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GEOMETRIC QUANTIZATION FOR THE MECHANICS ON SPHERES

KIYOTAKA II

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1. **Introduction.** The classical Hamiltonian (kinetic energy) H for the mechanical system consisting of a non-relativistic free particle of unit mass moving on a Riemannian manifold is given by one-half the Riemannian metric. Several authors say that the quantum Hamiltonian $\hat{H}$ corresponding to H is $2^{-1}\hbar^1\Delta$, where Δ is the Laplacian acting on functions and $\hbar = (2\pi)^{-1}h$, h = Planck's constant. See Blattner [4] and Simms and Woodhouse [20]. However, DeWitt [7] and Cheng [5] used the method of "Feynman's path integrals" to derive a different operator for the quantum Hamiltonian. See also Ben-Abraham and Lonke [1]. Elhadad [9] applied the method of "Maslov pairing" to the geodesic flow on the unit n-sphere S^n to obtain a quantum Hamiltonian. Also, Weinstein [21] has shown, in the case of S^n, that the N-th quasi-classical eigenvalue for H is $\lambda_N = 2^{-1}\hbar^2(N + 2^{-1}(n-1))^2$, which is not equal to the N-th eigenvalue $\mu_N = 2^{-1}\hbar^2 N(N + n - 1)$ of the operator $2^{-1}\hbar^2\Delta$ on S^n. Note that the multiplicity of λ_N is equal to that of μ_N. See Ii [11]. These results show that $2^{-1}\hbar^2\Delta$ may not necessarily be the quantum Hamiltonian corresponding to H. The correct quantum Hamiltonian $\hat{H}$ has to be determined from appropriate general principles. In the present paper, we apply the quantization procedure of Kostant to quantize the mechanical system consisting of a non-relativistic, positive energy free particle of unit mass moving on the unit n-sphere S^n ($n \geqq 3$). We construct a polarization which enables us to quantize H. The resulting quantum Hamiltonian $\hat{H}$ has λ_N as the N-th eigenvalue. Moreover using the same polarization, we define an operator $\tilde{L}^2$ which has $2\mu_N$ as the N-th eigenvalue and has the same eigenspaces as that of $\hat{H}$. From the construction $\tilde{L}^2$ may be identified with $\hbar^2\Delta$. Under this identification, the correct quantum Hamiltonian $\hat{H}$ on S^n is given by $2^{-1}\hbar^2(\Delta + 4^{-1}(n-1)^2)$. The referee pointed out that the idea of adding $4^{-1}(n-1)^2$ to the Laplacian was also found by Y. Akyildiz in

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his Berkeley thesis in connection with the representation of $SO(n+1, 2)$ on $L^2(S^n)$.

