GEOMETRIC PROPERTIES OF THE KERNEL, NUCLEOLUS, AND RELATED SOLUTION CONCEPTS

M. Maschler, B. Peleg, and L. S. Shapley

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SUMMARY

This paper explores the geometrical relationships between the kernel and nucleolus of an n-person game in characteristic-function form and a number of other cooperative solution concepts, most notably the core. As a result, many technical properties of all these solutions are clarified, and some new light is shed on their intuitive interpretations. The main technique that unifies the investigation is the study of the behavior of the strong ϵ-core as a function of ϵ.

In games that are "zero-monotonic" (including all superadditive games), the kernel coincides with the pre-kernel, which is analytically simpler and can be described as a multi-bilateral equilibrium in which every pair of players bisects the difference between the outcomes that they, with the support of their best allies, could separately impose on each other. One of the central results of this paper states that the part of the pre-kernel that falls within the core, or within any strong ϵ-core, depends only on the geometrical shape of that convex polyhedron. There is an analogous, but slightly more complicated statement for the kernel.

The smallest nonempty strong ϵ-core is called the least-core; it is contained in all other strong ϵ-cores and it always includes at least one pre-kernel point. By letting ϵ increase, the entire pre-kernel is eventually

included, but the strong ϵ-core may meanwhile acquire additional faces and become hard to keep track of. However, both the kernel and the pre-kernel can be proved to lie within the simplex of reasonable outcomes, which interlocks with the simplex of imputations in a curious way. A formula is given for the critical value of ϵ at which the strong ϵ-core just covers the intersection of these two simplices, and hence surely includes the kernel and pre-kernel. With the aid of this critical value, a notion of "quasi-zero-monotonicity" is developed that expands the class of games in which the pre-kernel can be used to determine the kernel.

The nucleolus is a special point in the kernel, and for zero-monotonic games it lies in the least-core as well. By letting ϵ decrease--even beyond the point where the strong ϵ-core vanishes--a finite nested sequence of sets can be constructed that leads to the nucleolus. This construction yields a very elementary proof of existence and uniqueness, and also provides a rationale for the nucleolus as a fair division point. An example shows, rather surprisingly, that the location of the nucleolus within the least-core (or core or kernel) cannot always be predicted merely from the geometrical shapes of these sets.

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CONTENTS

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1. INTRODUCTION

The kernel and nucleolus of a cooperative game with side payments were originally regarded as auxiliary solution concepts whose main task was to illuminate the properties of the bargaining set $\mathcal{M}_1(i)$. The latter is a solution concept which derives its justification from intuitive considerations, whereas the intuitive meaning of the kernel and nucleolus has been less clear; in fact, it is hard to justify their definitions without becoming involved with the obscure notion of "interpersonal comparison of utilities."

Nevertheless, both the kernel and the nucleolus possess interesting mathematical properties and reflect in many ways the structure of the game. A well known example is the seven-person "projective game," first described by von Neumann and Morgenstern [1944], which has an interesting kind of symmetry. In this game, the nucleolus consists of the payoff vector at the center of the imputation simplex, representing equal shares for all the players. This symmetric location is a consequence of the fact that under the group of permutations that leave the game invariant, all players belong to a single orbit. The kernel yields additional information in this case: it consists of seven radiating line segments that connect the nucleolus at the center with the seven points in the boundary that make up the "main simple solution" of von Neumann and Morgenstern (see Maschler & Peleg [1967]). This reflects the fact that the trio of players who are favored along any one of these rays form a minimal winning coalition, playing a role in the structure of the game exactly symmetric to that played by the trio favored along any other one of the rays. Thus, although the mathematical definitions of the kernel and the nucleolus are not particularly intuitive, their sensitivity to the structure of the game makes them appealing and worthy of further study.

It is well known that whenever the core of the game is not empty it contains the nucleolus. Since the nucleolus is also in the kernel, this means that the kernel and the core always have at least one point in common, provided of course that the core is not empty. The core itself is a concept that rests on strong intuitive grounds; specifically, it is the set of outcomes that no coalition can improve upon. If one could characterize in geometric terms the exact location of the nucleolus and of the relevant portion of the kernel within the core, then one would hope thereby to throw more light on the intuitive meaning of these solution concepts. This task is taken up in the present paper. We find such a geometric characterization for the relevant part of the kernel, i.e., the portion that is "visible" if we regard the core as a kind of "window" on the space of imputations, and we generalize this characterization from cores to strong ϵ-cores. The advantage is that the strong ϵ-cores always exist for sufficiently large ϵ even though the core itself--i.e., the strong 0-core--may be empty. Moreover, if ϵ is large enough the strong ϵ-core "window" will expose the entire kernel. (See Sec. 3.)

Roughly speaking, an outcome in the kernel represents the midpoint of a certain bargaining range for each pair of players. Each endpoint of this range is a point in the boundary of the core (or strong ϵ-core for fixed ϵ), representing a "maximum" demand made by one player beyond which the other player can find a coalition to support him in resisting any greater demand. An intuitive interpretation of the kernel is thereby provided that does not depend upon interpersonal utility comparisons. (See Sec. 5.)

A somewhat surprising but straightforward outcome of this midpoint property is that if two games have the same imputation space, and if the same geometrical set of imputations happens to be a strong ϵ-core of one game and a strong ϵ'-core of the other game, for some ϵ and ϵ', then the "visible" portions of the kernels of the two games must coincide. In particular, the portion of the kernel contained within the core of a game is completely determined by the geometrical shape and location of the core. (See Theorems 3.7 and 3.8 and Corollaries 3.10 and 3.11.)

The situation for the nucleolus is more complicated. We show in Sec. 6 how it can be characterized as a "lexicographic center," determined geometrically by certain hyperplanes, including those that define the core; yet its precise location depends on more than the shape of the core. In fact, an example is given in Sec. 6 of games having the same core but a whole range of different nucleoli. The general question of which points in a given convex polyhedron can be the nucleolus of a game having that polyhedron for its core is not settled.

In developing and exploiting our geometric approach, we were led to consider several auxiliary solution concepts in addition to the strong ϵ-core and the basic idea of an imputation. Specifically, we consider the <u>pre-kernel</u>, the <u>least-core</u>, and the notion of <u>reasonable outcome</u>. A number of interesting intersolutional relationships are established in Secs. 2 and 3. In particular, we determine explicitly several "critical values" of ϵ, at which the strong ϵ-core just barely contains one of the other solutions. In Sec. 4 we then employ these results, as well as the theory of balanced and totally balanced games, in a program of enlarging the class of games for which the pre-kernel (which has a simpler definition than the kernel) can be shown equivalent to the kernel; this section, which is rather technical, may be omitted at first reading, as what follows does not depend on it.

Section 5 is a brief discussion of the interpretation of the kernel as a fair division scheme for multilateral bargaining. Section 6 then applies our geometric methods to obtain a simplified approach to the theory of the nucleolus. A set temporarily called the "lexicographic center" of the game is defined. Elementary arguments show easily (1) that this set is non-empty, (2) that it is contained in any strong ϵ-core that is not empty, and (3) that it consists of precisely one point. We then show that the lexicographic center coincides with the nucleolus, as traditionally defined, and describe an example of a 4-person game in which the location of the nucleolus is not determined completely by the geometrical shape of the core. Finally, in Section 7, we use the new definition as the basis for an intuitive rationale for the nucleolus as a fair division scheme.

2. THE CORE AND ITS RELATIVES

The core and its immediate generalizations are among the simplest and most intuitive of the cooperative solution concepts for multiperson games. Their definitions are plausible and not too sophisticated, and, if we restrict ourselves (as we shall) to games with side payments, they consist geometrically of nothing more recondite than closed convex polyhedra in the space of the payoff vectors. For these reasons, the core concept is often regarded as basic to the theory, and other solutions gain some support if it can be shown that they are in some way related to it.

2.1. Definitions

Let us recall some basic definitions. We consider an n-person cooperative game with side payments, denoted $\Gamma \equiv (N; v)$, where $N = \{1, 2, \ldots, n\}$ represents the set of players and v is the characteristic function. We assume that v is a real-valued function from the coalitions (subsets of N) to the real numbers, satisfying

$$(2.1) \qquad v(N) \geqq \sum_{i=1}^{n} v(\{i\}) \quad \text{and} \quad v(\phi) = 0.$$

The first condition in (2.1) is needed in order to guarantee the existence of imputations (see (2.4)); in the interesting cases the inequality is strict. As to the second condition, we keep it because it is so often convenient in many parts of game theory.

Given a game $\Gamma \equiv (N; v)$, a vector $x = (x_1, x_2, \ldots, x_n)$ of real numbers, and a coalition S, we define

$$(2.2) \qquad x(S) = \begin{cases} \sum_{i \in S} x_i & \text{if } S \neq \phi \\ 0 & \text{if } S = \phi \end{cases}$$

and

$$(2.3) \qquad e(S, x) = v(S) - x(S).$$

The expression $e(S, x)$ is called the <u>excess</u> of S at x (in the game Γ). It represents the gain (or loss, if it is negative) to the coalition S if its members depart from an agreement that yields x in order to form their own coalition.

An <u>imputation</u> for $\Gamma \equiv (N; v)$ is a vector $x = (x_1, x_2, \ldots, x_n)$ that satisfies

$$(2.4) \quad x(N) = v(N) \quad \text{and} \quad x_i \geqq v(\{i\}), \quad \text{all } i \in N.$$

Equivalently, in terms of the excess,

$$(2.5) \quad e(N, x) = 0 \quad \text{and} \quad e(\{i\}, x) \leqq 0, \quad \text{all } i \in N.$$

The set of all imputations for the game Γ will be denoted by $\mathcal{X}(\Gamma)$. If the inequality in (2.1) is strict, $\mathcal{X}(\Gamma)$ is a simplex of n - 1 dimensions. We shall have occasion also to refer to "extended" imputations, i.e., vectors x satisfying $x(N) = v(N)$ without necessarily being elements of $\mathcal{X}(\Gamma)$; these are called pre-imputations and comprise an (n-1)-dimensional affine set that we shall denote by $\mathcal{X}^*(\Gamma)$.

DEFINITION 2.1. The core of the game $\Gamma \equiv (N; v)$, denoted $\mathcal{C}(\Gamma)$, is the set of all imputations that give rise only to non-positive excesses:

(2.6) $\mathcal{C}(\Gamma) = \{x \in \mathcal{X}(\Gamma): \quad e(S, x) \leqq 0 \quad \text{for all } S \subset N\}$.

Equivalently,

(2.7) $\mathcal{C}(\Gamma) = \{x \in \mathcal{X}^*(\Gamma): \quad e(S, x) \leqq 0 \quad \text{for all } S \subset N\}$.

DEFINITION 2.2. Let ϵ be a real number. The strong ϵ-core of the game $\Gamma \equiv (N; v)$, denoted $\mathcal{C}_\epsilon(\Gamma)$, is the set of all pre-imputations that give rise only to excesses not greater than ϵ, for all coalitions other than ϕ and N:

(2.8) $\mathcal{C}_\epsilon(\Gamma) = \{x \in \mathcal{X}^*(\Gamma): \quad e(S, x) \leqq \epsilon \quad \text{for all } S \neq N, \phi\}$.

Clearly, $\mathcal{C}_0(\Gamma) = \mathcal{C}(\Gamma)$. Also, $\mathcal{C}_\epsilon(\Gamma) \supset \mathcal{C}_{\epsilon'}(\Gamma)$ if $\epsilon > \epsilon'$, with strict inclusion if $\mathcal{C}_\epsilon(\Gamma) \neq \phi$. Obviously we have $\mathcal{C}_\epsilon(\Gamma) \neq \phi$ if ϵ is large enough and $\mathcal{C}_\epsilon(\Gamma) = \phi$ if ϵ is small enough.*

DEFINITION 2.3. The least-core of the game $\Gamma \equiv (N; v)$, denoted $\mathcal{LC}(\Gamma)$, is the intersection of all nonempty strong ϵ-cores. Equivalently, let $\epsilon_0(\Gamma)$ be the smallest ϵ such that $\mathcal{C}_\epsilon(\Gamma) \neq \phi$, i.e.,

$$(2.9) \qquad \epsilon_0(\Gamma) = \underset{x \in \mathcal{X}^*(\Gamma)}{\text{Min}} \; \underset{S \neq \phi, N}{\text{Max}} \; e(S, x);$$

this critical value may of course be negative. Then $\mathcal{LC}(\Gamma) = \mathcal{C}_{\epsilon_0(\Gamma)}(\Gamma)$. In other words, the least-core is the set of all pre-imputations that minimize the maximum excess.

The core was first studied by Gillies [1953, 1959]. It can be interpreted as the set of all payoff vectors that cannot be improved upon by any coalition. The strong ϵ-core, (as well as the "weak" ϵ-core, which will not concern us here), was introduced by Shapley and Shubik [1963, 1966]. It can be interpreted as the set of efficient payoff vectors that cannot be improved upon by any coalition if forming a coalition entails a "cost" of ϵ (or a "bonus" of $-\epsilon$, if ϵ is negative). The least-core, treated formally for the first time in this paper, combines these ideas in such a way as to ensure existence and uniqueness. If the core of a game is not empty (i.e., if $\epsilon_0 \leq 0$), then the least-core is a

*We are assuming here, and elsewhere whenever necessary, that $n > 1$.

centrally-located locus (or point) within the core. If the core is empty (i.e., if $\epsilon_0 > 0$), then the least-core may be regarded as revealing the "latent" position of the core.

The following critical value will also be of interest:

$$(2.10) \qquad \epsilon_1(\Gamma) = \operatorname*{Max}_{S \neq N, \phi} \left(v(S) - \sum_{i \in S} v(\{i\}) \right);$$

it represents the biggest gain that any group of individuals can ensure by forming a coalition.

LEMMA 2.4. $\mathcal{C}_\epsilon(\Gamma) \supset \mathcal{X}(\Gamma)$ if and only if $\epsilon \geq \epsilon_1(\Gamma)$.

Proof. Immediate from (2.5) and (2.8).

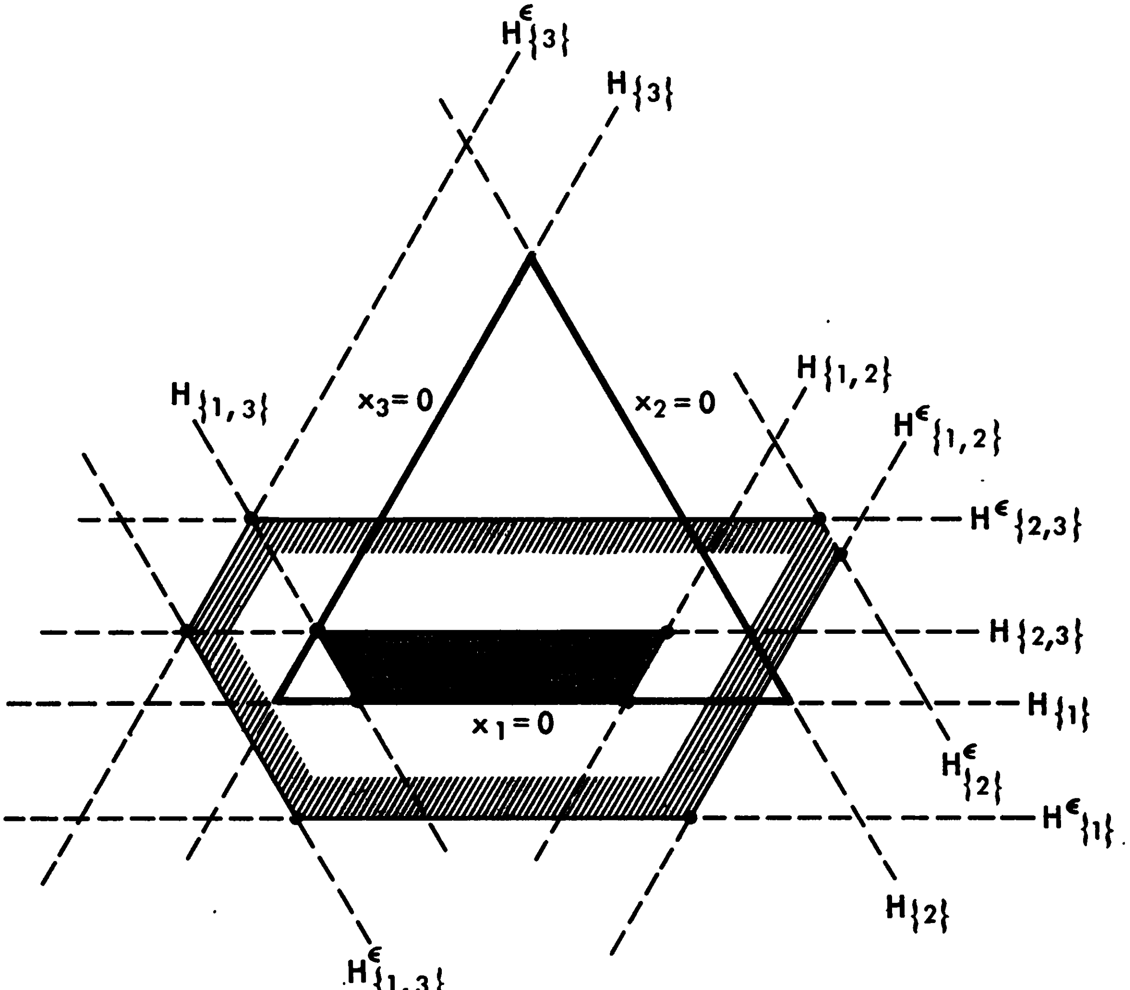
2.2. Geometric Properties of the Strong ε-Core

The strong ε-cores, including the least-core and the core itself, are compact convex polyhedra, bounded by no more than $2^n - 2$ hyperplanes of the form

$$(2.11) \quad H_S^\epsilon \equiv \{x \in \mathcal{X}^*(\Gamma): \ x(S) = v(S) - \epsilon\}, \ S \neq \phi, N.$$

We shall write H_S for H_S^0. Except for the least-core, all nonempty strong ε-cores have dimension $n - 1$, i.e., the dimension of $\mathcal{X}^*(\Gamma)$. The dimension of $\mathcal{LC}(\Gamma)$ is always $n - 2$ or less.

Figure 1 shows a typical 3-person game. The heavy black triangle represents $\mathcal{X}(\Gamma)$. The cross-hatched region is



$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0,$
$v(\{1,2\}) = 20,\ v(\{1,3\}) = 10,\ v((\{2,3\}) = 50,$
$v(\{1,2,3\}) = 60.$

Fig. 1--The core and the strong ϵ-core of a 3-person game

the core $\mathcal{C}$. The lightly bordered region represents the strong ϵ-core for ϵ=15. The least-core $\mathcal{LC}$, which is the strong ϵ-core for $\epsilon = \epsilon_0 = -5$, is the horizontal line inside the core--in fact it is the locus of the centers of the inscribed circles of maximum radius.

Observe that although H_S and H_S^ϵ are parallel and are, in a sense,* equally far apart for different S, the shapes of the various polyhedra $\mathcal{C}_\epsilon(\Gamma)$ may be quite different. In the figure, $\mathcal{C}(\Gamma)$ is a quadrilateral whereas $\mathcal{C}_{15}(\Gamma)$ is a hexagon. Indeed, whenever ϵ is large enough, all $2^n - 2$ of the hyperplanes (2.11) will appear in the boundary of the strong ϵ-core. The following lemma makes this precise.

LEMMA 2.5. *Let* $\Gamma \equiv (N; v)$ *be an* n-*person game and let* S_o *be a fixed coalition other than* ϕ *or* N. *Then there exists a number* $\epsilon(S_o)$ *such that* $H_{S_o}^\epsilon \cap \mathcal{C}_\epsilon(\Gamma) \neq \phi$ *if and only if* $\epsilon \geq \epsilon(S_o)$.

