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To my dear teacher
Victor Matveevich Buchstaber,
on the occasion of his 70*th birthday*

Geometric structures on moment-angle manifolds

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Abstract. A moment-angle complex $\mathscr{Z}_{\mathscr{K}}$ is a cell complex with a torus action constructed from a finite simplicial complex $\mathscr{K}$. When this construction is applied to a triangulated sphere $\mathscr{K}$ or, in particular, to the boundary of a simplicial polytope, the result is a manifold. Moment-angle manifolds and complexes are central objects in toric topology, and currently are gaining much interest in homotopy theory and complex and symplectic geometry. The geometric aspects of the theory of moment-angle complexes are the main theme of this survey. Constructions of non-Kähler complex-analytic structures on moment-angle manifolds corresponding to polytopes and complete simplicial fans are reviewed, and invariants of these structures such as the Hodge numbers and Dolbeault cohomology rings are described. Symplectic and Lagrangian aspects of the theory are also of considerable interest. Moment-angle manifolds appear as level sets for quadratic Hamiltonians of torus actions, and can be used to construct new families of Hamiltonian-minimal Lagrangian submanifolds in a complex space, complex projective space, or toric varieties.

Bibliography: 59 titles.

Keywords: moment-angle manifold, Hermitian quadrics, simplicial fans, simple polytopes, non-Kähler complex manifolds, Hamiltonian-minimal Lagrangian submanifolds.

Contents

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1. Introduction

A moment-angle complex $\mathscr{Z}_{\mathscr{K}}$ is a cell complex having a torus action and made up of products of disks D^2 and circles S^1 which are parametrized by faces of a simplicial complex $\mathscr{K}$. By replacing the pair (D^2, S^1) by an arbitrary cellular pair (X, A) we obtain the *polyhedral product* $(X, A)^{\mathscr{K}}$. Moment-angle complexes and polyhedral products are key players in the emerging field of *toric topology*, which lies on the borders between topology, algebraic and symplectic geometry, and combinatorics [15].

Both homotopical and geometric aspects of the theory of moment-angle complexes and polyhedral products have been actively studied recently. On the homotopy-theoretic side of the story, the stable and unstable decomposition techniques developed in [14], Chap. 6, [31], [4], [35], have led to an improved understanding of the topology of moment-angle complexes and related toric spaces.

In this survey we concentrate on the geometric aspects of the theory. The construction of moment-angle complexes has many interesting geometric interpretations. For example, the moment-angle complex $\mathscr{Z}_{\mathscr{K}}$ is homotopy equivalent to the complement $U(\mathscr{K})$ of the arrangement of coordinate subspaces in $\mathbb{C}^m$ defined by $\mathscr{K}$. The space $U(\mathscr{K})$ plays an important role in the geometry of toric varieties and the theory of configuration spaces.

The moment-angle complex $\mathscr{Z}_{\mathscr{K}}$ corresponding to a triangulated sphere $\mathscr{K}$ is a topological manifold. Moment-angle manifolds corresponding to simplicial polytopes or, more generally, complete simplicial fans, are smooth. In the polytopal case a smooth structure arises from the realization of $\mathscr{Z}_{\mathscr{K}}$ by a non-degenerate intersection of Hermitian quadrics in $\mathbb{C}^m$, similar to a level set of the moment map in the construction of symplectic quotients. The relationship between polytopes and systems of quadrics is described by the convex-geometric notion of Gale duality.

Another way to give $\mathscr{Z}_{\mathscr{K}}$ a smooth structure is to realize it as the quotient of the complement $U(\mathscr{K})$ of a coordinate subspace arrangement by an action of the multiplicative group $\mathbb{R}^{m-n}_{>}$. This is similar to the well-known quotient construction of toric varieties in algebraic geometry. The quotient of the non-compact manifold $U(\mathscr{K})$ by the action of the non-compact group $\mathbb{R}^{m-n}_{>}$ is Hausdorff precisely when $\mathscr{K}$ is the underlying complex of a simplicial fan.

If $m-n=2\ell$, then the action of the real group $\mathbb{R}^{m-n}_{>}$ on $U(\mathscr{K})$ can be turned into a holomorphic action of a complex (but not algebraic) group isomorphic to $\mathbb{C}^{\ell}$. In this way the moment-angle manifold $\mathscr{Z}_{\mathscr{K}}\cong U(\mathscr{K})/\mathbb{C}^{\ell}$ acquires a complex-analytic structure. The resulting family of non-Kähler complex manifolds generalizes the well-known series of Hopf and Calabi–Eckmann manifolds (see [10] and [54]).

Finally, the intersections of Hermitian quadrics defining polytopal moment-angle manifolds were also used in [46] to construct Lagrangian submanifolds in $\mathbb{C}^m$ with special minimality properties.

Different spaces with torus actions, or *toric spaces*, will feature throughout the paper. The most basic example of a toric space is the complex m-dimensional space $\mathbb{C}^m$ on which the *standard torus*

$$\mathbb{T}^m=\{\mathbf{t}=(t_1,\dots,t_m)\in\mathbb{C}^m\colon |t_i|=1 \text{ for } i=1,\dots,m\}$$

acts coordinatewise. That is, the action is given by

$$\mathbb{T}^m\times\mathbb{C}^m\to\mathbb{C}^m,$$
$$(t_1,\dots,t_m)\cdot(z_1,\dots,z_m)=(t_1z_1,\dots,t_mz_m).$$

The quotient $\mathbb{C}^m/\mathbb{T}^m$ of this action is the *positive orthant*

$$\mathbb{R}^m_{\geqslant}=\{(y_1,\dots,y_m)\in\mathbb{R}^m\colon y_i\geqslant 0 \text{ for } i=1,\dots,m\},$$

with the quotient projection given by

$$\mu\colon\mathbb{C}^m\to\mathbb{R}^m_{\geqslant},$$
$$(z_1,\dots,z_m)\mapsto(|z_1|^2,\dots,|z_m|^2).$$

We use the blackboard bold capitals in the notation $\mathbb{I}^m$, $\mathbb{T}^m$, $\mathbb{D}^m$ for the standard unit cube in $\mathbb{R}^m$, the standard (unit) torus, and the unit polydisk in $\mathbb{C}^m$, respectively. We use italic T^m to denote an abstract m-torus, that is, a compact Abelian Lie group isomorphic to a product of m circles. The underlying space of the unit disk $\mathbb{D}$ is a topological 2-disk, which we denote by D^2. We shall also denote the standard unit circle by $\mathbb{S}$ or $\mathbb{T}$ occasionally, to distinguish it from an abstract circle S^1.

2. Preliminaries: polytopes and Gale duality

Let $\mathbb{R}^n$ be a Euclidean space with scalar product $\langle\,\cdot\,,\cdot\,\rangle$. A convex *polyhedron* P is an intersection of finitely many half-spaces in $\mathbb{R}^n$. Bounded polyhedra are called *polytopes*. Alternatively, a polytope can be defined as the convex hull $\operatorname{conv}(\mathbf{v}_1,\dots,\mathbf{v}_q)$ of a finite set of points $\mathbf{v}_1,\dots,\mathbf{v}_q\in\mathbb{R}^n$.

A *supporting hyperplane* of P is a hyperplane H which has common points with P and for which the polyhedron is contained in one of the two closed half-spaces determined by H. The intersection $P\cap H$ with a supporting hyperplane is called a *face* of the polyhedron. Denote by ∂P and $\operatorname{int}P=P\setminus\partial P$ the topological boundary and interior of P, respectively. In the case $\dim P=n$ the boundary ∂P is the union of all the faces of P. Zero-dimensional faces are called *vertices*, one-dimensional faces are *edges*, and faces of codimension one are *facets*.

Two polytopes are *combinatorially equivalent* if there is a bijection between their faces preserving the inclusion relation. A *combinatorial polytope* is a class of combinatorially equivalent polytopes. Two polytopes are combinatorially equivalent if there is a homeomorphism between them preserving the face structure.

The faces of a given polytope P form a partially ordered set (a *poset*) with respect to inclusion. (It is called the *face poset* of P.) Two polytopes are combinatorially equivalent if and only if their face posets are isomorphic.

Consider a system of m linear inequalities defining a convex polyhedron in $\mathbb{R}^n$:

$$P=\{\mathbf{x}\in\mathbb{R}^n\colon \langle\mathbf{a}_i,\mathbf{x}\rangle+b_i\geqslant 0 \text{ for } i=1,\dots,m\},\tag{2.1}$$

where $\mathbf{a}_i\in\mathbb{R}^n$ and $b_i\in\mathbb{R}$. We refer to (2.1) as a *presentation* of the polyhedron P by inequalities. These inequalities contain more information than the polyhedron P itself, for the following reason. It may happen that some of the inequalities $\langle\mathbf{a}_i,\mathbf{x}\rangle+b_i\geqslant 0$ can be removed from the presentation without changing P; we refer to such inequalities as *redundant*. A presentation without redundant inequalities is called *irredundant*. An irredundant presentation of a given polyhedron is unique up to multiplication of the pairs $(\mathbf{a}_i,b_i)$ by positive numbers.

We shall assume (unless otherwise stated) that the polyhedron P defined by (2.1) has a vertex, which is equivalent to the vectors $\mathbf{a}_1,\dots,\mathbf{a}_m$ spanning the whole of $\mathbb{R}^n$. This condition is automatically satisfied for polytopes.

A presentation (2.1) is said to be *generic* if P is non-empty and the hyperplanes defined by the equations $\langle\mathbf{a}_i,\mathbf{x}\rangle+b_i=0$ are in general position at any point of P (that is, for any $\mathbf{x}\in P$ the normal vectors $\mathbf{a}_i$ of the hyperplanes containing $\mathbf{x}$ are linearly independent). If the presentation (2.1) is generic, then P is n-dimensional. If P is a polytope, then the existence of a generic presentation implies that P is *simple*, that is, exactly n facets meet at each vertex of P. A generic presentation may contain redundant inequalities, but, for any such inequality, the intersection of the corresponding hyperplane with P is empty (that is, the inequality is strict at any $\mathbf{x}\in P$). We set

$$F_i=\{\mathbf{x}\in P\colon \langle\mathbf{a}_i,\mathbf{x}\rangle+b_i=0\}.$$

If the presentation (2.1) is generic, then each F_i either is a facet of P or is empty.

The *polar set* of a polyhedron $P\subset\mathbb{R}^n$ is defined as

$$P^*=\{\mathbf{u}\in\mathbb{R}^n\colon \langle\mathbf{u},\mathbf{x}\rangle+1\geqslant 0 \text{ for all } \mathbf{x}\in P\}.\tag{2.2}$$

The set P^* is a convex polyhedron. (In fact, it is naturally a subset of the dual space $(\mathbb{R}^n)^*$, but we shall not make this distinction, assuming $\mathbb{R}^n$ to be Euclidean.)

