

$$\{x \in K: G(y, x) \not\subseteq \operatorname{int} C(x)\} = K \backslash P^{-1}(y)$$

is closed; see for example the proof of Theorem 2.1 in Ref. 31. This shows that, for each $y \in K$, $P^{-1}(y)$ is open. Suppose that $x \in P^{-1}(y)$; then,

$$x \in \operatorname{int} P^{-1}(y) = \operatorname{int} P^{-1}(\bar{y}), \qquad \text{with } \bar{y} = y.$$

Therefore, $P^{-1}: K \to 2^K$ is transfer open and (3) implies (ii).

Now, we establish some existence results for a solution to the GVEP without the maximal pseudomonotonicity or pseudomonotonicity assumption.

We define a multivalued map $Q: K \to 2^K$ by

$$Q(x) = \{y \in K: F(x, y) \subseteq -\operatorname{int} C(x)\}, \qquad \text{for all } x \in K.$$

Theorem 3.2. Let K, Z, C, T be the same as in Theorem 3.1. Let $F: K \times K \to 2^Z \backslash \{\emptyset\}$ be a multivalued map such that, for all $y \in K$, $x \in T(y)$, $F(x, y) \not\subseteq -\operatorname{int} C(x)$, and let $Q^{-1}: K \to 2^K$ be transfer open and $Q(x)$ be convex for all $x \in K$. Assume that there exist a nonempty compact subset D of K and, for each $M \in \langle K \rangle$, a compact convex subset L_M of K containing M such that $T(L_M) \backslash D \subseteq \bigcup \{\operatorname{int} Q^{-1}(y): y \in L_M\}$. Then, the GVEP (7) has a solution.

Proof. Suppose that the conclusion of this theorem does not hold. Then, for each $x \in K$, the set $Q(x) \neq \emptyset$. By Lemma 2.1,

$$K = \bigcup \{\operatorname{int} Q^{-1}(y): y \in K\}.$$

By Theorem 2.6, there exist $\bar{x} \in K$ and $\bar{y} \in K$ such that $\bar{x} \in T(\bar{y})$ and $\bar{y} \in Q(\bar{x})$. Then, $F(\bar{x}, \bar{y}) \not\subseteq -\operatorname{int} C(\bar{x})$, while $\bar{y} \in Q(\bar{x})$ implies $F(\bar{x}, \bar{y}) \subseteq -\operatorname{int} C(\bar{x})$, a contradiction. □

Remark 3.3. The assumption "Q^{-1} is transfer open" of Theorem 3.2 can be replaced by the condition that "for each $y \in K$, $F(\cdot, y)$ is upper semicontinuous with compact values on K together with the condition that the graph of the multivalued map $W: K \to 2^Z$, defined as $W(x) = Z \backslash \{-\operatorname{int} C(x)\}$, for all $x \in K$, is closed"; see for example the proof of Theorem 2.1 in Ref. 31.

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