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Geometric Realizations for Dyck's Regular Map on a Surface of Genus 3

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Abstract. Klein's and Dyck's regular maps on Riemann surfaces of genus 3 were one impetus for the theory of regular maps, automorphic functions, and algebraic curves. Recently a polyhedral realization in E^3 of Klein's map was discovered [18], thereby underlining the strong analogy to the icosahedron. In this paper we show that Dyck's map can be realized in E^3 as a polyhedron of Kepler–Poinsot-type, i.e., with maximal symmetry and minimal self-intersections. Furthermore some closely related polyhedra and a Kepler–Poinsot-type realization of Sherk's regular map of genus 5 are discussed.

1. Introduction

More than 100 years ago Klein [16, 17] and Dyck [9–11] discovered two pairs of regular maps on Riemann surfaces of genus 3 that play an important role in the theory of regular maps, automorphic functions, and algebraic curves. In Coxeter's notation these maps are $\{3,7\}_8$ and $\{3,8\}_6$, respectively, and their duals. As for all compact Riemann surfaces of genus $g \geq 2$, their usual geometric representation is in terms of the Poincaré model of the hyperbolic plane (cf. [5, 8, 11, 16]). But they can also be represented by algebraic curves of order 4 with homogeneous complex variables as follows (cf. [9, 16]).

Klein's quartic: $x^3y + y^3z + z^3x = 0$

Dyck's quartic: $x^4 + y^4 + z^4 = 0.$

According to Klein's ideas developed in [16, 17], namely to allow interaction of different branches of mathematics such as algebra, geometry, topology, and complex analysis, it seems to be also of interest to look for polyhedral realizations of the maps in ordinary Euclidean 3-space $\mathbb{E}^3$. As the maps share the most important property of the Platonic solids to admit a flag-transitive automorphism group (which is just the symmetry group in the case of the Platonic solids), this approach would lead in a natural way to polyhedral analogues of higher genus. For Klein's map such a polyhedron was recently discovered by the authors in [18]; see also [19, 22], where some of Coxeter's regular skew polyhedra [1, 3] are realized in $\mathbb{E}^3$.

By a polyhedron P we mean a finite family of plane (Jordan-) polygons in $\mathbb{E}^3$ (or $\mathbb{E}^d$ with $d \geq 3$), whose union is a 2-manifold without boundary (and without self-intersections). Sometimes we will also call this union the polyhedron P. If the plane polygons are disjoint except for their edges, then they are said to be the faces of P. The vertices and edges of P are the vertices and edges of the faces of P. We do also require adjacent faces not being coplanar.

The realizability of a regular map as a polyhedron in $\mathbb{E}^3$ seems to be a very rare property; so there is also an interest in polyhedral realizations of Kepler–Poinsot-type. By a polyhedron of Kepler–Poinsot-type (KP-type) we mean a finite family P of plane (Jordan-) polygons that fit together like the faces of a map on a surface but fail to give a polyhedron (that is, self-intersections occur), and whose union has the symmetry group of some Platonic solid. Besides "maximal" symmetry we do also require "minimal" self-intersections; that is: If F and G are vertices, edges or faces of P with $F \cap G \neq \emptyset$, then $F \cap G$ shall either be a vertex, an edge, or a face of P, or be a point-set in $\mathbb{E}^3$ of dimension less than $\dim(F)$ and $\dim(G)$.

In this paper we describe a polyhedron of KP-type in $\mathbb{E}^3$ for Dyck's map $\{3,8\}_6$. It is derived from a pair of octahedra and has tetrahedral symmetry group. Because of the combinatorial structure of the map we conjecture there is no realization as a polyhedron in $\mathbb{E}^3$ without self-intersections, and we would not be surprised if the same is also true for $\mathbb{E}^4$. We will give a proof that $\{3,8\}_6$ cannot be represented by a polyhedron in $\mathbb{E}^3$ (with or without self-intersections) that has an octahedral symmetry group, though this is possible in $\mathbb{E}^4$. These results together with the descriptions of another three-dimensional model and higher-dimensional realizations of Dyck's map are contained in Sections 3 and 4. At the end of Section 3 we briefly describe a KP-type realization of a regular map by Sherk [21], which is to some extent related to Dyck's map.

Our attempts to find a polyhedron for $\{3,8\}_6$ exhibited another map of type $\{3,8\}$ and genus 3. It is given by a polyhedron with exactly the same number of vertices, edges, and faces as $\{3,8\}_6$, namely 12, 48, and 32, respectively. This map together with another polyhedral realization was first discovered by Grünbaum and Shephard (cf. [14]; compare also [22, 23]).