Proof. We shall show (a) that there exists a number ϵ' such that

$$H_{S_o}^{\epsilon'} \cap \mathcal{C}_{\epsilon'}(\Gamma) \neq \phi,$$

and (b) that for any such ϵ', we have

*This would not be true in the Euclidean sense for $n > 3$, since the hyperplanes in n-space defined by $x(S) = v(S)$ do not make equal angles with the hyperplane $\mathcal{X}^*(\Gamma)$.

$$H_{S_o}^{\epsilon''} \cap \mathcal{C}_{\epsilon''}(\Gamma) \neq \phi$$

whenever $\epsilon'' \geqq \epsilon'$. The lemma then follows from the fact that the set of ϵ' satisfying (a) is bounded below and closed.

(a) Choose an arbitrary pre-imputation y satisfying

$$y(S_o) = v(S_o). \tag{2.12}$$

For each nonnegative ϵ, consider the pre-imputation y^ϵ obtained from y by having the members of S_o pay the amount ϵ to the members of $N - S_o$, in such a way that the payers pay equal amounts and the receivers receive equal amounts. Thus,*

$$y_i^\epsilon = \begin{cases} y_i - \frac{\epsilon}{|S_o|}, & \text{if } i \in S_o, \\ y_i + \frac{\epsilon}{|N - S_o|}, & \text{if } i \in N - S_o. \end{cases} \tag{2.13}$$

Denote

$$\mathfrak{T}^\epsilon = \{T: \ y^\epsilon(T) \geqq v(T) - \epsilon, \quad T \neq \phi, N\} \tag{2.14}$$

$$\mathcal{U}^\epsilon = \{U: \ y^\epsilon(U) < v(U) - \epsilon, \quad U \neq \phi, N\}. \tag{2.15}$$

Clearly, $T \in \mathfrak{T}^\epsilon$ implies $T \in \mathfrak{T}^{\epsilon'}$ whenever $\epsilon' > \epsilon$, because any decrease in the left-hand side of the inequality in [redacted]

*$|S|$ denotes the number of elements in S.

(2.14) will be offset by the decrease from $-\epsilon$ to $-\epsilon'$ in the right-hand side. On the other hand, if $U \in \mathcal{U}^{\epsilon}$, then $U \neq S_o$ by (2.12) and (2.13), and so we have

$$y^{\epsilon'}(U) - y^{\epsilon}(U) = \left(\frac{|U - S_o|}{|N - S_o|} - \frac{|U \cap S_o|}{|S_o|}\right)(\epsilon' - \epsilon)$$

$$\geq \left\{\begin{matrix} \frac{1}{|N - S_o|} - 1, & \text{if } U \supset S_o \\ 0 - \frac{|S_o| - 1}{|S_o|}, & \text{if } U \not\supset S_o \end{matrix}\right\}(\epsilon' - \epsilon)$$

$$\geq -\rho(\epsilon' - \epsilon)$$

where

$$\rho = \text{Max}\left(1 - \frac{1}{|S_o|}, 1 - \frac{1}{|N - S_o|}\right) < 1.$$

Hence the decrease in the left-hand side in (2.15) is more than offset by the decrease in the right-hand side, and so, if ϵ is made large enough,* U will belong to $\mathfrak{J}^{\epsilon}$. In fact, since there are only finitely many coalitions in $\mathcal{U}^{o}$, for some sufficiently large ϵ' $\mathcal{U}^{\epsilon'}$ is the empty set and so $y^{\epsilon'}$ belongs to $\mathcal{C}_{\epsilon'}(\Gamma)$ by (2.14), (2.8), and (2.3). On the other hand, we have $y^{\epsilon'} \in H^{\epsilon'}_{S_o}$ by (2.12) and (2.13). This proves (a).

(b) Start with $y^{\epsilon'}$ in $H^{\epsilon''}_{S_o} \cap \mathcal{C}_{\epsilon''}(\Gamma)$ and proceed as above, for $\epsilon'' \geq \epsilon'$; clearly the set $\mathcal{U}^{\epsilon''}$ will remain empty. This proves (b). Q.E.D.

*It suffices to take $\epsilon > (v(U) - y(U))/(1 - \rho)$. Note that ρ does not depend on ϵ.

This lemma tells us something about the behavior of the strong ϵ-core of a game as ϵ varies. If ϵ is very large, the boundary of $\mathcal{C}_\epsilon(\Gamma)$ will consist of $2^n - 2$ nonempty subsets* of the hyperplanes H_S^ϵ, $S \neq \phi$, N. When ϵ becomes smaller, some of these hyperplanes may cease to touch the strong ϵ-core, and when any one of them moves away it will never come back again to touch $\mathcal{C}_\epsilon(\Gamma)$ at some smaller value of ϵ. Of course, if ϵ is small enough, all the intersections will vanish and $\mathcal{C}_\epsilon(\Gamma)$ will become the empty set.

The proof of Lemma 2.5 yields the following important corollary:

COROLLARY 2.6. *If two games* $\Gamma \equiv (N, v)$ *and* $\Gamma' \equiv (N, v')$ *have nonempty strong epsilon-cores that match, say*

$$\mathcal{C}_\epsilon(\Gamma) = \mathcal{C}_{\epsilon'}(\Gamma') \neq \phi,$$

then their smaller strong epsilon-cores will also match, i.e.,

$$\mathcal{C}_{\epsilon-\delta}(\Gamma) = \mathcal{C}_{\epsilon'-\delta}(\Gamma') \text{ whenever } \delta > 0.$$

In particular, $\mathcal{LC}(\Gamma) = \mathcal{LC}(\Gamma')$.

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*One can show that the set $H_S^\epsilon \cap \mathcal{C}_\epsilon(\Gamma)$ is a true "facet" of $\mathcal{C}_\epsilon(\Gamma)$ whenever $\epsilon > \epsilon(S)$, that is, it has dimension $n - 2$. For $\epsilon = \epsilon(S)$ it is of lower dimension, unless perhaps if $\epsilon(S) = \epsilon_0(\Gamma)$, while of course for $\epsilon < \epsilon(S)$ it is empty.

Proof. Suppose the conclusion is false, and assume without loss of generality that $y \in \mathcal{C}_{\epsilon-\delta}(\Gamma) - \mathcal{C}_{\epsilon'-\delta}(\Gamma')$. Then

$$y(R) \geqq v(R) - \epsilon + \delta \qquad \text{for all } R \neq \phi, N,$$

and

$$y(S_o) < v'(S_o) - \epsilon' + \delta \qquad \text{for at least one } S_o \neq \phi, N.$$

Let the players of S_o pay an amount δ to the players of $N - S_o$, as in (2.13); then the resulting y^δ will satisfy

$$y^\delta(R) \geqq v(R) - \epsilon, \qquad \text{for all } R \neq \phi, N,$$

and

$$y^\delta(S_o) < v'(S_o) - \epsilon'.$$

Since, obviously, $y(N) = v(N) = v'(N) = y^\delta(N)$, we find that $y^\delta \in \mathcal{C}_\epsilon(\Gamma) - \mathcal{C}_{\epsilon'}(\Gamma')$, a contradiction.

2.3. Monotonicity

It is apparent that the core, and most other concepts treated in this paper as well, are relative invariants under "strategic equivalence," i.e., the addition of an arbitrary additive function to the characteristic function of the game.* When relations among such invariant concepts are discussed there will be no loss of generality in assuming that the underlying game (N; v) is 0-normalized, i.e.,

(2.16) $v(\{i\}) = 0, \qquad i = 1, 2, \ldots, n.$

We shall assume this whenever convenient.

A game (N; v) is called monotonic if $S \supset T$ implies $v(S) \geqq v(T)$ for all $S, T \subset N$. It is called zero-monotonic if the unique 0-normalized game that is strategically equivalent to (N; v) is monotonic.** Monotonicity is obviously not an invariant concept; in fact, it is easy to show that any game is strategically equivalent to some monotonic game. The property of zero-monotonicity, however, is invariant.

THEOREM 2.7. If $\Gamma \equiv (N; v)$ is zero-monotonic, then $\mathcal{LC}(\Gamma) \subset \mathcal{K}(\Gamma)$.

Proof. Let $x \in \mathcal{LC}(\Gamma) = \mathcal{C}_{\epsilon_0}(\Gamma)$ (see (2.9)), and suppose that $x_i < v(\{i\})$. Then $\epsilon_0 \geqq v(\{i\}) - x_i > 0$. For every $S \neq \phi$ not containing i, we have ***

*Thus, if a is an additive set function, then the core, etc. of (N; v + a) is obtained from the core, etc., of (N; v) by the transformation $x_i \to x_i + a(\{i\})$, all $i \in N$. We also have invariance under multiplication of v by a positive constant, but we shall stick to the original, limited notion of strategic equivalence defined by von Neumann and Morgenstern [1944, p. 245].

**Note that every superadditive game, i.e., a game (N; v) where $S \cap T = \phi$ implies $v(S) + v(T) \leqq v(S \cup T)$, is zero-monotonic.

***Note that when $S = N - \{i\}$ the first inequality depends on the fact that ϵ_0 is positive.

$$\begin{aligned} x(S) &= x(S \cup \{i\}) - x_i \\ &\geqq v(S \cup \{i\}) - \varepsilon_o - x_i \\ &\geqq v(S) + v(\{i\}) - \varepsilon_o - x_i \\ &> v(S) - \varepsilon_o. \end{aligned}$$

Therefore, we may take $y \in \mathcal{X}^*$ to be just slightly smaller than x in all coordinates except i, preserving the strictness of the inequality:

$$y(S) > v(S) - \epsilon_o,$$

for all $S \neq \phi$ not containing i. On the other hand, for all $T \neq N$ containing i, we have

$$y(T) > x(T) \geqq v(T) - \epsilon_o.$$

Hence y belongs to $\mathcal{C}_\epsilon(\Gamma)$ for some $\epsilon < \epsilon_o$, contradicting the definition of $\mathcal{LC}(\Gamma)$. So $x_i < v(\{i\})$ is impossible, and we have $x \in \mathcal{X}(\Gamma)$, as required.

2.4. Reasonable Outcomes

Another locus in the space of pre-imputations is of fundamental interest.*

*The reasonable set was first introduced by Milnor [1952]; it corresponds to what Luce and Raiffa [1957; Chap. 11] called "Class B".

DEFINITION 2.8. The reasonable set of a game $\Gamma \equiv (N; v)$, denoted $\mathcal{R}(\Gamma)$, is the set of all pre-imputations that give no player more than the largest amount he can contribute to a coalition. Thus,

$$(2.17) \quad \mathcal{R}(\Gamma) = \{x \in \mathcal{X}^*(\Gamma) : x_i \leqq r_i, \quad \text{all } i \in N\},$$

where

$$(2.18) \qquad r_i = \operatorname*{Max}_{S: i \in S} \Big(v(S) - v(S - \{i\}) \Big).$$

The next theorem details the intimate, interlocking relationship that exists between the sets $\mathcal{X}(\Gamma)$ and $\mathcal{R}(\Gamma)$ (see Fig. 2).

THEOREM 2.9. For any game $\Gamma \equiv (N; v)$, we have:

(a) $\mathcal{R}(\Gamma) \neq \phi$ --in fact, it is a simplex of $n - 1$ dimensions unless v is additive, in which case it is a single point coinciding with $\mathcal{X}(\Gamma)$;

(b) $\mathcal{R}(\Gamma) \cap \mathcal{X}(\Gamma) \neq \phi$--in fact, the two sets have interior* points in common unless either $r_i = v(\{i\})$ for some $i \in N$, or $\mathcal{X}(\Gamma)$ is a single point;

(c) no vertex of $\mathcal{R}(\Gamma)$ is interior* to $\mathcal{X}(\Gamma)$;

*The term "interior" is taken with respect to the $(n - 1)$- dimensional space $\mathcal{X}^*(\Gamma)$.

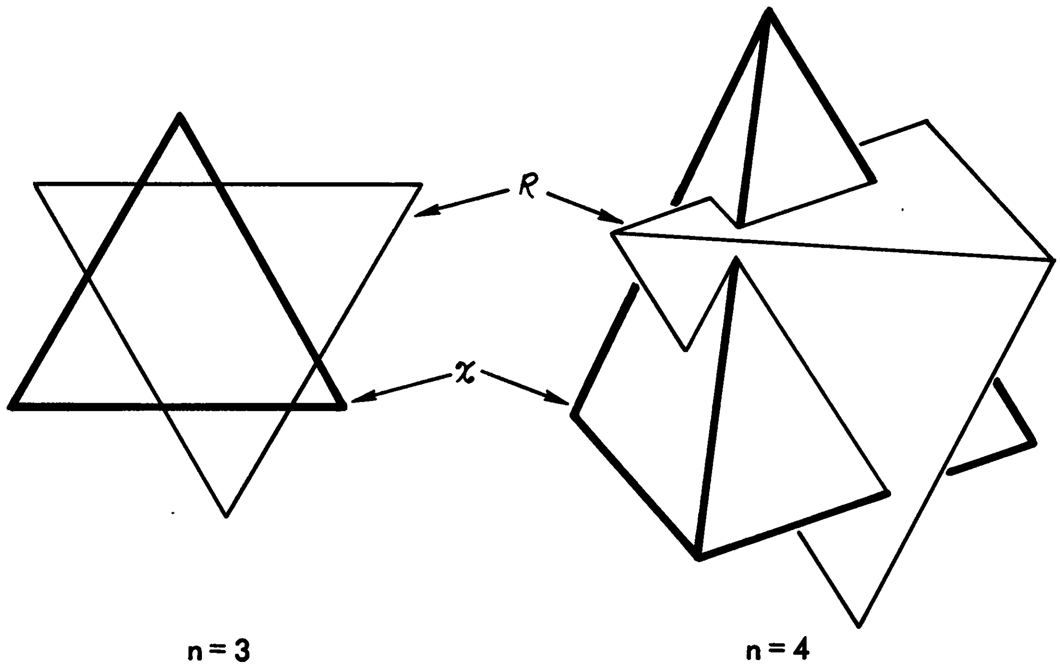

Fig. 2—Illustrating the interlocking of $\mathcal{X}$ and R

(d) <u>if</u> Γ <u>is zero-monotonic, then no vertex of</u> $\mathcal{X}(\Gamma)$ <u>is interior to</u> $\mathcal{R}(\Gamma)$.

<u>Proof</u>. (a) Let $(i_1, \ldots, i_n)$ be any ordering of N. By (2.18) we have

$$r_{i_1} \geqq v(\{i_1\}) \tag{2.19}$$

and, for k = 2, ..., n,

$$r_{i_k} \geqq v(\{i_1, \ldots, i_k\}) - v(\{i_1, \ldots, i_{k-1}\}). \tag{2.20}$$

Hence, summing, we obtain*

$$r(N) \geqq v(N), \tag{2.21}$$

which shows that the conditions of (2.17), i.e., $x(N) = v(N)$ and $x_i \leqq r_i$, $i \in N$, are consistent. $\mathcal{R}(\Gamma)$ is therefore nonempty, as claimed, and clearly is an (n - 1)-dimensional simplex unless $r(N) = v(N)$. But that would require equality throughout (2.19) and (2.20), for all orderings of N, which makes v additive and gives us $\mathcal{R}(\Gamma) = \mathcal{X}(\Gamma) = \{(r_1, \ldots, r_n)\}$.

(b) By (2.21) and (2.1) we have

$$r(N) \geqq v(N) \geqq \sum_{i \in N} v(\{i\}),$$

---

*We write $r(S)$ for $\Sigma_{i \in S}\, r_i$, as at (2.2).

so there is a number α satisfying $0 \leqq \alpha \leqq 1$ and

$$\alpha r(N) + (1 - \alpha) \sum_{i \in N} v(\{i\}) = v(N).$$

Define the vector y by $y_i = \alpha r_i + (1 - \alpha)v(\{i\})$, $i = 1, \ldots, n$. Then $y \in \mathcal{X}^*(\Gamma)$ and we have

$$r_i \geqq y_i \geqq v(\{i\}), \qquad i = 1, \ldots, n,$$

by (2.19). Hence y belongs to both $\mathcal{R}(\Gamma)$ and $\mathcal{X}(\Gamma)$. Moreover, if $r_i > v(\{i\})$, all $i \in N$, we can conclude that $r_i > y_i$, all $i \in N$, since $r(N) > v(N)$ implies that $\alpha < 1$; in other words, y is interior to $\mathcal{R}(\Gamma)$. Similarly, if in addition $\mathcal{X}(\Gamma)$ is not a single point, we can conclude that $y_i > v(\{i\})$, all $i \in N$, since $v(N) > \Sigma\, v(\{i\})$ implies that $\alpha > 0$; in other words, y is interior to $\mathcal{X}(\Gamma)$ as well.

(c) Take a typical vertex of $\mathcal{R}(\Gamma)$:

$$(2.22) \qquad x^{(j)} = (r_1, \ldots, r_{j-1}, x_j^{(j)}, r_{j+1}, \ldots, r_n),$$

where $x_j^{(j)} = v(N) - r(N - \{j\})$. Summing the $n - 1$ inequalities (2.20) with $i_1 = j$ gives us

$$r(N - \{j\}) \geqq v(N) - v(\{j\}).$$

Hence $x_j^{(j)} \leqq v(\{j\})$, showing that $x^{(j)}$ is not interior to $\mathcal{X}(\Gamma)$.

(d) Take a typical vertex of $\mathcal{X}(\Gamma)$:

$$y^{(j)} = (v(\{1\}), \ldots, v(\{j - 1\}), y_j^{(j)}, v(\{j + 1\}), \ldots, v(\{n\}))$$

where $y_j^{(j)} = v(N) - \Sigma_{i \neq j} v(\{i\})$. Zero-monotonicity implies the following two inequalities for any $S \subset N$:

$$(2.23) \quad v(S) \geqq \sum_{i \in S} v(\{i\}); \qquad v(N - S) \leqq v(N) - \sum_{i \in S} v(\{i\}).$$

For some T containing j we have $r_j = v(T) - v(T - \{j\})$. Hence, applying (2.23), we have

$$r_j \leqq v(N) - \sum_{i \in N-T} v(\{i\}) - \sum_{i \in T-\{j\}} v(\{i\}) = y_j^{(j)},$$

showing that $y^{(j)}$ is not interior to $\mathcal{R}(\Gamma)$. Q.E.D.

Next we show that the core and the least-core are "reasonable." In Sec. 3 we shall do the same for the kernel and pre-kernel (Theorem 3.13).*

*Milnor [1952] originally showed that the Shapley value and the von Neumann-Morgenstern solutions are "reasonable" in this sense (see also Luce and Raiffa, loc. cit.). The proof for the kernel was first given by Wesley [1971].

THEOREM 2.10. If $\Gamma \equiv (N; v)$ is any game, then $\mathcal{C}(\Gamma) \subset \mathcal{R}(\Gamma)$ and $\mathcal{LC}(\Gamma) \subset \mathcal{R}(\Gamma)$.

Proof. (a) Let $x \in \mathcal{C}(\Gamma)$; then in particular

$$x(N - \{i\}) \geqq v(N - \{i\}), \qquad i = 1, 2, \ldots, n.$$

Since $x(N) = v(N)$, this implies

$$x_i \leqq v(N) - v(N - \{i\}) \leqq r_i, \qquad i = 1, 2, \ldots, n.$$

by (2.18). Hence $x \in \mathcal{R}(\Gamma)$.