2. **Preliminaries.** Let $\boldsymbol{R}^{n+1}$ and $T^*\boldsymbol{R}^{n+1}$ be the $(n+1)$-space and its cotangent bundle with coordinates $x=(x_1, \cdots, x_{n+1})$ and $(x, y)=(x_1, \cdots, x_{n+1}, y_1, \cdots, y_{n+1})$, respectively. Let $|x|^2=\sum x_j^2$, $|y|^2=\sum y_j^2$ and $x\cdot y=\sum x_j y_j$, so that the cotangent bundle on the unit n-sphere S^n is represented by $T^*S^n=\{(x, y)\in T^*\boldsymbol{R}^{n+1} \mid |x|=1, x\cdot y=0\}$. The Hamiltonian of a free particle of unit mass on S^n is given by $H(x, y)=2^{-1}|y|^2$. The phase space of the positive energy free particle of unit mass on S^n is given by $M=T^*S^n-\{0\text{-section}\}=\{(x, y)\in T^*S^n \mid |y|>0\}$ with the action form $\omega=\sum y_j dx_j$ and the symplectic form $\Omega=-d\omega=\sum dx_j\wedge dy_j$. Let $C^\infty(M; \boldsymbol{R})$ be the space of real-valued smooth functions on M. Since Ω is real and non-degenerate, we can define for each $f\in C^\infty(M; \boldsymbol{R})$ a real vector field X_f on M by $X_f \rfloor \Omega=df$. X_f is called the Hamiltonian vector field generated by f. Let us define L_{jk} and $L^2\in C^\infty(M; \boldsymbol{R})$ by $L_{jk}(x, y)=x_j y_k-x_k y_j$ $(1\leqq j, k\leqq n+1)$ and $L^2=\sum_{j<k} L_{jk}^2$, which are called angular momenta and square of angular momenta, respectively. Note that $H=2^{-1}L^2$. Let us denote $X_j=\partial/\partial x_j$, $Y_j=\partial/\partial y_j$, $X=(X_1, \cdots, X_{n+1})$ and $Y=(Y_1, \cdots, Y_{n+1})$. For the sake of simplicity, we write $u\cdot X$ instead of $\sum u_j X_j$ for $u=(u_1, \cdots, u_{n+1})$. Then we have $X_H=\sum(y_j X_j-|y|^2 x_j Y_j)=y\cdot X-|y|^2 x\cdot Y$ and $X_{jk}\equiv X_{L_{jk}}=\sum_{i=1}^{n+1}\{(-\delta_{ij}x_k+\delta_{ik}x_j)X_i+(-\delta_{ij}y_k+\delta_{ik}y_j)Y_i$, which are complete vector fields on M. X_H is the geodesic flow vector field on M. See Moser [13].

3. **Quantum bundle $\boldsymbol{L}$.** Since the symplectic form Ω is exact, and M is simply connected $(n\geqq 3)$, there exists, up to isomorphism, unique quantum bundle $\boldsymbol{L}$ over M. For quantum bundles, see Kostant [12]. $\boldsymbol{L}$ is a trivial bundle; $\boldsymbol{L}=(M\times \boldsymbol{C}, p, M)$. Let $\Gamma(\boldsymbol{L})$ denote the space of smooth cross-sections of $\boldsymbol{L}$. Then $\Gamma(\boldsymbol{L})$ is identified with the space $C^\infty(M)$ of complex-valued smooth functions on M in a natural manner. The connection form θ on $\boldsymbol{L}$ is given by $\theta|_{(x,y,z)}=-\hbar^{-1}p^*\omega+i^{-1}z^{-1}dz$ $(z\in \boldsymbol{C}-\{0\})$, where $i=\sqrt{-1}$. The covariant differentiation ∇ corresponding to θ is given by $\nabla_Z\varphi=Z\varphi-i\hbar^{-1}\omega(Z)\varphi$ for any tangent vector Z to M and for any $\varphi\in C^\infty(M)$.

4. **Polarization P.** In this section, we construct a polarization of our symplectic manifold (M, Ω), which is invariant under the flows of X_H and X_{jk}. By means of this polarization, we quantize the "classical observables" H and L_{jk}. See Elhadad [8], Gawedzki [10], Kostant [12], Onofri [14], [15], [16], Simms [18], [19] and Simms and Woodhouse [20].

Let F_j $(1\leqq j\leqq n+1)$ be vector fields on $T^*\boldsymbol{R}^{n+1}$ defined by

$F_j|_{(x,y)} = X_j - i|y|Y_j$. For any $(x, y) \in M$ and for any $v = (v_1, \cdots, v_{n+1}) \in \boldsymbol{R}^{n+1}$, such that $v \cdot x = v \cdot y = 0$, the vector $v \cdot F|_{(x,y)} = \sum v_j F_j|_{(x,y)}$ is tangent to M. Let $P_{(x,y)}$ be the complexified tangent space spanned by the vectors $X_H|_{(x,y)}$ and $\{v \cdot F|_{(x,y)} \mid v \in \boldsymbol{R}^{n+1},\ v \cdot x = v \cdot y = 0\}$. Then $P\colon (x, y) \mapsto P_{(x,y)}$ defines a distribution on M, and we have:

LEMMA 1. *P is a polarization of (M, Ω), which is invariant under the flows of X_H and X_{jk}.*