2. Dyck's map $\{3,8\}_6$

According to Coxeter–Moser [5] a map $\mathcal{M}$ is a decomposition of a closed real 2-manifold into f_2 simply connected, nonoverlapping regions called faces by means of f_1 arcs called edges. The f_0 intersections of the edges are said to be the vertices of $\mathscr{M}$. The triplet $f := (f_0, f_1, f_2)$ is called the f-vector of $\mathscr{M}$. By a flag of $\mathscr{M}$ we mean a set consisting of one vertex, one edge incident with this vertex, and one face containing this edge. $\mathscr{M}$ is said to be of type $\{p, q\}$ if all the faces of $\mathscr{M}$ are topological p-gons, q meeting at each vertex.

With every map $\mathscr{M}$ is associated its dual map $\mathscr{M}^*$ having f_0 faces, one surrounding each vertex of $\mathscr{M}$, f_1 edges, one crossing each edge of $\mathscr{M}$, and f_2 vertices, one contained in the interior of each face of $\mathscr{M}$.

In Coxeter–Moser [5] a map $\mathscr{M}$ is called regular if its (combinatorial) automorphism group $A(\mathscr{M})$ contains two particular automorphisms ρ and σ: the automorphism ρ cyclicly permutes the edges bounding one face, and σ cyclicly permutes the edges surrounding one vertex of that face (these are the automorphisms R and S of [5]). The group of a regular map is transitive on the vertices, on the edges, and on the faces, but need not be flag-transitive. In the flag-transitive case the map $\mathscr{M}$ is also called a reflexible regular map in order to underline that the group contains sufficiently many (combinatorial) reflections, while in the other case $\mathscr{M}$ is said to be irreflexible. Note that this notion of regularity is not quite consistent with the more modern notion of regularity referring only to flag-transitive configurations (cf. Danzer–Schulte [6]).

A Petrie polygon of a regular map (or of a regular tessellation on the Euclidean 2-sphere, the Euclidean, or hyperbolic plane) is a zig-zag along its edges such that every two but no three consecutive edges of the polygon are edges of a single face.

A fruitful source for the construction of regular maps of type $\{p, q\}$ is the identification of those pairs of vertices of the regular tessellation $\{p, q\}$ which are separated by r steps along a Petrie polygon. For suitable values of r this gives a regular map denoted by $\{p, q\}_r$. It is reflexible, its Petrie polygons have length r, and its dual is $\{q, p\}_r$ (cf. [5]).

Dyck's regular maps $\{3,8\}_6$ and $\{8,3\}_6$ are particular instances obtained in this way from the hyperbolic tessellations $\{3,8\}$ and $\{8,3\}$, respectively. The map

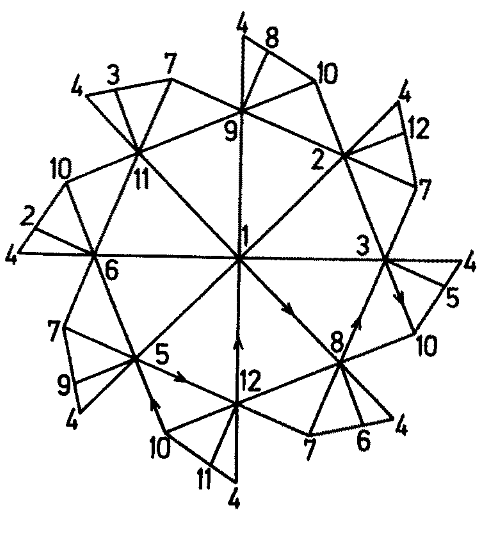

Fig. 1

$\{3,8\}_6$ is shown in Fig. 1 (with the usual convention that vertices with the same label are to be identified); one Petrie polygon of length 6 is indicated by means of arrows. The f-vector of $\{3,8\}_6$ is (12, 48, 32).

The group of $\{3,8\}_6$ is $G^{3,8,6}$ $(=G^{3,6,8})$ of order 192 and is abstractly defined by

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^3 = (\rho_1\rho_2)^8 = (\rho_0\rho_2)^2 = (\rho_0\rho_1\rho_2)^6 = 1.$$

Here, the automorphisms ρ and σ above are $\rho_0\rho_1$ and $\rho_1\rho_2$, respectively. They generate the rotation subgroup of order 96 (cf. [2, 5]).

In the following we point out some important properties of the map $\mathcal{M} = \{3,8\}_6$; the labelling is taken from Fig. 1.

Each vertex of $\mathcal{M}$ is joined by an edge to eight of the 11 remaining vertices. In particular, the 12 vertices split into three blocks of four, where in each block no two vertices are joined by an edge. One such block is given by the vertices 1, 4, 7, and 10.