(b) Let $x \in \mathcal{LC}(\Gamma)$ and let $\mathfrak{T}$ be the set of $T \subset N$, $T \neq \phi$, N such that

$$(2.24) \qquad e(T, x) = \operatorname*{Max}_{S \neq \phi, N} e(S, x) = \epsilon_0(\Gamma)$$

(see (2.9)). Then each $i \in N$ must belong to at least one $T \in \mathfrak{T}$, otherwise a smaller minimum could be obtained in (2.9) by increasing x_i slightly while diminishing all the other x_j. For each i, let $T_i \in \mathfrak{T}$ be such that $i \in T_i$. If $T_i \neq \{i\}$, then by (2.24)

$$e(T_i, x) \geqq e(T_i - \{i\}, x);$$

whence

$$x_i \leqq v(T_i) - v(T_i - \{i\}) \leqq r_i,$$

as required. If $T_i = \{i\}$ and $e(T_i, x) = \epsilon_0(\Gamma) \geqq 0$, then again

$$x_i \leqq v(\{i\}) = v(\{i\}) - v(\phi) \leqq r_i.$$

Finally, if $\epsilon_0(\Gamma) < 0$ then x is in the core and hence in $\mathcal{R}(\Gamma)$ by part (a). Q.E.D.

To finish this section, we identify yet another "critical value" for ϵ in connection with the strong ϵ-core. Define

$$(2.25) \qquad \epsilon_2(\Gamma) = \max_{S \neq \phi, N} \big(v(S) - v(N) + r(N - S)\big).$$

The following lemma is comparable to Lemma 2.4.

<u>LEMMA 2.11</u>. $\mathcal{C}_\epsilon(\Gamma) \supset \mathcal{R}(\Gamma)$ <u>if and only if</u> $\epsilon \geqq \epsilon_2(\Gamma)$.

<u>Proof</u>. Let $x^{(j)}$ be the j-th extreme point of $\mathcal{R}(\Gamma)$, as at (2.22). For any ϵ, we have

$$\mathcal{R}(\Gamma) \subset \mathcal{C}_\epsilon(\Gamma) \quad \Leftrightarrow x^{\overline{(j)}} \in \mathcal{C}_\epsilon(\Gamma), \text{ all } j \in N$$

$$\Leftrightarrow x^{(j)}(S) \geqq v(S) - \epsilon, \text{ all } j \in N \text{ \underline{and} } S \neq \emptyset, N$$

$$\Leftrightarrow \begin{Bmatrix} x_j^{(j)} + r(S - \{j\}) \geqq v(S) - \epsilon, & \text{all } j \in S, \\ \text{\underline{and} } r(S) \geqq v(S) - \epsilon, & \text{all } j \notin S \end{Bmatrix}$$

$$\text{all } S \neq \emptyset, N.$$

But the lower line in the brackets is superfluous, being implied by the first,* so we may conclude that

$$\mathcal{R}(\Gamma) \subset \mathcal{C}_\epsilon(\Gamma) \quad \Leftrightarrow \quad v(N) - r(N - \{j\}) + r(S - \{j\}) \geqq v(S) - \epsilon,$$

$$\text{all } j \in S,\ S \neq \phi,\ N$$

$$\Leftrightarrow \epsilon \geqq v(S) - v(N) + r(N - S), \text{ all } S \neq \phi, N$$

$$\Leftrightarrow \epsilon \geqq \epsilon_2(\Gamma). \qquad \text{Q.E.D.}$$

The critical values $\epsilon_1(\Gamma)$ and $\epsilon_2(\Gamma)$ may occur in either order, although of course, both are $\geqq \epsilon_0(\Gamma)$. In Sec. 3.5 we shall learn more about their relationship.

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*Proof: Since $S \neq \phi$ we may take $i \in S$ and obtain the lower inequality from the upper as follows:

$$\begin{aligned} r(S) &\geqq r(S - \{i\}) + r_i - r(N) + v(N) \\ &= r(S - \{i\}) + x_i^{(i)} \\ &\geqq v(S) - \epsilon. \end{aligned}$$

3. A GEOMETRIC CHARACTERIZATION OF THE KERNEL

The kernel of a cooperative game has been the subject of many studies, starting with those of Davis and Maschler [1965] and Maschler and Peleg [1966, 1967]. One of the surprising by-products of the 1966 paper was the discovery that if, for any ϵ, the set $\mathcal{C}_\epsilon \cap \mathcal{X}$ is not empty, then the kernel (for the grand coalition) intersects this set. The surprise stemmed from the fact that on the face of it, the definition of the kernel seemed to be completely unrelated to any idea connected with the core. Pursuing this observation, Schmeidler [1969] was led to the discovery of the nucleolus of a game, which is a unique point in the kernel on the one hand, and, on the other hand, is much more directly related in its definition to the strong ϵ-core concept. (See Sec. 6, below.)

In this section we shall show that the part of the kernel that is located in the strong ϵ-core of the game, for any ϵ such that $\mathcal{C}_\epsilon \cap \mathcal{X}$ is not empty, occupies there a well-defined central position. In view of Lemma 2.4 (or Lemma 2.11), we can guarantee, by choosing ϵ large enough, that the entire kernel is contained in $\mathcal{C}_\epsilon$; our results can therefore also be interpreted as a geometric characterization of the entire kernel.

3.1. Definitions

Let $\Gamma \equiv (N; v)$. For $i, j \in N$, $i \neq j$, we denote by $\mathcal{T}_{ij}$ the set of coalitions containing i but not j, thus:

(3.1) $$\mathcal{T}_{ij} = \{S: S \subset N,\ i \in S,\ j \notin S\}.$$

For each pre-imputation x in $\mathcal{X}^*(\Gamma)$ we define the maximum surplus of i over j to be

(3.2) $$s_{ij}(x) = \max_{S \in \mathcal{T}_{ij}} e(S, x).$$

We say that i outweighs j at x if $s_{ij}(x) > s_{ji}(x)$, and that i and j are in equilibrium at x if neither of these players outweighs the other, that is, if

(3.3) $$s_{ij}(x) = s_{ji}(x).$$

These concepts, however, are relative to $\mathcal{X}^*(\Gamma)$. More generally, let Y be any closed, convex polyhedron in $\mathcal{X}^*(\Gamma)$ and let x be any member of Y. Then we say that i outweighs j at x with respect to Y if $s_{ij}(x) > s_{ji}(x)$ and, for all sufficiently small $\delta > 0$, the pre-imputation obtained by taking δ from x_j and adding it to x_i lies in Y.* In particular, for $x \in \mathcal{X}(\Gamma)$, i outweighs j with respect to $\mathcal{X}(\Gamma)$ if and only if

(3.4) $$s_{ij}(x) > s_{ji}(x) \quad \text{and} \quad x_j > v(\{j\}),$$

so that the equilibrium condition for imputations becomes

*Intuitively, a player "with his back to the wall" cannot be pushed.

(3.5) $$[s_{ij}(x) - s_{ji}(x)][x_j - v(\{j\})] \leqq 0,$$

together with the same inequality with i and j transposed.

DEFINITION 3.1: Let $Y \subset \mathcal{X}^*(\Gamma)$. The kernel for Y of the game Γ (with respect to the grand coalition*) is the set of $x \in Y$ at which every two players are in equilibrium with respect to Y; it is denoted $\mathcal{K}_Y(\Gamma)$. The kernel for $\mathcal{X}(\Gamma)$ is called simply the kernel of Γ and is denoted $\mathcal{K}(\Gamma)$, while the kernel for $\mathcal{X}^*(\Gamma)$ is called the pre-kernel of Γ and is denoted $\mathcal{K}^*(\Gamma)$.

Thus, an imputation is in the kernel if and only if it satisfies (3.5) for all $i, j \in N$, $i \neq j$, while a pre-imputation is in the pre-kernel if and only if it satisfies the simpler requirement (3.3) for all $i, j \in N$, $i < j$.

It is known that the kernel and pre-kernel are always nonempty.** From the above we see that the pre-kernel is the simplest of all the kernels to compute and to work with in other ways. Moreover, it is easily seen that $\mathcal{K}^*(\Gamma) \cap Y$ is a subset of $\mathcal{K}_Y(\Gamma)$, for any $Y \subset \mathcal{X}^*(\Gamma)$. In particular, any imputation in the pre-kernel is in the kernel as well. We shall soon see that in many cases the pre-kernel gives us even more information about the kernel.

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*See Remark 3.19 at the end of this section.

**See Davis and Maschler [1965], Maschler and Peleg [1966, 1967], and Maschler, Peleg, and Shapley [1972]. Note that condition (2.1) is essential for a nonempty kernel, since otherwise $\mathcal{X}(\Gamma)$ itself is empty.

The pseudo-kernel, an auxiliary concept used in several of the earlier papers cited, is the kernel for the set $\mathcal{X}_+$ of nonnegative elements of $\mathcal{X}^*(\Gamma)$.

<u>THEOREM 3.2.</u> <u>If</u> $\Gamma \equiv (N; v)$ <u>is any game, then</u> $\mathcal{K}(\Gamma) \cap \mathcal{C}(\Gamma) = \mathcal{K}^*(\Gamma) \cap \mathcal{C}(\Gamma)$.

<u>Proof</u>. Clearly (3.3) implies (3.5); hence, every imputation, and in particular every core imputation, that satisfies (3.3) lies in the kernel. Conversely, suppose $x \in \mathcal{K}(\Gamma) \cap \mathcal{C}(\Gamma)$ and let $i, j \in N$, $i \neq j$. It is sufficient to show that $s_{ij}(x) \leq s_{ji}(x)$. Indeed, if not, then, by (3.5) and (2.7), $x_j = v(\{j\})$. Since $\{j\} \in \mathcal{T}_{ji}$ (see(3.1)), it follows from (3.2) and (2.3) that $s_{ji}(x) \geq v(\{j\}) - x_j = 0$. Consequently, $s_{ij}(x) > 0$. By (2.7), x does not belong to the core of Γ, which is a contradiction.

<u>THEOREM 3.3.</u> <u>If</u> Γ <u>is zero-monotonic, then</u> $\mathcal{K}(\Gamma) = \mathcal{K}^*(\Gamma)$.

This important theorem is proved in Maschler, Peleg, and Shapley [1972; Theorem 2.7]. In Section 4 we shall extend this result to a wider class of games.

3.2. The Bisection Property of the Kernel

We now begin the geometrical characterization of the kernel. Let ϵ be such that $\mathcal{C}_\epsilon(\Gamma) \neq \phi$, i.e., $\epsilon \geq \epsilon_0(\Gamma)$, and let x be a point in $\mathcal{C}_\epsilon(\Gamma)$. Consider the ray (half-line) emanating from x obtained by letting x_j increase and x_i decrease by the same amount, for a fixed pair of players $\{i, j\}$. Denote by $\delta^\epsilon_{ij}(x)$ the maximum amount which can be transferred from i to j in this way while remaining in $\mathcal{C}_\epsilon(\Gamma)$. Thus, if u^ν represents the ν-th unit vector, then

(3.6) $$\delta^{\epsilon}_{ij}(x) = \mathrm{Max}\ \{\delta\colon x - \delta u^i + \delta u^j \in \mathcal{C}_\epsilon(\Gamma)\}\ .$$

This is well defined for all x in $\mathcal{C}_\epsilon(\Gamma)$, and is obviously nonnegative.

<u>LEMMA 3.4</u>. <u>Let</u> $x \in \mathcal{C}_\epsilon(\Gamma)$. <u>Then</u>

(3.7) $$\delta^{\epsilon}_{ij}(x) = \epsilon - s_{ij}(x)$$

<u>for all</u> $i, j \in N$, $i \neq j$.

<u>Proof</u>. If x_i decreases by δ and x_j increases by δ, the following changes occur:

(i) $e(S, x)$ increases by δ whenever $S \in \mathcal{T}_{ij}$;
(ii) $e(S, x)$ decreases by δ whenever $S \in \mathcal{T}_{ji}$;
(iii) $e(S, x)$ remains unchanged otherwise.

Let S_0 be a coalition in $\mathcal{T}_{ij}$ for which $s_{ij}(x) = e(S_0, x)$. By (2.8), and because $x \in \mathcal{C}_\epsilon(\Gamma)$, we have $e(S_0, x) + \delta^{\epsilon}_{ij}(x) = \epsilon$, and so (3.7) follows. Q.E.D.

Observe that $s_{ij}(x)$ does not depend on ϵ. This fact enables us to conclude the following corollary concerning the "faces" of $\mathcal{C}_\epsilon$. First we define

(3.8) $$\mathcal{S}_{ij}(x) = \{S \in \mathcal{T}_{ij}\colon e(S, x) = s_{ij}(x)\},$$

and

$$\mathcal{S}(x) = \bigcup_{\substack{i,j \in N \\ i \neq j}} \mathcal{S}_{ij}(x). \tag{3.9}$$

That is, $\mathcal{S}(x)$ comprises all those coalitions that are significant at x, in that they enable some player to achieve his surplus over some other player.

COROLLARY 3.5. *Let* $x \in \mathcal{C}_\varepsilon(\Gamma)$. *Then*

$$H_S^\varepsilon \cap \mathcal{C}_\varepsilon(\Gamma) \neq \emptyset \tag{3.10}$$

holds for all $S \in \mathcal{S}(x)$.

The next two theorems set forth the bisection property in detail. Not surprisingly, the statement for the prekernel is simpler than that for the kernel. The geometry of the situation will be illustrated in Sec. 3.3.

DEFINITION 3.6: Let $x \in \mathcal{C}_\varepsilon(\Gamma)$. For each $i, j \in N$, $i \neq j$, denote by $R_{ij}(x, \varepsilon)$ the line segment with endpoints

$$x - \delta_{ij}^\varepsilon(x)u^i + \delta_{ij}^\varepsilon(x)u^j \text{ and } x + \delta_{ji}^\varepsilon(x)u^i - \delta_{ji}^\varepsilon(x)u^j. \tag{3.11}$$

This will be called the *segment through* x *in the* (i-j) *direction*; note that $R_{ij}(x, \varepsilon) = R_{ji}(x, \varepsilon)$.

THEOREM 3.7. *If* $\Gamma \equiv (N; v)$ *and if* $x \in \mathcal{C}_\varepsilon(\Gamma)$, *then* x *belongs to* $\mathcal{K}^*(\Gamma) \cap \mathcal{C}_\varepsilon(\Gamma)$

if and only if for each $i, j \in N$, $i \neq j$, x *bisects the segment* $R_{ij}(x, \epsilon)$.

Proof. Formula (3.3) and Lemma 3.4.

THEOREM 3.8. *Let* $\Gamma \equiv (N; v)$. *Then*:

(a) *If* $x \in \mathcal{C}(\Gamma)$, *then* x *belongs to* $\mathcal{K}(\Gamma) \cap \mathcal{C}(\Gamma)$ *if and only if for each* $i, j \in N$, $i \neq j$, x *bisects the segment* $R_{ij}(x, 0)$.

(b) *If* $x \in \mathcal{C}_\epsilon(\Gamma)$ *and* Γ *is zero-monotonic, then* x *belongs to* $\mathcal{K}(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma)$ *if and only if for each* $i, j \in N$, $i \neq j$, x *bisects the segment* $R_{ij}(x, \epsilon)$.

(c) *If* $x \in \mathcal{C}_\epsilon(\Gamma)$ (*but* Γ *is not necessarily zero-monotonic*), *then* x *belongs to* $\mathcal{K}(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma)$ *if and only if* $x \in \mathcal{X}(\Gamma)$ *and, for each* $i, j \in N$, $i \neq j$, *either* x *bisects the segment* $R_{ij}(x, \epsilon)$ *or* $x_j = v(\{j\})$, *and* $\delta^\epsilon_{ji}(x) > \delta^\epsilon_{ij}(x)$, *or* $x_i = v(\{i\})$ *and* $\delta^\epsilon_{ij}(x) > \delta^\epsilon_{ji}(x)$.

Proof. (a) Theorems 3.2 and 3.7. (b) Theorems 3.3 and 3.7. (c) Formulas (3.3) and (3.5) and Lemma 3.4.

REMARK 3.9. Suppose the core not empty, and consider the *bisecting hypersurfaces* B_{ij}, obtained by taking the mid-points of all the segments in the (i-j) direction through points of the core. Since $B_{ij} = B_{ji}$, there are $\binom{n}{2}$ such hypersurfaces, and according to Theorem 3.8(a) the set $\mathcal{C}(\Gamma) \cap \mathcal{K}(\Gamma)$ is their intersection. But this set is known to be nonempty (see Maschler and Peleg [1966, Theorem 5.4]).

Thus, we have discovered an interesting geometric property of the core itself, namely, whenever the core is not empty, its bisecting hypersurfaces $\bar{B}_{ij}$ must have a nonempty intersection. This is certainly not true in general for an arbitrary convex polyhedron in $\mathcal{X}^*(\Gamma)$, since the number of (i-j) directions and associated hyper-surfaces may greatly exceed the dimension of the space.

<u>COROLLARY 3.10</u>. <u>If</u> Γ <u>and</u> Γ' <u>are two games having the same core</u>: $\mathcal{C}(\Gamma) = \mathcal{C}(\Gamma')$, <u>then</u>

$$\mathcal{K}(\Gamma) \cap \mathcal{C}(\Gamma) = \mathcal{K}(\Gamma') \cap \mathcal{C}(\Gamma'). \tag{3.12}$$

<u>Proof</u>. By Theorem 3.8(a), $\mathcal{K}(\Gamma) \cap \mathcal{C}(\Gamma)$ depends only on the point set $\mathcal{C}(\Gamma)$, i.e., on its geometrical shape. Q.E.D.

Similarly, using Theorems 3.7 and 3.8(c), one can establish

<u>COROLLARY 3.11</u>. (a) <u>If</u> Γ, Γ', ϵ, ϵ' <u>are such that</u> $\mathcal{C}_\epsilon(\Gamma) = \mathcal{C}_{\epsilon'}(\Gamma')$, <u>then</u>

$$\mathcal{K}^*(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma) = \mathcal{K}^*(\Gamma') \cap \mathcal{C}_{\epsilon'}(\Gamma').$$

(b) <u>If</u> Γ, Γ', ϵ, ϵ' <u>are such that</u> $\mathcal{X}(\Gamma) = \mathcal{X}(\Gamma')$ <u>and</u> $\mathcal{C}_\epsilon(\Gamma) = \mathcal{C}_{\epsilon'}(\Gamma')$, <u>then</u>

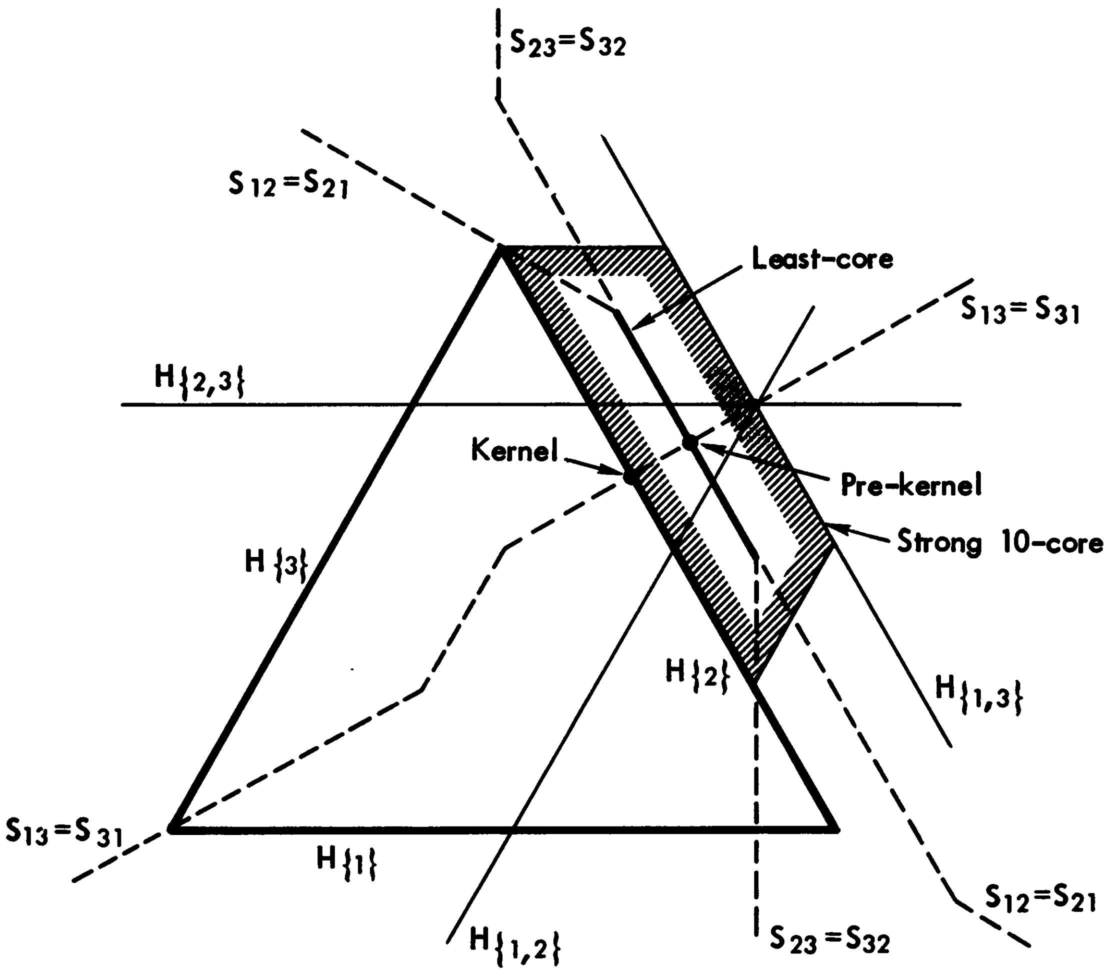
$$\mathcal{K}(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma) = \mathcal{K}(\Gamma') \cap \mathcal{C}_{\epsilon'}(\Gamma').$$

REMARK 3.12. Let Γ and Γ' be two games for which $\mathcal{X}(\Gamma) = \mathcal{X}(\Gamma')$ and $\mathcal{C}_\epsilon(\Gamma) = \mathcal{C}_{\epsilon'}(\Gamma')$ for some ϵ, ϵ', and suppose further that one of them, say Γ, is zero-monotonic. Then it follows from Corollary 3.11 that dropping the alternatives involving $\delta^{\epsilon}_{ji}(x) \neq \delta^{\epsilon}_{ij}(x)$ from Theorem 3.8(c) does not remove any points from the locus defined there. Moreover, for $x \in \mathcal{C}_{\epsilon'}(\Gamma')$, we can assert that $x \in \mathcal{K}(\Gamma')$ if and only if $s_{ij}(x) = s_{ji}(x)$ for all $i, j \in N$, $i \neq j$, where these surpluses may be taken with respect to either Γ or Γ'.