The following properties are well known in convex geometry:

Theorem 2.1 (see [11], § 2.9 or [59], Theorem 2.11). (a) *P^* is bounded if and only if $\mathbf{0} \in \operatorname{int} P$.*

(b) *$P \subset (P^*)^*$, and $(P^*)^* = P$ if and only if $\mathbf{0} \in P$.*

(c) *If a polytope Q is given as a convex hull, $Q = \operatorname{conv}(\mathbf{a}_1, \dots, \mathbf{a}_m)$, then Q^* is given by inequalities (2.1) with $b_i = 1$ for $1 \leqslant i \leqslant m$; in particular, Q^* is a convex polyhedron, but not necessarily bounded.*

(d) *If P is given by inequalities (2.1) with $b_i = 1$, then $P^* = \operatorname{conv}(\mathbf{a}_1, \dots, \mathbf{a}_m)$, and $\langle \mathbf{a}_i, \mathbf{x} \rangle + 1 \geqslant 0$ is a redundant inequality if and only if $\mathbf{a}_i \in \operatorname{conv}(\mathbf{a}_j : j \neq i)$.*

Remark. A polyhedron P admits a presentation (2.1) with $b_i = 1$ if and only if $\mathbf{0} \in \operatorname{int} P$. In general, $(P^*)^* = \operatorname{conv}(P, \mathbf{0})$.

Any combinatorial polytope P has a presentation (2.1) with $b_i = 1$ (take the origin to the interior of P by a parallel transform, and then divide each of the inequalities in (2.1) by the corresponding b_i). Then P^* is also a polytope with $\mathbf{0} \in P^*$, and $(P^*)^* = P$. We call the combinatorial polytope P^* the *dual* of the combinatorial polytope P. (We shall not introduce a new notation for the dual polytope, keeping in mind that polarity is a convex-geometric notion, while duality of polytopes is combinatorial.)

Theorem 2.2 (see [11], § 2.10). *If P and P^* are dual polytopes, then the face poset of P^* is obtained from the face poset of P by reversing the inclusion relation.*

If P is a simple polytope, then it follows from the theorem above that each face of P^* is a simplex. Such a polytope is said to be *simplicial*.

The following construction realizes any polytope (2.1) of dimension n as the intersection of the orthant $\mathbb{R}^m_{\geqslant}$ with an affine n-plane. It will be used in the next section to define intersections of quadrics and moment-angle manifolds.

Construction 2.3. We form the $n \times m$ matrix A whose columns are the vectors $\mathbf{a}_i$ written in the standard basis of $\mathbb{R}^n$. Note that A is of rank n if and only if the polyhedron P has a vertex. Also let $\mathbf{b} = (b_1, \dots, b_m)^t \in \mathbb{R}^m$ be the column vector of numbers b_i. Then we can write (2.1) as

$$P = P(A, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^n : (A^t\mathbf{x} + \mathbf{b})_i \geqslant 0 \text{ for } i = 1, \dots, m\},$$

where $\mathbf{x} = (x_1, \dots, x_n)^t$ is the column of coordinates. Consider the affine map

$$i_{A,\mathbf{b}} \colon \mathbb{R}^n \to \mathbb{R}^m, \qquad i_{A,\mathbf{b}}(\mathbf{x}) = A^t\mathbf{x} + \mathbf{b} = (\langle \mathbf{a}_1, \mathbf{x} \rangle + b_1, \dots, \langle \mathbf{a}_m, \mathbf{x} \rangle + b_m)^t.$$

If P has a vertex, then the image of $\mathbb{R}^n$ under $i_{A,\mathbf{b}}$ is an n-dimensional affine plane in $\mathbb{R}^m$, which we can write by $m - n$ linear equations:

$$\begin{aligned} i_{A,\mathbf{b}}(\mathbb{R}^n) &= \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = A^t\mathbf{x} + \mathbf{b} \text{ for some } \mathbf{x} \in \mathbb{R}^n\} \\ &= \{\mathbf{y} \in \mathbb{R}^m : \varGamma\mathbf{y} = \varGamma\mathbf{b}\}, \end{aligned} \tag{2.3}$$

where $\varGamma = (\gamma_{jk})$ is an $(m-n) \times m$ matrix whose rows form a basis of linear relations between the vectors $\mathbf{a}_i$. That is, $\varGamma$ is of full rank and satisfies the identity $\varGamma A^t = 0$.

We have $i_{A,\mathbf{b}}(P) = \mathbb{R}^m_{\geqslant} \cap i_{A,\mathbf{b}}(\mathbb{R}^n)$.

Construction 2.4 (Gale duality). Let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be a configuration of vectors that span the whole of $\mathbb{R}^n$. We form an $(m-n) \times m$ matrix $\varGamma = (\gamma_{jk})$ whose rows constitute a basis in the space of linear relations between the vectors $\mathbf{a}_i$. The set of columns $\gamma_1, \dots, \gamma_m$ of $\varGamma$ is called a *Gale dual* configuration of $\mathbf{a}_1, \dots, \mathbf{a}_m$. The transition from the configuration of vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ in $\mathbb{R}^n$ to the configuration of vectors $\gamma_1, \dots, \gamma_m$ in $\mathbb{R}^{m-n}$ is called the (linear) *Gale transform*. Each of the two configurations determines the other uniquely up to isomorphism of its ambient space. In other words, each of the matrices A and $\varGamma$ determines the other uniquely up to multiplication on the left by an invertible matrix.

Using the coordinate-free notation, we may think of $\mathbf{a}_1, \dots, \mathbf{a}_m$ as a set of linear functions on an n-dimensional space W. Then we have an exact sequence

$$0 \longrightarrow W \xrightarrow{A^t} \mathbb{R}^m \xrightarrow{\varGamma} L \longrightarrow 0,$$

where A^t is given by $\mathbf{x} \mapsto (\langle \mathbf{a}_1, \mathbf{x} \rangle, \dots, \langle \mathbf{a}_m, \mathbf{x} \rangle)$, and the map $\varGamma$ takes $\mathbf{e}_i$ to $\gamma_i \in L \cong \mathbb{R}^{m-n}$. Similarly, in the dual exact sequence

$$0 \longrightarrow L^* \xrightarrow{\varGamma^t} \mathbb{R}^m \xrightarrow{A} W^* \longrightarrow 0$$

the map A takes $\mathbf{e}_i$ to $\mathbf{a}_i \in W^* \cong \mathbb{R}^n$. Therefore, two configurations $\mathbf{a}_1, \dots, \mathbf{a}_m$ and $\gamma_1, \dots, \gamma_m$ are Gale dual if they are obtained as the images of the standard basis of $\mathbb{R}^m$ under the maps A and $\varGamma$ in a pair of dual short exact sequences.

Here is an important property of Gale dual configurations.

Theorem 2.5. *Let $\mathbf{a}_1, \dots, \mathbf{a}_m$ and $\gamma_1, \dots, \gamma_m$ be Gale dual configurations of vectors in $\mathbb{R}^n$ and $\mathbb{R}^{m-n}$, respectively, and let $I = \{i_1, \dots, i_k\}$. Then the subset $\{\mathbf{a}_i \colon i \in I\}$ is linearly independent if and only if the subset $\{\gamma_j \colon j \notin I\}$ spans the whole of $\mathbb{R}^{m-n}$.*

The proof uses an algebraic lemma.

Lemma 2.6. *Let $\mathbf{k}$ be a field or $\mathbb{Z}$, and assume as given a diagram*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & U \arrow[d, "i_1"] & & \\
0 \arrow[r] & R \arrow[r, "i_2"] & S \arrow[r, "p_2"] \arrow[d, "p_1"] & T \arrow[r] & 0 \\
 & & V \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

in which both the vertical and the horizontal lines are short exact sequences of vector spaces over $\mathbf{k}$ or free Abelian groups. Then $p_1 i_2$ is surjective (*respectively*, *injective* *or split injective) if and only if $p_2 i_1$ is surjective (respectively, injective or split injective).*

Proof. This is a simple diagram chase. Assume first that $p_1 i_2$ is surjective. Take an $\alpha \in T$; we need to cover it by an element in U. There is a $\beta \in S$ such that $p_2(\beta) = \alpha$. If $\beta \in i_1(U)$, then we are done. Otherwise let $\gamma = p_1(\beta) \neq 0$. Since $p_1 i_2$ is surjective, we can choose a $\delta \in R$ such that $p_1 i_2(\delta) = \gamma$. Let $\eta = i_2(\delta) \neq 0$. Hence, $p_1(\eta) = p_1(\beta)$ $(= \gamma)$ and there is a $\xi \in U$ such that $i_1(\xi) = \beta - \eta$. Then $p_2 i_1(\xi) = p_2(\beta - \eta) = \alpha - p_2 i_2(\delta) = \alpha$. Thus, $p_2 i_1$ is surjective.

Now assume that $p_1 i_2$ is injective. Suppose that $p_2 i_1(\alpha) = 0$ for a non-zero $\alpha \in U$. Let $\beta = i_1(\alpha) \neq 0$. Since $p_2(\beta) = 0$, there is a non-zero $\gamma \in R$ such that $i_2(\gamma) = \beta$. Then $p_1 i_2(\gamma) = p_1(\beta) = p_1 i_1(\alpha) = 0$. This contradicts the assumption that $p_1 i_2$ is injective. Thus, $p_2 i_1$ is injective.

Finally, if $p_1 i_2$ is split injective, then its dual map $i_2^* p_1^* \colon V^* \to R^*$ is surjective, and then $i_1^* p_2^* \colon T^* \to U^*$ is also surjective. Thus, $p_2 i_1$ is split injective. □

Proof of Theorem 2.5. Let A be the $n \times m$ matrix with column vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$, and let Γ be the $(m-n) \times m$ matrix with columns $\gamma_1, \dots, \gamma_m$. Denote by A_I the $n \times k$ submatrix formed by the columns $\{\mathbf{a}_i \colon i \in I\}$ and denote by $\Gamma_{\widehat{I}}$ the $(m-n) \times (m-k)$ submatrix formed by the columns $\{\gamma_j \colon j \notin I\}$. We also consider the corresponding maps $A_I \colon \mathbb{R}^k \to \mathbb{R}^n$ and $\Gamma_{\widehat{I}} \colon \mathbb{R}^{m-k} \to \mathbb{R}^{m-n}$.