PROOF. It is easy to see that P is an n-dimensional smooth, complex distribution on M. $\Omega(P, P) = 0$ is straightforward. Let $v = (v_1, \cdots, v_{n+1})$ and $w = (w_1, \cdots, w_{n+1})$ be $\boldsymbol{R}^{n+1}$-valued smooth functions on M such that $v \cdot x = v \cdot y = w \cdot x = w \cdot y = 0$. Then we have $[X_H, v \cdot F] = \sum (X_H v_j + i|y|v_j)F_j$, $[v \cdot F, w \cdot F] = \sum (a_j - i|y|b_j)F_j$, $[X_{jk}, X_H] = 0$ and $[X_{jk}, v \cdot F] = \sum c_i F_i$, where $a_j = (v \cdot X)w_j - (w \cdot X)v_j$, $b_j = (v \cdot Y)w_j - (w \cdot Y)v_j$ and $c_i = X_{jk}v_i + \delta_{ij}v_k - \delta_{ik}v_j$. a_j, b_j and c_i are real-valued and satisfy $\sum (X_H v_j)x_j = \sum (X_H v_j)y_j = a \cdot x = a \cdot y = b \cdot x = b \cdot y = 0$, $c \cdot x = X_{jk}(v \cdot x) = 0$ and $c \cdot y = X_{jk}(v \cdot y) = 0$. It follows that P is involutive and invariant under the flows of X_H and X_{jk}. $P \cap \bar{P}$ is a one-dimensional complex distribution spanned by X_H. $P + \bar{P}$ is a $(2n-1)$-dimensional involutive complex distribution spanned by X_H and the vectors of the form $v \cdot X$ and $v \cdot Y$. Thus we are done.

In the terminology of Gawedzki [10], P is a strongly admissible, positive polarization.

5. Half-P-forms. Let $\alpha = \sum y_j dy_j$ and $\beta_j = dx_j - i|y|^{-1}dy_j$ $(1 \leqq j \leqq n+1)$ be one-forms on M. Choose $\boldsymbol{R}^{n+1}$-valued (not necessarily continuous) functions $u_a = (u_a^1, \cdots, u_a^{n+1})$, $(1 \leqq a \leqq n-1)$, on M, such that the matrix ${}^t(x, |y|^{-1}y, u_1, \cdots, u_{n-1}) \in SO(n+1)$ at any point $(x, y) \in M$. Define $\beta = \bigwedge_{a=1}^{n-1} \sum_{1 \leqq j \leqq n+1} u_a^j \beta_j$. Then β is a smooth $(n-1)$-form on M, which is independent of the choice of $\{u_a\}$. Let $\mathscr{L}_Z$ denote the Lie derivation with respect to a vector field Z on M.

LEMMA 2. *$\mu = \alpha \wedge \beta$ is a nowhere-vanishing smooth n-form on M, which satisfies: (1) $Z \rfloor \mu = 0$ for any vector Z from P, (2) $\mathscr{L}_{X_H}\mu = i(n-1)|y|\mu$, (3) $\mathscr{L}_{v \cdot F}\mu = 0$ for any $\boldsymbol{R}^{n+1}$-valued smooth function v on M with $v \cdot x = v \cdot y = 0$ and (4) $\mathscr{L}_{X_{jk}}\mu = 0$.*

By the above lemma, it follows that the bundle $\boldsymbol{D}$ of complex n-forms on M, vanishing after contraction with any vector from P is a trivial bundle. Let $\boldsymbol{D}^{1/2} = (M \times \boldsymbol{C}, p, M)$ be another complex line bundle (trivial bundle) over M, and ν denote the cross-section $(x, y) \mapsto (x, y, 1)$ of $\boldsymbol{D}^{1/2}$. Let $\iota\colon \boldsymbol{D}^{1/2} \otimes \boldsymbol{D}^{1/2} \to \boldsymbol{D}$ be the vector bundle isomorphism defined by $\iota(\nu\otimes\nu)=\mu$. We call the pair $(\boldsymbol{D}^{1/2},\iota)$ the square root structure for P and sections of $\boldsymbol{D}^{1/2}$ half-P-forms on M. See Gawedzki [10] and Simms and Woodhouse [20]. Since $H^1(M, \boldsymbol{Z}_2)=0$ $(n\geqq 3)$, the square root structure is unique. For any smooth vector field Z on M, let us define a Z-derivation $\mathscr{L}_Z^{1/2}$ acting on the space $\Gamma(\boldsymbol{D}^{1/2})$ of smooth cross-sections of $\boldsymbol{D}^{1/2}$ by the following: $\iota(2(\mathscr{L}_Z^{1/2}\sigma)\otimes\sigma)=\mathscr{L}_Z(\iota(\sigma\otimes\sigma))$, for any $\sigma\in\Gamma(\boldsymbol{D}^{1/2})$. See Gawedzki [10].

