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Geometric Properties of a Certain Class of Mittag–Leffler-Type Functions

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Abstract: The main objective of this paper is to establish some sufficient conditions so that a class of normalized Mittag–Leffler-type functions satisfies several geometric properties such as starlikeness, convexity, close-to-convexity, and uniform convexity inside the unit disk. Moreover, pre-starlikeness and k-uniform convexity are discussed for these functions. Some sufficient conditions are also derived so that these functions belong to the Hardy spaces $\mathcal{H}^p$ and $\mathcal{H}^\infty$. Furthermore, the inclusion properties of some modified Mittag–Leffler-type functions are discussed. The various results, which are established in this paper, are presumably new, and their importance is illustrated by several interesting consequences and examples. Some potential directions for analogous further research on the subject of the present investigation are indicated in the concluding section.

Keywords: Mittag–Leffler-type functions; univalent functions; analytic functions; starlike functions; convex functions; close-to-convex functions; Fox-Wright function; Bessel-Wright function; general Wright function; Srivastava Mittag–Leffler-type functions

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1. Introduction and Motivation

Geometric Function Theory is one of the important branches of complex analysis. It deals with the geometric properties of analytic functions. The main foundation of Geometric Function Theory is the theory of univalent functions, but a number of new associated areas have emerged and led to various strong results and applications. Geometric Function Theory has applications in several fields of pure and applied mathematics such as mathematical physics, mathematical biology, fluid mechanics, fractional calculus, and mathematical chemistry. Recently, several researchers have constructed some new classes of functions involving fractional q-calculus operators, which are analytic in the unit disk and have established several interesting results with applications. It is remarkable to note that researchers in the field are interested nowadays in obtaining new theoretical methodologies and techniques with observational results together with their several applications. Let us now recall some known definitions and results in Geometric Function Theory.

Suppose that the class of analytic functions in open unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

is denoted by $\mathcal{H}$. Assume that $\mathcal{A}$ denotes the collection of all analytic functions $g(z)$ in $\mathcal{H}$, satisfying the normalization $g(0) = g'(0) - 1 = 0$ such that

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (z \in \mathbb{D}).$$

We denote the class of all univalent functions in $\mathcal{A}$ by $\mathcal{S}$. A function $g \in \mathcal{A}$ is known as a starlike function (with respect to the origin) in $\mathbb{D}$ if g is univalent in $\mathbb{D}$ and the domain $g(\mathbb{D})$ is starlike with respect to the origin in $\mathbb{C}$. Let us denote the class of starlike functions in $\mathbb{D}$ by $\mathcal{S}^*$. Then, the analytical description of $\mathcal{S}^*$ can be stated as follows (see, for details, [1]):

$$g \in \mathcal{S}^* \iff \Re\left(\frac{zg'(z)}{g(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{D}).$$

Moreover, we recall the class of starlike functions of order $\alpha(0 \leqq \alpha < 1)$, denoted by $\mathcal{S}^*(\alpha)$, which is defined as follows:

$$\mathcal{S}^*(\alpha) = \left\{g \in \mathcal{A} : \Re\left(\frac{zg'(z)}{g(z)}\right) > \alpha \qquad \forall\, z \in \mathbb{D};\ 0 \leqq \alpha < 1\right\}.$$

An analytic function $g(z)$ in $\mathcal{A}$ is said to be convex in $\mathbb{D}$, if $g(z)$ is a univalent function in $\mathbb{D}$ with $g(\mathbb{D})$ as a convex domain in $\mathbb{C}$. We denote this class of convex functions by $\mathcal{K}$, which can also be described as follows:

$$g \in \mathcal{K} \iff \Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0 \qquad (\forall\, z \in \mathbb{D}).$$

Moreover, if g satisfies the following condition:

$$\Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > \alpha \qquad (\forall\, z \in \mathbb{D};\ 0 \leqq \alpha < 1),$$

then g is known as a convex function of order α in $\mathbb{D}$. This class of functions is denoted by $\mathcal{K}(\alpha)$.

An analytic function g in $\mathcal{A}$ is called close-to-convex in the open unit disk $\mathbb{D}$ if there exists a function $h(z)$, which is starlike in $\mathbb{D}$ such that

$$\Re\left(\frac{zg'(z)}{h(z)}\right) > 0 \qquad (z \in \mathbb{D}).$$

It can be noted that every close-to-convex function in $\mathbb{D}$ is also univalent in $\mathbb{D}$.

A function g in class $\mathcal{A}$ is called uniformly convex (or uniformly starlike) in $\mathbb{D}$, if, for every circular arc ς contained in $\mathbb{D}$ with center $\eta \in \mathbb{D}$, the image arc $g(\varsigma)$ is convex $\big($or starlike with respect to $g(\eta)\big)$. This class of functions is denoted by UCV (or UST) (see, for details, [2]). It was introduced by Goodman (see [3,4]). On the other hand, Rønning [2] considered a newly-defined class of starlike functions $\mathcal{S}_p$ as follows:

$$\mathcal{S}_p := \{g : g(z) = zG'(z) \quad (G \in \mathrm{UCV})\}.$$

Assume that $g(z)$ and $h(z)$ are analytic in $\mathbb{D}$. Then, $g(z)$ is subordinate to $h(z)$ in $\mathbb{D}$, denoted by $f(z) \prec g(z)$ or $f \prec g$ $(z \in \mathbb{D})$, if there exists a Schwarz function $w(z)$, which is analytic in $\mathbb{D}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for any $z \in \mathbb{D}$, such that

$$g(z) = h\big(w(z)\big) \qquad (z \in \mathbb{D}).$$

It can be verified that, if $g(z) \prec h(z)$ $(z \in \mathbb{D})$, then $g(0) = h(0)$ and $g(\mathbb{D}) \subset h(\mathbb{D})$. Moreover, if $h(z)$ is univalent in $\mathbb{D}$, then $g(z) \prec h(z)$ if and only if $g(0) = h(0)$ and $g(\mathbb{D}) \subset h(\mathbb{D})$. For more information on the various geometric properties involving subordination between analytic functions, we refer the reader to the earlier works [1,5–11] and also to the references cited therein.

The celebrated and widely used Mittag–Leffler function $E_\alpha(z)$ and its two-parameter version $E_{\alpha,\beta}(z)$ are defined by (see [12–14])

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \quad \text{and} \quad E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{1}$$

$$\big(z,\alpha,\beta \in \mathbb{C};\ \Re(\alpha)>0\big),$$

respectively.

The above-defined Mittag–Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, which obviously provide extensions of the exponential, hyperbolic, and trigonometric functions, are contained (as a very specialized case) in the Fox-Wright function ${}_p\Psi_q \ (p,q \in \mathbb{N}_0)$, which was investigated by Fox [15] in the year 1928 and, subsequently, by Wright (see [16,17]; see, for details, an article on the legacy of Charles Fox by Srivastava [18]). By definition, we have

$$\begin{aligned} {}_p\Psi_q\left[\begin{array}{c} (a_1,A_1),\cdots,(a_p,A_p); \\ \\ (b_1,B_1),\cdots,(b_q,B_q); \end{array} z\right] &= {}_p\Psi_q\left[\begin{array}{c} (a_j,A_j)_{j=1,\cdots,p}; \\ \\ (b_j,B_j)_{j=1,\cdots,q}; \end{array} z\right] \\ &:= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j+A_j\,n)}{\prod_{j=1}^{q}\Gamma(b_j+B_j\,n)}\,\frac{z^n}{n!} \end{aligned} \tag{2}$$

$$\big(a_j, b_k \in \mathbb{C} \ \text{ and } \ A_j, B_k \in \mathbb{R}^+ \ (j=1,\cdots,p;\ k=1,\cdots,q)\big).$$

The series in (2) converges uniformly and absolutely for any bounded $|z| \ \ (z \in \mathbb{C})$ when

$$\Delta = 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0.$$

Recently, Al-Bassam and Luchko [19] introduced a multi-index (or vector-index) Mittag–Leffler function involving $2m$ parameters, which we recall below:

$$\begin{aligned} E^{(m)}_{(\alpha,\beta)}(z) \equiv E_{(\alpha_1,\beta_1),\cdots,(\alpha_m,\beta_m)}(z) &:= \sum_{n=0}^{\infty} \frac{z^n}{\prod_{k=1}^{m}\Gamma(\alpha_k\,n+\beta_k)} \\ &= {}_1\Psi_m\left[\begin{array}{c} (1,1); \\ \\ (\beta_1,\alpha_1),\cdots,(\beta_m,\alpha_m); \end{array} z\right], \end{aligned} \tag{3}$$

$$\big(\alpha_k, \beta_k \in \mathbb{R}^+ \ \ (k=1,\cdots,m);\ m \in \mathbb{N};\ z \in \mathbb{C}\big),$$

which they applied to solve a Cauchy-type problem for a fractional differential equation and obtained explicit solution in terms of $E^{(m)}_{(\alpha,\beta)}(z)$. It is easily seen that $E^{(m)}_{(\alpha,\beta)}(z)$ is a generalization of both the classical Mittag–Leffler function $E_\alpha(z)$ and the two-parametric Mittag–Leffler function $E_{\alpha,\beta}(z)$

$$E_\alpha(z) = E^{(1)}_{\alpha,1}(z) \quad \text{and} \quad E_{\alpha,\beta}(z) = E^{(1)}_{\alpha,\beta}(z),$$

as well as the Bessel-Wright function $J_\nu^\mu(z)$ given by (see [20])

$$J_\nu^\mu(z) = E^{(2)}_{(1,1),(\mu,\nu+1)}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(\mu n+\nu+1)}$$
$$= {}_0\Psi_1\left[\begin{array}{c} \overline{\qquad\qquad}\,; \\ (\nu+1,\mu); \end{array}\; -z\right]. \tag{4}$$

In the year 2009, the following inequality was established by Pogány and Srivastava ([21], p. 133, Theorem 4)

$$\psi_0\,\exp\left(\frac{\psi_1}{\psi_0}\,|z|\right) \leqq {}_p\Psi_q\left[\begin{array}{c} (a_j,A_j)_{j=1,\cdots,p}; \\ (b_j,B_j)_{j=1,\cdots,q}; \end{array}\; z\right] \leqq \psi_0 - \left(1-e^{|z|}\right)\psi_1 \tag{5}$$

for all suitably restricted $z, a_j, A_j, b_\ell, B_\ell \in \mathbb{R}$ $(j=1,\cdots,p;\ \ell=1,\cdots,q)$ and for all ${}_p\Psi_q[z]$ satisfying the following inequalities:

$$\psi_1 > \psi_2 \quad \text{and} \quad \psi_1^2 < \psi_0\,\psi_2,$$

where

$$\psi_k := \frac{\prod_{j=1}^{p}\Gamma(a_j+A_j\,k)}{\prod_{j=1}^{q}\Gamma(b_j+B_j\,k)} \qquad (k=0,1,2).$$

The general Wright function $\mathfrak{E}_{\alpha,\beta}(\phi;z)$ emerged from a systematic study of the asymptotic expansion of the following Taylor-Maclaurin series (see [22], p. 424):

$$\mathfrak{E}_{\alpha,\beta}(\phi;z) := \sum_{n=0}^{\infty} \frac{\phi(n)}{\Gamma(\alpha n+\beta)}\,z^n \qquad (\Re(\alpha)>0;\ \alpha,\beta\in\mathbb{C}), \tag{6}$$

where $\phi(t)$ is a function of t satisfying appropriate conditions. For a reasonably detailed historical background and other details about the following interesting unification of the definition (6) and some multi-parameter extensions of several functions occurring in Analytic Number Theory, the reader is referred, for example, to the recent works [23–25]:

$$\mathcal{E}_{\alpha,\beta}(\varphi;z,s,\kappa) := \sum_{n=0}^{\infty} \frac{\varphi(n)}{(\kappa+n)^s\,\Gamma(\alpha n+\beta)}\,z^n \qquad (\Re(\alpha)>0;\ \alpha,\beta\in\mathbb{C}), \tag{7}$$

where for a suitably restricted function $\varphi(\tau)$ of the argument τ, the parameters α, β, s, and κ satisfy suitable conditions.

