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Geometric properties of the tetrablock

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Abstract. In this short note, we show that the tetrablock is a $\mathbb{C}$-convex domain. In the proof of this fact, a new class of ($\mathbb{C}$-convex) domains is studied. The domains are natural candidates to study on them the behavior of holomorphically invariant functions.

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1. Introduction. Recently, two domains, the symmetrized bidisc and the tetrablock, arising from the μ-synthesis, turned out to be interesting examples in the geometric function theory. In particular, both domains are non-convex and even more, they cannot be exhausted by domains biholomorphic to convex ones, and yet the Lempert function and the Carathéodory distance coincide on them (see [2,5–7]). However, we have a more detailed knowledge on geometric properties in the case of the symmetrized bidisc. In particular, it is known that the symmetrized bidisc is $\mathbb{C}$-convex and may be exhausted by strongly linearly convex domains (see [13,14]). All these facts show the importance of the domains from the point of view of the Lempert theorem on the equality of the Lempert function and the Carathéodory distance (see the two papers of Lempert: [11,12]). We shall deal with analogous properties of the tetrablock. More precisely, we show that the tetrablock is $\mathbb{C}$-convex (see Corollary 4.2), which corrects the claim stated in [10] where, due to the typing error made in the formula describing the tetrablock, the converse was claimed.

It is interesting that the study of geometric properties of the tetrablock can be reduced to considering domains being generalizations of the symmetrized bidisc, which may lead in the future to the study of other domains arising

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in the process of symmetrization of a 'nice' pseudoconvex complete Reinhardt domain. The domains $\mathbb{G}_{2,\rho}$, which are $\mathbb{C}$-convex (see Theorem 4.1) and approximate the symmetrized bidisc, are natural candidates for the further study on the equality between the Lempert function and the Carathéodory distance as well as on the possibility of exhausting them with strongly linearly convex domains.

Basic notions, definitions, and properties from the theory of invariant functions, linearly and $\mathbb{C}$-convex domains that we shall use in the paper may be found in [3,8,9].

2. Preliminary results. Below we present analytic definitions of both domains that will be of interest to us.

Recall that the *tetrablock* may be defined as follows (see [1])

$$\mathbb{E} = \{x \in \mathbb{C}^3 : |x_1 - \bar{x}_2 x_3| + |x_2 - \bar{x}_1 x_3| + |x_3|^2 < 1\} \tag{2.1}$$

and the *symmetrized bidisc* as follows (see e.g. [2])

$$\mathbb{G}_2 = \{(s,p) \in \mathbb{C}^2 : |s - \bar{s}p| + |p|^2 < 1\}. \tag{2.2}$$

Let us begin our study with presenting a close relation between $\mathbb{E}$ and $\mathbb{G}_2$, which may be a good starting point for us.

Lemma 2.1. (*see* [4]) *For any* $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ *the following are equivalent.*

(1) $x \in \mathbb{E}$,
(2) *for any* $\omega \in \mathbb{C}$ *with* $|\omega| = 1$ *we have* $(x_1 + \omega x_2, \omega x_3) \in \mathbb{G}_2$,
(3) *for any* $\omega \in \mathbb{C}$ *with* $|\omega| \le 1$ *we have* $(x_1 + \omega x_2, \omega x_3) \in \mathbb{G}_2$.

Actually, the equivalence of the first two properties follows from the following observation which holds for all $|\omega| = 1$,

$$\begin{aligned} |x_1 + \omega x_2 - (\bar{x}_1 + \bar{\omega}\bar{x}_2)\omega x_3| &= |x_1 - \bar{x}_2 x_3 + \omega(x_2 - \bar{x}_1 x_3)| \\ &\le |x_1 - \bar{x}_2 x_3| + |x_2 - \bar{x}_1 x_3|, \end{aligned} \tag{2.3}$$

together with the fact that the inequality above becomes an equality for some $|\omega| = 1$. Note that the hyperconvexity of $\mathbb{G}_2$ (see e.g. [13]) together with the maximum principle for subharmonic functions gives the equivalence of the two latter conditions.