Each of the vertices 1, 4, 7, and 10 is surrounded by eight triangular faces forming the star of the vertex in $\mathcal{M}$. This gives all the 32 faces of $\mathcal{M}$. The crucial point is then that the links (that are the polygons bounding the stars) of the vertices 1, 4, 7, and 10 share the remaining eight vertices of $\mathcal{M}$. In particular, the links of the vertices 1 and 4 fit together like a pair of octagons, one regular convex octagon and another regular star-octagon $\{8/3\}$ with the same vertices. The same is true for the links of 7 and 10.

Furthermore, following the edges along a Petrie polygon we observe that the faces incident with these edges fit together like the faces in the mantle of a triangular antiprism, or what is the same, like the faces of an octahedron with one pair of antipodal faces removed. This fact will become particularly evident in our model for $\{3,8\}_6$.

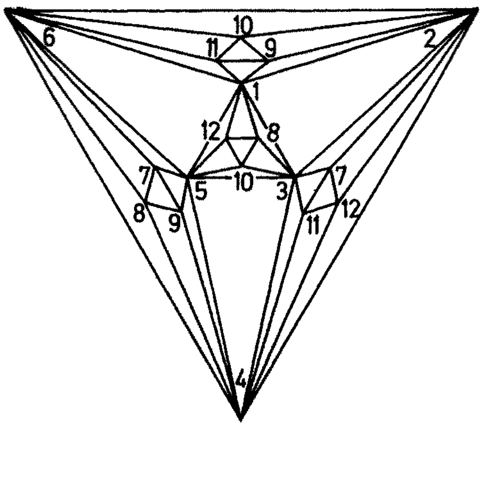

Fig. 2

3. The Polyhedron of Kepler-Poinsot-Type

Our Figs. 3 and 4 show how Dyck's map $\{3,8\}_6$ can be realized in $\mathbb{E}^3$ as a polyhedron of KP-type. In Fig. 3 the edges of the two octahedra and the "tunnel" in front are drawn with heavy lines, though some are not visible. Assume the faces of the two concentric regular octahedra Q_1 and Q_2, say Q_1 contained in Q_2, are colored in white and red such that any two adjacent faces of each octahedron as well as corresponding faces of the two octahedra are colored differently. Then, each face F of Q_2 is colored the same as the antipode G of the face of Q_1 corresponding to F. For simplicity we will call F and G antipodes (with respect to Q_1 and Q_2).

Now joining each of the four white pairs of antipodes by six red triangles forming the mantle of a triangular antiprism gives 24 triangles which, together with the eight red-colored faces of Q_1 and Q_2, give the 32 faces of a polyhedron with self-intersections representing a map of genus 3. This polyhedron P, or more exactly, the respective map $\mathcal{M}$ is in fact Dyck's regular map $\{3,8\}_6$.

To check this let us assume the vertices are labelled as in Figure 3 and the coloring is such that the face F of Q_2 with vertices 1, 3, and 5 is colored white. Representing Q_2 by a Schlegel diagram $\mathcal{D}'$ in the face of Q_2 opposite to F (cf. Grünbaum [13]) and each of the four triangular antiprisms by a Schlegel diagram in the respective triangular face in $\mathcal{D}'$ gives a diagram $\mathcal{D}$, where vertices are labelled by $1,\ldots,12$ (see Fig. 2). Here, as usual, vertices with the same labels are to be identified. Observing that the four red faces of Q_1 (that is, the faces 7 9 11, 7 8 12, 8 9 10, and 10 11 12) are not yet covered by $\mathcal{D}$ and have to be added in mind we can easily see that our polyhedron of KP-type belongs to the map shown in Fig. 1.

