Geometric structure for the tangent bundle of direct limit manifolds

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Abstract. We equip the direct limit of tangent bundles of paracompact finite dimensional manifolds with a structure of convenient vector bundle with structural group $GL(\infty, \mathbb{R}) = \varinjlim GL(\mathbb{R}^n)$.

On munit la limite directe des fibrés tangents à des variétés paracompactes de dimensions finies d'une structure de fibré vectoriel 'convenient' (au sens de Kriegel et Michor) de groupe structural $GL(\infty, \mathbb{R}) = \varinjlim GL(\mathbb{R}^n)$.

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1 Introduction

G. Galanis proved in [4] that the tangent bundle of a projective limit of Banach manifolds can be equipped with a Fréchet vector bundle structure with structural group a topological subgroup of the general linear group of the fiber type. Various problems were studied in this framework: connections, ordinary differential equations, ... ([1], [2], ...).

Here we consider the situation for direct (or inductive) limit of tangent bundles TM_i where M_i is a finite dimensional manifold: we first have (Proposition 4.1) that $M = \varinjlim M_i$ can be endowed with a structure of convenient manifold modelled on the convenient vector space $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ of finite sequences, equipped with the finite topology (cf [7]). We then prove (Theorem 4.2) that TM can be endowed with a convenient structure of vector bundle whose structural group is $GL(\infty, \mathbb{R}) = \varinjlim GL(\mathbb{R}^n)$ (the group of invertible matrices of countable size, differing from the identity matrix at only finitely many places, first described by Milnor in [10]). As an example we consider the tangent bundle to $\mathbb{S}^\infty$. Other examples can be found in the framework of manifolds for algebraic topology, such as Grassmannians ([8]) or Lie groups ([5], [6]).

The paper is organized as follows: We first recall the framework of convenient calculus (part 2). In part 3, we review direct limit in different categories. We obtain the main result (theorem 4.2) in the last part.

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2 Convenient calculus

Classical differential calculus is perfectly adapted to finite dimensional or even Banach manifolds (cf. [9]).

On the other hand, convenient analysis, developed in [8], provides a satisfactory solution of the question how to do analysis on a large class of locally convex spaces and in particular on strict inductive limits of Banach manifolds or fiber bundles.

In order to endow some locally convex vector spaces (l.c.v.s.) E, which will be assumed Hausdorff, with a differentiable structure we first use the notion of smooth curves $c : \mathbb{R} \to E$, which poses no problems.

We denote the space $C^\infty(\mathbb{R}, E)$ by $\mathcal{C}$; the set of continuous linear functionals is denoted by $E^{'}$.

We then have the following characterization: a subset B of E is bounded iff $l(B)$ is bounded for any $l \in E^{'}$.

Definition 2.1. A sequence (x_n) in E is called Mackey-Cauchy if there exists a bounded absolutely convex set B and for every $\varepsilon > 0$ an integer $n_\varepsilon \in N$ s.t. $a_n - a_m \in \varepsilon B$ whenever $n > m > n_\varepsilon$

Definition 2.2. A locally convex vector space is said to be c^∞-complete or *convenient* if one of the following (equivalent) conditions is satisfied :

1. if $c : \mathbb{R} \to E$ is a curve such that $l \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all continuous linear functionnal l, then c is smooth.
2. Any Mackey-Cauchy sequence converges[1] (i.e. E is Mackey complete)
3. For any $c \in \mathcal{C}$ there exists $\gamma \in \mathcal{C}$ such that $\gamma' = c$.

The c^∞–topology on a l.c.v.s. is the final topology with respect to all smooth curves $\mathbb{R} \to E$; it is denoted by $c^\infty E$. Its open sets will be called $c^\infty-$open.

Note that the c^∞–topology is finer than the original topology. For Fréchet spaces, this topology coincides with the given locally convex topology.

In general, $c^\infty E$ is not a topological vector space.

The following theorem gives some constructions inheriting of c^∞-completeness.

Theorem 2.1. *The following constructions preserve c^∞-completeness: limits, direct sums, strict inductive limits of sequences of closed embeddings.*

The category $\mathbb{CON}$ of convenient vector spaces and continuous linear maps forms a symmetric monoidal closed category.