Let us also denote $\Phi_\omega(x) := (x_1 + \omega x_2, \omega x_3)$, $x \in \mathbb{E}$, $\omega \in \overline{\mathbb{D}}$. Define $\sigma(x) = (x_2, x_1, x_3)$, $x \in \mathbb{C}^3$, $\Psi_\omega := \Phi_\omega \circ \sigma$.

Remark. The equality (2.1) defining the tetrablock allows us to find out that any point $x \in \partial\mathbb{E}$ such that $x_1 \neq \bar{x}_2 x_3$ and $x_2 \neq \bar{x}_1 x_3$ is a smooth boundary point of $\partial\mathbb{E}$. Moreover, the condition $x \in \partial\mathbb{E}$ and $x_1 = \bar{x}_2 x_3$ or $x_2 = \bar{x}_1 x_3$ means that $x = (re^{i\theta}, re^{i\tau}, e^{i(\theta+\tau)})$ or $x = (e^{i\theta}, re^{i\tau}, re^{i(\theta+\tau)})$ or $x = (re^{i\theta}, e^{i\tau}, re^{i(\theta+\tau)})$ for some $r \in [0,1]$, $\theta, \tau \in \mathbb{R}$. Composing with the automorphism of $\mathbb{E}$ given by the formula $(e^{-i\theta} y_1, e^{-i\tau} y_2, e^{-i(\theta+\tau)} y_3)$, we shall often be able to reduce the problem to special cases: $x = (r, r, 1)$ or $x = (1, r, r)$ or $x = (r, 1, r)$, $r \in [0,1]$.

At first we shall prove that for any $x \notin \mathbb{E}$ there is a hyperplane passing through x and omitting $\mathbb{E}$. Thus we show the following.

Lemma 2.2. $\mathbb{E}$ *is linearly convex.*

Proof. Let $x \notin \mathbb{E}$. Making use of the description of $\mathbb{E}$ from Lemma 2.1 for such an x, we find an $\omega \in \overline{\mathbb{D}}$ with $\Phi_\omega(x) = (x_1 + \omega x_2, \omega x_3) \notin \mathbb{G}_2$. The linear convexity of $\mathbb{G}_2$ (see e.g. [13]) implies that there is a line $l = \{(s,p) \in \mathbb{C}^2 : as + bp = c\}$ with $\Phi_\omega(x) \in l$ and $l \cap \mathbb{G}_2 = \emptyset$. Then $x \in L := \{y \in \mathbb{C}^3 : ay_1 + a\omega y_2 + b\omega y_3 = c\}$ and, as one may easily check, $L \cap \mathbb{E} = \emptyset$. □

As we shall see, a more refined procedure than the one described above will lead us to the precise description of supporting hyperplanes, which will finally lead to the proof of the $\mathbb{C}$-convexity of $\mathbb{E}$. But before that we need some notations and auxiliary results.

3. Two-dimensional symmetrized domains characterizing the tetrablock. Motivated by Lemma 2.1, we shall see that the symmetrized images of special domains of the form

$$D_\rho := \{z \in \mathbb{C}^2 : |z_1|, |z_2| < 1, |z_1 z_2| < \rho\}, \tag{3.1}$$

where $\rho \in (0,1]$, will play a special role in the study of the geometry of $\mathbb{E}$.

Let us denote

$$\mathbb{G}_{2,\rho} := \pi(D_\rho),\ \rho \in (0,1], \tag{3.2}$$

where $\pi(z) := (z_1 + z_2, z_1 z_2)$, $z \in \mathbb{C}^2$. Recall that $\mathbb{G}_2 = \mathbb{G}_{2,1}$.

Then it follows from Lemma 2.1 that $\Phi_\omega(\mathbb{E}) \subset \mathbb{G}_{2,|\omega|}$, $0 < |\omega| \leq 1$. We shall see that we even have the equality.