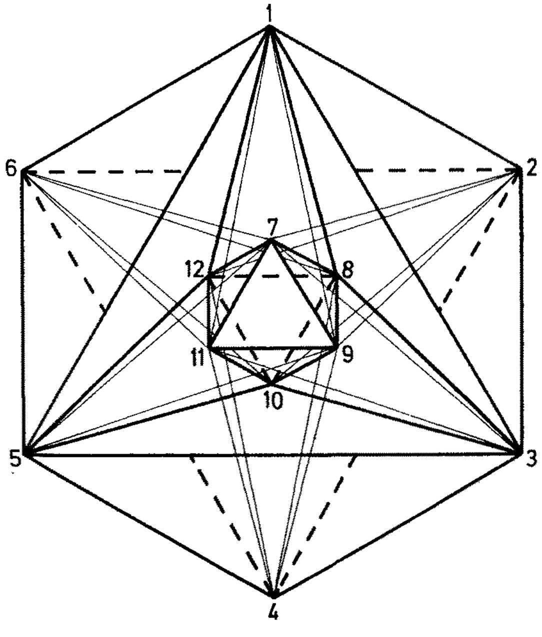

Fig. 3

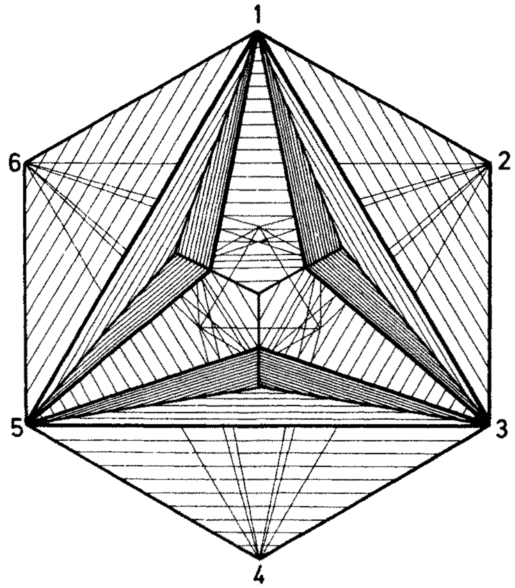

Fig. 4

This map is actually $\{3,8\}_6$. The automorphisms ρ and σ involved in the definition of regularity are easily determined: The Euclidean rotation in the face 1 2 3 of the octahedron Q_2 induces

$$\rho = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12),$$

and the automorphism σ permuting the neighboring vertices of 1 is

$$\sigma = (2\ 3\ 8\ 12\ 5\ 6\ 11\ 9)(7\ 10)$$

and keeps the vertices 1 and 4 fixed. From Sherk [20] we know that a regular map of type $\{3,8\}$ on a surface of genus 3 must be $\{3,8\}_6$.

In our model the faces in each of the four tunnels (that are the mantles of the triangular antiprisms) do not intersect except for vertices or edges, but there are self-intersections arising from faces of different tunnels. So in Fig. 4 the visible edges are drawn in heavy lines and the visible "false" edges in less heavy lines.

The symmetry group $S(P)$ of P is the color-preserving subgroup of the symmetry group of Q_2, hence is isomorphic to S_4. That shows that P is in fact a polyhedron of KP-type with tetrahedral symmetry group. This group is transitive on the vertices of Q_2 as well as on the vertices of Q_1. There is an obvious inside–outside automorphism τ, not a symmetry, that interchanges the vertices of Q_1 and Q_2, namely

$$\tau = (1\ 10)(2\ 11)(3\ 12)(4\ 7)(5\ 8)(6\ 9).$$

Together with $S(P)$ this automorphism τ generates a subgroup of the automor-

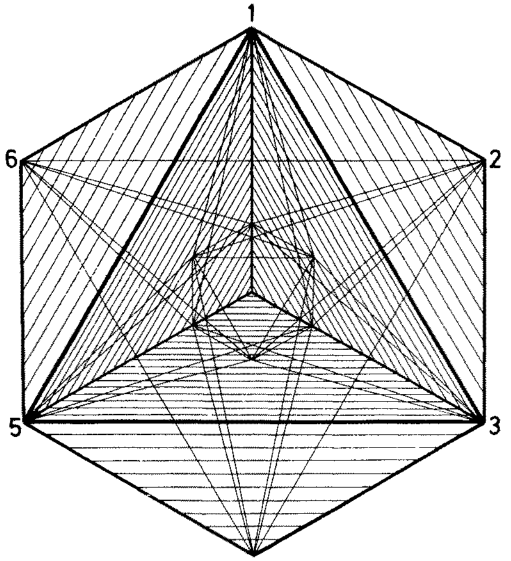

Fig. 5

phism group of P isomorphic to $S_4 \times C_2$, that is, to the octahedral symmetry group.

Our model does also emphasize the significance of the Petrie polygon in Dyck's map. Particular specimens of the Petrie polygon are provided by the six edges joining the six faces of the tunnels.

Interpreting our results purely combinatorially we have proved so far that $\{3,8\}_6$ can be derived from a pair of octahedra Q_1 and Q_2 by suitably coloring their faces white and red, defining a relation "x and y correspond to each other" between the vertices x of Q_1 and y of Q_2 that respects the antipode relation for vertices of Q_1 (or Q_2, respectively), and finally joining antipodal faces of Q_1 and Q_2 by a tunnel.

This shows that no matter how two octahedra (endowed with the respective coloring and the relation) are placed in space, not necessarily of three dimensions, we can always turn them into Dyck's model $\{3,8\}_6$ by the above procedure. In fact it is sufficient to start from two (possibly nonconvex) polyhedra isomorphic to octahedra.