We remark in passing that Prabhakar [26] considered a singular integral equation involving a three-parameter Mittag–Leffler-type function in its kernel, which happens to be a special case of the general Wright function $\mathfrak{E}_{\alpha,\beta}(\phi;z)$ in (6) when

$$\phi(n) = \frac{\Gamma(\gamma+n)}{n!\,\Gamma(\gamma)} \qquad (n\in\mathbb{N}_0;\ \gamma\in\mathbb{C}),$$

so that

$$\mathfrak{E}_{\alpha,\beta}(\phi;z) = \frac{1}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} = \frac{1}{\Gamma(\gamma)}\, {}_1\Psi_1\left[\begin{array}{c}(\gamma,1);\\ \\ (\beta,\alpha);\end{array} z\right] \qquad (\Re(\alpha)>0;\ \beta,\gamma\in\mathbb{C}) \tag{8}$$

in terms of the Fox-Wright function ${}_p\Psi_q$ $(p,q\in\mathbb{N}_0)$ defined by (2). For a potentially useful further investigation of this three-parameter Mittag–Leffler-type function, the reader is referred to a recent article by Garra and Garrappa [27]. Other Mittag–Leffler-type functions of the above class were used by Gorenflo et al. [28] and Srivastava et al. [29,30] as (for example) the kernel of some fractional integral operators.

If, in the general Wright function $\mathfrak{E}_{\alpha,\beta}(\phi;z)$ defined by (6), we set

$$\phi(n) = \frac{1}{[\Gamma(\alpha n+\beta)]^{\gamma-1}} \qquad (n\in\mathbb{N}_0)$$

or, alternatively, if we set

$$\varphi(n) = \frac{(n+\kappa)^s}{[\Gamma(\alpha n+\beta)]^{\gamma-1}} \qquad (n\in\mathbb{N}_0)$$

in the definition (7), we immediately lead to the following Mittag–Leffler-type function considered by Gerhold [31] and, subsequently, by Garra and Polito [32]:

$$E_{\alpha,\beta}^{(\gamma)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[\Gamma(\alpha n+\beta)]^{\gamma}} \qquad (\alpha,\beta,\gamma>0;\ z\in\mathbb{C}). \tag{9}$$

In particular, when $\alpha=\beta=1$, $E_{\alpha,\beta}^{(\gamma)}(z)$ leads to the following function studied by Le Roy [33] more than one century ago:

$$R_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^\gamma} \qquad (\gamma>0;\ z\in\mathbb{C}).$$

Some other special cases of the function $E_{\alpha,\beta}^{(\gamma)}(z)$ are given below:

$$\begin{aligned}
&E_{\alpha,\beta}^{(1)}(z) = E_{\alpha,\beta}(z), \quad E_{1,1}^{(\gamma)}(z) = R_\gamma(z),\\
&E_{2,2}^{(1)}(z) = \frac{\sinh\sqrt{z}}{z}, \quad E_{1,1}^{(1)}(z) = \exp(z),\\
&E_{1,2}^{(1)}(z) = \frac{\exp(z)-1}{z}, \quad E_{2,1}^{(1)}(z) = \cosh\sqrt{z},\\
&E_{1,1}^{(2)}(z) = J_0(2\sqrt{z}), \quad E_{\alpha,\beta}^{(n)}(z) = E_{(\alpha,\beta),\cdots,(\alpha,\beta)}^{(n)}(z) \ \ (n\in\mathbb{N}),\\
&E_{1,1}^{(\nu)}(\lambda) = Z(\lambda,\nu), \quad E_{1,1}^{(\alpha+1)}(z) = \mathfrak{e}_\alpha(z),
\end{aligned}$$

where $J_\nu(z) = J_\nu^1(z)$ is known as the Bessel function of the first kind of order ν, $Z(\lambda,\nu)$ is the COM-Poisson renormalization constant (see [34]), and $\mathfrak{e}_\alpha(z)$ denotes the αL-exponential function (see [32]).

The rest of the paper is organized as follows. We recollect some known results in Section 2, which will be helpful to establish the main theorems of this investigation. In Section 3, we derive some sufficient conditions so that the normalized Mittag–Leffler-type function $\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)$ satisfies such geometric properties as starlikeness, convexity, close-to-convexity, and uniform convexity inside the unit disk. In Section 4, some sufficient conditions are established so that these functions belong to the Hardy spaces $\mathcal{H}^p$ and $\mathcal{H}^\infty$. Moreover, the inclusion properties of some modified Mittag–Leffler-type functions are discussed in Section 5. As an application of our results, the inclusion properties of the Mittag–Leffler-type functions are also studied in this section. Pre-starlikeness and k-uniform convexity are studied in Section 6. Finally, in Section 7, we present several potential directions for analogous further research on the subject of the present investigation.

2. A Set of Useful Lemmas

In this section, we recall the following lemmas, each of which will be helpful to derive the main results.

Lemma 1 (see [5]). *If $g \in \mathcal{A}$ satisfies the condition*:

$$\left|\left(\frac{g(z)}{z}\right) - 1\right| < 1 \qquad (\forall\, z \in \mathbb{D}),$$

then $g \in \mathcal{S}^*$ in

$$\mathbb{D}_{\frac{1}{2}} = \left\{ z : z \in \mathbb{C} \quad \text{and} \quad |z| < \frac{1}{2} \right\}.$$

Lemma 2 (see [6]). *If $g \in \mathcal{A}$ and $|g'(z) - 1| < 1 \qquad (\forall\, z \in \mathbb{D})$, then $f \in \mathcal{K}$ in $\mathbb{D}_{\frac{1}{2}}$.*

Lemma 3 (see [35]). *Suppose that $g \in \mathcal{A}$ and the following inequality holds true*:

$$|g'(z) - 1| < \frac{2}{\sqrt{5}} \qquad (\forall\, z \in \mathbb{D}).$$

Then, the function $g(z)$ belongs to the class $\mathcal{S}^*$ in $\mathbb{D}$.

Lemma 4 (see [36]). *Assume that $g \in \mathcal{A}$.*

(i) *If* $\left|\dfrac{zg'(z)}{g(z)} - 1\right| < \dfrac{1}{2}$, *then* $g \in \mathcal{S}_p$.

(ii) *If* $\left|\dfrac{zg''(z)}{g'(z)}\right| < \dfrac{1}{2}$, *then* $g \in \mathrm{UCV}$.

Lemma 5 (see [37]). *Assume that $g(z)$ can be expressed as follows*:

$$g(z) = z + \sum_{k=2}^{\infty} B_k z^k.$$

If

$$1 \geqq 2B_2 \geqq \cdots \geqq nB_n \geqq (n+1)B_{n+1} \geqq \cdots \geqq 0$$

or if

$$1 \leqq 2B_2 \leqq \cdots \leqq nB_n \leqq (n+1)B_{n+1} \leqq \cdots \leqq 2,$$

then, $g(z)$ is close-to-convex with respect to $-\log(1-z)$.

Lemma 6 (see [38]). *Suppose that*

$$g(z) = z + \sum_{k=2}^{\infty} B_{2k-1} z^{2k-1}$$

is analytic in $\mathbb{D}$. If

$$1 \leqq 3B_3 \leqq \cdots \leqq (2k-1)B_{2k-1} \leqq \cdots \leqq 2$$

or if

$$1 \geqq 3B_3 \geqq \cdots \geqq (2k-1)B_{2k-1} \geqq \cdots \geqq 0,$$

then, $g(z)$ is univalent in $\mathbb{D}$.

Lemma 7 below can be proved fairly easily, so we omit the proof.

Lemma 7. *If $c, d \geqq 1$, then*

$$\frac{k}{(c)_k} \leqq \frac{1}{c(c+1)^{k-2}} \qquad (k \in \mathbb{N} \setminus \{1\}), \tag{10}$$

$$\frac{1}{(d)_k} \leqq \frac{1}{d(d+1)^{k-1}} \qquad (k \in \mathbb{N}), \tag{11}$$

and

$$\frac{\Gamma(d)}{\Gamma(ck+d)} \leqq \frac{1}{(d)_k} \qquad (k \in \mathbb{N}), \tag{12}$$

where

$$\begin{aligned}(\lambda)_k &= \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} \\ &= \begin{cases} 1 & (k=0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k \in \mathbb{N};\ \lambda \in \mathbb{C}) \end{cases}\end{aligned}$$

denotes the Pochhammer symbol [39].

Lemma 8 (see [40])**.** *Suppose that $p = p_1 + ip_2$ and $q = q_1 + iq_2$ with $p_1, p_2, q_1, q_2 \in \mathbb{R}$ and $\Lambda \in \mathbb{C}^2$. Suppose that $\Theta : \Lambda \to \mathbb{C}$ admits the following assertions*:

(i) $\Theta(p, q)$ *is continuous in* Λ.
(ii) $\Re\{\Theta(1, 0)\}$ *is positive and* $(1, 0) \in \Lambda$.
(iii) $\Re\{\Phi(ip_2, q_1)\} \leqq 0$ *for all* $(ip_2, q_1) \in \Lambda$ *such that*

$$q_1 \leqq -\frac{(1+p_2)^2}{2}.$$

Consider the analytic function $G(z)$ in $\mathbb{D}$, which satisfies the conditions $G(0) = 1$ and

$$\big(G(z), zG'(z)\big) \in \Lambda \qquad (\forall\, z \in \mathbb{D}).$$

If $\Re\{\Phi\big(G(z), zG'(z)\big)\}$ is positive in $\mathbb{D}$, then $\Re\{G(z)\}$ is also positive in $\mathbb{D}$.

Lemma 9 (see [41])**.** *Suppose that $g(z)$ is convex univalent in $\mathbb{D}$ and also assume that $w(z)$ and $h(z)$ are analytic in $\mathbb{D}$ with $h(0) = g(0)$ and a real part of $w(z)$ is non-negative in $\mathbb{D}$. Then, for any $z \in \mathbb{D}$, the following subordination*:

$$h(z) + zw(z)h'(z) \prec g(z)$$

yields

$$h(z) \prec g(z).$$

3. Starlikeness and Convexity of Normalized Mittag–Leffler-Type Functions

Since $E_{\alpha,\beta}^{(\gamma)}(z) \notin \mathcal{A}$, we consider the following normalization of $E_{\alpha,\beta}^{(\gamma)}(z)$:

$$\begin{aligned}\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z) &= \sum_{k=0}^{\infty}\left(\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha k)}\right)^{\gamma} z^{k+1} \\ &= \sum_{k=1}^{\infty} A_k(\alpha,\beta,\gamma) z^k \qquad (\alpha,\beta,\gamma>0,\ z\in\mathbb{C}),\end{aligned} \tag{13}$$

where

$$A_k(\alpha,\beta,\gamma) = \left(\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+\alpha k)}\right)^{\gamma}.$$

Although Formula (13) holds true for $\alpha,\beta,\gamma>0$ and $z\in\mathbb{C}$, yet in this article, we will restrict our attention to the case involving positive real-valued parameters α, β, and γ, and the argument $z\in\mathbb{D}$.

Theorem 1. *Suppose that one of the following hypotheses holds true:*

(a) $\alpha,\beta>0$, $\gamma \geqq 1$ *and*

$$(\mathrm{H}_1): \begin{cases} (\mathrm{i}) & \Gamma(3\alpha+\beta) > 2\Gamma(2\alpha+\beta); \\ (\mathrm{ii}) & 2[\Gamma(2\alpha+\beta)]^2 > \Gamma(\alpha+\beta)\Gamma(3\alpha+\beta); \\ (\mathrm{iii}) & \frac{(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} < 1. \end{cases}$$

(b) $\alpha,\beta \geqq 1$, $\gamma>0$ *and* $\beta^{\gamma} \leqq (\beta+1)^{\gamma}(\beta^{\gamma}-1)$.

Then, $\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}_{\frac{1}{2}}$.