Proposition 3.1.

$$\Phi_\omega(\mathbb{E}) = \mathbb{G}_{2,|\omega|},\ \omega \in \overline{\mathbb{D}} \setminus \{0\}. \tag{3.3}$$

Proof. Actually, let $\rho := |\omega|$ and take $(s,p) \in \mathbb{G}_{2,\rho}$. Put

$$x := \left(\frac{s - \overline{s}p}{1 - |p|^2}, \frac{\overline{s} - s\overline{p}}{1 - |p|^2}\frac{p}{\omega}, \frac{p}{\omega}\right). \tag{3.4}$$

It easily follows from (2.1) that $x \in \mathbb{E}$ and $\Phi_\omega(x) = (s,p)$. □

A domain $D \subset \mathbb{C}^n$ is called $\mathbb{C}$-*convex* if for any affine complex line l such that $l \cap D \neq \emptyset$, the set $l \cap D$ is connected and simply connected.

For a domain $D \subset \mathbb{C}^n$ and a point $a \in \mathbb{C}^n$, we denote by $\Gamma_D(a)$ the set of all complex hyperplanes L such that $(a + L) \cap D = \emptyset$. We shall often understand this set as a subset of $\mathbb{P}^{n-1}$: $L = \{x \in \mathbb{C}^n : \langle x, b\rangle = 0\}$ is identified with $[b] \in \mathbb{P}^{n-1}$.

Recall the basic criterion on $\mathbb{C}$-convexity that we shall use: the bounded domain $D \subset \mathbb{C}^n$, $n > 1$, is $\mathbb{C}$-convex iff for an $x \in \partial D$ the set $\Gamma_D(x)$ is non-empty and connected (cf. e.g. Theorem 2.5.2 in [3]).

Remark. It is elementary to see that for $n \geq 2$

$$\Gamma_{\mathbb{D}^n}(1,\ldots,1) = \{[(t_1,\ldots,t_n)] : (t_1,\ldots,t_n) \in [0,\infty)^n \setminus \{0\}\}. \tag{3.5}$$

Remark. The boundary point $(s,p) = \pi(\lambda_1,\lambda_2)$ of the domain $\mathbb{G}_{2,\rho}$, $\rho \in (0,1)$ is not smooth iff $\{|\lambda_1|, |\lambda_2|\} = \{1, \rho\}$.

One may also easily see that

- if $|\lambda_1| \le \rho$, $|\lambda_2| = 1$, then $\Gamma_{\mathbb{G}_{2,\rho}}(\pi(\lambda_1,\lambda_2)) \supset \{[(-\lambda_2,1)]\}$ (if $|\lambda_1| < \rho$, then the inclusion becomes the equality),
- if $|\lambda_1||\lambda_2| = \rho$, $\rho \le |\lambda_j| \le 1$, $j = 1,2$, then $\Gamma_{\mathbb{G}_{2,\rho}}(\pi(\lambda_1,\lambda_2)) \supset \{[(0,1)]\}$ (if $\rho < |\lambda_j| < 1$, $j = 1,2$, then the inclusion becomes the equality).

Let us formulate a result which essentially reduces the problem of describing $\Gamma_{\mathbb{E}}$ to that of $\Gamma_{\mathbb{G}_{2,\rho}}$ (and $\Gamma_{\mathbb{D}^2}$).

Lemma 3.2. *Let $x \in \mathbb{C}^3$ and let $0 < |\omega| \le 1$. Then the following are equivalent.*

- $[(a,c)] \in \Gamma_{\mathbb{G}_{2,|\omega|}}(\Phi_\omega(x))$ *(respectively,* $[(a,c)] \in \Gamma_{\mathbb{G}_{2,|\omega|}}(\Psi_\omega(x))$*),*
- $[(a,\omega a,\omega c)] \in \Gamma_{\mathbb{E}}(x)$ *(respectively,* $[(\omega a, a, \omega c)] \in \Gamma_{\mathbb{E}}(x)$*).*

Proof. Let l be such that $l - \Phi_\omega(x) \in \Gamma_{\mathbb{G}_{2,|\omega|}}\Phi_\omega(x)$ is given by the equality $as+cp = d$ (i.e. $[(a,c)] \in \Gamma_{\mathbb{G}_{2,|\omega|}}(\Phi_\omega(x))$). Then the equality $ay_1+\omega a y_2+\omega c y_3 = d$ defines a hyperplane omitting $\mathbb{E}$ and thus $[(a,\omega a,\omega c)] \in \Gamma_{\mathbb{E}}(\Phi_\omega(x))$.