In this way another model in $\mathbb{E}^3$ is obtained by inverting the first model in the sense that now two vertices x of Q_1 and y of Q_2 become 'antipodal' if and only if they correspond to each other in the first model. Then self-intersections occur in each tunnel, but there are no intersections between faces of different tunnels except for vertices. Again, we have a polyhedron of KP-type with tetrahedral symmetry group (Fig. 5).

This antipodal interchange of vertices leaves the three blocks of four vertices each unchanged. So also the edges remain unchanged, that is, both KP-type realizations have the same 1-skeleton, but clearly different faces. Hence, as for the classical Kepler–Poinsot polyhedra $\{5,\frac{5}{2}\}$ and $\{\frac{5}{2},5\}$ there are two isomorphic polyhedra of KP-type with different kinds of self-intersections.

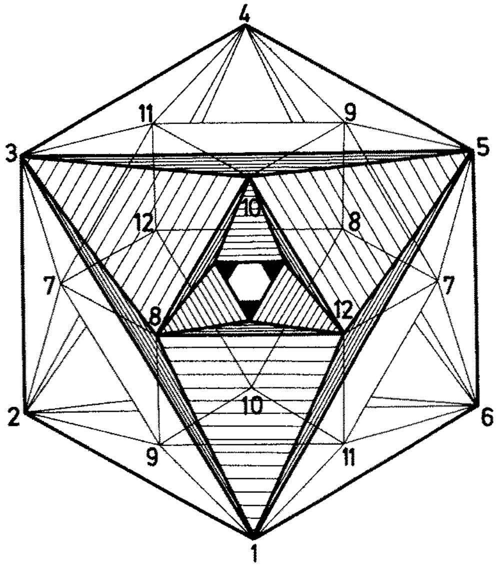

Fig. 6

In Fig. 5 the visible edges are drawn with heavy lines, the visible "false" edges with less heavy lines. The visible faces are shaded.

Our first model can also be regarded as the projection of a polyhedron of KP-type in $\mathbb{E}^4$, derived from a pair of opposite octahedral facets of the regular 24-cell. Again, self-intersections occur but this time the inside-outside automorphism τ is realizable by a symmetry, namely the central inversion in $\mathbb{E}^4$. Thus, the symmetry group is $S_4 \times C_2$.

Taking into account the number of self-intersections in our three-dimensional models and the way how the links of certain vertices are related to each other (see Section 2) we conjecture there is no realization of $\{3,8\}_6$ as a polyhedron (without self-intersections) in $\mathbb{E}^3$. We would not be surprised if the same holds for $\mathbb{E}^4$. The proof that realizations with octahedral symmetry group do not exist in $\mathbb{E}^3$ is postponed to Section 4.

An interesting three-dimensional model without self-intersections, some kind of complex but not quite a polyhedron, can be derived from the Schlegel diagram of the 24-cell (cf. Hilbert and Cohn-Vossen [15]). In this diagram the 24 vertices split into three blocks: two blocks of six each giving the vertices of a regular octahedron, say T_1 and T_3 with $T_1 \subset T_3$, and another block of 12, that is the vertex set of an Archimedean cuboctahedron T_2 with $T_1 \subset T_2 \subset T_3$ (cf. Fig. 6). The octahedra are concentric, and T_2 is a concentric copy of the cuboctahedron whose vertices are the midpoints of the edges of T_3. Again, we color the faces of T_1 and T_3 red and white, this time in such a way that corresponding faces of T_1 and T_3 (in the usual meaning) are colored the same. Then we join each pair (F, G) of corresponding white faces of T_1 and T_3 by a tunnel consisting of the mantles of two triangular antriprisms with a common basis. This basis is the triangular face of T_2 next to F (and G, respectively). Adding the eight red triangular faces of T_1

and T_3 to the 48 triangular faces in the four tunnels gives a polyhedron P of genus 3 with 56 triangular faces. The valence of each vertex of P is either 6 or 8.

This polyhedron can be turned into Dyck's map $\{3,8\}_6$ by suitably identifying the vertices of P and adding four triangular faces (namely 7 9 11, 7 8 12, 8 9 10, and 10 11 12 in Fig. 6). In fact identifying any two vertices of P if and only if they are either corresponding vertices of T_1 and T_3 or antipodal vertices of T_2 provides a map with four triangular faces removed. Note that the identification process (restricted to the vertices of T_2) turns the four triangular faces of T_2 used as holes in the construction of P into four faces of some octahedron, which fit together like the white faces of T_3. Hence, by our combinatorial considerations above, we get the map $\{3,8\}_6$ by adding the remaining four faces of this octahedron. Note that we must not think of these four faces as being the four triangular faces of T_2 not used in the construction of P.