Proof. (a) For any $z\in\mathbb{D}$ and $\gamma\geqq 1$, we obtain

$$\begin{aligned}\left|\frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z} - 1\right| &< \sum_{k=1}^{\infty}\left(\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha k)}\right)^{\gamma} \\ &= \sum_{k=0}^{\infty}\Gamma(k+1)\left(\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta+\alpha k)}\right)^{\gamma}\frac{1}{k!} \\ &\leqq \left(\Gamma(\beta)\sum_{k=0}^{\infty}\frac{\Gamma(k+1)}{\Gamma(\alpha+\beta+\alpha k)}\frac{1}{k!}\right)^{\gamma} \\ &= \left(\Gamma(\beta)\, {}_1\Psi_1\left[\begin{matrix}(1,1); \\ (\alpha+\beta,\alpha);\end{matrix}\ 1\right]\right)^{\gamma}.\end{aligned} \tag{14}$$

In this case, we have

$$\psi_0 = \frac{1}{\Gamma(\alpha+\beta)},\quad \psi_1 = \frac{1}{\Gamma(2\alpha+\beta)} \quad\text{and}\quad \psi_2 = \frac{2}{\Gamma(3\alpha+\beta)}.$$

It can be noted that the assertions given by (H_1) : (i) and (H_1) : (ii) are equivalent to $\psi_2<\psi_1$ and $\psi_1^2<\psi_0\psi_2$. Therefore, by (5), we find that

$${}_1\Psi_1\left[\begin{matrix}(1,1); \\ (\alpha+\beta,\alpha);\end{matrix}\ 1\right] \leqq \frac{e-1}{\Gamma(2\alpha+\beta)} + \frac{1}{\Gamma(\alpha+\beta)}. \tag{15}$$

Using (14), (15), and (H_1) : (iii), for any $z \in \mathbb{D}$, we have

$$\left| \frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z} - 1 \right| < 1.$$

Furthermore, with the help of Lemma 1, we get the required result.

(b) Using Lemma 7, for any $z \in \mathbb{D}$, it follows that

$$\begin{aligned} \left| \frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z} - 1 \right| &< \sum_{k=1}^{\infty} \left(\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha k)} \right)^{\gamma} \\ &\leqq \sum_{k=1}^{\infty} \left(\frac{1}{(\beta)_k} \right)^{\gamma} \\ &< \frac{1}{\beta^{\gamma}} \sum_{k=0}^{\infty} \left(\frac{1}{(\beta+1)^{\gamma}} \right)^{k} \\ &= \frac{(\beta+1)^{\gamma}}{\beta^{\gamma}[(\beta+1)^{\gamma}-1]} \leqq 1, \end{aligned}$$

under the given hypothesis. Finally, applying Lemma 1, we conclude that $\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z) \in S^*$ in $\mathbb{D}_{\frac{1}{2}}$.

□

Remark 1. *From Part* (a) *of Theorem* 1, *we note that the functions* $\mathbb{E}_{\alpha_2,2}^{(1)}$, $\mathbb{E}_{\alpha_3,3}^{(1)}$, *and* $\mathbb{E}_{\alpha_4,4}^{(1)}$ *belong to* S^* *in* $\mathbb{D}_{\frac{1}{2}}$ *if* $\alpha_2 \in [0.839, 1.897]$, $\alpha_3 \in [0.59, 2.156]$, *and* $\alpha_4 \in [0.474, 2.376]$, *respectively. Similarly, we can observe that for any* $\beta \geqq 2$, $\exists\, \alpha_n \in (0,1]$ *s.t.* $\mathbb{E}_{\alpha_n,\beta}^{(1)} \in \mathcal{S}^*$ *in* $\mathbb{D}_{\frac{1}{2}}$ *for any* $n \geqq 2$. *Again, upon setting* $\gamma = 1$ *in Part* (b) *of Theorem* 1, *we obtain that* $\mathbb{E}_{\alpha,\beta}^{(1)}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}_{\frac{1}{2}}$, *if* $\beta \geqq \frac{1+\sqrt{5}}{2}$ *and* $\alpha \geqq 1$, *which is the same condition as that given in* ([42], *Theorem* 2.4). *However,* ([42], *Theorem* 2.4) *can study the starlikeness of* $\mathbb{E}_{\alpha,\beta}^{(1)}(z)$ *for the case when* $\alpha \geqq 1$. *On the other hand, Part (a) of Theorem* 1 *can also discuss the case for* $\alpha \in (0,1)$. *Hence, Theorem* 1 *improves the corresponding results given in* ([42], Theorem 2.4). Other appropriate normalizations of the Mittag–Leffler-type functions and their applications in Geometric Function Theory of Complex Analysis can be found in (for example) [43].

Example 1. *The following functions belong to the class* $\mathcal{S}^*$ *in* $\mathbb{D}_{\frac{1}{2}}$*:*

$$\mathbb{E}_{2,2}^{(1)}(z) = \frac{\sinh(z)}{z},$$

$$\mathbb{E}_{1,2}^{(1)}(z) = e^z - 1,$$

$$\mathbb{E}_{1,3}^{(1)}(z) = \frac{2(e^z - z - 1)}{z},$$

$$\mathbb{E}_{1,4}^{(1)}(z) = \frac{6(e^z - 1 - z) - 3z^2}{z^2}$$

and

$$\mathbb{E}^{(1)}_{1,\frac{5}{2}}(z) = \frac{3}{4}\left(\frac{\sqrt{\pi}e^{z}\operatorname{erf}(\sqrt{z}) - 2\sqrt{z}}{\sqrt{z}}\right),$$

where $\operatorname{erf}(z)$ *denotes the error function, which is known also as the probability integral* $\Phi(z)$*, which is defined by (see* [39]*)*

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt = \Phi(z).$$

Theorem 2. *Suppose that one of the following assertions holds true*:

(a) $\alpha, \beta > 0$, $\gamma \geqq 1$ *and*

$$(\mathrm{H}_2) : \begin{cases} \text{(i)} & 3\Gamma(3\alpha+\beta) > 8\Gamma(2\alpha+\beta); \\ \text{(ii)} & 16[\Gamma(2\alpha+\beta)]^2 > 9\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta); \\ \text{(iii)} & \frac{3(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} + \frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} < 1. \end{cases}$$

(b) $\alpha, \beta \geqq 1$, $\gamma > 0$ *and*

$$\frac{(\beta+1)^{\gamma}[2(\beta+1)^{\gamma}-1]}{\beta^{\gamma}((\beta+1)^{\gamma}-1)^2} < 1.$$

Then, $\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z) \in \mathcal{K}$ *in* $\mathbb{D}_{\frac{1}{2}}$.

Proof. (a) For any $z \in \mathbb{D}$, under the given condition (b), we get

$$\begin{aligned} \left|\left(\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)\right)' - 1\right| &= \left|\sum_{k=0}^{\infty}(k+2)\left(\frac{\Gamma(\beta)}{\Gamma(\alpha k+\alpha+\beta)}\right)^{\gamma} z^{k+1}\right| \\ &< \left(\sum_{k=0}^{\infty}\frac{\Gamma(\beta)\Gamma(k+1)\Gamma(k+3)}{\Gamma(k+2)\Gamma(\alpha k+\alpha+\beta)}\frac{1}{k!}\right)^{\gamma} \\ &= \left(\Gamma(\beta)\,{}_2\Psi_2\left[\begin{matrix}(1,1),(3,1); \\ (2,1),(\alpha+\beta,\alpha);\end{matrix}\;1\right]\right)^{\gamma}. \end{aligned} \tag{16}$$

Moreover, it can be seen that the asertions (H_2) : (i) and (H_2) : (ii) are equivalent to $\psi_2 < \psi_1$ and $\psi_1^2 < \psi_0\psi_2$, where

$$\psi_0 = \frac{2}{\Gamma(\alpha+\beta)}, \quad \psi_1 = \frac{3}{\Gamma(2\alpha+\beta)} \quad \text{and} \quad \psi_2 = \frac{8}{\Gamma(3\alpha+\beta)}.$$

Now, by using (5), we get

$${}_2\Psi_2\left[\begin{matrix}(1,1),(3,1); \\ (2,1),(\alpha+\beta,\alpha);\end{matrix}\;1\right] < \frac{2}{\Gamma(\alpha+\beta)} + \frac{3(e-1)}{\Gamma(2\alpha+\beta)}. \tag{17}$$

Using (16), (17), and (H_2) : (iii), we obtain

$$\left|[\mathbb{E}^{(\gamma)}_{\alpha,\beta}]'(z) - 1\right| < 1 \qquad (z \in \mathbb{D}).$$

Again, by using Lemma 2, the desired result can be established.

(b) By using Lemma 7, for any $z \in \mathbb{D}$, we get

$$\begin{aligned}
\left|\left[\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)\right]' - 1\right| &< \sum_{k=1}^{\infty}(k+1)\left(\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha k)}\right)^{\gamma} \\
&\leqq \sum_{k=1}^{\infty}\frac{k+1}{[(\beta)_k]^{\gamma}} \\
&< \frac{1}{\beta^{\gamma}}\sum_{k=0}^{\infty}\frac{k}{[(\beta+1)^{\gamma}]^{k}} + \frac{2}{\beta^{\gamma}}\sum_{k=0}^{\infty}\left(\frac{1}{(\beta+1)^{\gamma}}\right)^{k} \\
&= \frac{(\beta+1)^{\gamma}}{\beta^{\gamma}\,[(\beta+1)^{\gamma}-1]^2} + \frac{2(\beta+1)^{\gamma}}{\beta^{\gamma}\,[(\beta+1)^{\gamma}-1]} \\
&= \frac{(\beta+1)^{\gamma}(2(\beta+1)^{\gamma}-1)}{\beta^{\gamma}\,[(\beta+1)^{\gamma}-1]^2} < 1,
\end{aligned}$$

under the given hypothesis (b). Finally, Lemma 2 helps us to establish the desired result.

The proof of Theorem 2 is thus completed. □

Remark 2. *It can be shown from Part* (a) *of Theorem 2 that* $\mathbb{E}^{(1)}_{\alpha_4,4}$, $\mathbb{E}^{(1)}_{\alpha_5,5}$, *and* $\mathbb{E}^{(1)}_{\alpha_6,6}$ *belong to* $\mathcal{K}$ *in* $\mathbb{D}_{\frac{1}{2}}$ *if* $\alpha_4 \in [0.88, 2.09]$, $\alpha_5 \in [0.77, 2.26]$, *and* $\alpha_6 \in [0.699, 2.43]$, *respectively. Similarly, it can be easily proved that for each* $\beta \geqq 4$, *there exist* $\alpha_n \in (0,1]$ *such that* $\mathbb{E}^{(1)}_{\alpha_n,\beta}$ *belongs to* $\mathcal{K}$ *in* $\mathbb{D}_{\frac{1}{2}}$ *for any* $n \geqq 4$. *Moreover, by putting* $\gamma = 1$ *in Part* (b) *of Theorem 2, we see that* $\mathbb{E}^{(1)}_{\alpha,\beta}(z)$ *belongs to* $\mathcal{K}$ *in* $\mathbb{D}_{\frac{1}{2}}$, *if* $\beta \geqq 3.0796$ *and* $\alpha \geqq 1$. *In ([42], Theorem 2.4), it was derived that the function* $\mathbb{E}^{(1)}_{\alpha,\beta}(z)$ *belongs to* $\mathcal{K}$ *in* $\mathbb{D}_{\frac{1}{2}}$, *if*

$$\beta \geqq \frac{3+\sqrt{17}}{2} \approx 3.561552813 \text{ and } \alpha \geqq 1.$$

Thus, clearly, Theorem 2 improves the corresponding result available in the literature ([42], Theorem 2.4).

Example 2. *The following functions belong to the class* $\mathcal{K}$ *in* $\mathbb{D}_{\frac{1}{2}}$:

$$\mathbb{E}^{(1)}_{1,4}(z) = \frac{6(e^z - 1 - z) - 3z^2}{z^2},$$

$$\mathbb{E}^{(1)}_{1,\frac{7}{2}}(z) = \frac{5}{8}\left(\frac{3\sqrt{\pi}e^z\,\mathrm{erf}(\sqrt{z})}{z^{\frac{3}{2}}} - \frac{6}{z} - 4\right),$$

$$\mathbb{E}^{(1)}_{1,5}(z) = \frac{-4(z^3 + 3z^2 + 6z - 6e^z + 6)}{z^3}$$

and

$$\mathbb{E}^{(1)}_{1,\frac{9}{2}}(z) = \frac{105\sqrt{\pi}\,e^z\ \mathrm{erf}(\sqrt{z})}{16z^{\frac{5}{2}}} - \frac{7(4z^2 + 10z + 15)}{8z^2}.$$

Theorem 3. *Consider that one of the following assertions is valid:*

(a) *The parameters* $\alpha, \beta > 0$, *and* $\gamma \geqq 1$ *satisfy the following inequalities*:

$$(\mathrm{H}_3): \begin{cases} \text{(i)} & \Gamma(3\alpha+\beta) > 3\Gamma(2\alpha+\beta); \\ \text{(ii)} & 3[\Gamma(2\alpha+\beta)]^2 > 2\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta); \\ \text{(iii)} & \left(\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)^\gamma + \left(\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{2(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)^\gamma < 1. \end{cases}$$

(b) *The parameters* $\alpha, \beta \geqq 1$, *and* $\gamma > 0$ *satisfy the conditions given by*

$$\frac{(\beta+1)^{2\gamma}}{[(\beta+1)^\gamma - 1]\left[\beta^\gamma\left((\beta+1)^\gamma - 1\right) - (\beta+1)^\gamma\right]} < 1.$$

Then, $\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}$.