3.3. Examples

The first example illustrates the difference between the kernel and the pre-kernel, using a three-person game that is not zero-monotonic (Fig. 3). The *core* happens to be empty, but at $\epsilon = 5$ the strong ϵ-core comes into view, situated outside the imputation simplex $\mathcal{X}$. (Compare Theorem 2.7.) The pre-kernel $\mathcal{K}^*$ turns out to be its midpoint. If we continue to increase ϵ, the strong ϵ-core ($\mathcal{C}_{10}$, shaded) at last touches $\mathcal{X}$, and the kernel $\mathcal{K}$ turns out to be the midpoint of $\mathcal{X} \cap \mathcal{C}_{10}$. The determination of $\mathcal{K}$ and $\mathcal{K}^*$ may be visualized from the broken lines in the figure, which indicate the bisection hypersurfaces where $s_{ij} = s_{ji}$ for the three pairs $\{i, j\}$. At $\mathcal{K}^*$ all three equalities are satisfied, while at $\mathcal{K}$ two of them are replaced by inequalities, in view of the fact that $x_1 = 0$.

The next example shows that two games may have the same core and yet have kernels that differ outside the core. (Cf. Corollary 3.10.) Consider the five-person game $\Gamma_1 \equiv (\{1, 2, 3, 4, 5\}; v_1)$ where



$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0,$
$v(\{1,2\}) = 20, \quad v(\{1,3\}) = 50, \quad v(\{2,3\}) = 30,$
$v(\{1,2,3\}) = 40.$
$\epsilon_0 = 5$
$\mathcal{K} = \{(25, 0, 15)\}, \quad \mathcal{K}^* = \{(-5, 17.5, 27.5)\}$

Fig. 3—The kernel and pre-kernel in a game that is not zero-monotonic, showing the bisecting hypersurfaces

$$v_1(\{1,2,3,4,5\}) = 7,$$

$$v_1(S) = 4 \text{ for } S = \{1,2,4\}, \{1,2,5\}, \{1,3,4\}, \{1,3,5\}, \{2,3,4\}, \{2,3,5\}, \text{and } \{4,5\},$$

$$v_1(S) = 0 \text{ otherwise.*}$$

It is easy to see that the core of Γ_1 consists of just the point (1, 1, 1, 2, 2). But the kernel turns out to be a line segment extending from the core to a point in the boundary of $\mathcal{X}(\Gamma_1)$; in fact it is

$$(3.13) \quad \mathcal{K}(\Gamma_1) = \{(t, t, t, \frac{7-3t}{2}, \frac{7-3t}{2}) : 0 \leqq t \leqq 1\}.$$

The pre-kernel is similar, but extends a short distance outside $\mathcal{X}(\Gamma_1)$ to the point given by $t = -.2$ in (3.13). (We omit the calculations.)

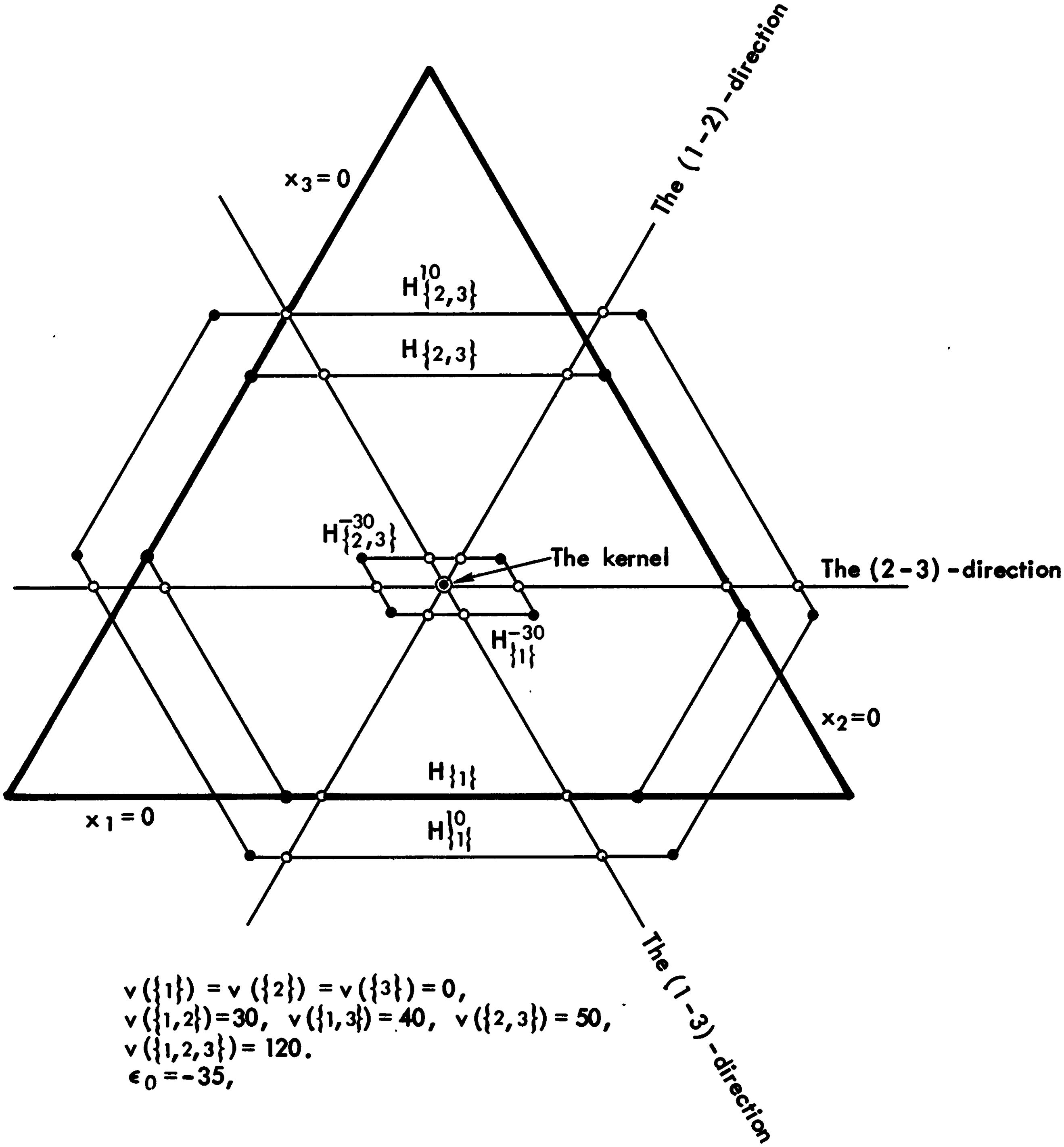
Now consider the game $\Gamma_2 = (\{1,2,3,4,5\}; v_2)$, where

$$v_2(\{2,3,5\}) = 0,$$

$$v_2(S) = v_1(S) \text{ otherwise.}$$

Γ_1 and Γ_2 have the same core and the same space of imputations; nevertheless it can be shown that the kernel and pre-kernel of Γ_2 now merely coincide with the core, i.e.,

*This is a modified form of an example in Peleg [1966].



$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0,$
$v(\{1,2\}) = 30, \quad v(\{1,3\}) = 40, \quad v(\{2,3\}) = 50,$
$v(\{1,2,3\}) = 120.$
$\epsilon_0 = -35,$

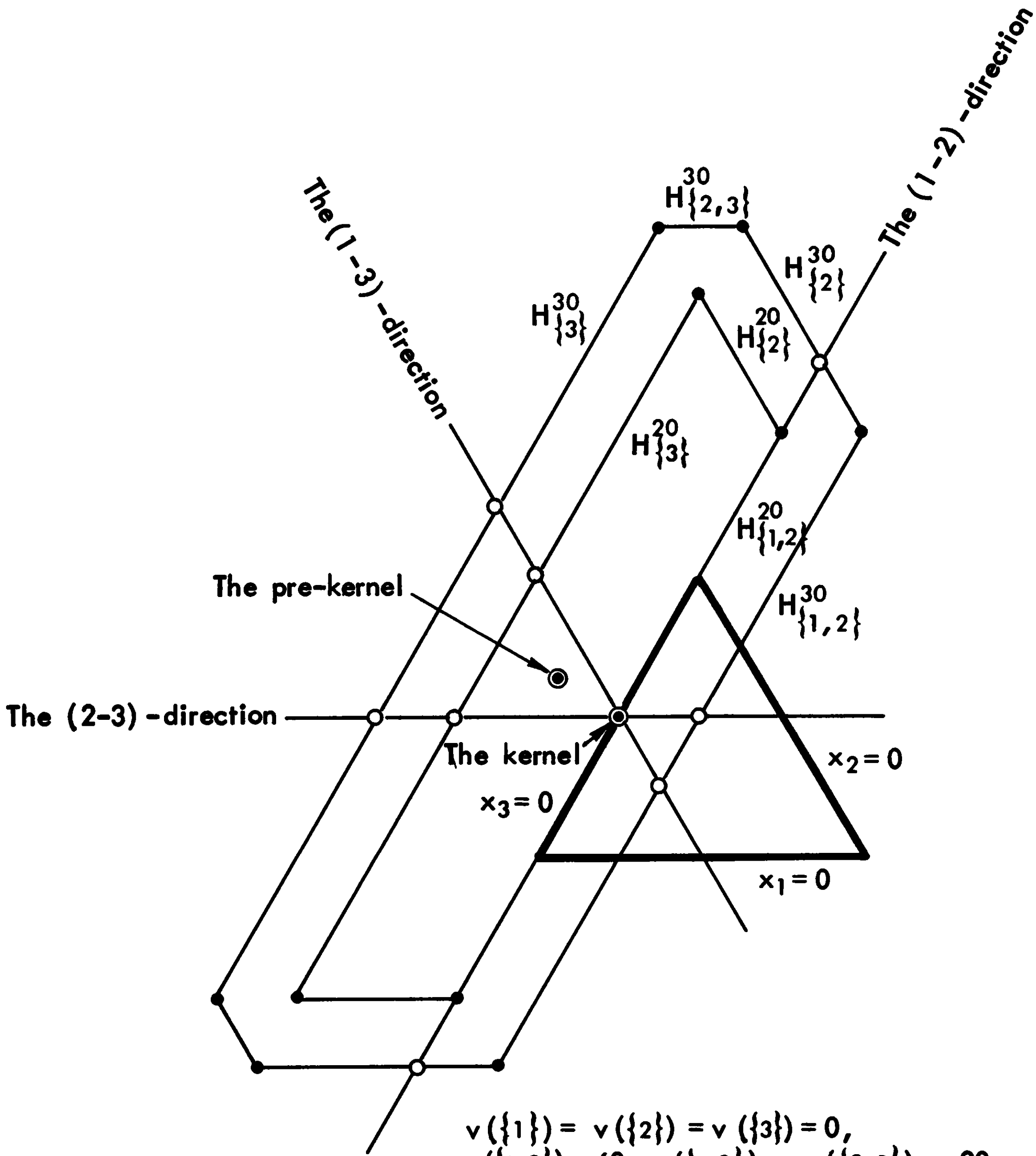
Fig. 4—The bisection property in a game that is zero-monotonic

consist of the single point (1,1,1,2,2).* So they are different from the kernel and pre-kernel of Γ_1.

The final two examples illustrate the bisection property (Theorem 3.8). The three-person game shown in Fig. 4 is zero-monotonic and its core is the large hexagon that just fits inside the imputation simplex. The kernel consists of the single point (35, 40, 45). We have chosen to illustrate $\mathcal{C}_\epsilon$ for $\epsilon = 10$, 0, and -30. The least-core $\mathcal{C}_{-35}$ is a short line segment, not shown. Although these strong ϵ-cores differ in shape, the kernel in each case is seen to bisect the three segments representing transfers between pairs of players.

Finally, Fig. 5 illustrates Theorem 3.8(c). The game is not unlike that of Fig. 3. The kernel consists of the single point (20, 20, 0) and the pre-kernel consists of the single point (25, 25, -10). We have illustrated $\mathcal{C}_\epsilon$ for $\epsilon = 20$ (the first value for which $\mathcal{C}_\epsilon(\Gamma) \cap \mathcal{X}(\Gamma) \neq \phi$) and $\epsilon = 30$. Note that the kernel bisects only the segments in the (1-2) direction. For the other directions, the parts which are outside the imputation simplex are longer than the parts inside.

*One way to prove this is to take a hypothetical x in $\mathcal{K} - \mathcal{C}$ or $\mathcal{K}^* - \mathcal{C}$ and show that the coalitions of highest excess are precisely $\{S: v(S) = 4\}$, thereby obtaining a contradiction. We omit the details.



$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0,$
$v(\{1,2\}) = 60, \ v(\{1,3\}) = v(\{2,3\}) = -20,$
$v(\{1,2,3\}) = 40.$
$\epsilon_0 = 20.$

Fig. 5—The bisection property in a game that is not zero-monotonic

3.4. Reasonableness of the Kernel and Pre-Kernel

The following theorem may be compared with Theorem 2.10 concerning the core and the least-core. It was first proved (for the kernel) by Eugene Wesley [1971].

THEOREM 3.13. *If* $\Gamma \equiv (N; v)$ *is any game, then* $\mathcal{K}(\Gamma) \subset \mathcal{R}(\Gamma)$ *and* $\mathcal{K}^*(\Gamma) \subset \mathcal{R}(\Gamma)$.

Proof. We shall prove both statements simultaneously. Suppose x is in $\mathcal{K}(\Gamma) \cup \mathcal{K}^*(\Gamma)$ but not in $\mathcal{R}(\Gamma)$. Then for some i_o we have $x_{i_o} > r_{i_o}$ (see (2.17)). That is,

$$x_{i_o} > v(S) - v(S - \{i_o\}) \tag{3.14}$$

holds for every $S \subset N$ containing i_o. In particular, we have $x_{i_o} > v(\{i_o\})$. Hence, for each $j \neq i_o$,

$$s_{i_o j}(x) \geqq s_{j i_o}(x) \tag{3.15}$$

holds if $x \in \mathcal{K}(\Gamma)$, by (3.5), while if $x \in \mathcal{K}^*(\Gamma)$ then it holds by (3.3). Moreover, by (3.2) we have, for each $j \neq i_o$,

$$\begin{aligned} s_{j i_o}(x) &\geqq v(N - \{i_o\}) - x(N - \{i_o\}) \\ &> v(N) - x_{i_o} - x(N - \{i_o\}) = 0, \end{aligned} \tag{3.16}$$

by (3.14) with $S = N$. Choose $k \neq i_o$ so that

$$(3.17) \qquad s_{ki_o}(x) = \max_{j:j \neq i_o} s_{ji_o}(x),$$

and choose $S_o \in \mathcal{T}_{i_o k}$ so that (see (3.2))

$$(3.18) \qquad s_{i_o k}(x) = e(S_o, x).$$

Now (3.14) implies that $e(S_o - \{i_o\}, x) > e(S_o, x)$. There are two cases. If $S_o \neq \{i_o\}$ we have a contradiction, since $j \in S_o - \{i_o\}$ implies

$$\begin{aligned} s_{ji_o}(x) &\geqq e(S_o - \{i_o\}, x) \\ &> e(S_o, x) = s_{i_o k}(x) \\ &\geqq s_{ki_o}(x) \geqq s_{ji_o}(x), \end{aligned}$$

by (3.2), (3.14), (3.18), (3.15), and (3.17) respectively. But if $S_o = \{i_o\}$ we also have a contradiction, since then

$$0 < s_{ki_o}(x) \leqq s_{i_o k}(x) = v(\{i_o\}) - x_{i_o} < 0,$$

by (3.16), (3.15), (3.18), and (3.14) respectively. This completes the proof.

3.5. The Bound $\epsilon_*(\Gamma)$

Our geometric characterizations thus far have not referred directly to the kernel, but only to its intersections with the strong ϵ-cores. It will be useful to have conditions on ϵ that guarantee that the strong ϵ-core will be so large that it surely contains the kernel. Of course, we already have two such conditions, namely

$$\epsilon \geqq \epsilon_1(\Gamma) = \underset{S\neq\phi,N}{\text{Max}} \; [v(S) - \sum_{i\in S} v(\{i\})],$$

which implies $\mathcal{C}_\epsilon(\Gamma) \supset \mathcal{X}(\Gamma) \supset \mathcal{K}(\Gamma)$ (see Lemma 2.4 and Definition 3.1), and

$$\epsilon \geqq \epsilon_2(\Gamma) = \underset{S\neq\phi,N}{\text{Max}} \; [v(S) + r(N - S) - v(N)],$$

which implies $\mathcal{C}_\epsilon(\Gamma) \supset \mathcal{R}(\Gamma) \supset \mathcal{K}(\Gamma)$ (see Lemma 2.11 and Theorem 3.13). These conditions are very weak. Of course, we can trivially combine them:

$$(3.19) \quad \epsilon \geqq \text{Min}\Big(\epsilon_1(\Gamma),\ \epsilon_2(\Gamma)\Big) = \text{Min}\Big(\text{Max} \;\ldots,\ \text{Max} \;\ldots\Big),$$

which implies __either__ $\mathcal{C}_\epsilon(\Gamma) \supset \mathcal{X}(\Gamma)$ __or__ $\mathcal{C}_\epsilon(\Gamma) \supset \mathcal{R}(\Gamma)$, but this represents no real progress. We would like a stronger condition on ϵ that guarantees $\mathcal{C}_\epsilon(\Gamma) \supset \mathcal{X}(\Gamma) \cap \mathcal{R}(\Gamma)$. Rather suprisingly, it is possible to invert the order of "Max" and "Min" in (3.19) and obtain a substantial improvement. In fact, the resulting bound $\epsilon_*(\Gamma)$, defined by

(3.20) $$\epsilon_*(\Gamma) = \underset{S \neq \phi, N}{\text{Max}} \ \text{Min}\left(v(S) - \sum_{i \in S} v(\{i\}), v(S) + r(N-S) - v(N)\right).$$

proves to be exactly the "critical value" of ϵ at which the strong ϵ-core just contains the intersection of $\mathcal{X}(\Gamma)$ and $\mathcal{R}(\Gamma)$.*

LEMMA 3.14. $\mathcal{C}_\epsilon(\Gamma) \supset \mathcal{X}(\Gamma) \cap \mathcal{R}(\Gamma)$ if and only if $\epsilon \geq \epsilon_*(\Gamma)$.