Another way of thinking about this model is the following. Consider only those 28 triangular faces of P not contained in T_2 and add four triangles Δ with vertices among the vertices of T_2 which, after identifying antipodal vertices in T_2, become the four additional triangular faces in the construction above. The four triangles Δ are not faces of T_2 but are skew in T_2, and they can be chosen in such a way that any two do not intersect. Then, starting from the 28 faces of P and the four triangles Δ the identification of antipodal vertices in T_2 gives again $\{3,8\}_6$.

Finally we mention some realizations of $\{3,8\}_6$ as polyhedra (without self-intersections) in higher dimensions. As the map $\{3,8\}_6$ has triangular faces, we can clearly realize it in the 2-skeleton of every three-neighbourly d-polytope in $\mathbb{E}^d$ (cf. Grünbaum [13]). This gives realizations in $\mathbb{E}^d$ with $6 \le d \le 11$ via the cyclic d-polytope $C(12,d)$ (which is the 11-simplex if $d=11$), and from there via Schlegel diagrams realizations in $\mathbb{E}^{d-1}$.

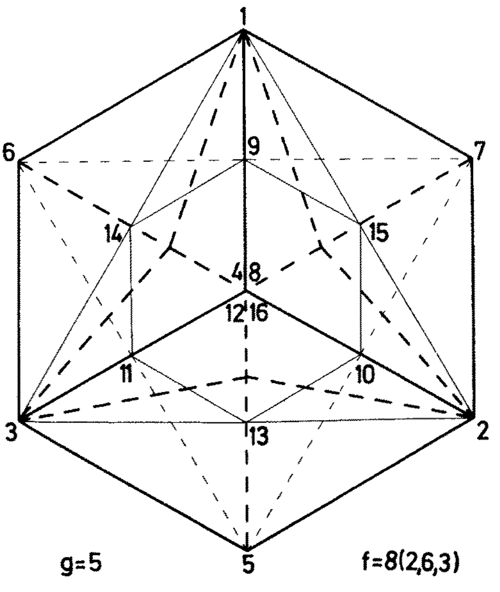

Fig. 7

Remark. The results of this section suggest looking for polyhedra of KP-type derived from a pair of homothetic Platonic solids. Figure 7 illustrates how such a polyhedron is obtained from a pair of cubes, namely by joining each face of the outer cube to the antipode of the corresponding face of the inner cube. Heavy lines in Fig. 7 show the visible edges; broken heavy lines show the "false" edges of self-intersections. A careful analysis shows that this is a realization of the (only) regular map of type $\{6,4\}$ on a surface of genus 5 which was discovered by Sherk [21]. This map has an automorphism group of order 192 and is not a lattice as, e.g., the Kepler–Poinsot polyhedra $\{\frac{5}{2},5\}$ and $\{5,\frac{5}{2}\}$. The symmetry group of our model is the octahedral group and is face-transitive. The latter is even true for the octahedral rotation subgroup.

4. A Realization in $\mathbb{E}^3$ Cannot Have Octahedral Symmetry

Our proof that a three-dimensional polyhedral realization (with or without self-intersections) of $\{3,8\}_6$ cannot have octahedral symmetry is based on the following simple observation. The orbit of each point x in $\mathbb{E}^3$ under the octahedral symmetry group contains either exactly 1, 6, 8, 12, or more points. The number is 1 if and only if x is the centroid of the underlying octahedron Q, 6 if and only if the orbit is the vertex set of an octahedron concentric to Q, 8 if and only if the orbit is the vertex set of a cube given by the centres of the faces of a concentric copy of Q, and 12 if and only if it is the vertex set of a cuboctahedron given by the midpoints of the edges of a concentric copy of Q.

Let us assume to the contrary there is a realization P with octahedral symmetry group. Considering the orbits of the 12 vertices of P under the group shows that either the vertex set of P splits into two blocks of 6 giving the vertices of two concentric octahedra Q_1 and Q_2, say $Q_1 \subset Q_2$, or the vertices of P are the vertices of a cuboctahedron R. Next we make use of the fact that the vertex set of P can be decomposed into three blocks of four such that in each block no two vertices are joined by an edge, and each triangular face of P has exactly one vertex in each block.

Let us consider the first case. For symmetry reasons, the three blocks must be given by pairs of antipodal vertices of Q_2 together with the corresponding vertices of Q_1. By the properties of the block dissection, the eight faces of P incident with a vertex z of Q_2 are among the 16 triangles, whose vertices different from z lie in the two blocks not containing z. These 16 triangles split into three classes according to the number of their vertices in Q_1. There are four triangles with two vertices, eight with one vertex, and four with no vertex in Q_1. For symmetry reasons, from each class either each or no triangle is among the faces of P incident with z. Hence, the star of z in P consists either of eight triangles with one vertex in Q_1, or of eight triangles with two vertices in Q_1 or in Q_2. In any case the star is decomposed into two discs so that P cannot be a polyhedron.