Proof. Suppose that

$$\mathcal{E}_{\alpha,\beta}^{(\gamma)}(z) = \frac{z\left[\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)\right]'}{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)} \qquad (z \in \mathbb{D}).$$

Then, clearly, $\mathcal{E}_{\alpha,\beta}^{(\gamma)}(z)$ satisfies the condition $\mathcal{E}_{\alpha,\beta}^{(\gamma)}(0) = 1$, and it is analytic in $\mathbb{D}$.

In order to establish the required result, it suffices to show that

$$\Re\left(\mathcal{E}_{\alpha,\beta}^{(\gamma)}(z)\right) > 0 \qquad (\forall\, z \in \mathbb{D}).$$

For this objective in view, it suffices to establish that

$$\left|\mathcal{E}_{\alpha,\beta}^{(\gamma)}(z) - 1\right| = \left|\frac{z[\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)]'}{[\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)]} - 1\right| = \frac{\left|\left[\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)\right]' - \frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right|}{\left|\frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right|} < 1 \qquad (\forall\, z \in \mathbb{D}).$$

(a) A simple computation leads us to

$$\begin{aligned}\left[\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)\right]' - \frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z} &= \sum_{k=1}^{\infty} k\left(\frac{\Gamma(\beta)}{\Gamma(\alpha k+\beta)}\right)^\gamma z^k \\ &= \sum_{k=0}^{\infty} \Gamma(k+2)\left(\frac{\Gamma(\beta)}{\Gamma(\alpha k+\alpha+\beta)}\right)^\gamma \frac{z^{k+1}}{k!},\end{aligned}$$

which yields

$$\left|[\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)]' - \frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right| < \left(\Gamma(\beta)\,{}_1\Psi_1\left[\begin{array}{cc} (2,1); & \\ & 1 \\ (\alpha+\beta,\alpha); & \end{array}\right]\right)^\gamma \qquad (z \in \mathbb{D}). \tag{18}$$

In our case, we get

$$\psi_0 = \frac{1}{\Gamma(\alpha+\beta)}, \quad \psi_1 = \frac{2}{\Gamma(2\alpha+\beta)} \quad \text{and} \quad \psi_2 = \frac{6}{\Gamma(3\alpha+\beta)}.$$

We see that the assertions $(\mathrm{H}_3):(\mathrm{i})$ and $(\mathrm{H}_3):(\mathrm{ii})$ are equivalent to $\psi_2 < \psi_1$ and $\psi_1^2 < \psi_0\psi_2$. Therefore, by (5), it follows that

$$ {}_1\Psi_1\left[\begin{array}{c}(2,1);\\ \\ (\alpha+\beta,\alpha);\end{array}\ 1\right] \leqq \frac{2(e-1)}{\Gamma(2\alpha+\beta)} + \frac{1}{\Gamma(\alpha+\beta)}. \tag{19} $$

Moreover, with the help of the inequality:

$$ |z_1+z_2| \geqq \big||z_1|-|z_2|\big|, $$

we get

$$ \begin{aligned} \left|\frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right| &\geqq 1 - \left|\sum_{k=1}^{\infty}\left(\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha k)}\right)^{\gamma} z^k\right| \\ &\geqq 1 - \left(\sum_{k=1}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha k)}\right)^{\gamma} = 1 - \left(\Gamma(\beta)\ {}_1\Psi_1\left[\begin{array}{c}(1,1);\\ \\ (\alpha+\beta,\alpha);\end{array}\ 1\right]\right)^{\gamma}. \end{aligned} $$

Furthermore, by applying the inequality (5), we have

$$ {}_1\Psi_1\left[\begin{array}{c}(1,1);\\ \\ (\alpha+\beta,\alpha);\end{array}\ 1\right] \leqq \frac{e-1}{\Gamma(2\alpha+\beta)} + \frac{1}{\Gamma(\alpha+\beta)}, $$

where $\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta) < 2[\Gamma(2\alpha+\beta)]^2$ and $2\Gamma(2\alpha+\beta) < \Gamma(3\alpha+\beta)$. Hence, we find that

$$ \left|\frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right| \geqq 1 - \left(\frac{(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right)^{\gamma} > 0 \qquad (\forall\, z \in \mathbb{D}). \tag{20} $$

Using (18) and (19), together with (20), for any $z \in \mathbb{D}$, we have

$$ \begin{aligned} \left|\frac{z[\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)]'}{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)} - 1\right| &= \frac{\left|\mathbb{E}'_{\alpha,\beta}(z) - \frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right|}{\left|\frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right|} \\ &< \left(\frac{2(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right)^{\gamma} \\ &\quad \cdot \left[1 - \left(\frac{(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)} + \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right)^{\gamma}\right]^{-1} < 1, \end{aligned} $$

where we have made use of the given hypothesis.

(b) Under the given hypothesis, by using Lemma 7, we obtain

$$\begin{aligned}
\left|\left[\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)\right]' - \frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right| &= \left|\sum_{k=1}^{\infty} \frac{k[\Gamma(\beta)]^{\gamma} z^k}{[\Gamma(\alpha k+\beta)]^{\gamma}}\right| \\
&\leqq \sum_{k=0}^{\infty} \frac{k+1}{[\beta(\beta+1)\cdots(\beta+k)]^{\gamma}} \\
&< \frac{1}{\beta^{\gamma}} \sum_{k=0}^{\infty} \frac{k}{[(\beta+1)^{\gamma}]^k} + \frac{1}{\beta^{\gamma}} \sum_{k=0}^{\infty} \left(\frac{1}{(\beta+1)^{\gamma}}\right)^k \\
&= \frac{(\beta+1)^{\gamma}}{\beta^{\gamma}\,[(\beta+1)^{\gamma}-1]^2} + \frac{(\beta+1)^{\gamma}}{\beta^{\gamma}\,[(\beta+1)^{\gamma}-1]} \\
&= \frac{(\beta+1)^{2\gamma}}{\beta^{\gamma}\,[(\beta+1)^{\gamma}-1]^2}
\end{aligned}$$

and

$$\begin{aligned}
\left|\frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}\right| &\geqq 1 - \left|\sum_{k=1}^{\infty} \frac{[\Gamma(\beta)]^{\gamma}\, z^k}{[\Gamma(\alpha k+\beta)]^{\gamma}}\right| \\
&> 1 - \frac{1}{\beta^{\gamma}} \sum_{k=0}^{\infty} \frac{1}{(\beta+1)^{\gamma k}} \\
&= \frac{\beta^{\gamma}((\beta+1)^{\gamma}-1) - (\beta+1)^{\gamma}}{\beta^{\gamma}((\beta+1)^{\gamma}-1)}.
\end{aligned}$$

Using the above inequalities, we get

$$\begin{aligned}
\left|\frac{[\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)]' - \frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}}{\frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z}}\right| &= \frac{\frac{(\beta+1)^{2\gamma}}{\beta^{\gamma}\,[(\beta+1)^{\gamma}-1]^2}}{\frac{\beta^{\gamma}((\beta+1)^{\gamma}-1)-(\beta+1)^{\gamma}}{\beta^{\gamma}((\beta+1)^{\gamma}-1)}} \\
&= \frac{(\beta+1)^{2\gamma}}{[(\beta+1)^{\gamma}-1]\left(\beta^{\gamma}\,[(\beta+1)^{\gamma}-1] - (\beta+1)^{\gamma}\right)} < 1 \qquad (21)
\end{aligned}$$

under the given Condition (b).

□

Remark 3. *It can be verified from Part* (a) *of Theorem 3 that* $\mathbb{E}_{\alpha_4,4}^{(1)}$ *and* $\mathbb{E}_{\alpha_5,5}^{(1)}$ *belong to* $\mathcal{S}^*$ *in* $\mathbb{D}$ *if* $\alpha_4 \in [0.88, 1.65]$ *and* $\alpha_5 \in [0.766, 1.805]$, *respectively. In the same manner, it can be concluded that for each* $\beta \geqq 4$, *there exist* $\alpha_n \in (0,1]$ *such that the function* $\mathbb{E}_{\alpha_n,\beta}^{(1)}$ *belongs to the class* $\mathcal{S}^*$ *in* $\mathbb{D}$ *for all* $n \geqq 4$. *Now, putting* $\gamma = 1$ *in Part* (b) *of Theorem 3, we claim that* $\mathbb{E}_{\alpha,\beta}^{(1)}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}$, *if* $\beta \geqq 3.0796$ *and* $\alpha \geqq 1$. *In* ([42], *Theorem* 2.2), *it is established that* $\mathbb{E}_{\alpha,\beta}^{(1)}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}$ *if*

$$\beta \geqq \frac{3+\sqrt{17}}{2} \approx 3.561552813 \text{ and } \alpha \geqq 1.$$

Again, in ([44], *Theorem 6), it is derived that* $\mathbb{E}_{\alpha,\beta}^{(1)}(z) \in \mathcal{S}^*$ *in* $\mathbb{D}$, *if* $\beta \geqq 3.214319744$ *and* $\alpha \geqq 1$. *Consequently, Theorem 3 improves the corresponding results available in* [42,44].

Example 3. *The following functions belong to the class $\mathcal{S}^*$ in $\mathbb{D}$:*

$$\mathbb{E}^{(1)}_{1,\frac{7}{2}}(z) = \frac{5}{8}\left(\frac{3\sqrt{\pi}e^z\,\mathrm{erf}(\sqrt{z})}{z^{\frac{3}{2}}} - \frac{6}{z} - 4\right)$$

and

$$\mathbb{E}^{(1)}_{1,\frac{9}{2}}(z) = \frac{105\sqrt{\pi}\,e^z\,\mathrm{erf}(\sqrt{z})}{16z^{\frac{5}{2}}} - \frac{7(4z^2+10z+15)}{8z^2}.$$

Similarly, using Lemma 4, the following result can be established.

Corollary 1. *Assume that one of the following hypotheses holds true*:

(a) $\alpha, \beta > 0$ *and* $\gamma \geqq 1$ *such that*

$$(\mathrm{H}_3'): \begin{cases} \text{(i)} & \Gamma(3\alpha+\beta) > 3\Gamma(2\alpha+\beta); \\ \text{(ii)} & 3[\Gamma(2\alpha+\beta)]^2 > 2\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta); \\ \text{(iii)} & \left(\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)^\gamma + 2\left(\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{2(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)^\gamma < 1. \end{cases}$$

(b) $\alpha, \beta \geqq 1$, *and* $\gamma > 0$ *with*

$$\frac{(\beta+1)^{2\gamma}}{\beta^\gamma\,[(\beta+1)^\gamma-1][(\beta+1)^\gamma-1]-(\beta+1)^\gamma} < \frac{1}{2}.$$

Then, $\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z) \in \mathcal{S}_p$.

Each of the following results can be proved in a manner that is analogous to the proofs of the earlier results in this section. Therefore, we omit the details involved.

Theorem 4. *Suppose that one of the following assertions holds true*:

(a) *The parameters* α, β, *and* γ *satisfy the hypothesis (a) of Theorem 2, together with the following hypotheses*:

$$(\mathrm{H}_4): \begin{cases} \text{(i)} & \Gamma(3\alpha+\beta) > 4\Gamma(2\alpha+\beta); \\ \text{(ii)} & 4[\Gamma(2\alpha+\beta)]^2 > 3\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta); \\ \text{(iii)} & \left(\frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{3(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)^\gamma + \left(\frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{6(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)^\gamma < 1. \end{cases}$$

(b) $\alpha, \beta \geqq 1, \gamma > 0$, *and*

$$\frac{(\beta+1)^\gamma[(\beta+1)^\gamma-1]^2}{2[(\beta+1)^\gamma-4]\left[\beta^\gamma[(\beta+1)^\gamma-1]^2-(\beta+1)^\gamma[2(\beta+1)^\gamma-1]\right]} < 1.$$

Then, $\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z) \in \mathcal{K}$ *in* $\mathbb{D}$.