Let E and F be two convenient spaces and let $U \subset E$ be a $c^\infty-$open. A map $f : E \supset U \to F$ is said to be smooth if $f \circ c \in C^\infty(\mathbb{R}, F)$ for any $c \in C^\infty(\mathbb{R}, U)$. Moreover, the space $C^\infty(U, F)$ may be endowed with a structure of convenient vector space.

[1]This condition is equivalent to:
For every absolutely convex closed bounded set B the linear span E_B of B in E, equipped with the Minkowski functional $p_B(v) = \inf\{\lambda > 0 : v \in \lambda.B\}$, is complete.

Let $L(E,F)$ be the space of all bounded linear mappings. We can define the differential operator

$$d : C^{\infty}(E,F) \to C^{\infty}(E, L(E,F))$$
$$df(x)\,v = \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}$$

which is linear and bounded (and so smooth).

3 Direct (or inductive) limits

3.1 Direct limit in a category

The references are [3] and [6].

Definition 3.1. A direct sequence in a category $\mathbb{A}$ is a pair $\mathcal{S} = (X_i, \varepsilon_{ij})_{(i,j)\in\mathbb{N}^2,\ i\leq j}$ where X_i is an object of $\mathbb{A}$ and each $\varepsilon_{ij} : X_i \to X_j$ is a morphism, called *bonding map*, such that:

- $\varepsilon_{ii} = \mathrm{Id}_{X_i}$
- $\varepsilon_{jk} \circ \varepsilon_{ij} = \varepsilon_{ik}$ if $i \leq j \leq k$

Definition 3.2. A cone over $\mathcal{S}$ is a pair $(X, \varepsilon_i)_{i\in\mathbb{N}}$ where X is an object of $\mathbb{A}$ and $\varepsilon_i : X_i \to X$ is a morphism of this category such that

$$\varepsilon_j \circ \varepsilon_{ij} = \varepsilon_i \text{ if } i \leq j$$

A cone $(X, \varepsilon_i)_{i\in\mathbb{N}}$ is a direct limit cone over $\mathcal{S}$ in the category $\mathbb{A}$ if for every cone (Y, ψ_i) over $\mathcal{S}$ there exists a unique morphism $\psi : X \to Y$ such that $\psi \circ \varepsilon_i = \psi_i$ for each i.
We then write $X = \varinjlim X_i$ and we call X the direct limit of $\mathcal{S}$.

3.2 Direct limit of sets

Let $\mathcal{S} = (X_i, \varepsilon_{ij})_{(i,j)\in\mathbb{N}^2,\ i\leq j}$ be a direct sequence of sets.

The *direct sum* $\bigoplus\limits_{n\in\mathbb{N}} X_n$ also called the *coproduct* $\coprod\limits_{n\in\mathbb{N}} X_n$ is the subspace of the cartesian product $\prod\limits_{n\in\mathbb{N}} X_n$ formed by all the points with only finetely many non-vanishing coordinates.

In this space we introduce the following binary relation (where $x \in X_i$ and $y \in Y_j$)

$$(i,x) \sim (j,y) \quad \Longleftrightarrow \quad \left| \begin{array}{c} y = \varepsilon_{ij}(x) \text{ if } i \leq j \\ \text{or} \\ x = \varepsilon_{ji}(y) \text{ if } i \geq j \end{array} \right.$$

which is an equivalence relation.

Then the set $X = \coprod_{n \in \mathbb{N}} X_n \,/\sim$ together with the maps

$$\begin{array}{rccc} \varepsilon_i : & X_i & \longrightarrow & X \\ & x & \mapsto & \widetilde{(i,x)} \end{array}$$

where $\widetilde{(i,x)}$ is the equivalence class of (i,x), is the *direct limit* of $\mathcal{S}$ in the category $\mathbb{SET}$.

We have $X = \bigcup_{i \in N} \varepsilon_i(X_i)$. If each ε_{ij} is injective then so is ε_i. $\mathcal{S}$ is then equivalent to the sequence of the subsets $\varepsilon_i(X_i) \subset X$ with the inclusion maps.