The second case is treated in a similar fashion. There are three mutually orthogonal hyperplanes through the centre of R each containing exactly four vertices of R. For symmetry reasons there cannot be any other block dissection of the vertex set of P of the required type. Again, the eight faces of P incident with a vertex z of P are among the 16 triangles with z as one vertex and the two other vertices in the two blocks not containing z. This time, these 16 triangles split into four classes of 2, 6, 6, and 2 triangles with 3, 2, 1, and 0, respectively, edges in common with the cuboctahedron R. Again, from each class either each or no triangle is a face of P incident with z. This leaves two possibilities.

If the second class gives faces of P, then for each of its triangles the edges in R incident with z belong to another triangle in the group. Hence, the two other faces of P incident with z must be members of the fourth class. But then the star of z in P splits into two discs.

Similarly, if the third group gives faces of P, then the edges in R are edges of only one triangle in the group. Hence, the other two faces of the star of z lie in the first class. Again, the star splits into two discs, completing the proof that P cannot exist.

Together with the fact that a polyhedral realization of Dyck's map cannot have icosahedral symmetry, this result shows that our model of Dyck's map has maximal symmetry.

5. A Polyhedron of Type $\{3,8\}$

Our attempt to realize $\{3,8\}_6$ in $\mathbb{E}^3$ exhibited another map of type $\{3,8\}$ and genus 3 with exactly the same f-vector as $\{3,8\}_6$. It is given by a polyhedron (without self-intersections) in $\mathbb{E}^3$.

This map together with another polyhedral realization was first discovered by Grünbaum and Shephard, who announced they could generate this polyhedron by starting from the icosahedron (cf. [14]; compare also [22, 23, 24]). The construction of our polyhedron is based on the truncated tetrahedron rather than the icosahedron.

First we start from one truncated tetrahedron T and generate two other polyhedra, say T_1 and T_2 (see Figs. 8 and 9).

When the four triangular faces (1 2 3 etc.) of T are slightly rotated clockwise with the same angle α, say $\alpha = \frac{1}{10}\pi$, about the four symmetry axes, then each of the four hexagonal faces of T are broken along three of its diagonals into four triangles, turning T into a (convex) polyhedron T_1 isomorphic to the icosahedron (see Fig. 8).

In a similar manner the four triangular faces (1 2 3 etc.) of T can be rotated counterclockwise with the angle $2\pi/3 - \alpha$. Again the four hexagonal faces are broken along diagonals, this time in a reverse sense (see Fig. 9). Because of the strong rotation all the edges of the resulting polyhedron T_2 but the 12 edges of the rotated triangles pass through the interior of the convex hull of T_2.

Now, deleting the four rotated triangular faces from T_1 as well as from T_2 gives two spheres each with four holes. By our construction, the convex hulls of T_1 and T_2 coincide so that the two spheres fit together along corresponding holes, thereby providing a polyhedron P of type $\{3,8\}$ and genus 3.