Corollary 2. *The normalized Mittag–Leffler-type function* $\mathbb{E}^{(1)}_{\alpha,\beta} \in \mathcal{K}$ *in* $\mathbb{D}$ *if* $\beta \geqq 4.52416$ *and* $\alpha \geqq 1$.

Remark 4. *It can be observed from Part* (a) *of Theorem 4 that* $\mathbb{E}^{(1)}_{\alpha_9,9} \in \mathcal{K}$ *in* $\mathbb{D}$ *if* $\alpha_9 \in [0.84, 1.87]$. *Similarly, we can verify that for each* $\beta \geqq 9$, *there exist* $\alpha_n \in (0,1]$ *s.t.* $\mathbb{E}^{(1)}_{\alpha_n,\beta} \in \mathcal{K}$ *in* $\mathbb{D}$ *for any* $n \geqq 9$. *In* ([44], *Theorem* 7), *the condition for convexity of* $\mathbb{E}^{(1)}_{\alpha,\beta}$ *in* $\mathbb{D}$ *was given by* $\beta \geqq 3.56155281$

and $\alpha \geqq 1$. However, Theorem 4 studies the case for $\alpha \in (0,1)$ also. As a consequence, Theorem 4 improves the result available in [44].

Example 4. *The function $\mathbb{E}^{(1)}_{1,5}(z) = \frac{-4(z^3+3z^2+6z-6e^z+6)}{z^3}$ is convex in $\mathbb{D}$.*

Proceeding in a similar way and using Lemma 4, we obtain the following result.

Corollary 3. *Assume that one of the following assertions hold*:

(a) α *and* β *satisfy the hypothesis* H_2 *of Theorem 2 as well as the following hypothesis*:

$$(\mathrm{H}_4') : \begin{cases} \text{(i)} & \Gamma(3\alpha+\beta) > 4\Gamma(2\alpha+\beta); \\ \text{(ii)} & 4[\Gamma(2\alpha+\beta)]^2 > 3\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta); \\ \text{(iii)} & \left(\frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{3(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)^{\gamma} + 2\left(\frac{2\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{6(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)^{\gamma} < 1. \end{cases}$$

(b) $\alpha, \beta \geqq 1$, $\gamma > 0$, *and*

$$\frac{(\beta+1)^{\gamma}\,[(\beta+1)^{\gamma}-1]^2}{[(\beta+1)^{\gamma}-4]\,[\beta^{\gamma}[(\beta+1)^{\gamma}-1]^2-(\beta+1)^{\gamma}[2(\beta+1)^{\gamma}-1]]} < 1.$$

Then, $\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z) \in \mathrm{UCV}$.

Remark 5. *Using Corollary 2 and proceeding similarly as in Remark 4, we observe that the functions $\mathbb{E}^{(1)}_{\alpha_8,8}(z)$ and $\mathbb{E}^{(1)}_{\alpha_9,9}(z)$ are in* UCV, *if $\alpha_8 \in [0.885, 1.78]$ and $\alpha_9 \in [0.84, 1.87]$, respectively. In* ([45], *Theorem 2.6) it is derived that the function $\mathbb{E}^{(1)}_{\alpha,\beta}(z) \in \mathrm{UCV}$, if $\beta \geqq 9.111259774$ and $\alpha \geqq 1$. As a consequence, Corollary 3 improves the known result* ([45], *Theorem* 2.6).

Corollary 4. *If $\alpha, \beta \geqq 1$, $\gamma > 0$, $\delta \in [0,1)$, and if the following condition is satisfied*:

$$\frac{(\beta+1)^{\gamma}\,[(\beta+1)^{\gamma}-1]^2}{2[(\beta+1)^{\gamma}-4]\,[\beta^{\gamma}((\beta+1)^{\gamma}-1)^2-(\beta+1)^{\gamma}[2(\beta+1)^{\gamma}-1]]} < 1-\delta.$$

Then, $\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)$ is convex of order δ in $\mathbb{D}$.

Theorem 5. *Suppose that $\alpha, \beta, \gamma \geqq 1$ and the following inequalities are satisfied*:

$$(\mathrm{H}_5) : \begin{cases} \text{(i)} & \Gamma(3+\beta) > 3\Gamma(2+\beta); \\ \text{(ii)} & 3[\Gamma(\beta+2)]^2 > 2\Gamma(\beta+1)\Gamma(\beta+3); \\ \text{(iii)} & \left(\frac{e-1}{\beta(\beta+1)} + \frac{1}{\beta}\right)^{\gamma} + \left(\frac{2(e-1)}{\beta(\beta+1)} + \frac{1}{\beta}\right)^{\gamma} < 1. \end{cases}$$

Then, the normalized Mittag–Leffler-type function $\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)$ is close-to-convex with respect to the starlike function $\mathbb{E}^{(\gamma)}_{1,\beta}(z)$ in $\mathbb{D}$.

4. Hardy Space of the Mittag–Leffler-Type Functions

Let $\mathcal{H}^{\infty}$ denote the space of all bounded functions in $\mathbb{D}$. We also assume that $h \in \mathcal{H}$ and set

$$\mathcal{M}_p(r,h) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |h(re^{(i\theta)}|^p \,\mathrm{d}\theta\right)^{\frac{1}{p}} & (0 < p < \infty) \\ \max\{|h(z)| : |z| \leqq r\} & (p = \infty). \end{cases}$$

It is known from [46] that $h \in \mathcal{H}^p$ if $\mathcal{M}_p(r, h)$ is bounded for $r \in [0, 1)$ and

$$\mathcal{H}^\infty \subset \mathcal{H}^q \subset \mathcal{H}^p \qquad (0 < q < p < 1).$$

Let us consider the following known result [47] for the Hardy space $\mathcal{H}^p$:

$$\begin{aligned} \Re\{h'(z)\} > 0 &\implies h' \in \mathcal{H}^p \quad (\forall\, p < 1) \\ &\implies h \in \mathcal{H}^{\frac{q}{1-q}} \quad (\forall\, 0 < q < 1). \end{aligned} \tag{22}$$

Suppose that the following power series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < R_1)$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \qquad (|z| < R_2)$$

have R_1 and R_2 as their radii of convergence, respectively. Then, their Hadamard product is given by (see, for example, [48])

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \qquad (|z| < R_1 \cdot R_2).$$

The following lemmas will be useful in proving the main results in this section.

Lemma 10 (see [49])**.** *If $0 \leqq \delta, \lambda < 1$ and $\mu = 1 - 2(1-\delta)(1-\lambda)$, then*

$$\mathcal{R}_o(\delta) * \mathcal{R}_o(\lambda) \subset \mathcal{R}_o(\mu)$$

or, equivalently,

$$\mathcal{P}_o(\delta) * \mathcal{P}_o(\lambda) \subset \mathcal{P}_o(\mu).$$

Lemma 11 (see [50])**.** *If the function h, convex of order δ $(0 \leqq \delta < 1)$, is not of the following form*:

$$h(z) = \begin{cases} m + d \cdot z(1 - ze^{i\eta})^{2\delta - 1} & \left(\delta \neq \frac{1}{2}\right) \\ \\ m + d \cdot \log(1 - ze^{i\eta}) & \left(\delta = \frac{1}{2}\right) \end{cases} \tag{23}$$

for $d, m \in \mathbb{C}$, and for $\eta \in \mathbb{R}$, then each of the following statements holds true:

(i) *If $\delta \in [0, \frac{1}{2})$, then $\exists\, \sigma = \sigma(h) > 0$ such that $h \in \mathcal{H}^{\sigma + \frac{1}{1-2\delta}}$.*

(ii) *If $\delta \geqq \frac{1}{2}$, then $h \in \mathcal{H}^\infty$.*

(iii) *$\exists\, \rho = \rho(h)$ such that $h' \in \mathcal{H}^{\rho + \frac{1}{2(1-\delta)}}$.*

Our first main result in this section is now given below.

Theorem 6. *If $\alpha \geqq 1, \beta \geqq 1, \gamma > 0, 0 \leqq \delta < 1, z \in \mathbb{D}$ and*

$$\frac{(\beta+1)^\gamma}{\beta^\gamma (1-\delta)[(\beta+1)^\gamma - 1]} < 1,$$

then $\frac{\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)}{z} \in \mathcal{P}(\delta)$.

Proof. We establish the result asserted by Theorem 6 by proving that $|h(z)-1|<1$, where

$$h(z)=\frac{1}{1-\delta}\left(\frac{\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)}{z}-\delta\right).$$

Indeed, by using Lemma 7, we get

$$\begin{aligned}\left|\frac{1}{1-\delta}\left(\frac{\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)}{z}-\delta\right)-1\right| &= \frac{1}{1-\delta}\left|\sum_{n=1}^{\infty}\frac{[\Gamma(\beta)]^{\gamma}z^{n}}{[\Gamma(\alpha n+\beta)]^{\gamma}}\right| \\ &\leqq \frac{1}{\beta^{\gamma}(1-\delta)}\sum_{n=0}^{\infty}\frac{1}{(\beta+1)^{\gamma}\cdots(\beta+n)^{\gamma}} \\ &= \frac{1}{\beta^{\gamma}(1-\delta)}\sum_{n=0}^{\infty}\frac{1}{(\beta+1)^{n\gamma}}=\frac{1}{\beta^{\gamma}(1-\delta)}\frac{1}{1-\frac{1}{(\beta+1)^{\gamma}}}<1,\end{aligned}$$

under the hypothesis of Theorem 6. □

Theorem 7. *Assume that* $\alpha\geqq 1$, $\beta\geqq 1$, $\gamma>0$, $0\leqq\delta<1$, *and*

$$\frac{(\beta+1)^{\gamma}\,[(\beta+1)^{\gamma}-1]^{2}}{2[(\beta+1)^{\gamma}-4]\left(\beta^{\gamma}\,[(\beta+1)^{\gamma}-1]^{2}-(\beta+1)^{\gamma}\,[2(\beta+1)^{\gamma}-1]\right)}<1-\delta.$$

Then,

$$\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)\in\begin{cases}\mathcal{H}^{\frac{1}{1-2\delta}} & \left(0\leqq\delta<\frac{1}{2}\right)\\ \\ \mathcal{H}^{\infty} & \left(\delta\geqq\frac{1}{2}\right).\end{cases} \tag{24}$$

Proof. From the definition of the hypergeometric function ${}_2F_1(a,b;c;z)$, we have

$$m+\frac{d\cdot z}{(1-ze^{i\eta})^{1-2\delta}}=m+d\cdot z{}_2F_1\left(1,1-2\delta;1;ze^{i\eta}\right)\qquad\left(\delta\neq\frac{1}{2}\right)$$

and

$$m+d\cdot\log(1-ze^{i\eta})=k+d\cdot z{}_2F_1\left(1,1;2;ze^{i\eta}\right)\qquad\left(\delta=\frac{1}{2}\right).$$

Hence, clearly, the normalized Mittag–Leffler-type function $\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)$ is not of the following forms:

$$m+d\cdot z(1-ze^{i\eta})^{2\delta-1}\qquad\left(\delta\neq\frac{1}{2}\right)$$

and

$$m+d\cdot\log(1-ze^{i\eta})\qquad\left(\delta=\frac{1}{2}\right).$$

Thus, by applying Corollary 4, we observe that $\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)$ is convex of order δ in $\mathbb{D}$. Finally, if we apply Lemma 11, the desired result would follow readily. □

Theorem 8. *Consider that* $\alpha\geqq 1$, $\beta\geqq 1$, $\gamma>0$, *and*

$$2(\beta+1)^{\gamma}-\beta^{\gamma}[(\beta+1)^{\gamma}-1]<0.$$

If $g \in \mathcal{R}$, then, the convolution $\mathbb{E}_{\alpha,\beta}^{(\gamma)} * g$ is in $\mathcal{H}^{\infty} \cap \mathcal{R}$.