3.3 Direct limit of topological spaces

Let $\mathcal{S} = (X_i, \varepsilon_{ij})_{(i,j) \in \mathbb{N}^2,\ i \leq j}$ be a direct sequence of topological spaces where the bonding maps are continuous.

We then endow X with the direct sum topology, i.e. is the final topology with respect to the family $(\varepsilon_i)_{i \in \mathbb{N}}$ which is the finest topology for which the maps ε_i are continuous. Then $U \subset X$ is open if and only if $(\varepsilon_i)^{-1}(U)$ is open in X_i for each i.

If the bonding maps are topological embeddings we call $\mathcal{S}$ *strict direct limi*t. For any $i \in \mathbb{N}$, ε_i is then a topological embedding.

3.4 Fundamental example of $\mathbb{R}^\infty$

The space $\mathbb{R}^\infty$ also denoted by $\mathbb{R}^{(\mathbb{N})}$ of all finite sequences is the direct limit of $(\mathbb{R}^i, \varepsilon_{ij})_{(i,j) \in \mathbb{N}^2,\ i \leq j}$ where $\varepsilon_{ij} : (x_1, \dots, x_i) \mapsto (x_1, \dots, x_i, 0, \dots, 0)$.

It is a convenient vector space ([8], 47.1).

3.5 Direct limit of finite dimensional manifolds

Let $\mathcal{M} = (M_i, \phi_{ij})_{\ i \leq j}$ be a direct sequence of paracompact finite dimensional smooth real manifolds where the bonding maps $\phi_{ij} : M_i \longrightarrow M_j$ are injective smooth immersions and $\sup_{i \in \mathbb{N}} \{\dim_{\mathbb{R}} M_i\} = \infty$. Adapting a result of Glöckner ([6], Theorem 3.1) to the convenient framework (using Proposition 3.6) we have:

Theorem 3.1. *There exists a uniquely determined $c^\infty-$manifold structure on the direct limit M of $\mathcal{M}$ modelled on the convenient vector space $\mathbb{R}^\infty$.*

Example 3.3. The sphere $\mathbb{S}^\infty$ ([8], 47.2).– The convenient vector space $\mathbb{R}^\infty$ is equipped with the weak inner product given by the finite sum $\langle x, y \rangle = \sum_i x_i y_i$ and is bilinear and bounded, therefore smooth. The topological inductive limit of $\mathbb{S}^1 \subset \mathbb{S}^2 \subset \cdots$ is the closed subset $\mathbb{S}^\infty = \{x \in \mathbb{R}^\infty : \langle x, x \rangle = 1\}$ of $\mathbb{R}^\infty$.

Choose $a \in \mathbb{S}^\infty$. We can define the stereographic atlas corresponding to the equivalence class of the two charts $\{(U_+, u_+), (U_-, u_-)\}$ where $U_+ = \mathbb{S}^\infty \setminus \{a\}$ (resp. $U_- = \mathbb{S}^\infty \setminus \{-a\}$) and $\begin{array}{rccc} u_+ : & U_+ & \longrightarrow & \{a\}^\perp \\ & x & \mapsto & \dfrac{x - \langle x, a \rangle a}{1 - \langle x, a \rangle} \end{array}$ (resp. $\begin{array}{rccc} u_- : & U_- & \longrightarrow & \{a\}^\perp \\ & x & \mapsto & \dfrac{x - \langle x, a \rangle a}{1 + \langle x, a \rangle} \end{array}$).

Then $\mathbb{S}^\infty$ is a convenient manifold modelled on $\mathbb{R}^\infty$.

4 Tangent bundle of direct limit of manifolds

4.1 Structure of manifold on direct limit of $\{TM_i\}_{i\in\mathbb{N}}$

Let $p \geq 4$ and $\{M_i, \phi_{ij}\}_{i\leq j}$ be a direct sequence of C^p paracompact finite dimensional manifolds for which the connecting morphisms are C^p embeddings with closed image. Without loss of generality (cf. 3.2) we may assume that $M_1 \subseteq M_1 \subseteq \cdots \subseteq M$ where $\{M, \phi_i\}$ is the direct limit of $\{M_i, \phi_{ij}\}_{i\leq j}$ in the category of topological spaces and the maps $\phi_i : M_i \longrightarrow M$ are inclusions [6]. Suppose that $\dim M_i = d_i$ and consider for $i \leq j$,

$$\begin{array}{cccc} \lambda_{ij}: & \mathbb{R}^{d_i} & \longrightarrow & \mathbb{R}^{d_j} \\ & (x_1,\ldots,x_{d_i}) & \mapsto & (x_1,\ldots,x_{d_i},0,\ldots,0) \end{array}$$