Proof. Without loss of generality, Γ is zero-normalized, making the r_i nonnegative. Then $x \in \mathcal{X}(\Gamma) \cap \mathcal{R}(\Gamma)$ implies that $0 \leq x_i \leq r_i$ for all $i \in N$, and of course $x(N) = v(N)$. Hence we have both

$$x(S) \geq 0$$

and

$$x(S) = x(N) - x(N - S) \geq v(N) - r(N - S)$$

for all $S \subset N$. Therefore,

*The first "Max" in (3.19) is likely to be attained when S is large, the second when N - S is large. Picture the strong $\epsilon_*(\Gamma)$-core snugly enclosing the set of reasonable imputations, but with some or all of the vertices of $\mathcal{R}(\Gamma)$ and $\mathcal{X}(\Gamma)$ "sticking out." (Fig. 2.)

$$v(S) - x(S) \leqq v(S) - \text{Max}\,[0,\ v(N) - r(N - S)]$$

$$= \text{Min}\,[v(S),\ v(S) - v(N) + r(N - S)]$$

$$\leqq \epsilon_*(\Gamma),$$

for all $S \neq \phi$, N. Hence $x \in \mathcal{C}_\epsilon(\Gamma)$ for all $\epsilon \geqq \epsilon_*(\Gamma)$, proving the assertion in one direction.

For the other direction, let the maximum in (3.20) be attained at S = T. Suppose first that $r(N - T) \geqq v(N)$. Then $\epsilon_*(\Gamma) = v(T)$. Define x by*

$$x_i = \begin{cases} 0, & \text{if } i \in T, \\ \left(\frac{v(N)}{r(N - T)}\right) r_i, & \text{if } i \in N - T. \end{cases}$$

It is easily verified that $x \in \mathcal{X}(\Gamma) \cap \mathcal{R}(\Gamma)$. However,

$$x(T) = 0 = v(T) - \epsilon_*(\Gamma),$$

so $x \notin \mathcal{C}_\epsilon$ for every $\epsilon < \epsilon_*(\Gamma)$, as we wished to prove.

Suppose finally that $r(N - T) < v(N)$. Then $\epsilon_*(\Gamma) = v(T) + r(N - T) - v(N)$. This time define x by

---

*In case $r(N - T) = v(N) = 0$ (trivializing $\mathcal{X}(\Gamma)$), we define $x_i = 0$ for all $i \in N$.

$$x_i = \begin{cases} \left(\dfrac{v(N) - r(N - T)}{r(N) - r(N - T)}\right) r_i, & \text{if } i \in T \\ r_i, & \text{if } i \in N - T. \end{cases}$$

Again it is clear that $x \in \mathcal{X}(\Gamma) \cap \mathcal{R}(\Gamma)$, since $r(N) \geqq v(N)$. However,

$$x(T) = v(N) - r(N - T) = v(T) - \epsilon_*(\Gamma),$$

so again $x \notin \mathcal{C}_\epsilon$ for every $\epsilon < \epsilon_*(\Gamma)$. This completes the proof.

THEOREM 3.15. *$\mathcal{K}(\Gamma) \subset \mathcal{C}_\epsilon(\Gamma)$ for all $\epsilon \geqq \epsilon_*(\Gamma)$.*

Proof. Definition 3.1, Theorem 3.13, and Lemma 3.14.

It would be interesting if a still better lower bound on ϵ could be found that ensures $\mathcal{K}(\Gamma) \subset \mathcal{C}_\epsilon(\Gamma)$. Any improvement that could be made would be reflected in sharper results in Sec. 4, and would add to our general knowledge of the kernel.

COROLLARY 3.16. *If Γ is zero-monotonic, then $\mathcal{K}^*(\Gamma) \subset \mathcal{C}_\epsilon(\Gamma)$ for all $\epsilon \geqq \epsilon_*(\Gamma)$.*

Proof. Theorems 3.3 and 3.15.

It may be that zero-monotonicity is not needed here. In other words, it may be that $\mathcal{K}^*$ is always in $\mathcal{C}_{\epsilon_*}$, even when it is not in $\mathcal{X} \cap \mathcal{R}$. But we have no proof.

COROLLARY 3.17. If Γ, Γ', ϵ, ϵ', are such that $\mathcal{X}(\Gamma) = \mathcal{X}(\Gamma')$ and $\mathcal{C}_\epsilon(\Gamma) = \mathcal{C}_{\epsilon'}(\Gamma')$, and if $\epsilon \geqq \epsilon_*(\Gamma)$ and $\epsilon' \geqq \epsilon_*(\Gamma')$, then $\mathcal{K}(\Gamma) = \mathcal{K}(\Gamma')$.

Proof. Corollary 3.11(b) and Theorem 3.15.

REMARK 3.18. Let $B(\Gamma, \epsilon)$ denote the intersection of the $\binom{n}{2}$ bisecting hypersurfaces of the strong ϵ-core of Γ.* By Theorem 3.8(b) it follows that for a game Γ that is zero-monotonic we have

$$\mathcal{K}(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma) = B(\Gamma, \epsilon); \tag{3.21}$$

moreover, for all $\epsilon \geqq \epsilon_*(\Gamma)$ we have

$$\mathcal{K}(\Gamma) = B(\Gamma, \epsilon). \tag{3.22}$$

But if Γ is not zero-monotonic, we can conclude only that the left-hand sides of (3.21) and (3.22) contain the respective right-hand sides.

REMARK 3.19. The discussions in this section have been limited to the particular kernels $\mathcal{K}_\mathcal{X} = \mathcal{K}$ and $\mathcal{K}_{\mathcal{X}*} = \mathcal{K}^*$, and have also been limited to the case of games without restrictive "coalition structures." It is clear that there are interesting extensions to be pursued. Similar results

*Compare Remark 3.9. Since any strong ϵ-core of a game is the core of another game (see Lemma 4.1), it follows that $B(\Gamma, \epsilon)$ is not empty if $\mathcal{C}_\epsilon(\Gamma)$ is not empty.

presumably can be obtained for other kernels, such as the <u>pseudo-kernel</u> $\mathcal{K}_{\mathcal{X}_+}$, about which quite a bit is already known.* Similar considerations can presumably also be developed for kernels with respect to coalition structures other than the "grand coalition" case that we treat here.** We may remark that in dealing with arbitrary coalition structures, however, it seems to be necessary to strengthen the condition of zero-monotonicity, which is the key to the equivalence of the kernel and pre-kernel (see Theorem 3.3), to a sweeping requirement that all of the reduced games that can arise, with respect to every payoff vector for the coalition structure in question, be zero-monotonic.***

---

*See Aumann, Peleg, and Rabinowitz [1965], Davis and Maschler [1965], and Maschler and Peleg [1966, 1967].

**See Aumann and Dreze [1974] for a recent survey of the coalition-structure approach as applied to various co-operative solution concepts in game theory.

***See Maschler and Peleg [1967], esp. Corollary 3.9.

4. SHIFTS, COVERS, AND QUASI-ZERO-MONOTONICITY

In this section we shall apply the results of Sec. 3 to extend the class of games for which the kernel can be shown to coincide with the pre-kernel or with a well-defined part of the pre-kernel.* The idea will be to take an arbitrary game that is not zero-monotonic, and try to find a zero-monotonic game that has the same strong ϵ-core, for some value of ϵ. If we succeed in this, then Corollary 3.11 and Theorem 3.3 will enable us to conclude that $\mathcal{K} \cap \mathcal{C}_\epsilon = \mathcal{K}^* \cap \mathcal{C}_\epsilon$ for the original game. Moreover, if we succeed in doing this with ϵ sufficiently large, in the sense of Theorem 3.15 (or any other criterion that ensures $\mathcal{C}_\epsilon \supset \mathcal{K}$), then we will be able to conclude that $\mathcal{K} = \mathcal{K}^* \cap \mathcal{C}_\epsilon$ for the original game, or perhaps even $\mathcal{K} = \mathcal{K}^* \cap \mathcal{X}$ or $\mathcal{K} = \mathcal{K}^*$. This makes the entire kernel easier to describe and compute with the aid of the bisection property.

For any game $\Gamma \equiv (N; v)$, we define a related family of "shifted" games $\Gamma_\epsilon \equiv (N; v_\epsilon)$ by

$$(4.1) \qquad v_\epsilon(S) = \begin{cases} v(S), & \text{if } S = \phi, N \\ v(S) - \epsilon, & \text{if } S \neq \phi, N. \end{cases}$$

Shifting a game obviously doesn't affect the pre-kernel, but it could change the kernel because the imputation space

*As this section is independent of what follows and is rather technical, some readers will prefer to turn directly to Section 5.

is different. Note that for (4.1) to define a game, we need

(4.2) $$\epsilon \geqq -\frac{1}{n}\left(v(N) - \sum_{i=1}^{n} v(\{i\})\right);$$

since otherwise (2.1) fails for v_ϵ. However, either of the conditions $\epsilon \geqq 0$ or $\epsilon \geqq \epsilon_0(\Gamma)$ suffices for (4.2), as short arguments based on (2.1) and (2.9) reveal.*

LEMMA 4.1. For all ϵ satisfying (4.2), we have $\mathcal{C}_\epsilon(\Gamma) = \mathcal{C}(\Gamma_\epsilon)$.

Proof. Definition 2.2 and (4.1).

4.1. The Cover of a Game

Let T be a non-empty set of players. A collection $\mathfrak{T} = \{T_1, T_2, \ldots, T_p\}$ of non-empty subsets of T is called balanced over T if there exist positive constants $\gamma_1, \gamma_2, \ldots, \gamma_p$ such that

(4.3) $$\sum_{j:i\in T_j} \gamma_j = 1$$

holds for each $i \in T$. The set of all such p-vectors of "balancing coefficients" for $\mathfrak{T}$ will be denoted $B(\mathfrak{T})$. $\mathfrak{T}$ is called minimal balanced over T if it is balanced over T and no proper subcollection of $\mathfrak{T}$ is balanced over T. It can be shown (see Shapley [1967]) that the balancing coeffi-

*Thus, (2.9) implies that $\epsilon_0(\Gamma) \geqq \mathrm{Min}_{x\in\mathcal{X}^*} \mathrm{Max}_{i\in N} [v(\{i\})-x_i]$, and it is clear that the minimum is achieved at the center of $\mathcal{X}(\Gamma)$, showing that $\epsilon_0(\Gamma)$ satisfies (4.2).

cients γ_j for $\mathfrak{T}$ are unique if and only if $\mathfrak{T}$ is minimal. We shall denote by τ the set of all balanced collections over T and by τ^m the set of all minimal balanced collections over T. Note that the inequality

$$(4.4) \qquad \sum_{j=1}^{p} \gamma_j \geqq 1$$

always holds, and holds strictly unless $\mathfrak{T}$ is the trivial collection {T}.*

DEFINITION 4.2. A game (T; u) that satisfies**

$$(4.5) \qquad u(T) \geqq \operatorname*{Max}_{\substack{\mathfrak{T}\in\tau \\ \gamma\in B(\mathfrak{T})}} \sum_{j=1}^{p} \gamma_j u(T_j)$$

is said to be balanced. A game (N; v) such that all of its restrictions (T; v), $\phi \neq T \subset N$, are balanced is said to be totally balanced. The cover*** of a game $\Gamma \equiv (N; v)$ is defined to be the game $\hat{\Gamma} \equiv (N; \hat{v})$, where $\hat{v}(\phi) = 0$ and

$$(4.6) \qquad \hat{v}(T) = \operatorname*{Max}_{\substack{\mathfrak{T}\in\tau \\ \gamma\in B(\mathfrak{T})}} \sum_{j=1}^{p} \gamma_j v(T_j)$$

*In contrast to the original definition in Shapley [1967], our present definition makes {T} a member of τ. Note also that we are requiring positive coefficients, not merely nonnegative as in Scarf [1967], Shapley and Shubik [1969], and elsewhere. For a simple geometric representation of the "balance" property, see Shapley [1973].

**Note that we could replace "τ" by "τ^m" in (4.5). Also, since $\{T\} \in \tau^m$, we could replace "$\geqq$" by "=".

***See Shapley and Shubik [1969]. This "totally balanced cover" is to be distinguished from the "monotonic cover," "superadditive cover," "exact cover," etc.

for each $\phi \neq T \subset N$. It is clear that $\hat{\Gamma}$ is totally balanced, and that it is the least totally balanced game that majorizes Γ; in other words, if (N; w) is totally balanced and $w \geqq v$, then $w \geqq \hat{v}$.

THEOREM 4.3. *A necessary and sufficient condition that a game have a nonempty core is that it be balanced.*

This theorem is basic to the study of the core. A proof may be found in Bondareva [1963] or Shapley [1967]; see also Gillies [1959, p. 71].

LEMMA 4.4. *Let* $\Gamma \equiv (N; v)$ *be a balanced game, and let* $\tilde{\Gamma} \equiv (N; \tilde{v})$ *satisfy*

$$(4.7) \qquad v(S) \leqq \tilde{v}(S) \leqq \hat{v}(S), \qquad \text{all } S \subset N.$$

Then $\mathcal{X}(\Gamma) = \mathcal{X}(\tilde{\Gamma}) = \mathcal{X}(\hat{\Gamma})$ *and* $\mathcal{C}(\Gamma) = \mathcal{C}(\tilde{\Gamma}) = \mathcal{C}(\hat{\Gamma})$.

Proof. We have $\hat{v}(N) = v(N)$ by (4.5) and (4.6), and $\hat{v}(\{i\}) = v(\{i\})$ for all $i \in N$ by (4.6); hence the imputation spaces agree: $\mathcal{X}(\Gamma) = \mathcal{X}(\hat{\Gamma}) = \mathcal{X}(\tilde{\Gamma})$. For a proof that $\mathcal{C}(\Gamma) = \mathcal{C}(\hat{\Gamma})$ see Shapley and Shubik [1969; Lemma 1]. The proof is completed by observing that $\mathcal{C}(\Gamma) \supset \mathcal{C}(\tilde{\Gamma}) \supset \mathcal{C}(\hat{\Gamma}) = \mathcal{C}(\Gamma)$.

Finally, we remark that "shifting" a game by a sufficiently large ϵ always makes it totally balanced. This may be seen by applying (4.1) and the strict form of (4.4) to the inequality (4.5); by increasing ϵ we make the right side of (4.5) go down faster than the left. Hence a new "critical value" for ϵ is born: we denote by $\epsilon_3(\Gamma)$ the *smallest value of* ϵ *for which* Γ_ϵ *is totally balanced*. Clearly $\epsilon_3(\Gamma) \geqq \epsilon_0(\Gamma)$, since $\epsilon_0(\Gamma)$ represents the smallest value of ϵ for which Γ_ϵ is balanced.

4.2. The Double Shift $\Delta(\Gamma, \epsilon)$

Now consider the following sequence of transformations which, for a given ϵ, take an arbitrary game Γ into a new game $\Delta = \Delta(\Gamma, \epsilon)$:

$$
(4.8) \qquad
\begin{array}{ccc}
\Gamma \equiv (N;\, v) & & \Delta \equiv (N;\, w) \\
\downarrow & & \uparrow \\
\Gamma_\epsilon & \longrightarrow & \hat{\Gamma}_\epsilon = \Delta_\epsilon
\end{array}
$$

In this process, we first "shift down" by ϵ, then we "take the cover," then we "shift up" by ϵ. The function w in (4.8) is given by

$$
(4.9) \qquad w(S) = \begin{cases} \hat{v}_\epsilon(S), & \text{if } S = \phi,\, N, \\ \hat{v}_\epsilon(S) + \epsilon, & \text{if } S \neq \phi,\, N \end{cases}
$$

(see (4.1) and (4.6)). More concisely, we could have written

$w = (\widehat{v_\epsilon})_{-\epsilon}$, revealing w as a kind of generalized "cover" of v. Note that always $w(S) \geq v(S)$.

To illustrate the double shift operation, we shall apply it to a five-person game $\Gamma \equiv (N; v)$ which is symmetric, in that the worth of a coalition S depends only on its size $|S|$.* (See Table 1.) This game is not balanced,

Table 1

$\|S\|$:	0	1	2	3	4	5
$v(S)$:	0	0	3	0	3	6
$v_1(S)$:	0	-1	2	-1	2	6
$w_1(S)$:	0	-1	2	3	4	6
$w(S)$:	0	0	3	4	5	6

as the two-person coalitions are too strong. Taking the cover of Γ itself would "push up" the value of $v(N)$ to $\hat{v}(N) = 7.5$ and would alter the imputation space.** But making the double shift with $\epsilon = 1$ (or indeed with any $\epsilon \geq 0.6$) avoids this problem and produces a game $\Delta = (N; w)$ which has the same imputation space as Γ and the same strong ϵ-core. In this case Δ turns out to be zero-monotonic, whereas Γ was not.

Several considerations enter into a suitable choice of ϵ for the double shift. Certainly we must have $\epsilon_0(\Gamma) \leq \epsilon < \epsilon_3(\Gamma)$; in other words, the game Γ_ϵ must be balanced

*See Remark 4.15, below.

**Taking the cover of a symmetric game simply means increasing v sufficiently to ensure that the ratio $v(S)/|S|$ never decreases. Thus, for each T, $\hat{v}(T) = \max\{|T|\, v(S)/|S| : 1 \leq |S| \leq |T|\}$.

but not totally balanced. If ϵ were less than $\epsilon_0(\Gamma)$ then $v(N) < w(N)$ and the imputation spaces would not agree, while if ϵ were not less than $\epsilon_3(\Gamma)$ then $\Gamma = \Delta$ and we would have accomplished nothing by the double shift. Indeed, if ϵ is only a little smaller than $\epsilon_3(\Gamma)$ the games Γ and Δ may be too similar to each other to reveal anything new about the kernel, while if ϵ is only a little larger than $\epsilon_0(\Gamma)$ the strong ϵ-core of Γ may give us too small a "window" on the region in which the kernel is known to lie. In fact, to be able to apply Theorem 3.15 we must have $\epsilon \geqq \epsilon_*(\Gamma)$, and this may not be possible, as $\epsilon_*(\Gamma)$ is not only $\geqq \epsilon_0(\Gamma)$ but may even be $> \epsilon_3(\Gamma)$.*

A further complication arises from the way the various critical values for ϵ change when we pass from Γ to Δ. It can be shown rather easily that $\epsilon_0(\Delta) = \text{Min}\ (\epsilon, \epsilon_0(\Gamma))$ and that $\epsilon_3(\Delta) = \text{Min}\ (\epsilon, \epsilon_3(\Gamma))$. But the behavior of ϵ_* is less predictable; we shall have more to say about this presently.