Proof. If $g \in \mathcal{R}$, then $g' \in \mathcal{P}$. Upon setting

$$w(z) = \mathbb{E}_{\alpha,\beta}^{(\gamma)}(z) * g(z),$$

we have

$$w'(z) = \frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z} * g'(z).$$

Using Theorem 6, we obtain $\frac{\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)}{z} \in \mathcal{P}(\frac{1}{2})$ under the given hypothesis.

Now, using Lemma 10 and (22), we see that $w'(z) \in \mathcal{P}$, which yields $w'(z) \in \mathcal{H}^q$ for all $q < 1$. Therefore, we get $w(z) \in \mathcal{H}^{\frac{q}{1-q}}$ for all $0 < q < 1$ or, equivalently, $w(z) \in \mathcal{H}^p$ for $0 < p < \infty$.

Applying the known bound for the Carathéodory function, it can be observed from ([51], p. 533, Theorem 1) that if

$$g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R},$$

then $|a_n| \leqq \frac{2}{n}$. Moreover, applying Lemma 7, we find that

$$\begin{aligned}
|w(z)| &\leqq |z| + \sum_{n=2}^{\infty} \frac{[\Gamma(\beta)]^{\gamma} a_n |z|^n}{[\beta + \Gamma(\alpha(n-1)]^{\gamma}} \\
&\leqq 1 + \sum_{n=2}^{\infty} \frac{[\Gamma(\beta)]^{\gamma}}{\left[\beta + \Gamma\big(\alpha(n-1)\big)\right]^{\gamma}} \frac{2}{n} \\
&\leqq 1 + \sum_{n=1}^{\infty} \frac{[\Gamma(\beta)]^{\gamma}}{\left[\beta + \Gamma(\alpha n)\right]^{\gamma}} \frac{2}{(n+1)} \\
&\leqq 1 + \sum_{n=1}^{\infty} \frac{2}{(n+1)} \frac{1}{[(\beta)_n]^{\gamma}} < \infty,
\end{aligned}$$

which yields that the power series for $w(z)$ converges absolutely for $|z| = 1$.

Next, by using a known result ([46], p. 42, Theorem 3.11), we see that $w'(z) \in \mathcal{H}^q$, which implies that $w(z)$ is continuous in the closure $\overline{\mathbb{D}}$ of $\mathbb{D}$. Since continuous functions on the compact set $\overline{\mathbb{D}}$ are bounded, the function $w(z)$ is a bounded analytic function in $\mathbb{D}$. Hence, $w(z) \in \mathcal{H}^{\infty}$. □

Remark 6. *Setting $\gamma = 1$ in Theorem 8, we observe that $\mathbb{E}_{\alpha,\beta}^{(1)} * g$ is in the Hardy class $\mathcal{H}^{\infty} \cap \mathcal{R}$, if $\beta > 1 + \sqrt{3}$. This leads to the known result* ([52], *Theorem* 4.5)*. Hence, Theorem 8 generalizes the result given in* [52]*.*

Example 5. *Suppose that $g \in \mathcal{R}$. Then the functions $\mathbb{E}_{1,5}^{(1)} * g$ and $\mathbb{E}_{1,\frac{9}{2}}^{(1)} * g$ are in the Hardy class $\mathcal{H}^{\infty} \cap \mathcal{R}$.*

Finally, in this section, we prove the following theorem.

Theorem 9. *Suppose that $\alpha \geqq 1$, $\beta \geqq 1$, $\gamma > 0$ and*

$$\frac{(\beta+1)^{\gamma}}{\beta^{\gamma}(1-\delta)[(\beta+1)^{\gamma} - 1]} < 1.$$

If $g \in \mathcal{R}(\lambda)$ $(\lambda < 1)$, then $\mathbb{E}_{\alpha,\beta}^{(\gamma)} * g \in \mathcal{R}(\mu)$, with $\mu = 1 - 2(1-\delta)(1-\lambda)$.

Proof. If $g \in \mathcal{R}(\lambda)$ $(\lambda < 1)$, then $g' \in \mathcal{P}(\lambda)$. Assuming that

$$w(z) = \mathbb{E}^{(\gamma)}_{\alpha,\beta}(z) * g(z),$$

we have

$$w'(z) = \frac{\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)}{z} * g'(z).$$

Now, if we apply Theorem 6, we observe that the normalized Mittag–Leffler-type function $\frac{\mathbb{E}^{(\gamma)}_{\alpha,\beta}(z)}{z} \in \mathcal{P}(\delta)$ under the given hypothesis. Therefore, by using Lemma 10, we find that $w'(z) \in \mathcal{P}(\mu)$, which is equivalent to $w(z) \in \mathcal{R}(\mu)$. This completes the proof of Theorem 9. □

Remark 7. *Upon setting $\gamma = 1$ in Theorem 9, we deduce that $\mathbb{E}^{(1)}_{\alpha,\beta} * g \in \mathcal{R}(\preceq)$, if*

$$\beta > \frac{1 + \sqrt{(5 - 4\delta)}}{2(1 - \delta)}.$$

This leads to the result given in ([52], Theorem 4.6). Hence, Theorem 9 generalizes the corresponding known result ([52], Theorem 4.6).

5. Inclusion Properties

Let us consider a modified Mittag–Leffler-type function defined as follows:

$$G^{(\gamma)}_{\alpha,\beta,A,B}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(Bk + A)[\Gamma(\beta k + \alpha)]^{\gamma - 1}}, \tag{25}$$

where $A, B, \alpha, \beta, z \in \mathbb{C}$ with $\min\{\Re(B), \Re(\beta)\} > 0$ and $\gamma \geqq 1$.

Remark 8. *Setting $A = \alpha$ and $B = \beta$ in (25), we obtain the Mittag–Leffler-type function as defined in (9), which is also known as the Le Roy-type Mittag–Leffler function [33]. Similarly, for $A = \alpha$, $B = \beta$ and $\gamma = 1$, $G^{(\gamma)}_{\alpha,\beta,A,B}(z)$ reduces to the Mittag–Leffler function. Moreover, the Bessel-Wright function $J^{\mu}_{\nu}(z)$ given by (4) can be derived as a particular case if set $A = B = 1$, $\alpha = \nu + 1$, $\beta = \mu$ and $\gamma = 2$, and replace z by $-z$, in the definition (25). In particular, upon setting $A = \alpha$, $B = \beta$, and $\gamma = n$ in (25), a multi-index Mittag–Leffler function can be obtained. These are important special functions, which have several applications in fractional calculus [53–56], mathematical physics and related branches of science and engineering [27,57–59]. Hence, clearly, the modified Mittag–Leffler-type function $G^{(\gamma)}_{\alpha,\beta,A,B}(z)$ defined by (25) has the potential for applications in physics, biology, fractional dynamics, and other branches of science and engineering.*

Since $G^{(\gamma)}_{\alpha,\beta,A,B}(z) \notin \mathcal{A}$, we consider the following normalized form of (25):

$$\mathbb{G}^{(\gamma)}_{\alpha,\beta,A,B}(z) = \sum_{k=1}^{\infty} \frac{\Gamma(A + B)[\Gamma(\alpha + \beta)]^{\gamma - 1} z^k}{\Gamma(A + Bk)[\Gamma(\alpha + \beta k)]^{\gamma - 1}} \in \mathcal{A}. \tag{26}$$

We now consider a linear convolution operator $\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B} : \mathcal{A} \to \mathcal{A}$ defined, in terms of the Hadamard product (or convolution), by

$$\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B} f(z) = \mathbb{G}^{(\gamma)}_{\alpha,\beta,A,B}(z) * f(z) \qquad (z \in \mathbb{D}, f(z) \in \mathcal{A}). \tag{27}$$

Then, by using (26) and (27), we get

$$z\big[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}f(z)\big]' = \frac{A+B}{B}\,\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z) - \frac{A}{B}\,\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}f(z). \tag{28}$$

With the help of the linear operator $\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}$, we define new subclasses as follows:

$$S^*_{\alpha,\beta,A,B}(\delta) = \left\{f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{z\big[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)\big]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)}\right) > \delta\right.$$

$$\left.(0 \leqq \delta < 1;\; z \in \mathbb{D})\right\}, \tag{29}$$

$$\kappa_{\alpha,\beta,A,B}(\delta) = \left\{f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{z\big[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)\big]''}{[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)]'}\right) > \delta\right.$$

$$\left.(0 \leqq \delta < 1;\; z \in \mathbb{D})\right\}, \tag{30}$$

$$C_{\alpha,\beta,A,B}(\rho,\delta) = \left\{f : f \in \mathcal{A} \quad \text{and} \quad \exists\, g \in S^*_{\alpha,\beta,A,B}(\delta) \text{ such that } \Re\left(\frac{z\big[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)\big]'}{[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}g(z)]}\right) > \rho\right.$$

$$\left.(0 \leqq \delta;\; \rho < 1;\; z \in \mathbb{D})\right\} \tag{31}$$

and

$$C^*_{\alpha,\beta,A,B}(\rho,\delta) = \left\{f : f \in \mathcal{A} \quad \text{and} \quad \exists\, g \in S^*_{\alpha,\beta,A,B}(\delta) \text{ such that } \Re\left(\frac{\left(z\big[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)\big]'\right)'}{\big[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}g(z)\big]'}\right) > \rho\right.$$

$$\left.(0 \leqq \delta;\; \rho < 1;\; z \in \mathbb{D})\right\}. \tag{32}$$

The linear operator $\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}$ satisfies the following properties:

$$f(z) \in \kappa_{\alpha,\beta,A,B}(\delta) \iff zf'(z) \in S^*_{\alpha,\beta,A,B}(\delta) \tag{33}$$

and

$$f(z) \in C^*_{\alpha,\beta,A,B}(\rho,\delta) \iff zf'(z) \in C_{\alpha,\beta,A,B}(\rho,\delta). \tag{34}$$

Theorem 10. *Let $A \in \mathbb{R}$, $B > 0$, $\alpha, \beta, z \in \mathbb{C}$, $\Re(\beta) > 0$, $0 \leqq \delta < 1$ and $B(B\delta + A) \geqq 0$. Then,*

$$S^*_{\alpha,\beta,A,B}(\delta) \subset S^*_{\alpha,\beta,A+1,B}(\delta).$$

Proof. Suppose that $f \in S^*_{\alpha,\beta,A,B}(\delta)$ and the function $\phi : \mathbb{C} \to \mathbb{C}$ is defined by

$$\phi(z) = \frac{1}{1-\delta}\left(\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}f(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}f(z)]} - \delta\right). \tag{35}$$

Then, $\phi(0) = 1$ and ϕ is analytic in $\mathbb{D}$. From (28), we get

$$\frac{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}f(z)} = \frac{A}{A+B} + \frac{B}{A+B}\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}f(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}f(z)}. \tag{36}$$

Combining (35) and (36), we obtain

$$\frac{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}f(z)} = \frac{A}{A+B} + \frac{B}{A+B}[(1-\delta)\phi(z)+\delta], \tag{37}$$

which leads to

$$\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)} = \frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}f(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}f(z)} + \frac{B(1-\delta)z\phi'(z)}{B[(1-\delta)\phi(z)+\delta]+A},$$

which, in view of (35), yields

$$\frac{1}{1-\delta}\left(\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)]} - \delta\right) = \frac{Bz\phi'(z)}{B[(1-\delta)\phi(z)+\delta]+A} + \phi(z). \tag{38}$$

Next, we consider another new function $\Phi : \mathbb{C}^2 \to \mathbb{C}$ defined by

$$\Phi(u,v) = u + \frac{Bv}{B[(1-\delta)u+\delta]+A},$$

where $u = u_1 + iu_2$ and $v = v_1 + iv_2$ with $u_1, u_2, v_1, v_2 \in \mathbb{R}$. It can be easily observed that Φ is continuous on $\Lambda = \left\{\mathbb{C} \setminus \frac{A+B\delta}{B(\delta-1)}\right\} \times \mathbb{C}$ with $(1,0) \in \Lambda$ and $\Re\{\Phi(1,0)\} > 0$. Since $f \in S^*_{\alpha,\beta,A,B}(\delta)$, it follows that

$$\Re\{\Phi(\phi(z)), z(\phi(z))\} > 0 \qquad (\forall\, z \in \mathbb{D}).$$

Moreover, for $(iu_2, v_1) \in \Lambda$, with

$$u_2, v_1 \in \mathbb{R} \qquad \text{and} \qquad v_1 \leqq -\frac{(1+u_2)^2}{2},$$

we have

$$\begin{aligned}\Re(\Phi(iu_2, v_1)) &= \Re\left\{iu_2 + \frac{Bv_1}{B[(1-\delta)iu_2 + \delta] + A}\right\} \\ &= \frac{Bv_1(B\delta + A)}{B^2(1-\delta)^2 {u_2}^2 + (B\delta + A)^2} \\ &\leqq -\frac{(1+u_2)^2}{2}\frac{B(B\delta + A)}{B^2(1-\delta)^2 {u_2}^2 + (B\delta + A)^2} \\ &\leqq 0,\end{aligned}$$

which leads to $\Re\{\Phi(iu_2, v_1)\} \leqq 0$. Hence, by Lemma 8, we claim that $\Re\{\phi(z)\} > 0$.