Let $i \colon \mathbb{R}^k \to \mathbb{R}^m$ be the inclusion of the coordinate subspace spanned by the vectors $\mathbf{e}_i$, $i \in I$, and let $p \colon \mathbb{R}^m \to \mathbb{R}^{m-k}$ be the projection sending every such $\mathbf{e}_i$ to zero. Then $A_I = A \cdot i$ and $\Gamma^t_{\widehat{I}} = p \cdot \Gamma^t$. The vectors $\{\mathbf{a}_i \colon i \in I\}$ are linearly independent if and only if $A_I = A \cdot i$ is injective, and the vectors $\{\gamma_j \colon j \notin I\}$ span $\mathbb{R}^{m-n}$ if and only if $\Gamma^t_{\widehat{I}} = p \cdot \Gamma^t$ is injective. These two conditions are equivalent by Lemma 2.6. □

Construction 2.7 (Gale diagram). Let P be a polytope (2.1) with $b_i = 1$ and let $P^* = \operatorname{conv}(\mathbf{a}_1, \dots, \mathbf{a}_m)$ be the polar polytope. Let $\widetilde{A}^t = (A^t\ \mathbf{1})$, be the $m \times (n+1)$ matrix obtained by appending a column of 1s to A^t. The matrix $\widetilde{A}^t$ has full rank $n+1$ (indeed, otherwise there is an $\mathbf{x} \in \mathbb{R}^n$ such that $\langle \mathbf{a}_i, \mathbf{x} \rangle = 1$ for all i, and then $\lambda\mathbf{x}$ is in P for any $\lambda \geqslant 1$, so that P is unbounded). A configuration of vectors $G = (\mathbf{g}_1, \dots, \mathbf{g}_m)$ in $\mathbb{R}^{m-n-1}$ which is Gale dual to $\widetilde{A}$ is called a *Gale diagram* of P^*. A Gale diagram $G = (\mathbf{g}_1, \dots, \mathbf{g}_m)$ of P^* is therefore determined by the conditions

$$GA^t = 0, \quad \operatorname{rank} G = m - n - 1, \quad \text{and} \quad \sum_{i=1}^m \mathbf{g}_i = \mathbf{0}.$$

The rows of the matrix G from a basis of *affine dependencies* between the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$, that is, a basis in the space of $\mathbf{y} = (y_1, \dots, y_m)^t$ satisfying

$$y_1 \mathbf{a}_1 + \dots + y_m \mathbf{a}_m = \mathbf{0}, \quad y_1 + \dots + y_m = 0.$$

Proposition 2.8. *The polyhedron $P = P(A, \mathbf{b})$ is bounded if and only if the matrix $\Gamma = (\gamma_{jk})$ can be chosen so that the affine plane $i_{A,\mathbf{b}}(\mathbb{R}^n)$ is given by*

$$i_{A,\mathbf{b}}(\mathbb{R}^n) = \left\{ \begin{array}{ll} \mathbf{y} \in \mathbb{R}^m \colon & \gamma_{11} y_1 + \dots + \gamma_{1m} y_m = c, \\ & \gamma_{j1} y_1 + \dots + \gamma_{jm} y_m = 0, \quad 2 \leqslant j \leqslant m-n \end{array} \right\}, \tag{2.4}$$

where $c > 0$ and $\gamma_{1k} > 0$ for all k.

Furthermore, if $b_i = 1$ *in* (2.1), *then the vectors* $\mathbf{g}_i = (\gamma_{2i}, \dots, \gamma_{m-n,i})^t$, $i = 1, \dots, m$, *form a Gale diagram of the polar polytope* $P^* = \operatorname{conv}(\mathbf{a}_1, \dots, \mathbf{a}_m)$.

Proof. The image $i_{A,\mathbf{b}}(P)$ is the intersection of the n-plane $L = i_{A,\mathbf{b}}(\mathbb{R}^n)$ with $\mathbb{R}^m_{\geqslant}$. It is bounded if and only if $L_0 \cap \mathbb{R}^m_{\geqslant} = \{\mathbf{0}\}$, where L_0 is the n-plane through $\mathbf{0}$ parallel to L. Choose a hyperplane H_0 through $\mathbf{0}$ such that $L_0 \subset H_0$ and $H_0 \cap \mathbb{R}^m_{\geqslant} = \{\mathbf{0}\}$. Let H be the affine hyperplane parallel to H_0 and containing L. Since $L \subset H$, we may take the equation defining H as the first equation in the system $\varGamma\mathbf{y} = \varGamma\mathbf{b}$ defining L. The conditions on H_0 imply that $H \cap \mathbb{R}^m_{\geqslant}$ is non-empty and bounded, that is, $c > 0$ and $\gamma_{1k} > 0$ for all k. By subtracting the first equation from the other equations of the system $\varGamma\mathbf{y} = \varGamma\mathbf{b}$ with appropriate coefficients, we now get that the right-hand sides of the last $m-n-1$ equations become zero.

To prove the last statement, we observe that in our case

$$\varGamma = \begin{pmatrix} \gamma_{11} & \dots & \gamma_{1m} \\ \mathbf{g}_1 & \dots & \mathbf{g}_m \end{pmatrix}.$$

The conditions $\varGamma A^t = 0$ and $\operatorname{rank} \varGamma = m - n$ imply that $GA^t = 0$ and $\operatorname{rank} G = m - n - 1$. Finally, comparing (2.3) with (2.4), we see that $\varGamma\mathbf{b} = \binom{c}{\mathbf{0}}$. Since $b_i = 1$, $\sum_{i=1}^m \mathbf{g}_i = \mathbf{0}$. Thus, $G = (\mathbf{g}_1, \dots, \mathbf{g}_m)$ is a Gale diagram of P^*. □

Corollary 2.9. *A polyhedron* $P = P(A, \mathbf{b})$ *is bounded if and only if the vectors* $\mathbf{a}_1, \dots, \mathbf{a}_m$ *satisfy* $\alpha_1\mathbf{a}_1 + \dots + \alpha_m\mathbf{a}_m = \mathbf{0}$ *for some positive numbers* α_k.

Proof. If $\mathbf{a}_1, \dots, \mathbf{a}_m$ satisfy $\sum_{k=1}^m \alpha_k\mathbf{a}_k = \mathbf{0}$ with positive α_k, then we can take $\sum_{k=1}^m \alpha_k y_k = \sum_{k=1}^m \alpha_k b_k$ as the first equation defining the n-plane $i_{A,\mathbf{b}}(\mathbb{R}^n)$ in $\mathbb{R}^m$. Thus, $i_{A,\mathbf{b}}(P)$ is in the intersection of the hyperplane $\sum_{k=1}^m \alpha_k y_k = \sum_{k=1}^m \alpha_k b_k$ with $\mathbb{R}^m_{\geqslant}$, which is bounded since all the α_k are positive. Therefore, P is bounded.

Conversely, if P is bounded, then it follows from Proposition 2.8 and Gale duality that $\mathbf{a}_1, \dots, \mathbf{a}_m$ satisfy $\gamma_{11}\mathbf{a}_1 + \dots + \gamma_{1m}\mathbf{a}_m = \mathbf{0}$ with $\gamma_{1k} > 0$. □

A Gale diagram of P^* encodes its combinatorics (and the combinatorics of P) completely. We give the corresponding statement in the generic case only.

Proposition 2.10. *Assume that* (2.1) *is a generic presentation with* $b_i = 1$. *Let* $P^* = \operatorname{conv}(\mathbf{a}_1, \dots, \mathbf{a}_m)$ *be the polar simplicial polytope and let* $G = (\mathbf{g}_1, \dots, \mathbf{g}_m)$ *be its Gale diagram. Then the following conditions are equivalent*:

(a) $F_{i_1} \cap \dots \cap F_{i_k} \neq \varnothing$ *in* $P = P(A, \mathbf{1})$;

(b) $\operatorname{conv}(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k})$ *is a face of* P^*;

(c) $\mathbf{0} \in \operatorname{conv}(\mathbf{g}_j \colon j \notin \{i_1, \dots, i_k\})$.

Proof. The equivalence (a) ⇔ (b) follows from Theorems 2.1 and 2.2.

(b) ⇒ (c). Let $\operatorname{conv}(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k})$ be a face of P^*. We first observe that each of $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k}$ is a vertex of this face, since otherwise the presentation (2.1) is not generic. By the definition of a face, there exists a linear function ξ such that $\xi(\mathbf{a}_j) = 0$ for $j \in \{i_1, \dots, i_k\}$ and $\xi(\mathbf{a}_j) > 0$ otherwise. The condition $\mathbf{0} \in \operatorname{int} P^*$ implies that $\xi(\mathbf{0}) > 0$, and we may assume that ξ has the form $\xi(\mathbf{u}) = \langle \mathbf{u}, \mathbf{x} \rangle + 1$

for some $\mathbf{x} \in \mathbb{R}^n$. Let $y_j = \xi(\mathbf{a}_j) = \langle \mathbf{a}_j, \mathbf{x} \rangle + 1$, that is, $\mathbf{y} = A^t\mathbf{x} + \mathbf{1}$. We have

$$\sum_{j \notin \{i_1,\dots,i_k\}} \mathbf{g}_j y_j = \sum_{j=1}^{m} \mathbf{g}_j y_j = G\mathbf{y} = G(A^t\mathbf{x} + \mathbf{1}) = G\mathbf{1} = \sum_{j=1}^{m} \mathbf{g}_j = \mathbf{0},$$

where $y_j > 0$ for $j \notin \{i_1, \dots, i_k\}$. It follows that $\mathbf{0} \in \operatorname{conv}(\mathbf{g}_j : j \notin \{i_1, \dots, i_k\})$.

(c) $\Rightarrow$ (b). Let $\sum_{j \notin \{i_1,\dots,i_k\}} \mathbf{g}_j y_j = \mathbf{0}$, with $y_j \geqslant 0$ and at least one y_j non-zero. This is a linear relation between the vectors $\mathbf{g}_j$. The space of such linear relations has a basis formed by the columns of the matrix $\widetilde{A}^t = (A^t\ \mathbf{1})$. Hence there exist $\mathbf{x} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, such that $y_j = \langle \mathbf{a}_j, \mathbf{x} \rangle + b$. The linear function $\xi(\mathbf{u}) = \langle \mathbf{u}, \mathbf{x} \rangle + b$ takes zero values on the vectors $\mathbf{a}_j$ with $j \in \{i_1, \dots, i_k\}$ and non-negative values on the other vectors $\mathbf{a}_j$. Hence, $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k}$ is a subset of the vertex set of some face. Since P^* is simplicial, $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k}$ is a vertex set of a face. $\square$

Remark. We allow redundant inequalities in Proposition (2.10). In this case we obtain the equivalences

$$F_i = \varnothing \quad \Leftrightarrow \quad \mathbf{a}_i \in \operatorname{conv}(\mathbf{a}_j : j \neq i) \quad \Leftrightarrow \quad \mathbf{0} \notin \operatorname{conv}(\mathbf{g}_j : j \neq i).$$

A configuration of vectors $G = (\mathbf{g}_1, \dots, \mathbf{g}_m)$ in $\mathbb{R}^{m-n-1}$ with the property

$$\mathbf{0} \in \operatorname{conv}(\mathbf{g}_j : j \notin \{i_1, \dots, i_k\}) \quad \Leftrightarrow \quad \operatorname{conv}(\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k}) \text{ is a face of } P^*$$

is called a *combinatorial Gale diagram* of $P^* = \operatorname{conv}(\mathbf{a}_1, \dots, \mathbf{a}_m)$. For example, a configuration obtained by multiplying each vector in a Gale diagram by a positive number is a combinatorial Gale diagram. Furthermore, the vectors of a combinatorial Gale diagram can be moved as long as the origin does not cross the 'walls', that is, the affine hyperplanes spanned by subsets of $\mathbf{g}_1, \dots, \mathbf{g}_m$. A combinatorial Gale diagram of P^* is a Gale diagram of a polytope which is combinatorially equivalent to P^*.

Gale diagrams provide an efficient tool for studying the combinatorics of higher-dimensional polytopes with few vertices, because in this case a Gale diagram translates the higher-dimensional structure to a low-dimensional one. For example, Gale diagrams are used to classify n-polytopes with up to $n+3$ vertices and to find unusual examples when the number of vertices is $n+4$ (see [59], § 6.5).

3. Intersections of quadrics

Here we describe the correspondence between polyhedra (2.1) and intersections of quadrics.