LEMMA 4.5. *Let* $\Gamma \equiv (N; v)$, *let* $\epsilon \geqq \epsilon_0(\Gamma)$, *let* $\Delta = \Delta(\Gamma, \epsilon) \equiv (N; w)$, *and, finally, let* $\tilde{\Delta} \equiv (N; \tilde{w})$ *be such that*

$$v(S) \leqq \tilde{w}(S) \leqq w(S), \qquad \text{all } S \subset N \tag{4.10}$$

(compare Lemma 4.4). Then

*Thus, for the v in Table 1 we have $\epsilon_0 = 0.6$, $\epsilon_* = 3$, and $\epsilon_3 = 9$.

(4.11) $$\mathcal{X}(\Gamma) = \mathcal{X}(\tilde{\Delta}) = \mathcal{X}(\Delta)$$

[redacted] and for all $\epsilon' \leq \epsilon$,

(4.12) $$\mathcal{C}_{\epsilon'}(\Gamma) = \mathcal{C}_{\epsilon'}(\tilde{\Delta}) = \mathcal{C}_{\epsilon'}(\Delta),$$

(4.13) $$\mathcal{K}(\Gamma) \cap \mathcal{C}_{\epsilon'}(\Gamma) = \mathcal{K}(\tilde{\Delta}) \cap \mathcal{C}_{\epsilon'}(\tilde{\Delta}) = \mathcal{K}(\Delta) \cap \mathcal{C}_{\epsilon'}(\Delta),$$

and

(4.14) $$\mathcal{K}^*(\Gamma) \cap \mathcal{C}_{\epsilon'}(\Gamma) = \mathcal{K}^*(\tilde{\Delta}) \cap \mathcal{C}_{\epsilon'}(\tilde{\Delta}) = \mathcal{K}^*(\Delta) \cap \mathcal{C}_{\epsilon'}(\Delta).$$

Proof. Using Lemma 4.1 and Definition 2.3, we have $\mathcal{C}(\Gamma_\epsilon) = \mathcal{C}_\epsilon(\Gamma) \neq \phi$. Hence, by Lemma 4.1 and 4.4, we have $\mathcal{C}(\Gamma_\epsilon) = \mathcal{C}(\hat{\Gamma}_\epsilon) = \mathcal{C}_\epsilon(\Delta)$. Hence $\mathcal{C}_\epsilon(\Gamma) = \mathcal{C}_\epsilon(\Delta)$, and, in particular, $v(N) = w(N)$. Then (4.10) gives us $\mathcal{C}_\epsilon(\Gamma) \supset \mathcal{C}_\epsilon(\tilde{\Delta}) \supset \mathcal{C}_\epsilon(\Delta) = \mathcal{C}_\epsilon(\Gamma)$, so (4.12) follows with the aid of Corollary 2.6. To prove (4.11), since we already have $v(N) = w(N)$ we need only observe that $v(\{i\}) = w(\{i\})$ for all $i \in N$ and apply (4.10). Finally, (4.13) and (4.14) are direct applications of Corollary 3.11, using (4.11) and (4.12).

The next lemma shows that double-shifting tends to shrink the reasonable set. Note that nothing is asserted for "intermediate" games in the sense of (4.7) (Lemma 4.4) or (4.10) (Lemma 4.5); this omission will be explained below in Remark 4.10.

LEMMA 4.6. *Let* $\epsilon \geqq \epsilon_0(\Gamma)$ *and* $\Delta = \Delta(\Gamma, \epsilon)$. *Then* $\mathcal{R}(\Delta) \subset \mathcal{R}(\Gamma)$.

Proof. We must show that $r_i(\Delta) \leqq r_i(\Gamma)$ for all $i \in N$. Fix i, and let T be a maximizing coalition in the definition of $r_i(\Delta)$ (see (2.18)), so that we have

$$r_i(\Delta) = w(T) - w(T - \{i\}). \tag{4.15}$$

There are several cases; let us first dispose of the easy ones. If $T = \{i\}$, then we have at once

$$r_i(\Delta) = w(\{i\}) = v(\{i\}) \leqq r_i(\Gamma),$$

by (4.11) and (2.19). If $T = N$, then since $w(N) = v(N)$ and $w(N - \{i\}) \geqq v(N - \{i\})$, we have

$$r_i(\Delta) = w(N) - w(N - \{i\}) \leqq v(N) - v(N - \{i\}) \leqq r_i(\Gamma).$$

There remains the general case: $\{i\} \subsetneqq T \subsetneqq N$. Let $\mathfrak{J} = \{T_1, \ldots, T_p\}$ be a balanced collection of subsets of T that achieves the maximum in (4.6) for the game $(N; v_\epsilon)$, so that for a suitable choice of coefficients $(\gamma_1, \ldots, \gamma_p) \in B(\mathfrak{J})$ we have

$$\hat{v}_\epsilon(T) = \sum_{j=1}^{p} \gamma_j v_\epsilon(T_j). \tag{4.16}$$

Define $S_j = T_j - \{i\}$, $j = 1, \ldots, p$. Then it is easy to see that the collection $\{S_j\}$ is balanced over $T - \{i\}$, by virtue of the same coefficients. (Note that if any two of the S_j are equal, because the corresponding T_j differ only by $\{i\}$, we must add together the corresponding γ_j.) It follows that

$$(4.17) \qquad \hat{v}_\epsilon(T - \{i\}) \geq \sum_{j=1}^{p} \gamma_j v_\epsilon(T_j - \{i\}),$$

by the definition of cover. Hence, we have

$$\begin{aligned} r_i(\Delta) &= \hat{v}_\epsilon(T) - \hat{v}_\epsilon(T - \{i\}) \\ &\leq \sum_{j=1}^{p} \gamma_j \Big(v_\epsilon(T_j) - v_\epsilon(T_j - \{i\})\Big) \\ &= \sum_{j:i\in T_j} \gamma_j \Big(v(T_j) - v(T_j - \{i\})\Big) \\ &\leq \sum_{j:i\in T_j} \gamma_j r_i(\Gamma) \\ &= r_i(\Gamma), \end{aligned}$$

applying successively (4.15) and (4.9), (4.16) and (4.17), (4.1), (2.18), and (4.3). Q.E.D.

<u>COROLLARY 4.7</u>. <u>If</u> Γ <u>is balanced, then</u> $\mathcal{R}(\hat{\Gamma}) \subset \mathcal{R}(\Gamma)$.

Proof. Since Γ is balanced its core is not empty. Hence $\epsilon_0(\Gamma) \leqq 0$ and we can apply Lemma 4.6 with $\epsilon = 0$.

COROLLARY 4.8. *If* $\Delta = \Delta(\Gamma, \epsilon)$ *and if* $\epsilon \geqq \epsilon_*(\Gamma)$, *then* $\epsilon_*(\Delta) \leqq \epsilon_*(\Gamma)$.

Proof. By Lemma 3.14 we have

$$\mathcal{C}_{\epsilon_*(\Gamma)}(\Gamma) \supset \mathcal{X}(\Gamma) \cap \mathcal{R}(\Gamma).$$

Lemma 4.5 with $\epsilon \geqq \epsilon' = \epsilon_*(\Gamma) \geqq \epsilon_0(\Gamma)$ gives us $\mathcal{X}(\Delta) = \mathcal{X}(\Gamma)$ and $\mathcal{C}_{\epsilon_*(\Gamma)}(\Delta) = \mathcal{C}_{\epsilon_*(\Gamma)}(\Gamma)$, and Lemma 4.6 gives us $\mathcal{R}(\Delta) \subset \mathcal{R}(\Gamma)$. Hence

$$\mathcal{C}_{\epsilon_*(\Gamma)}(\Delta) \supset \mathcal{X}(\Delta) \cap \mathcal{R}(\Delta),$$

so Lemma 3.14, applied to Δ, tells us that $\epsilon_*(\Delta) \leqq \epsilon_*(\Gamma)$. Q.E.D.

THEOREM 4.9. *If* $\Delta = \Delta(\Gamma, \epsilon)$ *and if* $\epsilon \geqq \epsilon_*(\Gamma)$, *then* $\mathcal{K}(\Delta) = \mathcal{K}(\Gamma)$.

Proof. Since $\epsilon \geqq \epsilon_*(\Gamma)$, Corollary 4.8 gives us $\epsilon \geqq \epsilon_*(\Delta)$ and Theorem 3.15 gives us

$$\mathcal{K}(\Delta) \subset \mathcal{C}_\epsilon(\Delta) \qquad \text{and} \qquad \mathcal{K}(\Gamma) \subset \mathcal{C}_\epsilon(\Gamma).$$

The result now follows from (4.12) and (4.13) in Lemma 4.5.

REMARK 4.10. Unfortunately, the condition $\epsilon \geqq \epsilon_*(\Gamma)$ in Corollary 4.8 and Theorem 4.9 cannot be replaced by $\epsilon \geqq \epsilon_0(\Gamma)$. The double shift already shown in Table 1 provides a counterexample. Indeed, one may readily calculate from (3.20) that $\epsilon_*(\Gamma) = 3$ and $\epsilon_*(\Delta) = 4$. (The value of ϵ_0 in both games is 0.6.)

To see why Lemma 4.6 and its corollaries, unlike Lemmas 4.4 and 4.5, do not extend to "intermediate" games, consider the following example:

Table 2

$\|S\|$:	0	1	2	3	4	5	r_i	ϵ_*
$v(S)$:	0	0	2	0	3	5	3	2
$\hat{v}(S)$:	0	0	2	3	4	5	2	2
$\tilde{v}(S)$:	0	0	2	0	4	5	4	3

Here v is balanced and we are simply taking the cover, i.e., double-shifting with $\epsilon = 0$. To find the value of r_i we merely look for the biggest upward jump, reading from left to right. It is easy to see from Table 2 how a function like $\tilde{v}$, intermediate between v and $\hat{v}$, can have bigger upward jumps than either v or $\hat{v}$, even though $\hat{v}$ is in a sense always "smoother" than v. For the given $\tilde{v}$ we have $r_i = 4$, and so $\mathcal{R}((N; \tilde{v})) \not\subset \mathcal{R}((N; v))$, contradicting what the "intermediate" versions of Lemma 4.6 and Corollary 4.7 would say.

Corollary 4.8 survives this counterexample because the hypothesis $\epsilon \geqq \epsilon_*((N; v))$ is not satisfied. But we can construct a larger example on the same principle that will do the trick, as shown in Table 3.

Table 3

$\|S\|$:	0	1	2	3	4	5	6	7	8	9	10	11	r_i	ϵ_*
v(S):	0	0	1	2	3	0	1	2	3	4	5	6	1	3
v_3(S):	0	-3	-2	-1	0	-3	-2	-1	0	1	2	6		
w_3(S):	0	-3	-2	-1	0	0	0	0	0	1	2	6		
w(S):	0	0	1	2	3	3	3	3	3	4	5	6	1	3
$\tilde{w}$(S):	0	0	1	2	3	0	3	3	3	4	5	6	3	4

4.3. Quasi-Zero-Monotonicity

We can now reap the fruits of our "shifty" techniques. The general idea is to double-shift from a game that is not zero-monotonic to one that is, meanwhile trying to make ϵ, and hence the strong ϵ-core, as large as possible. Sometimes it will pay to use an "intermediate" game, in the sense of (4.10), as the following example shows:

Table 4

$\|S\|$:	0	1	2	3	4	5	6	7	8	9	ϵ_0	ϵ_*	ϵ_3
$v(S)$:	0	0	10	20	30	40	39	49	49	50	$12\frac{2}{9}$	30	49
$v_{35}(S)$:	0	-35	-25	-15	-5	5	4	14	14	50			
$w_{35}(S)$:	0	-35	-25	-15	-5	5	6	14	16	50			
$w(S)$:	0	0	10	20	30	40	41	49	51	50	$12\frac{2}{9}$	30	35
$\tilde{w}(S)$:	0	0	10	20	30	40	41	49	50	50	$12\frac{2}{9}$	30	42

Here $\tilde{w}$ is zero-monotonic, but w is not. Moreover, no other choice of ϵ would make w zero-monotonic, since any $\epsilon < 42$ raises $v(S)$ at $|S| = 8$ too much while any $\epsilon > 40$ does not raise $v(S)$ at $|S| = 6$ enough.

With this possibility in mind, we define Γ to be <u>quasi-zero-monotonic for ϵ</u> if there exists a zero-monotonic game $\tilde{\Delta}$ that lies "between" Γ and $\Delta(\Gamma, \epsilon)$, in the sense of (4.10). Since $\Delta(\Gamma, \epsilon)$ is a nonincreasing function of ϵ, it is easily seen that any game that is quasi-zero-monotonic for ϵ is also quasi-zero-monotonic for every $\epsilon' < \epsilon$. Every game is quasi-zero-monotonic for $\epsilon = 0$, and every zero-monotonic game is quasi-zero-monotonic for every ϵ. A game Γ that is quasi-zero-monotonic for $\epsilon_*(\Gamma)$ will be said simply to be <u>quasi-zero-monotonic</u>.

First, three simple conclusions from our previous results:

THEOREM 4.11. *If* Γ *is quasi-zero-monotonic for* ϵ *and if* $\epsilon \geqq \epsilon_0(\Gamma)$, *then*

$$(4.18) \qquad \mathcal{K}(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma) = \mathcal{K}^*(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma).$$

Proof. Theorem 3.3 and Lemma 4.5, especially (4.13) and (4.14).

COROLLARY 4.12. *If* Γ *is quasi-zero-monotonic for* ϵ *and if* $\epsilon \geqq \epsilon_*(\Gamma)$, *then*

$$(4.19) \qquad \mathcal{K}(\Gamma) = \mathcal{K}^*(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma).$$

Proof. Theorems 4.11 and 3.15.

COROLLARY 4.13. *If* Γ *is quasi-zero-monotonic for any* $\epsilon \geqq \epsilon_2(\Gamma)$, *then*

$$(4.20) \qquad \mathcal{K}(\Gamma) = \mathcal{K}^*(\Gamma).$$

Proof. By Theorem 3.13 and Lemma 2.11 we have

$$\mathcal{K}^*(\Gamma) \subset \mathcal{R}(\Gamma) \subset \mathcal{C}_\epsilon(\Gamma).$$

Since $\epsilon_2(\Gamma) \geqq \epsilon_*(\Gamma)$ (see Sec. 3.5), it follows from Corollary 4.12 that $\mathcal{K}^*(\Gamma) = \mathcal{K}^*(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma) = \mathcal{K}(\Gamma)$. Q.E.D.

We would of course like to be able to assert the conclusion of Corollary 4.13 for the hypotheses of Corollary 4.12; we do not know if this can be done.

We now come to the principal theorem of this section:

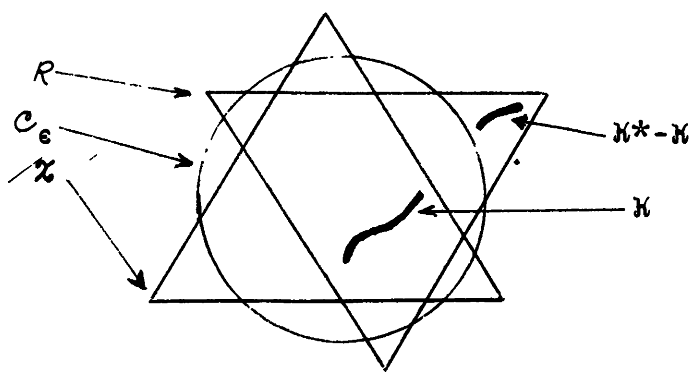
THEOREM 4.14. *If* Γ *is quasi-zero-monotonic, then*

(4.21) $$\mathcal{K}(\Gamma) = \mathcal{K}^*(\Gamma) \cap \mathcal{X}(\Gamma).$$

In other words, the kernel of Γ *is the set of solutions of the system*:

(4.22) $$\begin{cases} s_{ij}(x) = s_{ji}(x), & \text{all } i, j \in N, i < j, \\ x_i \geqq v(\{i\}), & \text{all } i \in N, \\ x(N) = v(N). & \end{cases}$$

The following diagram may help to clarify the logic of the proof; of course it does not represent an actual game:



Proof. By (4.19) we have $\mathcal{K}(\Gamma) = \mathcal{K}^*(\Gamma) \cap \mathcal{C}_{\epsilon_*}(\Gamma)$. Since by definition $\mathcal{K}(\Gamma) \subset \mathcal{X}(\Gamma)$, we can replace this by

(4.23) $$\mathcal{K}(\Gamma) = \mathcal{K}^*(\Gamma) \cap \mathcal{C}_{\epsilon_*}(\Gamma) \cap \mathcal{X}(\Gamma).$$

By Theorem 3.13 and Lemma 3.14 we have

(4.24) $$\mathcal{K}^*(\Gamma) \cap \mathcal{X}(\Gamma) \subset \mathcal{P}(\Gamma) \cap \mathcal{X}(\Gamma) \subset \mathcal{C}_{\epsilon_*}(\Gamma).$$

Hence we may omit the $\mathcal{C}_{\epsilon_*}(\Gamma)$ from the equation (4.23), obtaining (4.21). Q.E.D.

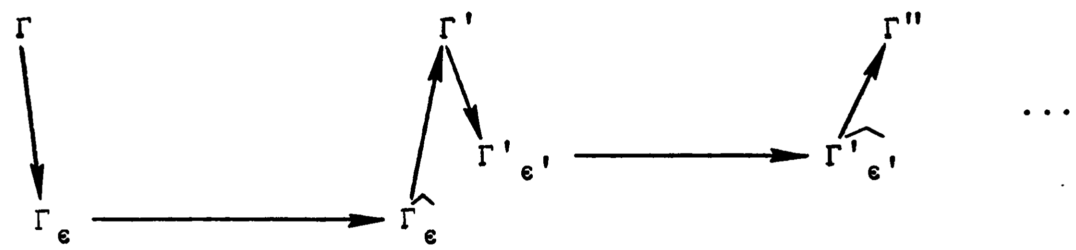
REMARK 4.15. While the symmetric games of Tables 1-4 (and 5) are very convenient for illustrative purposes, it should be pointed out that their kernel theories are trivial. It is well known that each point in the kernel of a game must give equal treatment to any two players who are substitutes, i.e., who are interchangeable in the characteristic function, since otherwise they would not be in equilibrium. The same holds for the pre-kernel. It follows that in a fully symmetric game the kernel and pre-kernel contain only the center of symmetry of $\mathcal{X}$, i.e., the imputation giving each player exactly $v(N)/|N|$. The reader may therefore rightly question our reliance on symmetric examples, and he may feel that the whole quasi-zero-monotonic concept is of little practical value because of the difficulty in recognizing it except in cases where we already know that $\mathcal{K}$ and $\mathcal{K}^*$ coincide in a trivial way.

A general necessary and sufficient condition that characterizes the quasi-zero-monotonic games might indeed be very difficult. But it is easy to give workable _suf-ficient_ conditions that enable us to describe many nonsymmetric games to which the results of this section can be applied.

For example, let $\Gamma = (N; v)$ be a 0-normalized game, let $\mathcal{E}$ denote the set of $S \neq \phi, N$ with $v(S) = \epsilon_1(\Gamma)$, and suppose that $v(N) \geqq \epsilon_1(\Gamma)$. (Recall that $\epsilon_1(\Gamma)$ in a 0-normalized game is just the maximum of $v(S)$ over all $S \neq \phi, N$.) Suppose further that whenever $v(S) > v(T)$ with $S \subset T$ we can find a subset $\mathcal{T}$ of $\mathcal{E}$ that is balanced over T. _Then Γ is quasi-zero-monotonic_. The proof is simple: the double-shift with $\epsilon = \epsilon_1(\Gamma)$ shows that Γ is quasi-zero-monotonic for $\epsilon_1(\Gamma)$, and the inequality $\epsilon_*(\Gamma) \leqq \epsilon_1(\Gamma)$ (see Sec. 3.5) completes the proof.