Let $\mathscr{D}'(M)$ be the space of generalized functions (distributions or 0-currents) on M. See de Rham [6] and Schwartz [17]. We call the tensor product $\mathscr{D}'(M)\otimes\Gamma(\boldsymbol{D}^{1/2})$, taken over the ring $C^\infty(M)$, the space of generalized half-P-forms on M. See Simms [19]. Finally, we have the space of generalized $\boldsymbol{L}$-valued half-P-forms on M, $\Gamma=\Gamma(\boldsymbol{L})\otimes D'(M)\otimes\Gamma(\boldsymbol{D}^{1/2})$. Note that Γ is naturally identified with $\mathscr{D}'(M)$ by the correspondence $1\otimes T\otimes\nu\leftrightarrow T$.

6. Quantum phase space $\mathscr{H}^P$. Let $\Gamma(P)$ denote the space of smooth, complex vector fields on M which belong to P at each point of M. A complex vector field Z on M is said to preserve the polarization P if $[Z, X]\in\Gamma(P)$ for any $X\in\Gamma(P)$. For each vector field Z, which preserves P, we define a linear operator δ_Z on Γ by $\delta_Z(\varphi\otimes T\otimes\sigma)=(\nabla_Z\varphi)\otimes T\otimes\sigma+\varphi\otimes ZT\otimes\sigma+\varphi\otimes T\otimes\mathscr{L}_Z^{1/2}\sigma$, where ZT is defined by $(ZT)(A)=-T(\mathscr{L}_ZA)$ for any smooth $2n$-form A on M of compact support. See Gawedzki [10] and Simms [19]. A cross-section $\gamma\in\Gamma$ is called P-horizontal if $\delta_Z(\gamma)=0$ for all $Z\in\Gamma(P)$. Then by Lemma 2, a cross-section $1\otimes T\otimes\nu\in\Gamma$ is P-horizontal if and only if $X_HT-i|y|(\hbar^{-1}|y|-2^{-1}(n-1))T=0$ and $(v\cdot F)T=0$, for any v as in Lemma 2.

For each integer N, $N>2^{-1}(n-1)$, let us denote $r_N=\hbar(N+2^{-1}(n-1))$. Define a submanifold: $M_N=\{(x,y)\in M\,|\,|y|=r_N\}$ of M with the inclusion $p_N\colon M_N\to M$. Let $\bigwedge^q(M)$ denote the space of smooth q-forms on M. Define $\eta=|y|^{-1}(y\cdot Y)\,\lrcorner\,\Omega^n\in\bigwedge^{2n-1}(M)$. Then η satisfies $(y\cdot Y)\,\lrcorner\,\eta=0$, $d(|y|)\wedge\eta=\Omega^n$ and $p_N^*(\mathscr{L}_{X_H}\eta)=0$. It follows that $\eta_N=p_N^*\eta\in\bigwedge^{2n-1}(M_N)$ is non-vanishing and invariant under the flow of X_H restricted to M_N. For any $A\in\bigwedge^{2n}(M)$, $A=a\Omega^n$ with $a\in C^\infty(M)$, let $A_N=p_N^*(a\eta)\in\bigwedge^{2n-1}(M_N)$. For any $T_N\in\mathscr{D}'(M_N)$, let us define $\tilde{T}_N\in\mathscr{D}'(M)$ by $\tilde{T}_N(A)=T_N(A_N)$, for any $A\in\bigwedge^{2n}(M)$ of compact support. In the following, we shall determine the subspace $\mathscr{H}^P$ of Γ composed of P-horizontal cross-sections of the form $1\otimes\sum_N\tilde{T}_N\otimes\nu$. If we write $\mathscr{H}_N^P=\{1\otimes\tilde{T}_N\otimes\nu\in\mathscr{H}^P\,|\,T_N\in\mathscr{D}'(M_N)\}$, then $\mathscr{H}^P=\bigoplus\mathscr{H}_N^P$.