For $x \in M$ there exists $n \in \mathbb{N}$ such that $x = \phi_n(x)$. Using tubular neighborhoods Glöckner proved that there exists an open neighborhood O_x of x in M and a sequence of C^{p-2} diffeomorphisms $\{h_i^{(x)} : \mathbb{R}^{d_i} \longrightarrow U_i\}_{i\geq n}$ (inverse of chart mappings) where $U_i = {\phi_i}^{-1}(O_x)$. Moreover for $j \geq i \geq n$ the compatibility condition

$$h_j^{(x)} \circ \lambda_{ij} = \phi_{ij}|_{U_i} \circ h_i^{(x)} \tag{4.1}$$

holds true ([5], Lemma 4.1).

Our first aim is to introduce appropriate connecting morphisms, say $\{\Phi_{ij}\}_{i\leq j}$, such that $\{TM_i, \Phi_{ij}\}$ form a direct system of manifolds in the sense of Glöckner.

For $i \leq j$ define

$$\begin{aligned} \Phi_{ij} : TM_i &\longrightarrow TM_j \\ [\alpha_i, x_i]_i &\longmapsto [\phi_{ij}\circ\alpha_i, \phi_{ij}(x_i)]_j \end{aligned}$$

where the bracket $[.,.]_i$ stands for the equivalence classes of TM_i with respect to the classical equivalence relations between paths

$$\alpha \sim_x \beta \quad \Leftrightarrow \quad \begin{cases} \alpha(0) = \beta(0) = x \\ \alpha'(0) = \beta'(0) \end{cases}$$

where $\alpha'(t) = [d\alpha(t)](1)$. Clearly $\Phi_{ii} = \mathrm{Id}_{TM_i}$ and $\Phi_{jk}\circ\Phi_{ij} = \Phi_{ik}$, for $i \leq j \leq k$, and $\{TM_i\}$ is a sequence of C^{p-1} finite dimensional paracompact manifolds. Moreover $\Phi_{ij}(TM_i)$ is diffeomorphic to a closed submanifold of TM_j.

Proposition 4.1. *Let $p \geq 4$ and $\{M_i, \phi_{ij}\}_{i\leq j}$ be a direct sequence of C^p paracompact finite dimensional manifolds for which the connecting morphisms are C^p embeddings with closed image.*
Then $\varinjlim TM_i$ is a C^{p-3} manifold modelled on $\mathbb{R}^\infty \times \mathbb{R}^\infty = \varinjlim(\mathbb{R}^i \times \mathbb{R}^i)$.

Proof. Let $[f, x] \in \varinjlim TM_i$. Then for some $n \in \mathbb{N}$, $[f, x] = \phi_n([f_n, x_n]) \in TM_n$. Without loss of generality suppose that $TM_1 \subseteq TM_2 \subseteq \cdots \subseteq TM$ and $[f, x] \in TM_{n(x)}$. This means that x belongs to M_n and $f : (-\epsilon, \epsilon) \longrightarrow M_{n(x)}$ is a smooth curve passing trough x. Since $\{M_i, \phi_{ij}\}_{i\leq j}$ is a directed system of manifolds satisfying Lemma 4.1. of [5], then there exists an open neighbourhood O_x of x in M and a family of C^{p-2} diffeomorphisms $\{h_i^{(x)} : \mathbb{R}^{d_i} \to U_i\}_{i \geq n(x)}$ where $U_i = {\phi_i}^{-1}(O_x)$ and (4.1) holds true. For $i \geq n(x)$ define

$$\begin{aligned} Th_i^{(x)} : \mathbb{R}^{d_i} \times \mathbb{R}^{d_i} &\longrightarrow TU_i \subseteq TM_i \\ (\bar{y}, \bar{v}) &\longmapsto [\gamma, y] \end{aligned}$$

where $({h_i^{(x)}}^{-1} \circ \gamma)(t) = \bar{y} + t\bar{v}$. For $i \leq j$ we get

$$\Phi_{ij} \circ Th_i^{(x)}(\bar{y}, \bar{v}) = \Phi_{ij}([\gamma, y]) = [\phi_{ij} \circ \gamma, \phi_{ij}(y)].$$