3.1. From polyhedra to quadrics.

Construction 3.1 ([14]; also [16], § 3). Let $P = P(A, \mathbf{b})$ be a presentation (2.1) of a polyhedron with a vertex. Recall the map $i_{A,\mathbf{b}} \colon \mathbb{R}^n \to \mathbb{R}^m$ with $\mathbf{x} \mapsto A^t\mathbf{x} + \mathbf{b}$ (see Construction 2.3). It embeds P into $\mathbb{R}^m_{\geqslant}$ (since the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ span $\mathbb{R}^n$). We define the space $\mathscr{Z}_{A,\mathbf{b}}$ from the commutative diagram

$$\begin{array}{ccc} \mathscr{Z}_{A,\mathbf{b}} & \xrightarrow{i_Z} & \mathbb{C}^m \\ \big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ P & \xrightarrow{i_{A,\mathbf{b}}} & \mathbb{R}^m_{\geqslant} \end{array} \tag{3.1}$$

where $\mu(z_1,\dots,z_m)=(|z_1|^2,\dots,|z_m|^2)$. The torus $\mathbb{T}^m$ acts on $\mathscr{Z}_{A,\mathbf{b}}$ with quotient P, and i_Z is a $\mathbb{T}^m$-equivariant embedding.

Replacing y_k by $|z_k|^2$ in the equations defining the affine plane $i_{A,\mathbf{b}}(\mathbb{R}^n)$ (see (2.3)), we get that $\mathscr{Z}_{A,\mathbf{b}}$ embeds into $\mathbb{C}^m$ as the set of common zeros of $m-n$ quadratic equations (*Hermitian quadrics*):

$$i_Z(\mathscr{Z}_{A,\mathbf{b}})=\biggl\{\mathbf{z}\in\mathbb{C}^m\colon \sum_{k=1}^m\gamma_{jk}|z_k|^2=\sum_{k=1}^m\gamma_{jk}b_k \text{ for } 1\leqslant j\leqslant m-n\biggr\}. \tag{3.2}$$

The following property of $\mathscr{Z}_{A,\mathbf{b}}$ is an easy consequence of its construction.

Proposition 3.2. *Given a point $z\in\mathscr{Z}_{A,\mathbf{b}}$, the jth coordinate of $i_Z(z)\in\mathbb{C}^m$ vanishes if and only if z projects onto a point $\mathbf{x}\in P$ such that $\mathbf{x}\in F_j$.*

Theorem 3.3. *The following conditions are equivalent*:

(a) *the presentation* (2.1) *determined by the data* $(A,\mathbf{b})$ *is generic*;

(b) *the intersection of quadrics in* (3.2) *is non-empty and non-degenerate, so that $\mathscr{Z}_{A,\mathbf{b}}$ is a smooth manifold of dimension $m+n$.*

Under these conditions the embedding $i_Z\colon\mathscr{Z}_{A,\mathbf{b}}\to\mathbb{C}^m$ has a $\mathbb{T}^m$-equivariantly trivial normal bundle, and a $\mathbb{T}^m$-framing is determined by a choice of the matrix $\varGamma$ in (2.3).

Proof. For simplicity we identify $\mathscr{Z}_{A,\mathbf{b}}$ with its embedding $i_Z(\mathscr{Z}_{A,\mathbf{b}})\subset\mathbb{C}^m$. We calculate the gradients of the $m-n$ quadrics in (3.2) at a point $\mathbf{z}=(x_1,y_1,\dots,x_m,y_m)\in\mathscr{Z}_{A,\mathbf{b}}$, where $z_k=x_k+iy_k$:

$$2(\gamma_{j1}x_1,\gamma_{j1}y_1,\dots,\gamma_{jm}x_m,\gamma_{jm}y_m),\qquad 1\leqslant j\leqslant m-n. \tag{3.3}$$

These gradients form the rows of the $(m-n)\times 2m$ matrix $2\varGamma\varDelta$, where

$$\varDelta=\begin{pmatrix} x_1 & y_1 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & x_m & y_m\end{pmatrix}.$$

Let $I=\{i_1,\dots,i_k\}=\{i\colon z_i=0\}$ be the set of zero coordinates of $\mathbf{z}$. Then the rank of the gradient matrix $2\varGamma\varDelta$ at $\mathbf{z}$ is equal to the rank of the $(m-n)\times(m-k)$ matrix $\varGamma_{\widehat{I}}$ obtained by deleting the columns with indices $i_1,\dots,i_k$ from $\varGamma$.

Now let (2.1) be a generic presentation. By Proposition 3.2, a point $\mathbf{z}$ with $z_{i_1}=\dots=z_{i_k}=0$ projects to a point in $F_{i_1}\cap\dots\cap F_{i_k}\ne\varnothing$. Hence the vectors $\mathbf{a}_{i_1},\dots,\mathbf{a}_{i_k}$ are linearly independent. By Theorem 2.5 the rank of $\varGamma_{\widehat{I}}$ is $m-n$. Therefore, the intersection of quadrics (3.2) is non-degenerate.

On the other hand, if (2.1) is not generic, then there is a point $\mathbf{z}\in\mathscr{Z}_{A,\mathbf{b}}$ such that the vectors $\{\mathbf{a}_{i_1},\dots,\mathbf{a}_{i_k}\colon z_{i_1}=\dots=z_{i_k}=0\}$ are linearly dependent. By Theorem 2.5, the columns of the corresponding matrix $\varGamma_{\widehat{I}}$ do not span $\mathbb{R}^{m-n}$, so its rank is less than $m-n$ and the intersection of quadrics (3.2) is degenerate at $\mathbf{z}$.

The last statement follows from the fact that $\mathscr{Z}_{A,\mathbf{b}}$ is a non-degenerate intersection of quadratic surfaces, each of which is $\mathbb{T}^m$-invariant. $\square$

3.2. From quadrics to polyhedra. This time we start with an intersection of $m-n$ Hermitian quadrics in $\mathbb{C}^m$:

$$\mathscr{Z}_{\Gamma,\delta} = \biggl\{\mathbf{z} = (z_1,\dots,z_m) \in \mathbb{C}^m \colon \sum_{k=1}^{m} \gamma_{jk}|z_k|^2 = \delta_j \text{ for } 1 \leqslant j \leqslant m-n\biggr\}. \qquad (3.4)$$

The coefficients of the quadrics form an $(m-n)\times m$ matrix $\Gamma = (\gamma_{jk})$, and we denote its column vectors by $\gamma_1,\dots,\gamma_m$. We also consider the column vector $\delta = (\delta_1,\dots,\delta_{m-n})^t \in \mathbb{R}^{m-n}$ of right-hand sides.

These intersections of quadrics are considered up to *linear equivalence*, which corresponds to applying a non-singular linear transformation of $\mathbb{R}^{m-n}$ to Γ and δ. Obviously, such a linear equivalence does not change the set $\mathscr{Z}_{\Gamma,\delta}$.

We denote by $\mathbb{R}_{\geqslant}\langle\gamma_1,\dots,\gamma_m\rangle$ the cone spanned by the vectors $\gamma_1,\dots,\gamma_m$ (that is, the set of linear combinations of these vectors with non-negative real coefficients).

A version of the following proposition appeared in [40], and the proof below is a modification of the argument in [10], Lemma 0.3. It allows us to determine the non-degeneracy of an intersection of quadrics directly from the data (Γ,δ).

Proposition 3.4. *The intersection of quadrics in* (3.4) *is non-empty and also non-degenerate if and only if the following two conditions are satisfied*:

(a) $\delta \in \mathbb{R}_{\geqslant}\langle\gamma_1,\dots,\gamma_m\rangle$;

(b) *if* $\delta \in \mathbb{R}_{\geqslant}\langle\gamma_{i_1},\dots\gamma_{i_k}\rangle$, *then* $k \geqslant m-n$.

Under these conditions $\mathscr{Z}_{\Gamma,\delta}$ *is a smooth submanifold of* $\mathbb{C}^m$ *of dimension* $m+n$, *and the vectors* $\gamma_1,\dots,\gamma_m$ *span* $\mathbb{R}^{m-n}$.

Proof. First, assume that (a) and (b) are satisfied. Then (a) implies that $\mathscr{Z}_{\Gamma,\delta} \neq \varnothing$. Let $\mathbf{z} \in \mathscr{Z}_{\Gamma,\delta}$. Then the rank of the matrix of gradients of the quadrics in (3.4) at $\mathbf{z}$ is $\operatorname{rk}\{\gamma_k \colon z_k \neq 0\}$. Since $\mathbf{z} \in \mathscr{Z}_{\Gamma,\delta}$, the vector δ is in the cone generated by those γ_k for which $z_k \neq 0$. By the Carathéodory Theorem, δ belongs to the cone generated by some $m-n$ of these vectors, that is, $\delta \in \mathbb{R}_{\geqslant}\langle\gamma_{k_1},\dots,\gamma_{k_{m-n}}\rangle$, where $z_{k_i} \neq 0$ for $i = 1,\dots,m-n$. Moreover, the vectors $\gamma_{k_1},\dots,\gamma_{k_{m-n}}$ are linearly independent (otherwise, again by the Carathéodory Theorem, we obtain a contradiction to (b)). This implies that the $m-n$ gradients of the quadrics in (3.4) are linearly independent at $\mathbf{z}$, and therefore $\mathscr{Z}_{\Gamma,\delta}$ is smooth and $(m+n)$-dimensional.

To prove the other implication we observe that if (b) fails, that is, δ is in the cone generated by some $m-n-1$ vectors among $\gamma_1,\dots,\gamma_m$, then there is a point $\mathbf{z} \in \mathscr{Z}_{\Gamma,\delta}$ with at least $n+1$ zero coordinates. The gradients of the quadrics in (3.4) cannot be linearly independent at such a point $\mathbf{z}$. $\square$

The torus $\mathbb{T}^m$ acts on $\mathscr{Z}_{\Gamma,\delta}$, and the quotient $\mathscr{Z}_{\Gamma,\delta}/\mathbb{T}^m$ is identified with the set of non-negative solutions of the system of $m-n$ linear equations

$$\sum_{k=1}^{m} \gamma_k y_k = \delta. \qquad (3.5)$$

This set may be described as a convex polyhedron $P(A,\mathbf{b})$ given by (2.1), where $(b_1,\dots,b_m)$ is any solution of (3.5) and the vectors $\mathbf{a}_1,\dots,\mathbf{a}_m \in \mathbb{R}^n$ form the transpose matrix of the matrix of a basis of solutions of the homogeneous system $\sum_{k=1}^{m}\gamma_k y_k = \mathbf{0}$. We call $P(A,\mathbf{b})$ the *associated polyhedron* of the intersection of quadrics $\mathscr{Z}_{\Gamma,\delta}$. If the vectors $\gamma_1,\dots,\gamma_m$ span $\mathbb{R}^{m-n}$, then $\mathbf{a}_1,\dots,\mathbf{a}_m$ span $\mathbb{R}^n$. In this case the two vector configurations are Gale dual.

We summarize the results and constructions of this section as follows.