To show the other implication, let L be such that $L - x \in \Gamma_{\mathbb{E}}(x)$. Let L be given by the equation $ay_1 + \omega a y_2 + \omega c y_3 = d$, which may be written as $a(y_1 + \omega y_2) + c\omega y_3 = d$. Then from the equality $\Phi_\omega(\mathbb{E}) = \mathbb{G}_{2,|\omega|}$ (Proposition 3.1), we get that the line given by the equality $as + cp = 0$ belongs to $\Gamma_{\mathbb{G}_{2,|\omega|}}(\Phi_\omega(x))$. □

Theorem 3.3. *Let $r \in [0,1]$. Then*

$$\Gamma_{\mathbb{E}}(r,r,1) = \bigcup_{0<|\omega|\le 1} \left\{[(\tilde{s},\omega\tilde{s},\omega\tilde{p})],[(\omega\tilde{s},\tilde{s},\omega\tilde{p})] : [(\tilde{s},\tilde{p})] \in \Gamma_{\mathbb{G}_{2,|\omega|}}(r+r\omega,\omega)\right\} \cup \{[(\tilde{s},0,\tilde{p})],[(0,\tilde{s},\tilde{p})] : [(\tilde{s},\tilde{p})] \in \Gamma_{\mathbb{D}^2}(r,1)\}. \tag{3.6}$$

Let $r \in [0,1)$. Then

$$\Gamma_{\mathbb{E}}(1,r,r) = \{[(-1,-\omega,\omega)] : \omega \in \overline{\mathbb{D}}\}. \tag{3.7}$$

Remark. Smoothness of the boundary point x together with the linear convexity means that $\Gamma_{\mathbb{E}}(x)$ is a singleton. Note also that it follows from the earlier remark that the cases of boundary points considered in Theorem 3.3 (i. e. $(r,r,1)$ and $(1,r,r)$) represent all (up to linear automorphisms of $\mathbb{E}$) non-smooth boundary points, and it means that Theorem 3.3 gives a complete description of $\Gamma_{\mathbb{E}}$.

Remark. Note that $\Psi_\omega(1,r,r) = (r+\omega, r\omega) \in \mathbb{G}_{2,|\omega|}$, $r \in [0,1)$ and $\Phi_\omega(1,r,r) = (1+r\omega, r\omega) \in \partial\mathbb{G}_{2,|\omega|}$, which implies that the equality in (3.7) may be expressed in the form which would also be applicable in (3.6). We may namely write the right hand side of (3.7) as follows:

$$\bigcup_{0<|\omega|\le 1} \left\{[(\tilde{s},\omega\tilde{s},\omega\tilde{p})] : [(\tilde{s},\tilde{p})] \in \Gamma_{\mathbb{G}_{2,|\omega|}}(\Phi_\omega(1,r,r))\right\} \cup \bigcup_{0<|\omega|\le 1} \left\{[(\omega\tilde{s},\tilde{s},\omega\tilde{p})] : [(\tilde{s},\tilde{p})] \in \Gamma_{\mathbb{G}_{2,|\omega|}}(\Psi_\omega(1,r,r))\right\} \cup \left\{[(\tilde{s},0,\tilde{p})] : [(\tilde{s},\tilde{p})] \in \Gamma_{\mathbb{D}^2}(1,r)\right\}. \tag{3.8}$$

Proof. Note that in the case of the point $(r, r, 1)$, we get that $\Phi_\omega(r, r, 1) = \Psi_\omega(r, r, 1) = (r + \omega r, \omega) \in \partial\mathbb{G}_{2,|\omega|}$, $0 < |\omega| \leq 1$. In this case the inclusion '$\supset$' follows from Lemma 3.2 and the fact that $\mathbb{E} \subset \mathbb{D}^3$.