On the other hand,

$$Th_j^{(x)} \circ (\lambda_{ij} \times \lambda_{ij})\,(y, v) = Th_j^{(x)}\,((\bar{y}, 0)\,, (\bar{v}, 0)) = [\gamma', y']$$

for which $({h_j^{(x)}}^{-1} \circ \gamma')(t) = (y, 0) + t(v, 0) = \lambda_{ij}(\bar{y} + t\bar{v})$. We claim that $[\phi_{ij} \circ \gamma, \phi_{ij}(y)] = [\gamma', y']$.

Using (4.1) we observe that

$$\begin{aligned} {h_j^{(x)}}^{-1} \circ (\phi_{ij} \circ \gamma(t)) &= ({h_j^{(x)}}^{-1} \circ \phi_{ij}) \circ \gamma(t) = (\lambda_{ij} \circ {h_i^{(x)}}^{-1}) \circ \gamma(t) \\ &= \lambda_{ij} \circ ({h_i^{(x)}}^{-1} \circ \gamma)(t) = \lambda_{ij}(\bar{y} + t\bar{v}), \end{aligned}$$

which proves the assertion.

Roughly speaking for any $[f, x] \in TM$, we constructed a family of C^{p-3} diffeomorphisms

$$\{Th_i^{(x)} : \mathbb{R}^{d_i} \times \mathbb{R}^{d_i} \longrightarrow TU_i \subseteq TM_i\}_{i \geq n(x)}$$

which satisfy the compatibility conditions

$$\Phi_{ij} \circ h_i^{(x)} = h_j^{(x)} \circ (\lambda_{ij} \times \lambda_{ij}) \;\; ; \;\; j \geq i \geq n(x).$$

As a consequence the limit map $Th^{(x)} = \varinjlim Th_i^{(x)} : \mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow TU^{(x)} := \bigcup_{i \geq n(x)} TU_i$ can be defined. The map $Th^{(x)}$ denotes the diffeomorphism whose restriction to $\mathbb{R}^{d_i} \times \mathbb{R}^{d_i}$ is $Th_i^{(x)}$.

The next step is to establish that the family $\mathcal{B} = \{{Th^{(x)}}^{-1}; x \in M\}$ is an atlas for TM. For $[f, x]$ and $[f', x']$ in TM define $n = \max\{n(x), n(x')\}$. Set $\tau := Th^{(x')} \circ {Th^{(x)}}^{-1}$. Since for $i \geq n$

$$\tau \circ \lambda_i = \lambda_i \circ Th_i^{(x')} \circ {Th_i^{(x)}}^{-1}$$

it follows that τ is a C^{p-3} diffeomorphism too. Moreover for every natural number i, TM_i is a locally compact topological space. This last means that $\varinjlim TM_i$ is Hausdorff ([7], [6]) which completes the proof. □

4.2 The Lie group $Gl\,(\infty, \mathbb{R})$

In the situation described in [4] (tangent bundle of projective limit of Banach manifolds), the general linear group $GL\,(\mathbb{F})$ cannot play the rôle of structural group and is replaced by $H_0\,(\mathbb{F})$ which is a projective limit of Banach Lie groups.

In our framework we are going to use the convenient Lie group $GL\,(\infty, \mathbb{R})$ as structural group. It is defined as follows. The canonical embeddings $\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ induce injections $GL\,(\mathbb{R}^n) \longrightarrow GL\left(\mathbb{R}^{n+1}\right)$. The inductive limit is given by

$$GL\,(\infty, \mathbb{R}) = \varinjlim GL\,(\mathbb{R}^n)$$

and can be endowed with a real analytic regular Lie group modeled on $\mathbb{R}^\infty$ (cf [8], Theorem 47.8).