Theorem 3.5. *A presentation of a polyhedron*

$$P=P(A,\mathbf{b})=\{\mathbf{x}\in\mathbb{R}^n\colon \langle\mathbf{a}_i,\mathbf{x}\rangle+b_i\geqslant 0 \textit{ for } i=1,\dots,m\}$$

(with $\mathbf{a}_1,\dots,\mathbf{a}_m$ spanning $\mathbb{R}^n$) defines an intersection of Hermitian quadrics

$$\mathscr{Z}_{\Gamma,\delta}=\biggl\{\mathbf{z}=(z_1,\dots,z_m)\in\mathbb{C}^m\colon \sum_{k=1}^m\gamma_{jk}|z_k|^2=\delta_j \textit{ for } j=1,\dots,m-n\biggr\}$$

(with $\gamma_1,\dots,\gamma_m$ spanning $\mathbb{R}^{m-n}$) uniquely up to a linear isomorphism of $\mathbb{R}^{m-n}$, and an intersection of quadrics $\mathscr{Z}_{\Gamma,\delta}$ defines a presentation $P(A,\mathbf{b})$ uniquely up to an isomorphism of $\mathbb{R}^n$.

The systems of vectors $\mathbf{a}_1,\dots,\mathbf{a}_m\in\mathbb{R}^n$ and $\gamma_1,\dots,\gamma_m\in\mathbb{R}^{m-n}$ are Gale dual, and the vectors $\mathbf{b}\in\mathbb{R}^m$ and $\delta\in\mathbb{R}^{m-n}$ are related by the equality $\delta=\Gamma\mathbf{b}$.

The intersection of quadrics $\mathscr{Z}_{\Gamma,\delta}$ is non-empty and non-degenerate if and only if the presentation $P(A,\mathbf{b})$ is generic.

Example 3.6 ($m=n+1$: one quadric). If the presentation (2.1) is generic and P is bounded, then $m\geqslant n+1$. The case $m=n+1$ corresponds to a simplex. The *standard* simplex is given by the following $n+1$ inequalities:

$$\Delta^n=\{\mathbf{x}\in\mathbb{R}^n\colon x_i\geqslant 0 \text{ for } i=1,\dots,n \text{ and } -x_1-\dots-x_n+1\geqslant 0\}.$$

We therefore have $\mathbf{a}_i=\mathbf{e}_i$ (the ith standard basis vector) for $i=1,\dots,n$ and $\mathbf{a}_{n+1}=-\mathbf{e}_1-\dots-\mathbf{e}_n$. Taking $\Gamma=(1\dots1)$, we get that

$$\mathscr{Z}_{A,\mathbf{b}}=\mathbb{S}^{2n+1}=\{\mathbf{z}\in\mathbb{C}^{n+1}\colon |z_1|^2+\dots+|z_{n+1}|^2=1\}.$$

More generally, a presentation (2.1) with $m=n+1$ and $\mathbf{a}_1,\dots,\mathbf{a}_n$ spanning $\mathbb{R}^n$ can be taken by an isomorphism of $\mathbb{R}^n$ to the form

$$P=\{\mathbf{x}\in\mathbb{R}^n\colon x_i+b_i\geqslant 0 \text{ for } i=1,\dots,n \text{ and } -c_1x_1-\dots-c_nx_n+b_{n+1}\geqslant 0\}.$$

We therefore have $\Gamma=(c_1\dots c_n\,1)$, and $\mathscr{Z}_{A,\mathbf{b}}$ is given by the single equation

$$c_1|z_1|^2+\dots+c_n|z_n|^2+|z_{n+1}|^2=c_1b_1+\dots+c_nb_n+b_{n+1}.$$

If the presentation is generic and bounded, then by Theorem 3.3 $\mathscr{Z}_{A,\mathbf{b}}$ is non-empty, non-degenerate, and bounded. This implies that all the c_i and the right-hand side above are positive, and $\mathscr{Z}_{A,\mathbf{b}}$ is an ellipsoid.

4. Moment-angle manifolds from polytopes

Here we consider the case when the polyhedron P defined by (2.1) (or equivalently, the intersection of quadrics (3.4)) is bounded. We also assume that (2.1) is a generic presentation, so that P is an n-dimensional simple polytope and $\mathscr{Z}_{A,\mathbf{b}}=\mathscr{Z}_{\Gamma,\delta}$ is an $(m+n)$-dimensional closed smooth manifold.

We start with the construction of an identification space which goes back to Vinberg's paper [58] on Coxeter groups and was presented in the form described below in the paper [21] of Davis and Januszkiewicz. It was the first construction of what later became known as a moment-angle manifold.

Construction 4.1. Let $[m]=\{1,\dots,m\}$ be the standard m-element set. For each $I\subset[m]$ we consider the coordinate subtorus

$$\mathbb{T}^I=\{(t_1,\dots,t_m)\in\mathbb{T}^m\colon t_j=1 \text{ for } j\notin I\}\subset\mathbb{T}^m.$$

In particular, $\mathbb{T}^\varnothing$ is the trivial subgroup $\{1\}\subset\mathbb{T}^m$.

We define the map $\mathbb{R}_\geqslant\times\mathbb{T}\to\mathbb{C}$ by $(y,t)\mapsto yt$. Taking the product, we obtain a map $\mathbb{R}^m_\geqslant\times\mathbb{T}^m\to\mathbb{C}^m$. The pre-image of a point $\mathbf{z}\in\mathbb{C}^m$ under this map is $\mathbf{y}\times\mathbb{T}^{\omega(\mathbf{z})}$, where $y_i=|z_i|$ for $1\leqslant i\leqslant m$ and

$$\omega(\mathbf{z})=\{i\colon z_i=0\}\subset[m]$$

is the set of zero coordinates of $\mathbf{z}$. Therefore, $\mathbb{C}^m$ can be identified with the quotient

$$\mathbb{R}^m_\geqslant\times\mathbb{T}^m/{\sim},\quad \text{where } (\mathbf{y},\mathbf{t}_1)\sim(\mathbf{y},\mathbf{t}_2) \text{ if } \mathbf{t}_1^{-1}\mathbf{t}_2\in\mathbb{T}^{\omega(\mathbf{y})}. \tag{4.1}$$

Given $\mathbf{x}\in P$, we let

$$I_\mathbf{x}=\{i\in[m]\colon \mathbf{x}\in F_i\}$$

(the set of facets containing $\mathbf{x}$).

Proposition 4.2. *$\mathscr{Z}_{A,\mathbf{b}}$ is $\mathbb{T}^m$-equivariantly homeomorphic to the quotient*

$$P\times\mathbb{T}^m/{\sim},\quad \textit{where } (\mathbf{x},\mathbf{t}_1)\sim(\mathbf{x},\mathbf{t}_2) \textit{ for } \mathbf{t}_1^{-1}\mathbf{t}_2\in\mathbb{T}^{I_\mathbf{x}}.$$

Proof. Using (3.1), we identify $\mathscr{Z}_{A,\mathbf{b}}$ with $i_{A,\mathbf{b}}(P)\times\mathbb{T}^m/{\sim}$, where $\sim$ is the equivalence relation in (4.1). A point $\mathbf{x}\in P$ is mapped by $i_{A,\mathbf{b}}$ to the point $\mathbf{y}\in\mathbb{R}^m_\geqslant$ with $I_\mathbf{x}=\omega(\mathbf{y})$. $\square$

An important consequence of this construction is that the topological type of $\mathscr{Z}_{A,\mathbf{b}}$ depends only on the combinatorics of P.

Proposition 4.3. *Assume as given two generic presentations*

$$P=\{\mathbf{x}\in\mathbb{R}^n\colon (A^t\mathbf{x}+\mathbf{b})_i\geqslant0\}\quad \textit{and}\quad P'=\{\mathbf{x}\in\mathbb{R}^n\colon (A'^t\mathbf{x}+\mathbf{b}')_i\geqslant0\}$$

such that P and P' are combinatorially equivalent simple polytopes.

(a) *If both presentations are irredundant, then the corresponding manifolds $\mathscr{Z}_{A,\mathbf{b}}$ and $\mathscr{Z}_{A',\mathbf{b}'}$ are $\mathbb{T}^m$-equivariantly homeomorphic.*

(b) *If the second presentation is obtained from the first by adding k redundant inequalities, then $\mathscr{Z}_{A',\mathbf{b}'}$ is homeomorphic to the product of $\mathscr{Z}_{A,\mathbf{b}}$ and a k-torus T^k.*

Proof. (a) By Proposition 4.2, $\mathscr{Z}_{A,\mathbf{b}}\cong P\times\mathbb{T}^m/{\sim}$ and $\mathscr{Z}_{A',\mathbf{b}'}\cong P'\times\mathbb{T}^m/{\sim}$. If both presentations are irredundant, then any F_i is a facet of P, and the equivalence relation $\sim$ depends only on the face structure of P. Therefore, any homeomorphism $P\to P'$ preserving the face structure extends to a $\mathbb{T}^m$-equivariant homeomorphism $P\times\mathbb{T}^m/{\sim}\to P'\times\mathbb{T}^m/{\sim}$.

(b) Suppose that the first presentation has m inequalities, and the second has m' inequalities, so that $m'-m=k$. Let $J\subset[m']$ be the subset corresponding to the k added redundant inequalities; we may assume that $J=\{m+1,\dots,m'\}$. Since $F_j=\varnothing$ for any $j\in J$, we have $I_{\mathbf x}\cap J=\varnothing$ for any $\mathbf x\in P'$. Therefore, the equivalence relation $\sim$ does not affect the factor $\mathbb T^J\subset\mathbb T^{m'}$, and we have

$$\mathscr Z_{A',\mathbf b'}\cong P'\times\mathbb T^{m'}/{\sim}\cong(P\times\mathbb T^m/{\sim})\times\mathbb T^J\cong\mathscr Z_{A,\mathbf b}\times T^k. \qquad \square$$

Remark. A $\mathbb T^m$-homeomorphism in Proposition 4.3 (a) can be replaced by a $\mathbb T^m$-diffeomorphism (with respect to the smooth structures in Theorem 3.3), but the proof is more technical. It follows from the fact that two simple polytopes are combinatorially equivalent if and only if they are diffeomorphic as 'smooth manifolds with corners'. For an alternative argument, see [10], Corollary 4.7.

The statement (a) remains valid without assuming that the presentation is generic, although $\mathscr Z_{A,\mathbf b}$ is not a manifold in this case.

Definition 4.4. The $(m+n)$-dimensional manifold $\mathscr Z_{A,\mathbf b}$ defined by any irredundant presentation (2.1) of an n-dimensional simple polytope P with m facets is called the *moment-angle manifold* corresponding to P, and denoted by $\mathscr Z_P$. Moment-angle manifolds appearing in this way are said to be *polytopal*; more general moment-angle manifolds will be considered later.

Proposition 4.5. *The moment-angle manifold $\mathscr Z_P$ is $\mathbb T^m$-equivariantly diffeomorphic to a non-degenerate intersection of quadrics of the following form*:

$$\left\{\begin{array}{ll}\mathbf z\in\mathbb C^m: & \displaystyle\sum_{k=1}^m|z_k|^2=1,\\ & \displaystyle\sum_{k=1}^m\mathbf g_k|z_k|^2=\mathbf 0\end{array}\right\}, \tag{4.2}$$

where $(\mathbf g_1,\dots,\mathbf g_m)\subset\mathbb R^{m-n-1}$ is a combinatorial Gale diagram of P^*.