To show the other inclusion, let L be such that $L - (r, r, 1) \in \Gamma_{\mathbb{E}}(r, r, 1)$. Let L be given by the equation $ay_1 + by_2 + cy_3 = d$. Let $b = \omega a$ where $|\omega| \leq 1$ (the other case will be dealt with analogously). If $\omega \neq 0$, then in view of Lemma 3.2, we get that $[(a, \frac{c}{\omega})] \in \Gamma_{\mathbb{G}_{2,|\omega|}}(r + r\omega, \omega)$.

Consider now $\omega = 0$. Then from the fact that $(r, r, 1) + \lambda(-c, 0, a) + \mu(0, 1, 0) \notin \mathbb{E}$ for any $\lambda, \mu \in \mathbb{C}$, we get that the point $(r - c\lambda, 1 + a\lambda) \notin \mathbb{D}^2$, $\lambda \in \mathbb{C}$—the last property follows from the fact that if the point (y_1, y_3) satisfies the inequality $|y_1 - \bar{\mu} y_3| + |\mu - \bar{y}_1 y_3| \geq 1 - |y_3|^2$ for any $\mu \in \mathbb{C}$, then $(y_1, y_3) \notin \mathbb{D}^2$ (take e. g. $\mu = \bar{y}_1 y_3$). This finishes the proof of the case (3.6).

Let us consider now the point $(1, r, r)$, $r \in [0, 1)$. Then $\Phi_\omega(1, r, r) = (1 + r\omega, r\omega) \in \partial\mathbb{G}_{2,|\omega|}$ and $\Psi_\omega(1, r, r) = (\omega + r, \omega r) \in \mathbb{G}_{2,|\omega|}$, $0 < |\omega| \leq 1$.

Note that $[(-1, 1)] \in \Gamma_{\mathbb{G}_{2,|\omega|}}(1 + r\omega, r\omega)$, $0 < |\omega| \leq 1$, which in view of Lemma 3.2 implies that $[(-1, -\omega, \omega)] \in \Gamma_{\mathbb{E}}(1, r, r)$, $0 < |\omega| \leq 1$. Certainly $[(-1, 0, 0)] \in \Gamma_{\mathbb{E}}(1, r, r)$. And this gives the inclusion '$\supset$' in the case (3.7).

To show the other inclusion, we proceed similarly as in the first case. Let L be such that $L - (1, r, r) \in \Gamma_{\mathbb{E}}(1, r, r)$. Let L be given by the equation $ay_1 + by_2 + cy_3 = d$. First note that $b = \omega a$ for some $|\omega| \leq 1$. Actually in the other case $a, b \neq 0$ and $a = \omega b$ for some $0 < |\omega| < 1$, and then in view of Lemma 3.2, we get that $[(b, \frac{c}{\omega})] \in \Gamma_{\mathbb{G}_{2,|\omega|}}(r + \omega, r\omega) = \emptyset$ —a contradiction.

If $\omega \neq 0$, then in view of Lemma 3.2, we get that $[(a, \frac{c}{\omega})] \in \Gamma_{\mathbb{G}_{2,|\omega|}}(1 + r\omega, r\omega) = \{[(-1, 1)]\}$, so $[(a, b, c)] = [(-1, -\omega, \omega)]$.

Consider now $\omega = 0$. Then, as in the first case $[(a, c)] \in \Gamma_{\mathbb{D}^2}(1, r)$, which equals $\{[(1, 0)]\}$, which finishes the proof. □

4. $\mathbb{C}$-convexity of $\mathbb{G}_{2,\rho}$ and the tetrablock. In view of the above results, we see that crucial for the proof of $\mathbb{C}$-convexity of the tetrablock is to find the description of $\Gamma_{\mathbb{G}_{2,\rho}}$. Recall that $\mathbb{G}_2 = \mathbb{G}_{2,1}$ is $\mathbb{C}$-convex (see [13]). We have the following.