4.3 Convenient vector bundle structure on TM

Theorem 4.2. *TM over M admits a convenient vector bundle structure with the structure group $GL\,(\infty, \mathbb{R})$.*

Proof. For any $i \in \mathbb{N}$ consider the natural projection $\pi_i : TM_i \longrightarrow M_i$ which maps $[\gamma, y]$ onto y. As a first step we show that the limit map $\pi := \varinjlim \pi_i$ exists. For $j \geq i$ and $[\gamma, x] \in TM_i$ we have

$$\phi_{ij} \circ \pi_i[\gamma, y] = \phi_{ij}\,(y)$$

On the other hand

$$\pi_j \circ \Phi_{ij}[\gamma, y] = \pi_j[\phi_{ij} \circ \gamma, \phi_{ij}\,(y)]$$

The compatibility condition $\phi_{ij} \circ \pi_i = \pi_j \circ \Phi_{ij}$ leads us to the limit (differentiable) map

$$\pi := \varinjlim \pi_i : \varinjlim TM_i \longrightarrow \varinjlim M_i$$

whose restriction to TM_i is given by $\phi_i \circ \pi_i = \pi \circ \Phi_i$.

For $[f, x] \in \varinjlim TM_i$ consider the family of diffeomorphisms $\{h_i^{(x)} : \mathbb{R}^{d_i} \longrightarrow U_i^{(x)}\}_{i \geq n(x)}$ as before. For any $i \geq n(x)$ define

$$\begin{aligned} \Psi_i : {\pi_i}^{-1}(U_i^{(x)}) &\longrightarrow U_i^{(x)} \times \mathbb{R}^{d_i} \\ [\gamma, y] &\longmapsto \left(y, ({h_i^{(x)}}^{-1} \circ \gamma)'(0)\right). \end{aligned}$$

With the standard calculation for the finite dimensional manifolds it is known that Ψ_i, $i \in \mathbb{N}$, is a diffeomorphism. For $j \geq i \geq n(x)$, we claim that the following diagram is commutative

$$\begin{array}{ccc} {\pi_i}^{-1}(U_i^{(x)}) & \overset{\Psi_i}{\to} & U_i^{(x)} \times \mathbb{R}^{d_i} \\ \Phi_{ij} \downarrow & & \downarrow \phi_{ij} \times \lambda_{ij} \\ {\pi_j}^{-1}(U_j^{(x)}) & \overset{\Psi_j}{\to} & U_j^{(x)} \times \mathbb{R}^{d_j} \end{array}$$

To see that we argue as follows.

$$\begin{aligned}
(\phi_{ij} \times \lambda_{ij}) \circ \Psi_i([\gamma, y]) &= (\phi_{ij} \times \lambda_{ij}) \Big(y, (h_i^{(x)^{-1}} \circ \gamma)'(0) \Big) \\
&= \Big(\phi_{ij}(y), \lambda_{ij} \circ \Big((h_i^{(x)^{-1}} \circ \gamma)'(0) \Big) \Big) \\
&\overset{(*)}{=} \Big(\phi_{ij}(y), (\lambda_{ij} \circ h_i^{(x)^{-1}} \circ \gamma)'(0) \Big) \\
&\overset{(**)}{=} \Big(\phi_{ij}(y), \Big((h_j^{(x)^{-1}} \circ \phi_{ij} \circ \gamma)'(0) \Big) \Big) \\
&= \Psi_j \left([\phi_{ij} \circ \gamma, \phi_{ij}(y)] \right) \\
&= (\Psi_j \circ \Phi_{ij}) \, [\gamma, y]
\end{aligned}$$

For $(**)$ we used the equation (4.1) and for $(*)$ using the linearity of λ_{ij} we get

$$\begin{aligned}
\lambda_{ij} \circ \Big((h_i^{(x)^{-1}} \circ \gamma)'(0) \Big) &= \lambda_{ij} \left(\lim_{t \to 0} \frac{(h_i^{(x)^{-1}} \circ \gamma)(t) - (h_i^{(x)^{-1}} \circ \gamma)(0)}{t} \right) \\
&= \lim_{t \to 0} \frac{(\lambda_{ij} \circ h_i^{(x)^{-1}} \circ \gamma)(t) - (\lambda_{ij} \circ h_i^{(x)^{-1}} \circ \gamma)(0)}{t} \\
&= (\lambda_{ij} \circ h_i^{(x)^{-1}} \circ \gamma)'(0).
\end{aligned}$$