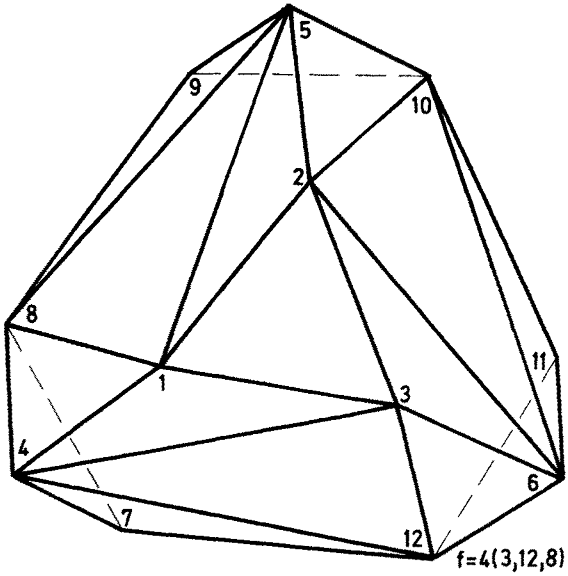

Fig. 8

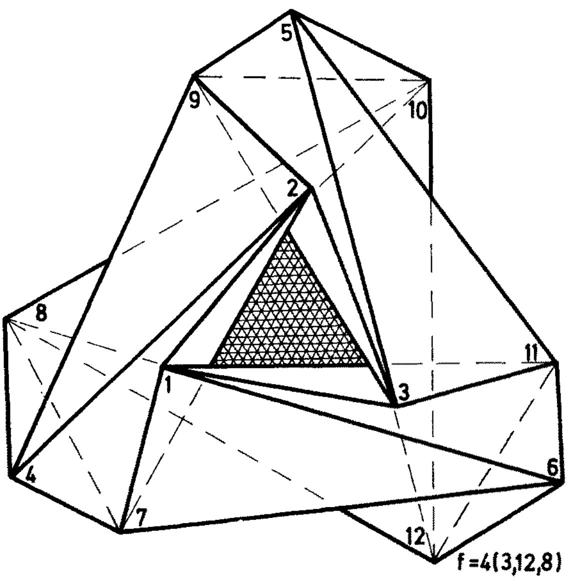

Fig. 9

However, P is not isomorphic to $\{3,8\}_6$ though it has the same f-vector. In fact, the 12 vertices of P do not split into three blocks of four, where in each block no two vertices are joined by an edge.

Recently the authors discovered also a Kepler–Poinsot-type realization for the regular map of type $\{3,8\}$ and genus 5 by Fricke and Klein. This is a twofold covering of Dyck's $\{3,8\}_6$. Details will be published.

References

1. H. S. M. Coxeter, Regular skew polyhedra in three and four dimensions and their topological analogues, Proc. London Math. Soc. (2), 43 (1937), 33–62.
2. H. S. M. Coxeter, The abstract groups $G^{m,n,p}$, Trans. Amer. Math. Soc. 45 (1939), 73–150.
3. H. S. M. Coxeter, Twelve Geometric Essays, 75–105, Southern Illinois Univ. Press, Carbondale, 1968.
4. H. S. M. Coxeter, Regular Polytopes, 3rd ed., Dover, New York, 1973.
5. H. S. M. Coxeter and W. O. J. Moser, Generators and relations for discrete groups, 4th ed., Springer-Verlag, Berlin, 1980.
6. L. Danzer and E. Schulte, Reguläre Inzidenzkomplexe I, Geom. ded. 13 (1982), 295–308.
7. A. Dress, A combinatorial theory of Grünbaum's new regular polyhedra I, Aequationes Mathematicae 23 (1981), 252–265.
8. A. Duma, Über die Automorphismen kompakter Riemannscher Flächen, Seminarberichte, Fernuniversität Hagen, 1984.
9. W. Dyck, Notiz über eine reguläre Riemannsche Fläche vom Geschlecht 3 und die zugehörige Normalkurve 4. Ordnung, Math. Ann. 17 (1880), 510–516.
10. W. Dyck, Über Aufstellung und Untersuchung von Gruppe und Irrationalität regulärer Riemannscher Flächen, Math. Ann. 17 (1880), 473–508.
11. W. Dyck, Gruppentheoretische Studien, Math. Ann. 20 (1882), 1–45.
12. L. Fejes Tóth, Reguläre Figuren, Verlag der Ungarischen Akademie der Wissenschaften, Budapest, 1965.
13. B. Grünbaum, Convex Polytopes, John Wiley and Sons, London, 1967.
14. B. Grünbaum and G. C. Shephard, Polyhedra with transitivity properties, C.R. Math. Rep. Acad. Sci. Canad. 6 (1984), 61–66.
15. D. Hilbert and S. Cohn-Vossen, Anschauliche Geometrie, Berlin, 1932.
16. F. Klein, Über die Transformation siebenter Ordnung der elliptischen Functionen, Math. Ann. 14 (1879), 428–471 (slightly revised version in: Gesammelte Math. Abh., 3. Band, Springer-Verlag, Berlin, 1923).
17. F. Klein, Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen fünften Grades, Leipzig, 1884.
18. E. Schulte and J. M. Wills, A polyhedral realization of Felix Klein's map $\{3,7\}_8$ on a Riemann surface of genus 3, J. London Math. Soc., in print.
19. E. Schulte and J. M. Wills, On Coxeter's regular skew polyhedra, Discrete Mathematics, in print.
20. F. A. Sherk, The regular maps on a surface of genus three, Canad. J. Math. 11 (1959), 452–480.
21. F. A. Sherk, A family of regular maps of type $\{6,6\}$, Canad. Math. Bull. 5 (1962), 13–20.
22. J. M. Wills, Semi-platonic manifolds, in Convexity and Its Applications, 413–421, ed. by P. Gruber and J. M. Wills, Birkhäuser, Basel, 1983.
23. J. M. Wills, Platonic manifolds, Mitt. Math. Sem. Giessen (Coxeter-Festschrift) 163 (1984), 45–60.
24. J. M. Wills, On polyhedra with transitivity properties, to appear.

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