Proof. It follows from Proposition 2.8 that $\mathscr Z_P$ is given by

$$\left\{\begin{array}{ll}\mathbf z\in\mathbb C^m: & \gamma_{11}|z_1|^2+\dots+\gamma_{1m}|z_m|^2=c,\\ & \mathbf g_1|z_1|^2+\dots+\mathbf g_m|z_m|^2=\mathbf 0\end{array}\right\},$$

where the numbers γ_{1k} and c are positive. Divide the first equation by c and then replace each z_k by $\sqrt{c/\gamma_{1k}}\,z_k$. As a result, each $\mathbf g_k$ is multiplied by a positive number, so that $(\mathbf g_1,\dots,\mathbf g_m)$ is still a combinatorial Gale diagram for P^*. $\square$

By adapting Proposition 3.4 to the special case of quadrics (4.2) we obtain the following result.

Proposition 4.6. *An intersection of quadrics* (4.2) *is non-empty and also non-degenerate if and only if the following two conditions are satisfied*:

(a) $\mathbf 0\in\operatorname{conv}(\mathbf g_1,\dots,\mathbf g_m)$;

(b) *if* $\mathbf 0\in\operatorname{conv}(\mathbf g_{i_1},\dots,\mathbf g_{i_k})$, *then* $k\geqslant m-n$.

Following [10], we call a non-degenerate intersection (4.2) of $m-n-1$ homogeneous quadrics with a unit sphere in $\mathbb{C}^m$ a *link*. We therefore get that any moment-angle manifold is diffeomorphic to a link, and any link is a product of a moment-angle manifold and a torus.

As we have seen in Example 3.6, the moment-angle manifold corresponding to an n-simplex is a sphere S^{2n+1}. This is also the link of an empty system of homogeneous quadrics, corresponding to the case $m=n+1$.

Example 4.7 ($m=n+2$: two quadrics). A polytope P defined by $m=n+2$ inequalities either is combinatorially equivalent to a product of two simplices (when there are no redundant inequalities), or is a simplex (when one inequality is redundant). In the case $m=n+2$ the link (4.2) has the form

$$\left\{\begin{array}{ll}\mathbf{z}\in\mathbb{C}^m\colon & |z_1|^2+\cdots+|z_m|^2=1,\\ & g_1|z_1|^2+\cdots+g_m|z_m|^2=0\end{array}\right\},$$

where $g_k\in\mathbb{R}$. The condition (b) in Proposition 4.6 implies that all the g_i are non-zero; assume that there are p positive and $q=m-p$ negative numbers among them. Then the condition (a) implies that $p>0$ and $q>0$. Therefore, the link is the intersection of the cone over a product of two ellipsoids of dimensions $2p-1$ and $2q-1$ (given by the second quadric) with a unit sphere of dimension $2m-1$ (given by the first quadric). Such a link is diffeomorphic to $S^{2p-1}\times S^{2q-1}$. The case $p=1$ or $q=1$ corresponds to one redundant inequality. In the irredundant case (P is a product $\Delta^{p-1}\times\Delta^{q-1}$, $p,q>1$) we get that $\mathscr{Z}_P\cong S^{2p-1}\times S^{2q-1}$.

5. Hamiltonian toric manifolds and moment maps

Particular examples of polytopal moment-angle manifolds $\mathscr{Z}_P$ appear as level sets for the moment maps used in the construction of Hamiltonian toric manifolds via symplectic reduction. In this case the left-hand sides of the equations in (3.2) are quadratic Hamiltonians of a torus action on $\mathbb{C}^m$.

5.1. Symplectic reduction. We briefly review the background material in symplectic geometry, referring the reader to the monographs by Audin [3] and Guillemin [33] for further details.

A *symplectic manifold* is a pair (W,ω) consisting of a smooth manifold W and a closed differential 2-form ω which is non-degenerate at each point. The dimension of a symplectic manifold W is necessarily even.

Assume now that a torus T acts on W while preserving the symplectic form ω. We denote the Lie algebra of the torus T by $\mathfrak{t}$ (since T is commutative, its Lie algebra is trivial, but the construction can be generalized to non-commutative Lie groups). Given an element $\mathbf{v}\in\mathfrak{t}$, we denote by $X_{\mathbf{v}}$ the corresponding T-invariant vector field on W. The torus action is said to be *Hamiltonian* if the 1-form $\omega(X_{\mathbf{v}},\,\cdot\,)$ is exact for any $\mathbf{v}\in\mathfrak{t}$. In other words, an action is Hamiltonian if for any $\mathbf{v}\in\mathfrak{t}$ there exists a function $H_{\mathbf{v}}$ on W (called a *Hamiltonian*) satisfying the condition

$$\omega(X_{\mathbf{v}},Y)=dH_{\mathbf{v}}(Y)$$

for any vector field Y on W. The function $H_{\mathbf{v}}$ is defined up to addition of a constant. Choose a basis $\{\mathbf{e}_i\}$ in $\mathfrak{t}$ and the corresponding Hamiltonians $\{H_{\mathbf{e}_i}\}$. Then the *moment map*

$$\mu\colon W\to\mathfrak{t}^*,\qquad (x,\mathbf{e}_i)\mapsto H_{\mathbf{e}_i}(x)$$

(where $x\in W$), is defined. Observe that adding constants to the Hamiltonians $H_{\mathbf{e}_i}$ results in shifting the image of μ by a vector in $\mathfrak{t}^*$. According to a theorem of Atiyah and Guillemin–Sternberg, the image $\mu(W)$ of the moment map is convex, and if W is compact, then $\mu(W)$ is a convex polytope in $\mathfrak{t}^*$.

Example 5.1. The most basic example is $W=\mathbb{C}^m$ with the symplectic form

$$\omega=i\sum_{k=1}^m dz_k\wedge d\overline{z}_k=2\sum_{k=1}^m dx_k\wedge dy_k,\qquad z_k=x_k+iy_k.$$

The coordinatewise action of $\mathbb{T}^m$ on $\mathbb{C}^m$ is Hamiltonian, and the moment map $\mu\colon\mathbb{C}^m\to\mathbb{R}^m$ is given by $\mu(z_1,\dots,z_m)=(|z_1|^2,\dots,|z_m|^2)$. The image of μ is the positive orthant $\mathbb{R}^m_{\geqslant}$.

Construction 5.2 (symplectic reduction). Assume as given a Hamiltonian action of a torus T on a symplectic manifold W. Assume further that the moment map $\mu\colon W\to\mathfrak{t}^*$ is *proper*, that is, $\mu^{-1}(V)$ is compact for any compact subset $V\subset\mathfrak{t}^*$ (this is always the case if W itself is compact). Let $\mathbf{u}\in\mathfrak{t}^*$ be a *regular value* of the moment map, that is, the differential $\mathscr{T}_xW\to\mathfrak{t}^*$ is surjective for all $x\in\mu^{-1}(\mathbf{u})$. Then the level set $\mu^{-1}(\mathbf{u})$ is a smooth compact T-invariant submanifold of W. Furthermore, the T-action on $\mu^{-1}(\mathbf{u})$ is almost free, that is, all the stabilizers are finite subgroups.

Assume now that the T-action on $\mu^{-1}(\mathbf{u})$ is free. The restriction of the symplectic form ω to $\mu^{-1}(\mathbf{u})$ may be degenerate. However, the quotient manifold $\mu^{-1}(\mathbf{u})/T$ is endowed with a unique symplectic form ω' such that

$$p^*\omega'=i^*\omega,$$

where $i\colon\mu^{-1}(\mathbf{u})\to W$ is the inclusion and $p\colon\mu^{-1}(\mathbf{u})\to\mu^{-1}(\mathbf{u})/T$ the projection.

We therefore obtain a new symplectic manifold $(\mu^{-1}(\mathbf{u})/T,\omega')$ which is called the *symplectic reduction*, or the *symplectic quotient* of (W,ω) by T.

The construction of a symplectic reduction works also under milder assumptions on the action (see [25] and additional references there), but the generality described here will be enough for our purposes.

5.2. The toric case. We want to study symplectic quotients of $\mathbb{C}^m$ by torus subgroups $T\subset\mathbb{T}^m$. Such a subgroup of dimension $m-n$ has the form

$$T_\Gamma=\{(e^{2\pi i\langle\gamma_1,\varphi\rangle},\dots,e^{2\pi i\langle\gamma_m,\varphi\rangle})\in\mathbb{T}^m\},\tag{5.1}$$

where $\varphi\in\mathbb{R}^{m-n}$ is an $(m-n)$-dimensional parameter, and $\Gamma=(\gamma_1,\dots,\gamma_m)$ is a set of m vectors in $\mathbb{R}^{m-n}$. In order for T_Γ to be an $(m-n)$-torus, the configuration of vectors $\gamma_1,\dots,\gamma_m$ must be *rational*, that is, the set $L=\mathbb{Z}\langle\gamma_1,\dots,\gamma_m\rangle$ of all their integral linear combinations must be an $(m-n)$-dimensional discrete subgroup (*lattice*) in $\mathbb{R}^{m-n}$. Let

$$L^*=\{\lambda^*\in\mathbb{R}^{m-n}\colon\langle\lambda^*,\lambda\rangle\in\mathbb{Z}\text{ for all }\lambda\in L\}$$

be the dual lattice. We shall occasionally represent the elements of T_Γ by vectors $\varphi\in\mathbb{R}^{m-n}$, so that T_Γ is identified with the quotient $\mathbb{R}^{m-n}/L^*$.

The restricted action of $T_\Gamma\subset\mathbb{T}^m$ on $\mathbb{C}^m$ is obviously Hamiltonian, and the corresponding moment map is the composition

$$\mu_\Gamma\colon \mathbb{C}^m\xrightarrow{\mu}\mathbb{R}^m\to\mathfrak{t}_\Gamma^*, \tag{5.2}$$

where $\mathbb{R}^m\to\mathfrak{t}_\Gamma^*$ is the map of the dual Lie algebras corresponding to the embedding $T_\Gamma\to\mathbb{T}^m$. The map $\mathbb{R}^m\to\mathfrak{t}_\Gamma^*$ takes the ith basis vector $\mathbf{e}_i\in\mathbb{R}^m$ to $\gamma_i\in\mathfrak{t}_\Gamma^*$. By choosing a basis in $L\subset\mathfrak{t}_\Gamma^*$ we can write the map $\mathbb{R}^m\to\mathfrak{t}_\Gamma^*$ as an *integer* matrix $\Gamma=(\gamma_{jk})$. The moment map (5.2) is then given by

$$(z_1,\dots,z_m)\mapsto\biggl(\sum_{k=1}^m\gamma_{1k}|z_k|^2,\dots,\sum_{k=1}^m\gamma_{m-n,k}|z_k|^2\biggr).$$

Its level set $\mu_\Gamma^{-1}(\delta)$ corresponding to a value $\delta=(\delta_1,\dots,\delta_{m-n})^t\in\mathfrak{t}_\Gamma^*$ is exactly the intersection of quadrics $\mathscr{Z}_{\Gamma,\delta}$ given by the system (3.4).

To apply the symplectic reduction we need to see when the moment map μ_Γ is proper, find its regular values δ, and finally identify when the action of T_Γ on $\mu_\Gamma^{-1}(\delta)=\mathscr{Z}_{\Gamma,\delta}$ is free. In Theorem 5.3 below, all these conditions are expressed in terms of the polyhedron P associated with $\mathscr{Z}_{\Gamma,\delta}$ as described in § 3. We need a couple more definitions before we state this theorem.