4.4. Repeated Double Shifts

The double shift operation can of course be repeated:



giving us the games $\Delta(\Delta(\Gamma, \epsilon), \epsilon')$, $\Delta(\Delta(\Delta(\Gamma, \epsilon), \epsilon'), \epsilon'')$, etc. It may seem that nothing is accomplished by this, since the final result of such a sequence of double shifts is [redacted] nothing but the game $\Delta(\Gamma, \bar{\epsilon})$, where $\bar{\epsilon}$ is the minimum of all the ϵ's used. Nevertheless, something may be gained from the viewpoint of quasi-zero-monotonicity. We have seen that the set $\mathcal{R}$ may shrink during a double shift. This means that the value of ϵ_*, which determines the lowest "admissible" value of ϵ, may decrease. (Recall that ϵ_* is the smallest ϵ such that $\mathcal{C}_\epsilon \supset \mathcal{X} \cap \mathcal{R}$.) In other words, it is possible that a game that is <u>not</u> quasi-zero-monotonic could be transformed, by an "admissible" double shift, into a game that <u>is</u> quasi-zero-monotonic.

Table 5

$\|S\|$	0	1	2	3	4	5	r_i	ϵ_*	ϵ_3	ϵ_2
v(S)	0	0	1	-3	4	8	7	3	9	20
v_3(S)	0	-3	-2	-6	1	8				
w_3(S)	0	-3	-2	-3	1	8				
w(S)	0	0	1	0	4	8	4	1	3	8
w_1(S)	0	-1	0	-1	3	8				
y_1(S)	0	-1	0	0	3	8				
y(S)	0	0	1	1	4	8	4	1	1	8

Table 5 shows that this is a real possibility. The first double shift uses $\epsilon = \epsilon_*((N; v)) = 3$, and we see that the resulting game is not zero-monotonic. No intermediate game is zero-monotonic either, so the original game (N; v) is not quasi-zero-monotonic. But the value of r_i has decreased from 7 to 4, and we are rewarded with a smaller ϵ_*. The second double shift, using $\epsilon = \epsilon_*((N; w)) = 1$, then has the desired result: (N; y) is zero-monotonic. Hence (N; w) is quasi-zero-monotonic, and it is tempting to say that (N; v) is "quasi-quasi-zero-monotonic"!

We have no doubt that we could construct similar examples requiring three or more double shifts, though we have not done so. The next theorem generalizes Corollary 4.12 (set p = 1) and also, in a sense, Theorem 3.3 (set p = 0, $\bar{\epsilon} = \infty$).

<u>THEOREM 4.16</u>. <u>Let</u> Γ <u>be any game that can be connected to a zero-monotonic game</u> Δ <u>by a sequence of "admissible" double shifts, namely</u>:

$$\begin{array}{ll} \epsilon^{(1)} \geqq \epsilon_*(\Gamma) & \Gamma^{(1)} = \Delta(\Gamma, \epsilon^{(1)}) \\ \epsilon^{(2)} \geqq \epsilon_*(\Gamma^{(1)}) & \Gamma^{(2)} = \Delta(\Gamma^{(1)}, \epsilon^{(2)}) \\ \dots & \dots \\ \epsilon^{(p)} \geqq \epsilon_*(\Gamma^{(p-1)}) & \Gamma^{(p)} = \Delta(\Gamma^{(p-1)}, \epsilon^{(p)}) = \Delta . \end{array}$$

<u>Then</u>

$$\mathcal{K}(\Gamma) = \mathcal{K}^*(\Gamma) \cap \mathcal{C}_{\bar{\epsilon}}(\Gamma), \tag{4.25}$$

<u>where</u> $\bar{\epsilon} = \text{Min}\,\{\epsilon^{(j)}:\ j = 1, \ldots, p\}$.

<u>Proof</u>. By (4.12) of Lemma 4.5 and Theorem 4.9, we have

$$\mathcal{C}_{\bar{\epsilon}}(\Gamma) = \ldots = \mathcal{C}_{\bar{\epsilon}}(\Gamma^{(j)}) = \ldots = \mathcal{C}_{\bar{\epsilon}}(\Delta)$$

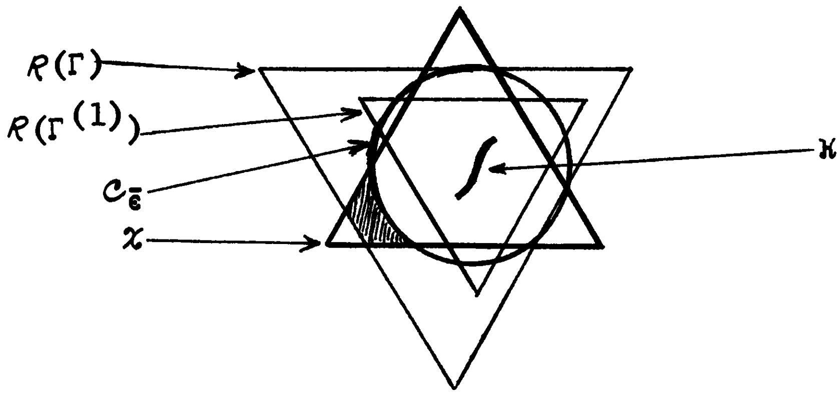
$$\mathcal{K}(\Gamma) = \ldots = \mathcal{K}(\Gamma^{(j)}) = \ldots = \mathcal{K}(\Delta);$$

call these sets respectively $\mathcal{C}_{\bar{\epsilon}}$ and $\mathcal{K}$. From Theorem 3.15, applied to $\Gamma^{(j-1)}$ where $\epsilon^{(j)} = \bar{\epsilon}$, we see that $\mathcal{K} \subset \mathcal{C}_{\bar{\epsilon}}$. Hence, applying (4.14) of Lemma 4.5 and Theorem 3.3., we obtain

$$\mathcal{K}^*(\Gamma) \cap \mathcal{C}_{\bar{\epsilon}} = \mathcal{K}^*(\Delta) \cap \mathcal{C}_{\bar{\epsilon}} = \mathcal{K}(\Delta) \cap \mathcal{C}_{\bar{\epsilon}} = \mathcal{K}.$$

Q.E.D.

We do not know if the corresponding generalization of Theorem 4.14 is true. We cannot argue as before, for we do not have $\mathcal{C}_{\bar{\epsilon}} \supset \mathcal{R}(\Gamma)$ if the value of $\bar{\epsilon}$ is less than $\epsilon_*(\Gamma)$, as it may well be. The diagram below shows the difficulty:



none of our results exclude the possibility that elements of $\mathcal{H}^*(\Gamma)$ might lie in the shaded region.

5. THE KERNEL AS A FAIR DIVISION SCHEME

Any attempt at providing an intuitive interpretation to the definition of the kernel, as given in Sec. 3, seems to rely on interpersonal comparison of utilities. The quantity $s_{ij}(x)$, which measures i's "strength" against j, would there be interpreted as, essentially, the maximum gain (or, if negative, the minimal loss) that i would obtain by "bribing" some players other than j to depart from x, giving each of them a very small bonus. If we compare $s_{ij}(x)$ with $s_{ji}(x)$, we in effect compare i's utility units with j's utility units and implicitly assume that the "intensity of feeling" of i toward's i's utility units are, in some sense, equal to the "intensity of feeling" of j towards j's utility units. Since no clear meaning of "intensity of feeling"--interpersonally comparable--is known at present, the kernel was never considered a satisfactory solution concept, from the intuitive point of view. Its study was pursued mainly because of its mathematical properties (being sensitive to various kinds of symmetry in the game) and because it provided important information on the bargaining set $\mathcal{M}_1^{(i)}$.

R. J. Aumann has suggested* another interpretation: to regard $s_{ij}(x)$ as the amount available to player i for "bribing" a certain coalition (which does not contain j),

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*Oral communication.

taking for himself only a small bonus. This interpretation does not make any use of interpersonal comparison of utilities, but it makes sense only if $s_{ij}(x) \geq 0$. Furthermore, intuitively it seems to be a ground for comparing i's strength against j only if the coalitions that are bribed by both players have a nonempty intersection.

In this section we shall provide yet another intuitive interpretation of the kernel, based on the bisection property--specifically, Theorem 3.8, which does not rely on interpersonal comparison of utilities. We shall then discuss the merits of the kernel under this interpretation.

Consider, at first, a game Γ with a nonempty core, and let $x \in \mathcal{C}(\Gamma)$. The line segments $R_{ij}(x)$ (see (3.11), with $\varepsilon = 0$) can be regarded as the bargaining range between i and j (given that the other players receive at least their amount in x). If player i presses player j for an amount greater than $\delta_{ji}(x)$ (see (3.6), $\varepsilon = 0$), then j will be able to find a coalition which can block i's demand. The middle point of $R_{ij}(x)$ represents, therefore, a situation in which both players are symmetric with respect to the bargaining range. By Theorem 3.8(a) we can therefore interpret $\mathcal{K}(\Gamma) \cap \mathcal{C}(\Gamma)$ as the set of payoff vectors for which every pair of players is situated symmetrically with respect to its bargaining range. This interpretation makes use of symmetries in the game situation without referring to interpersonal comparison of utilities.*

If Γ is zero-monotonic, a similar interpretation can be provided for $\mathcal{K}(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma)$ (see Theorem 3.8(b)) and--if ϵ is large enough, for $\mathcal{K}(\Gamma)$ itself. In this case we have to assume that once x is being considered, a constant penalty equal to ϵ is imposed on any coalition initiating a departure from x. The effect of such a penalty or cost is to lengthen the existing $R^{\epsilon}_{ij}(x)$, and enlarge the set of outcomes x at which these bargaining ranges can be defined.

If Γ is not zero-monotonic (or quasi-zero-monotonic; see Sec. 4.3), the interpretation must be modified due to the requirement that the outcome should be individually rational. Thus, as long as the middle point ξ of $R_{ij}(x, \epsilon)$ is individually rational, it is taken into account. But if ξ_i is less than $v(\{i\})$, it must be replaced by $v(\{i\})$ and ξ_j must be modified appropriately; similarly if $\xi_j < v(\{j\})$. This interpretation results from Theorem 3.8(c).

We have succeeded in providing an interpretation of the kernel which relies on symmetry considerations concerning the bargaining situation, rather than the use of

*Of course, any claim that a solution should reflect symmetries in the bargaining situation is strengthened if utility units are, in some sense, interpersonally equal. Similarly, any requirement of symmetry in the outcome can be attacked on the ground that it does not take into account the possibility that utility units might be interpersonally unequal.

interpersonal comparisons of utilities. However, we have paid a price, in that the outcomes in the kernel are not equally convincing. For outcomes outside of $\mathcal{K}(\Gamma) \cap \mathcal{C}(\Gamma)$ we have to assume a cost for departure from x which is the same for all departing coalitions, and which may have to be fairly heavy if we are to interpret the entire kernel. In practice, direct penalty arrangements seem unlikely to be adopted by players who contract for x, and yet, if we tried to interpret them as a "sum" of personal disutilities for committing treason, in many cases the cost would be too small and would probably not be the same for all coalitions. (Small coalitions that depart might be considered traitors, but if large coalitions depart the disutility might disappear.) To summarize, the points in $\mathcal{K}(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma)$ seem quite intuitive if $\epsilon = 0$, but become less so as ϵ becomes larger and larger.

6. A GEOMETRIC CHARACTERIZATION OF THE NUCLEOLUS

The nucleolus of a game was introduced by Schmeidler [1969]; it is a nonempty subset of the kernel which consists of a single point, depending continuously on the characteristic function. It is known that the nucleolus lies in $\mathcal{C}_\epsilon \cap \mathcal{X}$ whenever this set is not empty. In this section we shall characterize its location within $\mathcal{C}_\epsilon \cap \mathcal{X}$ by means of what amounts to an alternative definition of the nucleolus. In fact, to keep the logic of our argument in view, we shall define a set temporarily called the "lexicographic center" of the game; it is easily proved to exist, to lie within any nonempty $\mathcal{C}_\epsilon \cap \mathcal{X}$, and to consist of a single point. We then show that the lexicographic center coincides with the nucleolus as traditionally defined.*

Let $\Gamma \equiv (N;\ v)$ be an n-person cooperative game, satisfying as usual the condition (2.1). For each $x \in \mathcal{X}(\Gamma)$, let $\theta(x)$ be the 2^n-vector whose components are the numbers $e(S,\ x)$, $S \subset N$, arranged in nonincreasing order. That is, we have

$$(6.1) \qquad \theta_i(x) \geqq \theta_j(x) \quad \text{whenever } 1 \leqq i \leqq j \leqq 2^n,$$

and for each real number c, the number of integers i with $\theta_i(x) = c$ is equal to the number of sets S such that $e(S,\ x) = c$. The lexicographic order on such vectors is

*This approach also provides a method for computing the nucleolus via a sequence of linear programs; compare Kopelowitz [1967], Kohlberg [1972], and Owen [1974].

given by the relation $\theta(x) \prec \theta(y)$, which holds if and only if there is an index ν_o such that

$$(6.2) \qquad \theta_\nu(x) = \theta_\nu(y) \text{ for all } \nu < \nu_o, \text{ and } \theta_{\nu_o}(x) < \theta_{\nu_o}(y).$$

We shall write "$\theta(x) \precsim \theta(y)$" for "not $\theta(y) \prec \theta(x)$".

<u>DEFINITION 6.1</u>. Let $Y \subset \mathcal{X}^*(\Gamma)$. The <u>nucleolus for Y</u> of the game Γ (with respect to the grand coalition*) is the set $\mathcal{N}_Y(\Gamma)$ of payoff vectors in Y that minimize θ in the lexicographic ordering, that is,

$$(6.3) \qquad \mathcal{N}_Y(\Gamma) = \{x \in Y : \theta(x) \precsim \theta(y) \text{ for all } y \in Y\}.$$

The nucleolus for $\mathcal{X}(\Gamma)$ is called simply the <u>nucleolus</u> of Γ, and is denoted $\mathcal{N}(\Gamma)$, while the nucleolus for $\mathcal{X}^*(\Gamma)$ is called the <u>pre-nucleolus</u> of Γ and is denoted $\mathcal{N}^*(\Gamma)$. (Compare this definition with Definition 3.1.)

<u>6.1. The Lexicographic Center</u>

We now introduce the lexicographic center of a game. Intuitively, it is an extension of the idea leading to the least-core (see Definition 2.3.). The procedure is as follows: First we find all the imputations that minimize the maximum excess; in general they will form a nonempty compact

*See Remark 3.19.

convex set. Then we put aside those coalitions whose excess never goes below this minimum in this set* and "reminimize" the maximum excess over the remaining coalitions. This gives us in general a nonempty compact convex subset of the previous set, as well as some new coalitions whose excess cannot be further reduced. These coalitions in turn are put aside, and the process is repeated until there are no coalitions left. We now formalize this procedure.

DEFINITION 6.2. We shall construct a nested sequence $X^0 \supset X^1 \supset \ldots \supset X^\kappa$ of sets of payoff vectors, and a nested sequence $\Sigma^0 \supset \Sigma^1 \supset \ldots \supset \Sigma^\kappa$ of sets of coalitions. To initiate these sequences, define $X^0 = \mathcal{X}(\Gamma)$ and $\Sigma^0 = \{S \subset N: S \neq \emptyset, N\}$.** For k = 1, 2, ..., assume that $X^{k-1} \neq \emptyset$ and $\Sigma^{k-1} \neq \emptyset$ and define

$$(6.4) \qquad \epsilon^k = \min_{x \in X^{k-1}} \max_{S \in \Sigma^{k-1}} e(S, x),$$

$$(6.5) \qquad X^k = \{x \in X^{k-1}: \max_{S \in \Sigma^{k-1}} e(S, x) = \epsilon^k\},$$

$$(6.6) \qquad \Sigma_k = \{S \in \Sigma^{k-1}: e(S, x) = \epsilon^k, \text{ all } x \in X^k\},$$

*Alternatively, we could put aside all coalitions whose excess is constant on the minimum set. This could speed up the process by ensuring that the dimension of X^k decreases at every step; the final result, of course, is the same.

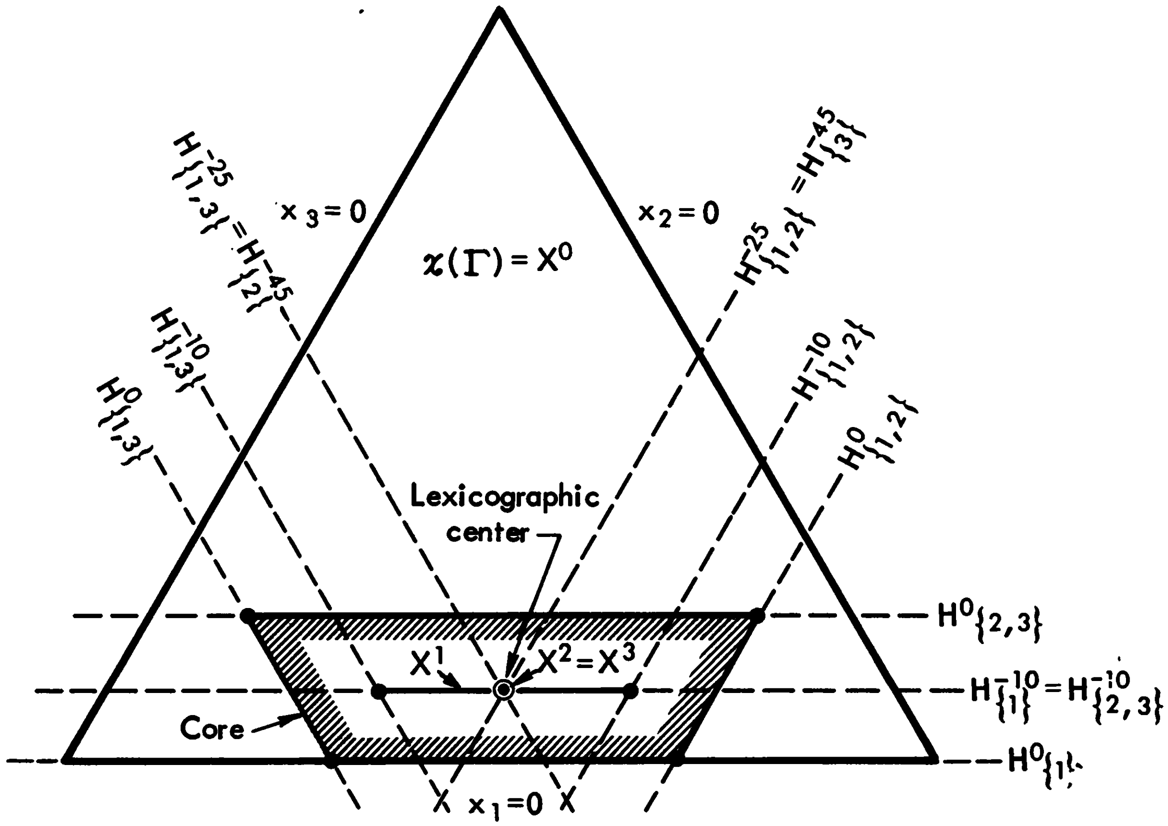
**More generally, we could set X^0 equal to any closed set $Y \subset \mathcal{X}^*(\Gamma)$; compare Definitions 2.1 and 6.1. If the game is zero-monotonic, Theorem 2.7 shows that this "lexicographic center for Y" would coincide with the lexicographic center whenever $Y \supset \mathcal{X}(\Gamma)$.

(6.7) $\Sigma^k = \Sigma^{k-1} \backslash \Sigma_k$.

Let κ be the first value of k for which either $X^k = \emptyset$ or $\Sigma^k = \emptyset$. (It will be shown presently that κ is well defined. The sequences $\{X^k\}$ and $\{\Sigma^k\}$ terminate at $k = \kappa$, and the set X^κ is called the lexicographic center of Γ.