Finally, by using (35), we see that $f \in S^*_{\alpha,\beta,A+1,B}(\delta)$, which is the desired result asserted by Theorem 10. □

Remark 9. *Setting* $A = \alpha$, $B = \beta$ *and* $\gamma = 1$ *in* (25) *and using Theorem* 10, *the inclusion relation* $S^*_{\alpha,\beta}(\delta) \subset S^*_{\alpha+1,\beta}(\delta)$ *for a subclass associated with the normalized Mittag–Leffler function* $\mathbb{E}_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z)$ *can be deduced.*

Theorem 11. *Let* $A \in \mathbb{R}$, $B > 0$, $\alpha, \beta, z \in \mathbb{C}$, $\Re(\beta) > 0$, $0 \leqq \delta < 1$ *and* $B(B\delta + A) \geqq 0$. *Then,*

$$\kappa^*_{\alpha,\beta,A,B}(\delta) \subset \kappa^*_{\alpha,\beta,A+1,B}(\delta).$$

Proof. Using (33) and Theorem 10, we get

$$\begin{aligned}f \in \kappa_{\alpha,\beta,A,B}(\delta) &\iff zf' \in S^*_{\alpha,\beta,A,B}(\delta) \\ &\implies zf' \in S^*_{\alpha,\beta,A+1,B}(\delta) \\ &\iff f \in \kappa^*_{\alpha,\beta,A+1,B}(\delta).\end{aligned}$$

Hence, the proof of Theorem 11 is completed. □

Theorem 12. *Let* $A \in \mathbb{R}$, $B > 0$, $\alpha, \beta, z \in \mathbb{C}$, $\Re(\beta) > 0$, $0 \leqq \rho, \delta < 1$ *and* $(B\delta + A) \geqq 0$. *Then,*

$$C_{\alpha,\beta,A,B}(\rho, \delta) \subset C_{\alpha,\beta,A+1,B}(\rho, \delta).$$

Proof. Let $f \in C_{\alpha,\beta,A,B}(\rho, \delta)$. Then, $\exists\, g \in S^*_{\alpha,\beta,A,B}(\delta)$ s.t.

$$\Re\left(\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B} f(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B} g(z)}\right) > \rho. \tag{39}$$

Define a function $\phi : \mathbb{C} \to \mathbb{C}$ by

$$\phi(z) = \frac{1}{1-\rho}\left(\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B} f(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B} g(z)} - \rho\right). \tag{40}$$

Then $\phi(0) = 1$ and ϕ is analytic in $\mathbb{D}$. Thus, by using (28), we have

$$\begin{aligned}
\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}g(z)]} &= \frac{\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}zf'(z)\right]}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}g(z)} \\
&= \frac{\frac{B}{A+B}z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}zf'(z)\right]' + \frac{A}{A+B}\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}zf'(z)}{\frac{B}{A+B}z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}g(z)\right]' + \frac{A}{A+B}\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}g(z)} \\
&= \frac{\frac{B}{A+B}\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}zf'(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}g(z)} + \frac{A}{A+B}\frac{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}zf'(z)}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}g(z)}}{\frac{B}{A+B}\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}g(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}g(z)} + \frac{A}{A+B}}.
\end{aligned} \tag{41}$$

Now, we consider a new function $q : \mathbb{D} \to \mathbb{C}$, defined by

$$q(z) = \frac{1}{1-\delta}\left(\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}g(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}g(z)} - \delta\right). \tag{42}$$

Since $g \in S^*_{\alpha,\beta,A,B}(\delta)$, by Theorem 10, we have $g \in S^*_{\alpha,\beta,A+1,B}(\delta)$. Hence, $\Re\{q(z)\} > 0$ in $\mathbb{D}$. Then, by using (40) and (42) into (41), we have

$$\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}g(z)]} = \frac{\frac{B}{A+B}\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}zf'(z)\right]'}{\mathbb{H}_{\alpha,\beta,A+1,B}g(z)} + \frac{A}{A+B}[(1-\rho)\phi(z)+\rho]}{\frac{B}{A+B}[(1-\delta)q(z)+\delta] + \frac{A}{A+B}}. \tag{43}$$

With the help of (40), we obtain

$$\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A+1,B}zf'(z)\right]'}{\mathbb{H}_{\alpha,\beta,A+1,B}g(z)} = (1-\rho)z\phi'(z) + (1-\rho)[(1-\rho)\phi(z)+\rho][(1-\delta)q(z)+\delta]. \tag{44}$$

Using (44) in the Equation (40), we get

$$\frac{1}{1-\rho}\left(\frac{z\left[\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}f(z)\right]'}{\mathbb{H}^{(\gamma)}_{\alpha,\beta,A,B}g(z)} - \rho\right) = \phi(z) + \frac{Bz\phi'(z)}{B(1-\delta)q(z)+B\delta+A}. \tag{45}$$

Assume that

$$w(z) = \frac{B}{B(1-\delta)q(z)+B\delta+A}, \tag{46}$$

which is analytic in $\mathbb{D}$. Therefore, using (39), we see that $\Re\{\phi(z) + w(z)z\phi'(z)\}$ is positive in $\mathbb{D}$.

Since $\Re\{q(z)\}$ is positive in $\mathbb{D}$, by using the inequality $(B\delta + A) > 0$, we observe that $\Re\{w(z)\}$ is positive in $\mathbb{D}$. Moreover, if we apply Lemma 9 with

$$g(z) = \frac{1+z}{1-z},$$

we find that $\Re\{\phi(z)\}$ is positive in $\mathbb{D}$.

Finally, by using (40), we see that $f \in C_{\alpha,\beta,A+1,B}$, which completes the proof of Theorem 12. □

Remark 10. *Putting $A = \alpha$, $B = \beta$ and $\gamma = 1$ in (25) and using Theorem 12, the inclusion relation $C^*_{\alpha,\beta}(\rho,\delta) \subset C^*_{\alpha+1,\beta}(\rho,\delta)$ for a subclass associated with the normalized Mittag–Leffler function given by*

$$\mathbb{E}_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z)$$

can be established.

Theorem 13. *Let $A \in \mathbb{R}$, $B > 0$, $\alpha,\beta,z \in \mathbb{C}$,, $\Re(\beta) > 0$, $0 \leqq \rho,\delta < 1$ and $(B\delta + A) \geqq 0$. Then,*

$$C^*_{\alpha,\beta,A,B}(\rho,\delta) \subset C^*_{\alpha,\beta,A+1,B}(\rho,\delta).$$

Proof. Using (32), (34), and Theorem 12, we get

$$\begin{aligned} f \in C^*_{\alpha,\beta,A,B}(\rho,\delta) &\Longleftrightarrow zf' \in C_{\alpha,\beta,A,B}(\rho,\delta) \\ &\Longrightarrow zf' \in C_{\alpha,\beta,A+1,B}(\rho,\delta) \\ &\Longleftrightarrow f \in C^*_{\alpha,\beta,A+1,B}(\rho,\delta), \end{aligned}$$

which completes proof of Theorem 13. □

6. Pre-Starlikeness and k-Uniform Convexity

In this section, we consider the class of pre-starlike functions, which was introduced by Ruscheweyh [60]. The class of pre-starlike functions of order μ is denoted by $\mathcal{L}_\mu$ and defined as follows:

$$\mathcal{L}_\mu = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \frac{z}{(1-z)^{2-2\mu}} * f \in \mathcal{S}^*(\mu) \ \ (z \in \mathbb{D};\ 0 \leqq \mu < 1) \right\}.$$

In particular, $\mathcal{L}_0 = \mathcal{C}$. An interesting generalization $\mathcal{L}[\rho,\mu]$ of the class $\mathcal{L}_\mu$ was considered in [61]. It is known that $f \in \mathcal{L}[\rho,\mu]$ if

$$f \in \mathcal{A} \qquad \text{and} \qquad \frac{z}{(1-z)^{2-2\rho}} * f \in \mathcal{S}^*(\mu) \qquad (0 \leqq \rho < 1;\ 0 \leqq \mu < 1).$$

Clearly, we have

$$\mathcal{L}[\mu,\mu] = \mathcal{L}_\mu.$$

A function $f \in \mathcal{A}$, which is real on $(-1,1)$, is said to be a typically real function, if it satisfies the following condition:

$$\Im(z)\Im\big(f(z)\big) > 0 \qquad (z \in \mathbb{D}).$$

A function $f \in \mathcal{A}$ is called convex in the direction of the imaginary axis, if $f(\mathbb{D})$ is convex in the direction of the imaginary axis, that is,

$$\Re\{u_1\} = \Re\{u_2\} \qquad \big(\forall\ u_1,u_2 \in f(\mathbb{D})\big).$$

It is seen from [62] that a function $f \in \mathcal{A}$ is convex in the direction of the imaginary axis with real coefficients if the function $zf'(z)$ is typically real. Equivalently, we have

$$\Re\{(1-z^2)f'(z)\} > 0 \qquad (z \in \mathbb{D}).$$

Let k-UCV and k-ST be the subclasses of the normalized univalent function class $\mathcal{S}$, which consist, respectively, of k-uniformly convex functions and k-starlike functions in $\mathbb{D}$. The classes k-UCV and k-ST were introduced and studied by Kanas et al. (see [63,64]; see also [65]) as follows:

$$k\text{-UCV} = \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \ \ (z \in \mathbb{D}) \right\}$$

and

$$k\text{-ST} = \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{D}) \right\}.$$

We now discuss some theorems related to the classes k-UCV and k-ST.

Theorem 14. *Assume that* $f \in \mathcal{H}$. *If*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leqq \frac{1}{k+2}$$

for some $k \ (0 \leqq k < \infty)$, *then* $f \in k$-UCV. *The number* $\frac{1}{k+2}$ *cannot be increased.*

Theorem 15. *Suppose that* $f \in \mathcal{H}$. *If*

$$\sum_{n=2}^{\infty} [n + k(n-1)]|a_n| > 1$$

for some $k \ (0 \leqq k < \infty)$, *then* $f \in k$-ST.

Let us consider the following class:

$$\mathcal{R}_\eta(\delta) = \left\{ f : f \in \mathcal{H}, \Re\{e^{i\eta}(f'(z) - \delta)\} > 0 \qquad \left(z \in \mathbb{D};\ \delta < 1;\ -\frac{\pi}{2} < \eta < \frac{\pi}{2}\right) \right\}.$$

If the function $f \in \mathcal{A}$ is in the class $\mathcal{R}_\eta(\delta)$, then

$$|a_n| \leqq \frac{2(1-\delta)\cos\eta}{n} \qquad (n \in \mathbb{N} \setminus \{1\}).$$

We now define a linear convolution operator, which is associated with the normalized Mittag–Leffler-type function $\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)$, as follows:

$$J_{\alpha,\beta,\gamma} : \mathcal{H} \longrightarrow \mathcal{H} \ \text{ such that}$$

$$\left[J_{\alpha,\beta,\gamma}(f)\right](z) = \mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)^{*} f(z) \qquad (f \in \mathcal{H}).$$

Theorem 16. *Let the parameters* $\alpha > 0, \beta > 0, \gamma \geqq 1$, *and* $k \in [0, \infty)$ *be such that* $f(z) \in \mathcal{R}_\eta(\delta)$. *In addition, let the following conditions hold true*:

$$(\mathrm{H}_6) : \begin{cases} (\mathrm{i}) & \frac{3}{\Gamma(3\alpha+\beta)} < \frac{1}{\Gamma(2\alpha+\beta)}; \\ (\mathrm{ii}) & \frac{4}{[\Gamma(2\alpha+\beta)]^2} < \frac{3}{\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta)}; \\ (\mathrm{iii}) & 2(1-\delta)\cos\eta\left[\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{2(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right]^{\gamma} \leqq \frac{1}{k+2}. \end{cases}$$

Then, $\left[J_{\alpha,\beta,\gamma}(f)\right](z) \in k$-UCV.