It follows from Gale duality that $\gamma_1,\dots,\gamma_m$ span a lattice L in $\mathbb{R}^{m-n}$ if and only if the dual configuration $\mathbf{a}_1,\dots,\mathbf{a}_m$ spans a lattice $N=\mathbb{Z}\langle\mathbf{a}_1,\dots,\mathbf{a}_m\rangle$ in $\mathbb{R}^n$. We say that a presentation (2.1) is *rational* if $\mathbb{Z}\langle\mathbf{a}_1,\dots,\mathbf{a}_m\rangle$ is a lattice.

Recall that for each $\mathbf{x}\in P$ we defined

$$I_\mathbf{x}=\{i\in[m]\colon\langle\mathbf{a}_i,\mathbf{x}\rangle+b_i=0\}=\{i\in[m]\colon\mathbf{x}\in F_i\}$$

(the set of facets containing $\mathbf{x}$). A polyhedron P is said to be *Delzant* if it has a rational presentation (2.1) such that for any $\mathbf{x}\in P$ the vectors $\{\mathbf{a}_i\colon i\in I_\mathbf{x}\}$ constitute part of a basis of $N=\mathbb{Z}\langle\mathbf{a}_1,\dots,\mathbf{a}_m\rangle$. Equivalently, P is Delzant if it is simple and for any vertex $\mathbf{x}\in P$ the vectors $\mathbf{a}_i$ normal to the n facets meeting at $\mathbf{x}$ form a basis of the lattice N. The term comes from the classification of Hamiltonian toric manifolds due to Delzant [22], which we shall briefly review later.

Now let $\delta\in\mathfrak{t}_\Gamma$ be a value of the moment map $\mu_\Gamma\colon\mathbb{C}^m\to\mathfrak{t}_\Gamma^*$, and let $\mu_\Gamma^{-1}(\delta)=\mathscr{Z}_{\Gamma,\delta}$ be the corresponding level set, which is an intersection of quadrics (3.4). We associate with $\mathscr{Z}_{\Gamma,\delta}$ a presentation (2.1) as described in § 3 (see Theorem 3.5).

Theorem 5.3. *Let $T_\Gamma\subset\mathbb{T}^m$ be a torus subgroup* (5.1) *determined by a rational configuration of vectors $\gamma_1,\dots,\gamma_m$.*

(a) *The moment map $\mu_\Gamma\colon\mathbb{C}^m\to\mathfrak{t}_\Gamma^*$ is proper if and only if its level set $\mu_\Gamma^{-1}(\delta)$ is bounded for some* (*and then for any*) *value $\delta\in\mathfrak{t}_\Gamma^*$. Equivalently, the map μ_Γ is proper if and only if the Gale dual configuration $\mathbf{a}_1,\dots,\mathbf{a}_m$ satisfies $\alpha_1\mathbf{a}_1+\dots+\alpha_m\mathbf{a}_m=\mathbf{0}$ for some positive numbers α_k.*

(b) *$\delta\in\mathfrak{t}_\Gamma^*$ is a regular value of μ_Γ if and only if the intersection of quadrics $\mu_\Gamma^{-1}(\delta)=\mathscr{Z}_{\Gamma,\delta}$ is non-empty and non-degenerate. Equivalently, δ is a regular value if and only if the associated presentation of $P=P(A,\mathbf{b})$ is generic.*

(c) *The action of T_Γ on $\mu_\Gamma^{-1}(\delta) = \mathscr{Z}_{\Gamma,\delta}$ is free if and only if the associated polyhedron P is Delzant.*

Proof. (a) If μ_Γ is proper, then $\mu_\Gamma^{-1}(\delta) \subset \mathfrak{t}_\Gamma^*$ is compact, so it is bounded.

Assume now that $\mu_\Gamma^{-1}(\delta) = \mathscr{Z}_{\Gamma,\delta}$ is bounded for some δ. Then the corresponding polyhedron P is also bounded. By Corollary 2.9, this is equivalent to the vanishing of some positive linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_m$. This condition is independent of δ, and we conclude that $\mu_\Gamma^{-1}(\delta)$ is bounded for any δ. Let $X \subset \mathfrak{t}_\Gamma^*$ be a compact subset. Since $\mu_\Gamma^{-1}(X)$ is closed, it is compact whenever it is bounded. By Proposition 2.8 we may assume that, for any $\delta \in X$, the first quadric defining $\mu_\Gamma^{-1}(\delta) = \mathscr{Z}_{\Gamma,\delta}$ is given by $\gamma_{11}|z_1|^2 + \dots + \gamma_{1m}|z_m|^2 = \delta_1$ with $\gamma_{1k} > 0$ for all k. Let $c = \max_{\delta \in X} \delta_1$. Then $\mu_\Gamma^{-1}(X)$ is contained in the bounded set

$$\{\mathbf{z} \in \mathbb{C}^m \colon \gamma_{11}|z_1|^2 + \dots + \gamma_{1m}|z_m|^2 \leqslant c\}$$

and is therefore bounded. Hence, $\mu_\Gamma^{-1}(X)$ is compact, and μ_Γ is proper.

(b) The first statement is the definition of a regular value. The equivalent statement was already proved in Theorem 3.3.

(c) We first need to identify the stabilizers of the T_Γ-action on $\mu_\Gamma^{-1}(\delta)$. Although the fact that these stabilizers are finite for a regular value δ follows from the general construction of a symplectic reduction, we can prove this directly.

Given a point $\mathbf{z} = (z_1, \dots, z_m) \in \mathscr{Z}_{\Gamma,\delta}$, we define the sublattice

$$L_{\mathbf{z}} = \mathbb{Z}\langle \gamma_i \colon z_i \neq 0 \rangle \subset L = \mathbb{Z}\langle \gamma_1, \dots, \gamma_m \rangle.$$

Lemma 5.4. *The stabilizer subgroup of a point $\mathbf{z} \in \mathscr{Z}_{\Gamma,\delta}$ under the action of T_Γ is given by $L_{\mathbf{z}}^*/L^*$. Furthermore, if $\mathscr{Z}_{\Gamma,\delta}$ is non-degenerate, then all these stabilizers are finite, that is, the action of T_Γ on $\mathscr{Z}_{\Gamma,\delta}$ is almost free.*

Proof. An element $(e^{2\pi i\langle \gamma_1, \varphi \rangle}, \dots, e^{2\pi i\langle \gamma_m, \varphi \rangle}) \in T_\Gamma$ fixes a point $\mathbf{z} \in \mathscr{Z}_\Gamma$ if and only if $e^{2\pi i\langle \gamma_k, \varphi \rangle} = 1$ whenever $z_k \neq 0$. In other words, $\varphi \in T_\Gamma$ fixes $\mathbf{z}$ if and only if $\langle \gamma_k, \varphi \rangle \in \mathbb{Z}$ whenever $z_k \neq 0$. The latter means that $\varphi \in L_{\mathbf{z}}^*$. Since $\varphi \in L^*$ maps to $1 \in T_\Gamma$, the stabilizer of $\mathbf{z}$ is $L_{\mathbf{z}}^*/L^*$.

Assume now that $\mathscr{Z}_{\Gamma,\delta}$ is non-degenerate. In order to see that $L_{\mathbf{z}}^*/L^*$ is finite we need to check that the sublattice $L_{\mathbf{z}} = \mathbb{Z}\langle \gamma_i \colon z_i \neq 0 \rangle \subset L$ has full rank $m-n$. Indeed, $\operatorname{rk}\{\gamma_i \colon z_i \neq 0\}$ is the rank of the matrix of gradients of the quadrics in (3.4) at $\mathbf{z}$. Since $\mathscr{Z}_{\Gamma,\delta}$ is non-degenerate, this rank is $m-n$, as needed. □

Now we can finish the proof of Theorem 5.3 (c). Assume that P is a Delzant polyhedron. By Lemma 5.4, the T_Γ-action on $\mathscr{Z}_{\Gamma,\delta}$ is free if and only if $L_{\mathbf{z}} = L$ for any $\mathbf{z} \in \mathscr{Z}_{\Gamma,\delta}$. Let $i \colon \mathbb{Z}^k \to \mathbb{Z}^m$ be the inclusion of the coordinate sublattice spanned by those $\mathbf{e}_i$ for which $z_i = 0$, and let $p \colon \mathbb{Z}^m \to \mathbb{Z}^{m-k}$ be the projection sending every such $\mathbf{e}_i$ to zero. We also have lattice maps

$$\Gamma^t \colon L^* \to \mathbb{Z}^m, \quad \mathbf{l} \mapsto (\langle \gamma_1, \mathbf{l} \rangle, \dots, \langle \gamma_m, \mathbf{l} \rangle) \quad \text{and} \quad A \colon \mathbb{Z}^m \to N, \quad \mathbf{e}_k \mapsto \mathbf{a}_k.$$

Consider the diagram

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & L^* \arrow[d, "\Gamma^t"] & & \\ 0 \arrow[r] & \mathbb{Z}^k \arrow[r, "i"] & \mathbb{Z}^m \arrow[r, "p"] \arrow[d, "A"] & \mathbb{Z}^{m-k} \arrow[r] & 0 \\ & & N \arrow[d] & & \\ & & 0 & & \end{tikzcd} \tag{5.3}$$

in which the vertical and horizontal sequences are exact. Then the Delzant condition is equivalent to the composition $A \cdot i$ being split injective. The condition $L_{\mathbf{z}} = L$ is equivalent to the composition $\Gamma \cdot p^*$ being surjective, or $p \cdot \Gamma^t$ being split injective. These two conditions are equivalent by Lemma 2.6. $\square$

Corollary 5.5. *Let $P = P(A, \mathbf{b})$ be a Delzant polytope, $\Gamma = (\gamma_1, \ldots, \gamma_m)$ the Gale dual configuration, and $\mathscr{Z}_P$ the corresponding moment-angle manifold. Then*

(a) *$\delta = \Gamma\mathbf{b}$ is a regular value of the moment map $\mu_\Gamma\colon \mathbb{C}^m \to \mathfrak{t}_\Gamma^*$ for the Hamiltonian action of $T_\Gamma \subset \mathbb{T}^m$ on $\mathbb{C}^m$;*

(b) *$\mathscr{Z}_P$ is the regular level set $\mu_\Gamma^{-1}(\Gamma\mathbf{b})$;*

(c) *the action of T_Γ on $\mathscr{Z}_P$ is free.*

We therefore may consider the symplectic quotient of $\mathbb{C}^m$ by T_Γ. It is a compact $2n$-dimensional symplectic manifold, which we denote $V_P = \mathscr{Z}_P/T_\Gamma$. This manifold has a 'residual' Hamiltonian action of the quotient n-torus $\mathbb{T}^m/T_\Gamma$. It follows from the vertical exact sequence in (5.3) that $\mathbb{T}^m/T_\Gamma$ can be identified canonically with $N \otimes_{\mathbb{Z}} \mathbb{S} = \mathbb{R}^n/N$, and we shall denote this torus by T_N. We therefore obtain an exact sequence of tori

$$1 \to T_\Gamma \to \mathbb{T}^m \xrightarrow{\exp A} T_N \to 1, \tag{5.4}$$

where $\exp A\colon \mathbb{T}^m \to T_N$ is the map of tori corresponding to the map of lattices $A\colon \mathbb{Z}^m \to N$.