Let us now consider two examples. Figure 6 shows a three-person game in 0-normalized form. It is easy to verify that $\epsilon^1 = -10$, $\epsilon^2 = -25$, and $\epsilon^3 = -45$; that $\Sigma_1 = \{\{1\}, \{2,3\}\}$, $\Sigma_2 = \{\{1,2\}, \{1,3\}\}$, and $\Sigma_3 = \{\{2\}, \{3\}\}$; and that $X^1 = \{(x_1, x_2, x_3): \; x_1 = 10, x_2 \leqq 60, x_3 \leqq 60, x_2 + x_3 = 90\}$ and $X^2 = X^3 = \{(10, 45, 45)\}$. The sets X^k are shown in the figure, as well as the core and a few of the hyperplanes $H_S^{\epsilon^k}$; the set X^1 is of course the least-core, as it will be for any game with a nonempty core.

Figure 7 shows the process of reaching the lexicographic center when there is no core. This time, $\epsilon^1 = 4$, $\epsilon^2 = 0$, $\epsilon^3 = -3$, $\Sigma_1 = \{\{2,3\}\}$, $\Sigma_2 = \{\{1\}\}$, $\Sigma_3 = \{\{2\}, \{3\}, \{1,2\}, \{1,3\}\}$, $X^1 = X^2 = \{x \in \mathcal{X}(\Gamma): \; x_1 = 0\}$, and $X^3 = \{(0, 3, 3)\}$. The strong 4- and 5-cores are also indicated; note that the least-core (which is the strong 2-core, not shown), it outside of $\mathcal{X}(\Gamma)$ in this non-zero-monotonic game.



$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0,$
$v(\{1,2\}) = v(\{1,3\}) = 30, \; v(\{2,3\}) = 80,$
$v(\{1,2,3\}) = 100.$

Fig. 6—Reaching the lexicographic center in a game with a non-empty core

With these examples in front of us, we can describe the geometry of the process in general. Starting with any ϵ large enough so that the strong ϵ-core intersects $\mathcal{X}(\Gamma)$, one "pushes in" all the hyperplanes H_S^ϵ, $S \neq \phi$, N. The push is performed at equal speeds (in the ℓ_1 norm) and is stopped either when the set enclosed would become empty (as in Fig. 6) or become disjoint from $\mathcal{X}(\Gamma)$ (as in Fig. 7). Thus, the amount of pushing depends both on the shape of the strong ϵ-core and on its relation to the space of imputations. The push brings us to the set X^1.

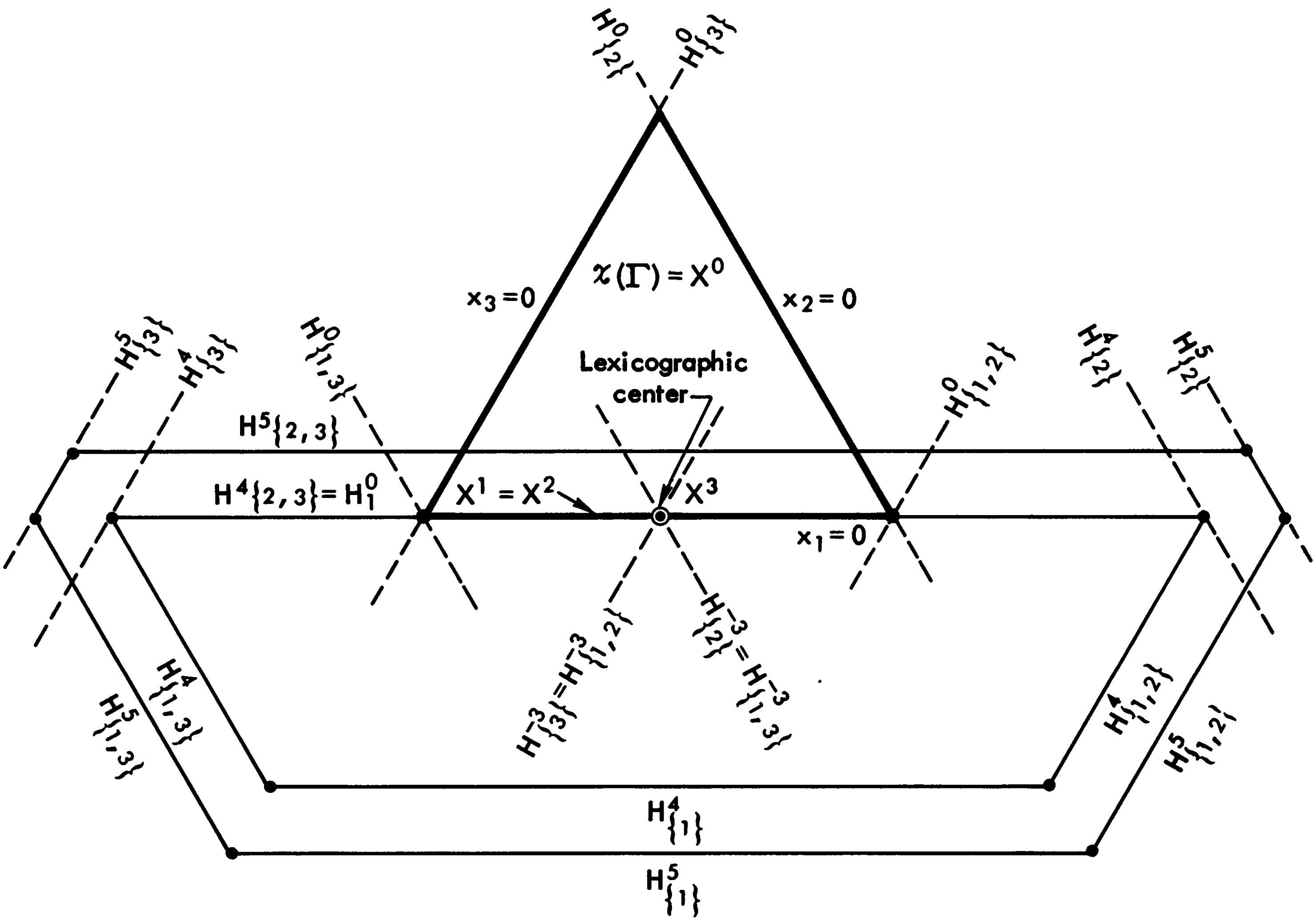
By (6.6), $H_S^{\epsilon^1}$ contains X^1 if and only if $S \in \Sigma_1$; any further push of such a hyperplane will render X^1 empty. We therefore continue to push only those hyperplanes H_S^ϵ where $S \in \Sigma^0 \setminus \Sigma_1 = \Sigma^1$. These we push at equal ℓ_1-speeds so long as the enclosed set modified in this fashion is neither empty nor disjoint from $\mathcal{X}(\Gamma)$. This brings us to X^2. The process continues in the same manner until all the hyperplanes H_S^ϵ have been pushed to their respective limits (that is, to $\epsilon = \epsilon^k$ where $S \in \Sigma_k$) where they will all intersect in the one-point set X^κ.

<u>LEMMA 6.3</u>. <u>The number</u> κ <u>of Definition 6.2 is well defined (i.e., finite), and we have, for all k</u>, $1 \leq k \leq \kappa$,

(i) <u>the</u> ϵ^k <u>are well defined</u>,

(ii) <u>the</u> X^k <u>are nonempty compact and convex sets</u>,

(iii) $\Sigma_k \neq \phi$, <u>and</u>



$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0,$
$v(\{1,2\}) = v(\{1,3\}) = 0, \ v(\{2,3\}) = 10,$
$v(\{1,2,3\}) = 6.$

Fig. 7—Reaching the lexicographic center in a coreless game

(iv) $\epsilon^k < \epsilon^{k-1}$.

<u>Proof</u>. Claim (i) implies claim (ii), and claim (ii) implies claim (i) when k + 1 replaces k, provided that $\Sigma^k \neq \phi$. Since X^0 satisfies (ii), both (i) and (ii) are proved by induction for k = 1, 2, ..., until $\Sigma^k = \phi$. Since $X^k \neq \phi$, the sequences terminate, if at all, by virtue of $\Sigma^k = \phi$, not $X^k = \phi$.

Assume now that $\Sigma_k = 0$, i.e., $\Sigma^k = \Sigma^{k-1}$, for some k before $\Sigma^k = \phi$. This means that for each S in Σ^{k-1} there exists a payoff $x^{(S)}$ in X^k such that $e(S, x^{(S)}) < \epsilon^k$. Let $m \geq 1$ be the number of coalitions in Σ^{k-1}. Then, by the convexity of X^k, the payoff vector

$$\tilde{x} = \frac{1}{m} \sum_{R \in \Sigma^{k-1}} x^{(R)} \tag{6.8}$$

lies in X^k, and hence in X^{k-1}. Clearly, for $S \in \Sigma^{k-1}$, we must have

$$\begin{aligned} e(S, \tilde{x}) &= v(S) - \frac{1}{m} \sum_{R \in \Sigma^{k-1}} x^{(R)}(S) \\ &= \frac{1}{m} \sum_{R \in \Sigma^{k-1}} e(S, x^{(R)}) < \epsilon^k, \end{aligned} \tag{6.9}$$

contrary to (6.4). This contradiction proves claim (iii). The well-definition of $\underline{K}$ now follows from claim (iii), since the $\Sigma_{\underline{K}}$ are disjoint and the number of coalitions is finite.

Finally, to prove claim (iv), observe that for each coalition R in Σ^k, there must exist a point $x^{(R)}$ in X^k for which $e(R, x^{(R)}) < \epsilon^k$. Thus, if Σ^k contains m coalitions and if $m > 0$, the payoff vector

$$\tilde{x} = \frac{1}{m} \sum_{R \epsilon \Sigma^k} x^{(R)} \tag{6.10}$$

belongs to X^k and satisfies $e(S, \tilde{x}) < \epsilon^k$ whenever $S \in \Sigma^k$. Consequently, for $k < \kappa$ we have $\epsilon^{k+1} \leqq \max_{S \epsilon \Sigma^k} e(S, \tilde{x}) < \epsilon^k$. Q.E.D.

THEOREM 6.4. The lexicographic center of a game consists of a single point.

Proof. By Lemma 6.3(ii) we see that the lexicographic center X^{κ} is nonempty. By (6.5), the excess of each coalition S is constant in X^k, if $S \in \Sigma_k$, and hence is constant in X^{κ}. As this holds in particular for the single-person coalitions, we see that it is impossible for X^{κ} to contain more than one point. Q.E.D.

6.2. Equivalence of the Lexicographic Center and the Nucleolus

LEMMA 6.5. For any k, $1 \leqq k \leqq \kappa$, if $x \in X^k$ and $y \in X^{k-1} \backslash X^k$, then $\theta(x) \prec \theta(y)$.

Proof. Consider the partition of 2^N into the sets $\Sigma_1, \ldots, \Sigma_{k-1}, \Sigma^{k-1}, \{\emptyset, N\}$. We may ignore $\emptyset$ and N in the lexicographic comparison between $\theta(x)$ and $\theta(y)$, since their excesses are always 0. By (6.6) and Lemma 6.3(iv), we have

$$e(S, x) = e(S, y) = \epsilon^h \geqq \epsilon^{k-1}$$

for all S in Σ_h, $h = 1, \ldots, k - 1$. Moreover, for all S in Σ^{k-1} we have

$$e(S, x) \leqq \epsilon^k < \epsilon^{k-1} \text{ and } e(S, y) \leqq \epsilon^{k-1},$$

by (6.5). However, since y is not in X^k, we must have $\epsilon^k < e(R, y) \leqq \epsilon^{k-1}$ for at least one R in Σ^{k-1}. Hence (6.2) is satisfied for the index ν_o corresponding to some such R, and we have $\theta(x) \prec \theta(y)$. Q.E.D.

THEOREM 6.6. The nucleolus of a game coincides with the lexicographic center, and hence consists of a single point.

Proof. Lemma 6.5 (with the fact that $X^0 = \mathcal{X}$) and Theorem 6.4.

COROLLARY 6.7. For any Γ, ϵ, if $\mathcal{C}_\epsilon(\Gamma) \cap \mathcal{X}(\Gamma)$ is not empty, then it contains $\mathcal{N}(\Gamma)$. In particular, if Γ is zero-monotonic, then the nucleolus is contained in every nonempty strong ϵ-core, and hence in the least-core.

Finally, for completeness, we state the following well-known result:*

THEOREM 6.8. $\mathcal{N}(\Gamma) \subset \mathcal{K}(\Gamma)$.

Similar considerations can obviously be applied to the nucleolus for sets Y other than $Y = \mathcal{X}(\Gamma)$; we will not pursue them here.

6.3. Discussion of a Counterexample

Our ability to describe the nucleolus in terms of geometric manipulations of the hyperplanes H_S^{ϵ} suggests that, in analogy with earlier results, the nucleolus might occupy a definite position within the core, or other strong ϵ-core, independently of the other data of the game. The formal conjecture would be: If Γ, Γ', ϵ, ϵ' are such that $\mathcal{X}(\Gamma) = \mathcal{X}(\Gamma')$ and $\mathcal{C}_{\epsilon}(\Gamma) = \mathcal{C}_{\epsilon'}(\Gamma') \neq \phi$, then $\mathcal{N}(\Gamma) = \mathcal{N}(\Gamma')$.**

Of course, this conjecture holds if ϵ and ϵ' are sufficiently large, for then the characteristic functions of the games Γ and Γ' are completely determined (see Lemma 2.5). In fact, they would be directly related by the "shifting" identity $\Gamma' \equiv \Gamma_{\epsilon-\epsilon'}$, and shifts obviously do not affect the location of the nucleolus. (Compare Theorem 4.1.) Also, the conjecture will be true whenever the geometrical shape of $\mathcal{C}_{\epsilon}(\Gamma)$ happens to imply that $\mathcal{LC}(\Gamma)$

*Schmeidler [1969; Theorem 3].

**Compare Corollaries 2.6 and 3.11.

is a single point, or that $\mathcal{K}(\Gamma) \cap \mathcal{C}_\epsilon(\Gamma)$ is a single point, for we know that $\mathcal{N}(\Gamma)$ is contained in each of these sets.

But the conjecture in general is false. Let Γ be the game (N; v) with N = {1,2,3,4} and v defined by

$$\begin{cases} v(N) = 2, \\ v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = 1, \\ v(\{1,2\}) = v(\{3,4\}) = v(\{1,4\}) = v(\{2,3\}) = 1, \\ v(\{1,3\}) = 1/2, \qquad v(\{2,4\}) = 0, \\ v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = v(\emptyset) = 0. \end{cases}$$

Let $\Gamma' \equiv (N; v')$ be the same as Γ, except that $v'(\{1,2,3\}) = 5/4$. It is easily determined that in both games, the core is the line segment joining the two points*

$$E_1 = (\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{3}{4}) \quad \text{and } E_2 = (1, 0, 1, 0).$$

Thus, the hypothesis of the conjecture is satisfied, with $\epsilon = \epsilon' = 0$. But it is easily verified that

$$\mathcal{N}(\Gamma) = \{(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\} \neq \mathcal{N}(\Gamma') = \{(\tfrac{5}{8}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{3}{8})\},$$

disproving the conjecture.

*This set is also the least-core and the kernel of both Γ and Γ'.

Geometrically, the "walls" of the core do not move at the same speeds in these games when we are squeezing X^1 (the least-core) down to X^2 (the nucleolus). The hyperplanes all move at the same speed in four-dimensional space, but they travel in different directions, so their intersections with the line through E_1 and E_2 move at different speeds. Specifically, in the game Γ, the wall at E_1 is being pushed by the coalition $\{1,3\}$ only half as fast as the wall at E_2 is being pushed by the coalitions $\{1,2,4\}$, $\{2,3,4\}$, $\{2\}$, or $\{4\}$ (all of which happen to define the same point on E_1E_2). Consequently, the nucleolus of Γ lies closer to E_1 than E_2. In the game Γ', however, the coalition $\{1,2,3\}$ is doing the pushing at the E_1 end, making that wall move at the same speed as the other wall and making the nucleolus of Γ' lie at the midpoint of E_1E_2.

We may extend this example by introducing a parameter α and defining $\Gamma(\alpha) \equiv (N; w_\alpha)$ for $1 \leqq \alpha \leq 5/4$ by

$$\begin{cases} w_\alpha(S) = v(S), & S \neq \{1,2,3\}, \\ w_\alpha(\{1,2,3\}) = \alpha. \end{cases}$$

Thus, $\Gamma(1) = \Gamma$ and $\Gamma(5/4) = \Gamma'$. (We might mention that all these games $\Gamma(\alpha)$ are zero-monotonic; in fact, they are super-additive and balanced, but not totally balanced.) The cores, least-cores, and kernels of these games are all the same segment E_1E_2.* But in squeezing down to the nucleolus, we find that the coalition $\{1,3\}$ pushes at the E_1 end for a while, then $\{1,2,3\}$ catches up and pushes faster. The catch-up point, and hence the nucleolus, depends on the value of α; in fact,

$$\mathcal{N}(\Gamma(\alpha)) = \{(\tfrac{\alpha}{2}, 1 - \tfrac{\alpha}{2}, \tfrac{\alpha}{2}, 1 - \tfrac{\alpha}{2})\}, \quad 1 \leqq \alpha \leqq 5/4.$$

This shows that there can be a continuum of locations for the nucleolus. It would be interesting to characterize the set of all possible nucleolus locations within a given core. Evidently it is convex and compact, and is properly contained in $\mathcal{LC}$ if the latter is not just a single point.

*For any $\varepsilon > 0$ the strong ε-cores also agree if α is in the range of $1 \leqq \alpha \leqq 5/4 - \varepsilon/2$. This shows (with the help of Lemma 4.1) that the counterexample is "robust" and does not depend on the core being of less than full dimension.

7. THE NUCLEOLUS AS A FAIR DIVISION SCHEME

In this section we shall discuss the possibility of regarding the nucleolus of a game as an outcome recommended by an arbitrator whom the players may wish to consult.

It has long been known that the nucleolus satisfies many properties desired for such a purpose. We list some of them:

1. It defines a unique payoff vector for each game.
2. It satisfies individual and group rationality.
3. Symmetric players receive in the nucleolus equal payments and, in fact, more desirable players receive at least as much as less desirable players.*
4. A dummy receives only his own value.**

But these properties are satisfied also by the Shapley value (for zero-monotonic games) and by any rule that selects a point in the kernel. Thus, they do not determine the nucleolus. It is still an open problem to find additional intuitively acceptable properties that, together with the above, will constitute a system of axioms that determine the nucleolus.

However, it seems that Theorem 6.6 gives a clue to an <u>operational procedure</u> that may lead the arbitrator to select the nucleolus. An arbitrator may wish to regard

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*See Maschler and Peleg [1966] and take into account the fact that the nucleolus is contained in the kernel.

**See the previous footnote.

the excess of a coalition at a solution point x as a measure of dissatisfaction of that coalition from x. He might therefore wish to select an imputation such that the maximum excess is minimal, i.e., an imputation in the least-core. If he has several choices, as is often the choice, he will "tell" some coalitions that he is unable to satisfy them any further, but he will still attempt to satisfy further the other coalitions, by looking at outcomes that, in addition, minimize the second highest excess. He will continue with this procedure until he is left with the lexicographic center, namely, the nucleolus.*

Naturally there arises the task of comparing possible advantages and disadvantages of the nucleolus over the Shapley value. They certainly have many properties in common. As a contribution to this task let us state two properties which distinguish between them: (1) The nucleolus (in the zero-monotonic case) is always in any non-empty ε-core; the Shapley value need not be. (2) The Shapley value for any player always responds monotonically to changes in the characteristic function v(S) (positively when S contains the player, negatively when it does not); the nucleolus does not have this property. Indeed, Megiddo [1974] has provided an example of a game whose nucleolus payoff <u>decreases</u> for some players when v(N) is increased, with nothing else changed.

*We refer the reader to Justman [1973], where a different procedure is described, leading the players to the nucleolus.

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