Proof. We make use of Theorem 14 to prove the result asserted by Theorem 16. It is sufficient to show that

$$\sum_{n=2}^{\infty} n(n-1)\left|\left(\frac{\Gamma(\beta)}{\Gamma(\alpha n - \alpha + \beta)}\right)^{\gamma} a_n\right| \leqq \frac{1}{k+2}.$$

Since $f \in \mathcal{R}_\eta(\delta)$, we have

$$|a_n| \leqq \frac{2(1-\beta)\cos\eta}{n}$$

and

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-1)\left|\left(\frac{\Gamma(\beta)}{\Gamma(\alpha n-\alpha+\beta)}\right)^{\gamma} a_n\right| \\
&\leqq \sum_{n=0}^{\infty}(n+2)(n+1)\left(\frac{\Gamma(\beta)}{\Gamma(\alpha n-\alpha+\beta)}\right)^{\gamma} \frac{2(1-\beta)cos\eta}{n+2} \\
&\qquad \cdot 2(1-\beta)\cos\eta \sum_{n=0}^{\infty}\Gamma(n+2)\left(\frac{\Gamma(\beta)}{\Gamma(\alpha n+\alpha+\beta)}\right)^{\gamma}\frac{1}{n!} \\
&\leqq 2(1-\beta)\cos\eta\left(\Gamma(\beta)\,{}_1\Psi_1\left[\begin{array}{c}(2,1);\\ \\ (\alpha+\beta,\alpha);\end{array} 1\right]\right)^{\gamma} \qquad (\forall\, z\in\mathbb{D}). \qquad (47)
\end{aligned}$$

In this case, we have

$$\psi_0=\frac{1}{\Gamma(\alpha+\beta)},\quad \psi_1=\frac{2}{\Gamma(2\alpha+\beta)} \quad\text{and}\quad \psi_2=\frac{6}{\Gamma(3\alpha+\beta)}.$$

We observe that the parametric conditions in $(\mathrm{H}_6):(\mathrm{i})$ and $(\mathrm{H}_6):(\mathrm{ii})$ are equivalent to $\psi_2<\psi_1$ and $\psi_1^2<\psi_0\psi_2$. Therefore, by (5), we have

$$\begin{aligned}
&2(1-\beta)\cos\eta\left(\Gamma(\beta)\,{}_1\Psi_1\left[\begin{array}{c}(2,1);\\ \\ (\alpha+\beta,\alpha);\end{array} 1\right]\right)^{\gamma} \\
&\qquad \leqq 2(1-\beta)\cos\eta\left(\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}+\frac{2(e-1)\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\right)^{\gamma} \leqq \frac{1}{k+2}.
\end{aligned}$$

Hence, the required result is proved. □

Theorem 17. *Assume that* $\alpha,\beta>0, \gamma\geqq 1$ *and*

$$(\mathrm{H}_7):\begin{cases}(\mathrm{i}) & \frac{4-2\rho}{\Gamma(3\alpha+\beta}<\frac{1}{\Gamma(2\alpha+\beta)};\\ (\mathrm{ii}) & \frac{3-2\rho}{[\Gamma(2\alpha+\beta)]^2}<\frac{4-2\rho}{\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta)};\\ (\mathrm{iii}) & \frac{2-\mu}{2-2\rho}\left(\frac{\Gamma(\beta)\Gamma(3-2\rho)}{\Gamma(\alpha+\beta)}-\frac{(e-1)\Gamma(\beta)\Gamma(4-2\rho)}{\Gamma(2\alpha+\beta)}\right)^{\gamma}<1-\mu.\end{cases}$$

Then $\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)\in\mathcal{L}[\rho,\mu]$ *for* $0\leqq\rho<1$ *and* $0\leqq\mu<1$.

Proof. Let the function g be given by

$$g(z)=\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)*\frac{z}{(1-z)^{2-2\rho}} \qquad (0\leqq\rho<1).$$

To prove the result asserted by Theorem 17, it is sufficient to show that

$$\Re\left(\frac{z[g(z)]'}{g(z)}\right)>\mu \qquad (z\in\mathbb{D}).$$

Thus, clearly, its sufficient to show that

$$\left|\frac{z[g(z)]'}{g(z)}-1\right|=\frac{\left|[g(z)]'-\frac{g(z)}{z}\right|}{\left|\frac{g(z)}{z}\right|}<1-\mu,$$

by using the following convolution:

$$g(z) = z + \sum_{k=2}^{\infty} \left(\frac{\Gamma(\beta)}{\Gamma(\alpha k - \alpha + \beta)} \right)^{\gamma} \frac{\Gamma(1-2\rho+k)}{\Gamma(2-2\rho)(k-1)!} \, z^k.$$

A simple computation leads to

$$\begin{aligned}
[g(z)]' - \frac{g(z)}{z} &= \sum_{k=2}^{\infty} (k-1) \left(\frac{\Gamma(\beta)}{\Gamma(\alpha k - \alpha + \beta)} \right)^{\gamma} \frac{\Gamma(1-2\rho+k)}{\Gamma(2-2\rho)(k-1)!} z^{k-1} \\
&= \sum_{k=2}^{\infty} \left(\frac{\Gamma(\beta)}{\Gamma(\alpha k - \alpha + \beta)} \right)^{\gamma} \frac{\Gamma(1-2\rho+k)}{\Gamma(2-2\rho)(k-2)!} z^{k-1} \\
&= \sum_{k=0}^{\infty} \left(\frac{\Gamma(\beta)}{\Gamma(\alpha k + \alpha + \beta)} \right)^{\gamma} \frac{\Gamma(3-2\rho+k)}{\Gamma(2-2\rho)(k)!} z^{k+1} \\
&\leqq \frac{1}{\Gamma(2-2\rho)} \left(\Gamma(\beta) \, {}_1\Psi_1 \left[\begin{matrix} (3-2\rho, 1); \\ \\ (\alpha+\beta, \alpha); \end{matrix} \; 1 \right] \right)^{\gamma} \qquad (\forall \, z \in \mathbb{D}). \qquad (48)
\end{aligned}$$

In this case, we have

$$\psi_0 = \frac{3-2\rho}{\Gamma(\alpha+\beta)}, \quad \psi_1 = \frac{4-2\rho}{\Gamma(2\alpha+\beta)} \quad \text{and} \quad \psi_2 = \frac{5-2\rho}{\Gamma(3\alpha+\beta)}.$$

We observe that the parametric conditions in (H_7) : (i) and (H_7) : (ii) are equivalent to $\psi_2 < \psi_1$ and $\psi_1^2 < \psi_0\psi_2$. Therefore, by means of (5), we have

$${}_1\Psi_1 \left[\begin{matrix} (3-2\rho, 1); \\ \\ (\alpha+\beta, \alpha); \end{matrix} \; 1 \right] \leqq \frac{\Gamma(3-2\rho)}{\Gamma(\alpha+\beta)} - \frac{(1-e)\Gamma(4-2\rho)}{\Gamma(2\alpha+\beta)}. \qquad (49)$$

Now, using the following inequality:

$$|z_1 + z_2| \geqq \big| |z_1| - |z_2| \big|,$$

we find that

$$\begin{aligned}
\left| \frac{g(z)}{z} \right| &\geqq 1 - \left| \sum_{k=2}^{\infty} \left(\frac{\Gamma(\beta)}{\Gamma(\alpha k - \alpha + \beta)} \right)^{\gamma} \frac{\Gamma(1-2\rho+k)}{\Gamma(2-2\rho)(k-1)!} z^{k-1} \right| \\
&= 1 - \left| \sum_{k=0}^{\infty} \left(\frac{\Gamma(\beta)}{\Gamma(\alpha k + \alpha + \beta)} \right)^{\gamma} \frac{\Gamma(3-2\rho+k)}{\Gamma(2-2\rho)(k+1)k!} z^{k+1} \right| \\
&> 1 - \frac{1}{\Gamma(2-2\rho)} \left(\Gamma(\beta) \, {}_1\Psi_1 \left[\begin{matrix} (3-2\rho, 1); \\ \\ (\alpha+\beta, \alpha); \end{matrix} \; 1 \right] \right)^{\gamma}.
\end{aligned}$$

Moreover, by applying the inequality (5), we have

$${}_1\Psi_1 \left[\begin{matrix} (3-2\rho, 1); \\ \\ (\alpha+\beta, \alpha); \end{matrix} \; 1 \right] \leqq \frac{\Gamma(3-2\rho)}{\Gamma(\alpha+\beta)} - \frac{(1-e)\Gamma(4-2\rho)}{\Gamma(2\alpha+\beta)},$$

where

$$\frac{4-2\rho}{\Gamma(3\alpha+\beta)} < \frac{1}{\Gamma(2\alpha+\beta)}$$

and

$$\frac{3-2\rho}{[\Gamma(2\alpha+\beta)]^2} < \frac{4-2\rho}{\Gamma(\alpha+\beta)\Gamma(3\alpha+\beta)}.$$

Therefore, we get

$$\left|\frac{g(z)}{z}\right| \geqq 1 - \frac{1}{2-2\rho}\left(\frac{\Gamma(\beta)\Gamma(3-2\rho)}{\Gamma(\alpha+\beta)} - \frac{(1-e)\Gamma(\beta)\Gamma(4-2\rho)}{\Gamma(2\alpha+\beta)}\right)^{\gamma} > 0 \qquad (\forall\, z \in \mathbb{D}). \quad (50)$$

Finally, by using (48)–(50) and (H_7) : (iii), we obtain

$$\begin{aligned}\left|\frac{z[g(z)]'}{g(z)} - 1\right| &= \frac{\left|[g(z)]' - \frac{g(z)}{z}\right|}{\left|\frac{g(z)}{z}\right|} \\ &< 1-\mu,\end{aligned}$$

which completes the proof of Theorem 17. □

Remark 11. *Upon setting $\rho = \mu$ in Theorem 17, it can be proved that $\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z) \in \mathcal{L}[\mu]$, that is, the function $\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)$ is pre-starlike of order μ for all $z \in \mathbb{D}$.*

Remark 12. *If we set $\rho = \mu = 0$ in Theorem 17, we obtain*

$$\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z) * \frac{z}{(1-z)^2} \in \mathcal{S}^*.$$

It is clear that $z\big(\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)\big)' \in \mathcal{S}^*$, which yields $\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z) \in \mathcal{C}$; that is, $\mathbb{E}_{\alpha,\beta}^{(\gamma)}(z)$ is a convex function.

7. Concluding Remarks and Observations

In our present investigation, we have established some sufficient conditions so that a class of Mittag–Leffler-type functions satisfies several geometric properties such as starlikeness, convexity, close-to-convexity, and uniform convexity inside the unit disk $\mathbb{D}$. For each of these functions, we also discuss pre-starlikeness and k-uniform convexity. Moreover, some sufficient conditions are derived so that these functions belong to the Hardy spaces $\mathcal{H}^p$ and $\mathcal{H}^\infty$. Moreover, we have derived the inclusion properties of the modified Mittag–Leffler-type functions. The various results, which we have established in this paper, are believed to be new, and their importance is illustrated by several interesting consequences and examples.

Several potential directions for further research on the subject of the present investigation can be based analogously upon Wright's general Mittag–Leffler-type function $\mathfrak{E}_{\alpha,\beta}(\phi; z)$, defined by (6), and Srivastava's unification $\mathcal{E}_{\alpha,\beta}(\varphi; z, s, \kappa)$ of the Mittag–Leffler-type functions as well as such important functions of Analytic Number Theory as the Hurwitz-Lerch-type functions, which are defined by (7). Yet another novel direction of research can possibly be motivated by some of the related developments on Analytic Function Theory of Complex Analysis, which are presented in the monograph by Alpay [66].

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