LEMMA 3. *$\mathscr{H}_N^P$ is non-trivial if and only if N is non-negative. In this case, $\mathscr{H}_N^P$ is given by $\mathscr{H}_N^P=\{1\otimes\tilde{T}_N\otimes\nu\,|\,T_N=\sum_{|K|=N}c_Kz^K\}$, where*

$c_K \in \boldsymbol{C}$, $z = (z_1, \cdots, z_{n+1})$, $z_j = x_j - ir_N^{-1}y_j \in C^\infty(M_N)$, $K = (k_1, \cdots, k_{n+1})$ *and* $|K| = \sum k_j$.

Note that

$$\dim \mathscr{H}_N^P = \frac{2N+n-1}{N}\binom{N+n-2}{n-1},$$

which is equal to the multiplicity of the N-th eigenvalue of the Laplacian Δ acting on functions on S^n. See Berger-Gauduchon-Mazet [2].

7. Kostant quantization for H and L_{jk}. Following the Kostant quantization prescription, we shall assign for H and L_{jk} linear operators $\hat{H} = i^{-1}\hbar\delta_{X_H} + H$ and $\hat{L}_{jk} = i^{-1}\hbar\delta_{X_{jk}} + L_{jk}$ on $\mathscr{H}^P$. We call $\hat{L}_{jk}$ the angular momentum operators. Furthermore, we define $\tilde{L}^2 = \sum_{i<k}(\hat{L}_{jk})^2$, which we call the square of angular momentum operators.

LEMMA 4. (1) $\hat{H}|_{\mathscr{H}_N^P} = 2^{-1}\hbar^2(N + 2^{-1}(n-1))^2$ (*multiplication operator*). (2) $\hat{L}_{jk}(1 \otimes z^A \otimes \nu) = 1 \otimes i^{-1}\hbar(a_k z^B - a_j z^C) \otimes \nu$, *where* $z = (z_1, \cdots, z_{n+1})$, $z_j = x_j - ir_N^{-1}y_k$, $A = (a_1, \cdots, a_{n+1})$, $\sum a_j = N$, $z^A = z_1^{a_1} \cdots z_{n+1}^{a_{n+1}}$, $B = (a_1, \cdots, a_j + 1, \cdots, a_k - 1, \cdots, a_{n+1})$ *and* $C = (a_1, \cdots, a_j - 1, \cdots, a_k + 1, \cdots, a_{n+1})$. (3) $\tilde{L}^2|_{\mathscr{H}_N^P} = \hbar^2 N(N+n-1)$ (*multiplication operator*).

PROOF. Since $\delta_{X_H} = 0$ on $\mathscr{H}^P$, we have $\hat{H}|_{\mathscr{H}_N^P} = H|_{\mathscr{H}_N^P} = 2^{-1}\hbar^2(N + 2^{-1}(n-1))^2$. Thus (1) is proved. To prove (2), it is sufficient to note $\nabla_{X_{jk}}1 = -i\hbar^{-1}L_{jk}$ and $\mathscr{L}_{X_{jk}}^{1/2}\nu = 0$, which follow from Lemma 2. To prove (3), it is sufficient to note $z \cdot z = 0$ on M_N.