Since $\pi_i^{-1}(U_i^{(x)})$, $i \geq n(x)$, is open and since $\pi^{-1}(U) = \varinjlim \pi_i^{-1}(U_i^{(x)})$, it follows that $\pi^{-1}(U) \subseteq TM$ is open. Furthermore $\Psi_x := \varinjlim \Psi_i : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^\infty$ exists and, as a direct limit of C^{p-3} diffeomorphisms, is a C^{p-3} diffeomorphism. On the other hand

$$\Psi_x|_{\pi^{-1}(y)} : \pi^{-1}(y) \longrightarrow \{y\} \times \mathbb{R}^\infty$$

is linear and $pr_1 \circ \Psi_x$ coincides with π. (pr_1 stands for projection to the first factor.)

Suppose that $[f, x], [g, y] \in TM$, $n = \max\{n(x), n(y)\}$ and the intersection $U_{xy} := U^{(x)} \cap U^{(y)}$ is not empty. Then

$$(\Psi_y)^{-1}|_{U_{xy} \times \mathbb{R}^\infty} \circ \Psi_x|_{U_{xy} \times \mathbb{R}^\infty} : U_{xy} \times \mathbb{R}^\infty \longrightarrow U_{xy} \times \mathbb{R}^\infty$$

arises as the inductive limit of the family

$$\begin{aligned}
(\Psi_i^y)^{-1}|_{U_i^{xy} \times \mathbb{R}^{d_i}} \circ \Psi_i^x|_{U_i^{xy} \times \mathbb{R}^{d_i}} : U_i^{xy} &\longrightarrow GL\left(\mathbb{R}^{d_i}\right) \\
\bar{y} &\longmapsto T_{xy}^i(\bar{y}).
\end{aligned}$$

Finally the family of maps $\{T_{xy}^i := (\Psi_i^y)^{-1}|_{U_i^{xy} \times \mathbb{R}^{d_i}} \circ \Psi_i^x|_{U_i^{xy} \times \mathbb{R}^{d_i}}\}_{i \geq n}$, satisfy the required compatibility condition and their limit $T_{xy} := \varinjlim T_{xy}^i$ belongs to $\varinjlim GL\left(\mathbb{R}^{d_i}\right) := GL\left(\infty, \mathbb{R}\right)$. Consequently $\varinjlim TM_i$ becomes a (convenient) vector bundle with the fibres of type $\mathbb{R}^\infty$ and the structure group $GL\left(\infty, \mathbb{R}\right)$. $\square$

Example 4.1. Tangent bundle to $\mathbb{S}^\infty$.– The tangent bundle $T\mathbb{S}^\infty$ to the sphere $\mathbb{S}^\infty$ is diffeomorphic to $\{(x, v) \in \mathbb{S}^\infty \times \mathbb{R}^\infty : \langle x, v \rangle = 0\}$.

Proposition 4.3. $\varinjlim TM_i$ *as a set is isomorphic to* TM.

Proof. Arguing as before, let $[f,x] \in \varinjlim TM_i$. Then there exists $n(x) \in \mathbb{N}$ such that, for $i \geq n(x)$, $[f,x]$ belongs to TM_i which means that $x \in M_i$ and $f : (-\epsilon, \epsilon) \longrightarrow M_i$ for some $\epsilon > 0$. This last means that $f : (-\epsilon, \epsilon) \longrightarrow \varinjlim M_i$ and consequently $[f,x]$ belongs to TM.

Conversely, suppose that $[f,x]$ belongs to TM that is $x \in M$ and f is a curve in $M = \varinjlim M_i$. Again there exists $n(x)$ such that $x \in M_i$ and $f : (-\epsilon, \epsilon) \longrightarrow M_i$ is a smooth curve for $i \geq n(x)$. Since $[f,x] \in TM_i$, $i \geq n(x)$, then $[f,x] \in \varinjlim TM_i$ which completes the proof. □

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