Theorem 4.1. *$\mathbb{G}_{2,\rho}$ is $\mathbb{C}$-convex for any $\rho \in (0, 1]$.*

Moreover, if $(s, p) = \pi_2(\lambda_1, \lambda_2) \in \partial\mathbb{G}_{2,\rho}$, then for any $\rho \in (0, 1)$ we have

- $\Gamma_{\mathbb{G}_{2,\rho}}(s, p) = \{[(-\lambda_2, 1)]\}$ *if* $|\lambda_1| < \rho$, $|\lambda_2| = 1$,
- $\Gamma_{\mathbb{G}_{2,\rho}}(s, p) = \{[(0, 1)]\}$, *if* $|\lambda_1 \lambda_2| = \rho$ *and* $|\lambda_1|, |\lambda_2| \in (\rho, 1)$,
- *the set* $\{\frac{\tilde{s}}{\tilde{p}} : [(\tilde{s}, \tilde{p})] \in \Gamma_{\mathbb{G}_{2,\rho}}(s, p)\}$ *contains* 0 *and is a convex set if* $\{|\lambda_1|, |\lambda_2|\} = \{1, \rho\}$.

Proof. Let $\rho \in (0, 1)$. The equalities in the first two cases follow from earlier remarks and the fact that the points considered there are smooth.

Let us consider the third case. Note that any complex line passing through (s, p) and through a point from $\mathbb{G}_{2,\rho}$ must contain a point from the set $\pi(D_\rho \cap \{(\mu_1, \mu) : \rho < |\mu_1|\})$, which implies that the set of points not in the set considered are the points of the form

$$\frac{\lambda_1 + \lambda_2 - \mu_1 - \mu}{\lambda_1\lambda_2 - \mu_1\mu}, \tag{4.1}$$

where $\rho < |\mu_1| < 1$, $|\mu\mu_1| < \rho$. The last may be given in the form

$$\frac{1}{\lambda_1\lambda_2}\left(\frac{\lambda_1\lambda_2}{\mu_1} + \frac{\lambda_1 + \lambda_2 - \mu_1 - \frac{\lambda_1\lambda_2}{\mu_1}}{1 - \frac{\mu_1\mu}{\lambda_1\lambda_2}}\right). \tag{4.2}$$

Since the function $z \to 1/(1-z)$ maps the unit disc to $\{\operatorname{Re} z > 1/2\}$, we easily get that for the fixed μ_1 the set of points of the previous form for all μ with $|\mu_1||\mu| < \rho$ is an open half plane. Now the set of numbers of the set in the theorem is the intersection of the complements of the sets of the last form. This implies that it is convex. We already know that $[(0,1)] \in \Gamma_{\mathbb{G}_{2,\rho}}(s,p)$, which finishes the proof. The fact that all sets $\Gamma_{\mathbb{G}_{2,\rho}}(x)$, $x \in \partial\mathbb{G}_{2,\rho}$, are non-empty and connected implies the $\mathbb{C}$-convexity of $\mathbb{G}_{2,\rho}$. □

Corollary 4.2. $\mathbb{E}$ *is* $\mathbb{C}$*-convex.*

Proof. Linear convexity of $\mathbb{E}$ implies that in the case of a smooth boundary point $x \in \partial\mathbb{E}$, the set $\Gamma_{\mathbb{E}}(x)$ is a singleton. Consider then the non-smooth point $x \in \partial\mathbb{E}$. It is sufficient to consider the cases

- $x = (r, r, 1)$, $r \in [0, 1]$,
- $x = (1, r, r)$, $r \in [0, 1)$.

Theorem 3.3 together with Theorem 4.1 (we also need to know that $\Gamma_{\mathbb{G}_2}(\pi(\lambda))$ is connected and contains the point $[(0,1)]$—this follows from [13]) imply that the set $\Gamma_{\mathbb{E}}(r, r, 1)$ is the union of connected sets whose intersection is non-empty (it contains the point $[(0,0,1)]$), so it is connected too.

The fact that $\Gamma_{\mathbb{E}}(1, r, r)$ is connected follows immediately from its description from Theorem 3.3. This finishes the proof. □

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