Summing up, we have the following:

THEOREM. *There exists a polarization P on $M = T^*S^n - \{0\text{-section}\}$, which is invariant under the geodesic flow and under the natural $SO(n+1)$-action on M. By means of this polarization, the classical Hamiltonian H and the functions L_{jk}'s are geometrically quantized. For $n \geqq 3$, the corresponding quantum Hamiltonian $\hat{H}$ has $2^{-1}\hbar^2(N + 2^{-1}(n-1))^2$ as the N-th eigenvalue ($N \geqq 0$) with the eigenspace $\mathscr{H}_N^P$ of dimension*

$$\frac{2N+n-1}{N}\binom{N+n-2}{n-1}.$$

Moreover, an operator $\tilde{L}^2$, defined by $\sum_{j<k}(\hat{L}_{jk})^2$, has $\hbar^2 N(N+n-1)$ as the N-th eigenvalue with the eigenspace $\mathscr{H}_N^P$.

As "classical observables", energy H and one-half the square of angular momenta, $2^{-1}\sum L_{jk}^2$, are equal, but as "quantum observables", $\hat{H}$ and $2^{-1}\sum(\hat{L}_{jk})^2$ are different by an additive constant; $\hat{H} = 2^{-1}(\tilde{L}^2 + \hbar^2(2^{-1}(n-1))^2)$.

A similar observation may be possible for such manifolds as compact symmetric spaces of rank one. See Besse [3].

8. **Appendix.** Let Q be the restriction to M of the vertical polarization of $p: T^*S^n \to S^n$, and E the bundle of complex n-forms on M, vanishing after contraction with any vector from Q. In the following, we use the same letter p for the restriction of p to M. E is a trivial bundle and $p^*\mu$ is a nowhere-vanishing cross-section of E, where $\mu \in \bigwedge^n(S^n)$ is the volume form on S^n. Let $(E^{1/2}, \iota)$ be the square root structure for Q and ν the cross-section of the trivial bundle $E^{1/2}$ such that $\iota(\nu \otimes \nu) = p^*\mu$. For each vector field Z on M, which preserves the polarization Q, we define a linear operator δ_Z on $\Gamma(L) \otimes \Gamma(E)$ by $\delta_Z(\varphi \otimes \nu) = (\nabla_Z \varphi) \otimes \nu + \varphi \otimes \mathscr{L}_Z^{1/2}\nu$. Q-horizontal sections are similarily defined, which are of the form $(f \circ p) \otimes \nu$ for $f \in C^\infty(S^n)$. The space $\mathscr{H}^Q$ of Q-horizontal sections is naturally identified with $C^\infty(S^n)$ by the correspondence $(f \circ p) \otimes \nu \leftrightarrow f$. Since X_{jk} preserves Q, we can define a linear operator $\hat{L}_{jk}$ by $\hat{L}_{jk} = i^{-1}\hbar\delta_{X_{jk}} + L_{jk}$ on $\mathscr{H}^Q$. We also call $\hat{L}_{jk}$ the angular momentum operator. Furthermore, if we define $\tilde{L}^2 = \sum_{j<k} (\hat{L}_{jk})^2$, then we have $\tilde{L}^2((f \circ p) \otimes \nu) = ((\hbar^2 \Delta f) \circ p) \otimes \nu$. Thus, under the identification of $\mathscr{H}^Q$ with $C^\infty(S^n)$, $\tilde{L}^2$ is nothing but $\hbar^2$ times the Laplacian Δ acting on functions on S^n, (the Casimir operator). Since X_H does not preserve Q, we cannot quantize H in the same way as above as a linear operator on $\mathscr{H}^Q$. But, by Lemma 4 and the above calculation, it is reasonable to say that if we quantize the classical Hamiltonian H as an operator on $\mathscr{H}^Q$, then we should have the operator $\hat{H} = 2^{-1}(\tilde{L}^2 + \hbar^2(2^{-1}(n-1))^2)$ as the corresponding quantum Hamiltonian. If we identify $\mathscr{H}^Q$ with $C^\infty(S^n)$, then $\hat{H}$ is given by $2^{-1}\hbar^2(\Delta + (2^{-1}(n-1))^2)$.

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Department of Mathematics
Yamagata University
Yamagata, 990
Japan