The symplectic $2n$-manifold $V_P = \mathscr{Z}_P/T_\Gamma$ with the Hamiltonian action of the n-torus $T_N = \mathbb{T}^m/T_\Gamma$ is called the *Hamiltonian toric manifold* corresponding to a Delzant polytope P.

We denote by $\mu_V\colon V_P \to \mathfrak{t}_N^*$ the moment map for the T_N-action on V_P, where $\mathfrak{t}_N = N_{\mathbb{R}}$ is the Lie algebra of T_N. The dual Lie algebra $\mathfrak{t}_N^*$ is naturally embedded as a subspace in $\mathbb{R}^m$ (the dual Lie algebra of $\mathbb{T}^m$), with the inclusion given by $A^t\colon \mathfrak{t}_N^* \cong \mathbb{R}^n \to \mathbb{R}^m$.

Proposition 5.6. *The image of the moment map $\mu_V\colon V_P \to \mathfrak{t}_N^*$ is the polytope P, up to a shift by a vector in $\mathfrak{t}_N^*$.*

Proof. Let ω be the standard symplectic form on $\mathbb{C}^m$ and let $\mu\colon \mathbb{C}^m \to \mathbb{R}^m$ be the moment map for the standard action of $\mathbb{T}^m$ (see Example 5.1). Let $p\colon \mathscr{Z}_P \to V_P$ be the quotient projection by the action of T_Γ, and let $i\colon \mathscr{Z}_P \to \mathbb{C}^m$ be the inclusion, so that the symplectic form ω' on V_P satisfies $p^*\omega' = i^*\omega$. Let $H_{\mathbf{e}_i}\colon \mathbb{C}^m \to \mathbb{R}$ be the Hamiltonian of the $\mathbb{T}^m$-action on $\mathbb{C}^m$ corresponding to the ith basis vector $\mathbf{e}_i$ (explicitly, $H_{\mathbf{e}_i}(\mathbf{z}) = |z_i|^2$), and let $H_{\mathbf{a}_i}\colon V_P \to \mathbb{R}$ be the Hamiltonian of the T_N-action on V_P corresponding to $\mathbf{a}_i \in \mathfrak{t}_N$. Denote by $X_{\mathbf{e}_i}$ the vector field on $\mathscr{Z}_P$ generated by $\mathbf{e}_i$, and denote by $Y_{\mathbf{a}_i}$ the vector field on V_P generated by $\mathbf{a}_i$. Observe that $p_*X_{\mathbf{e}_i} = Y_{\mathbf{a}_i}$. For any vector field Z on $\mathscr{Z}_P$ we have

$$
\begin{aligned}
dH_{\mathbf{e}_i}(Z) &= i^*\omega(X_{\mathbf{e}_i}, Z) = p^*\omega'(X_{\mathbf{e}_i}, Z)\\
&= \omega'(Y_{\mathbf{a}_i}, p_*Z) = dH_{\mathbf{a}_i}(p_*Z) = d(p^*H_{\mathbf{a}_i})(Z),
\end{aligned}
$$

hence $H_{\mathbf{e}_i} = p^*H_{\mathbf{a}_i}$ or $H_{\mathbf{e}_i}(\mathbf{z}) = H_{\mathbf{a}_i}(p(\mathbf{z}))$ up to a constant. By the definition of the moment map this implies that $\mu_V(V_P) \subset \mathfrak{t}_N^* \subset \mathbb{R}^m$ is identified with $\mu(\mathscr{Z}_P) \subset \mathbb{R}^m$ up to a shift by a vector in $\mathbb{R}^m$. The inclusion $\mathfrak{t}_N^* \subset \mathbb{R}^m$ is the map A^t, and $\mu(\mathscr{Z}_P) = i_{A,\mathbf{b}}(P) = A^t(P) + \mathbf{b}$ by the definition of $\mathscr{Z}_P$ (see (3.1)). We therefore get that there exists a vector $\mathbf{c} \in \mathbb{R}^m$ such that

$$
A^t(\mu_V(V_P)) + \mathbf{c} = A^t(P) + \mathbf{b},
$$

that is, $A^t(\mu_V(V_P))$ and $A^t(P)$ differ by $\mathbf{b} - \mathbf{c} \in A^t(\mathfrak{t}_N^*)$. Since A^t is monomorphic, the result follows. $\square$

We have described how to construct a Hamiltonian toric manifold from a Delzant polytope. A theorem of Delzant [22] says that *any* $2n$-dimensional compact connected symplectic manifold W with an effective Hamiltonian action of an n-torus T is equivariantly symplectomorphic to a Hamiltonian toric manifold V_P, where P is the image of the moment map $\mu\colon W \to \mathfrak{t}^*$ (whence the name 'Delzant polytope').

Example 5.7. Consider the case $m - n = 1$, that is, T_Γ is 1-dimensional and $\gamma_k \in \mathbb{R}$. By Theorem 5.3 (a) the moment map μ_Γ is proper whenever each of its level sets

$$
\mu_\Gamma^{-1}(\delta) = \{\mathbf{z} \in \mathbb{C}^m \colon \gamma_1|z_1|^2 + \dots + \gamma_m|z_m|^2 = \delta\}
$$

is bounded. By Theorem 5.3 (b), δ is a regular value whenever the quadratic hypersurface $\gamma_1|z_1|^2 + \dots + \gamma_m|z_m|^2 = \delta$ is non-empty and non-degenerate. These two conditions together imply that the hypersurface is an ellipsoid, and the associated polyhedron is an n-simplex (see Example 3.6). By Lemma 5.4 the T_Γ-action on $\mu_\Gamma^{-1}(\delta)$ is free if and only if $L_{\mathbf{z}} = L$ for any $\mathbf{z} \in \mu_\Gamma^{-1}(\delta)$. This means that each γ_k generates the same lattice as the whole set $\gamma_1, \dots, \gamma_m$, which implies that $\gamma_1 = \dots = \gamma_m$. The Gale dual configuration satisfies $\mathbf{a}_1 + \dots + \mathbf{a}_m = \mathbf{0}$. Then T_Γ is the diagonal circle in $\mathbb{T}^m$, the hypersurface $\mu_\Gamma^{-1}(\delta) = \mathscr{Z}_P$ is a sphere, and the associated polytope P is a standard simplex up to a shift and a multiplication by a positive factor δ. The Hamiltonian toric manifold $V_P = \mathscr{Z}_P/T_\Gamma$ is the complex projective space $\mathbb{C}P^n$.

6. Fans and toric varieties

A toric variety is a normal algebraic variety on which an *algebraic torus* $(\mathbb{C}^\times)^n$ acts with a dense (Zariski open) orbit. Toric varieties are described by combinatorial-geometric objects, rational fans.

A toric variety can be defined from a rational fan by using an algebraic version of symplectic reduction, also known as the 'Cox construction'. Different versions of this construction have appeared in the work of several authors since the early 1990s. In our exposition we mainly follow the paper [18] of Cox (and the modernized version [19], Chap. 5); the relationships between toric varieties and moment-angle manifolds will be explored further in the next sections.

6.1. Cones and fans. A set of vectors $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathbb{R}^n$ defines a *convex polyhedral cone* or simply *cone*

$$\sigma = \mathbb{R}_{\geqslant}\langle \mathbf{a}_1, \dots, \mathbf{a}_m \rangle = \{\mu_1 \mathbf{a}_1 + \dots + \mu_k \mathbf{a}_k \colon \mu_i \in \mathbb{R}_{\geqslant}\}.$$

Here $\mathbf{a}_1, \dots \mathbf{a}_k$ are called *generating vectors* (or *generators*) of σ. A *minimal* set of generators of a cone is defined up to multiplication of vectors by positive constants. A cone is *rational* if its generators can be chosen from the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. If σ is a rational cone, then its generators $\mathbf{a}_1, \dots \mathbf{a}_k$ are usually chosen to be *primitive*, that is, each $\mathbf{a}_i$ is the smallest lattice vector in the ray defined by it.

A cone is *strongly convex* if it does not contain a line. A cone is *simplicial* if it is generated by part of a basis of $\mathbb{R}^n$, and is *regular* if it is generated by part of a basis of $\mathbb{Z}^n$.

Any cone σ is an (unbounded) polyhedron, and *faces* of σ are defined as its intersections with supporting hyperplanes. Each face of a cone is itself a cone. If a cone is strongly convex, then it has a unique vertex $\mathbf{0}$; otherwise there are no vertices. A minimal generator set of a cone consists of non-zero vectors along its edges.

A *fan* is a finite collection $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ of strongly convex cones in some space $\mathbb{R}^n$ such that every face of a cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each. A fan Σ is *rational* (respectively, *simplicial* or *regular*) if every cone in Σ is rational (respectively, simplicial or regular). A fan $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ is said to be *complete* if $\sigma_1 \cup \dots \cup \sigma_s = \mathbb{R}^n$.

Cones in a fan can be separated by hyperplanes.

Lemma 6.1 (separation lemma). *Let σ and σ' be two cones whose intersection τ is a face of each. Then there exists a common supporting hyperplane H for σ and σ' such that*

$$\tau = \sigma \cap H = \sigma' \cap H.$$

For the proof, see, for instance, [29], § 1.2. Remarkably, this convex-geometrical separation property translates into topological separation (Hausdorffness) of algebraic varieties and topological spaces constructed from fans as described below.

Given a simplicial fan Σ with m edges generated by vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$, we define its *underlying simplicial complex* $\mathscr{K}_\Sigma$ on $[m] = \{1, \dots, m\}$ as the collection of subsets $I \subset [m]$ such that $\{\mathbf{a}_i \colon i \in I\}$ spans a cone in Σ.

A simplicial fan Σ in $\mathbb{R}^n$ is therefore determined by two pieces of data:

- a simplicial complex $\mathscr{K}$ on $[m]$;
- a configuration of vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ in $\mathbb{R}^n$ such that the subset $\{\mathbf{a}_i \colon i \in I\}$ is linearly independent for any simplex $I \in \mathscr{K}$.

Then for each $I \in \mathscr{K}$ we can define the simplicial cone σ_I spanned by the vectors $\mathbf{a}_i$ with $i \in I$. The 'bunch of cones' $\{\sigma_I \colon I \in \mathscr{K}\}$ patches into a fan Σ whenever any two cones σ_I and σ_J intersect in a common face (which has to be $\sigma_{I\cap J}$). Equivalently, the relative interiors of the cones σ_I are pairwise disjoint. Under this condition, we say that the data $\{\mathscr{K};\mathbf{a}_1,\dots,\mathbf{a}_m\}$ *define the fan* Σ.

The next construction assigns a complete fan to every convex polytope.

Construction 6.2 (normal fan). Let P be a polytope (2.1) with m facets $F_1,\dots,F_m$ and normal vectors $\mathbf{a}_1,\dots,\mathbf{a}_m$. Given a face $Q\subset P$, we say that a vector $\mathbf{a}_i$ is *normal* to Q if $Q\subset F_i$. Define the *normal* cone σ_Q as the cone generated by those $\mathbf{a}_i$ which are normal to Q. It can be given by

$$\sigma_Q=\{\mathbf{u}\in\mathbb{R}^n\colon \langle\mathbf{u},\mathbf{x}'\rangle\leqslant\langle\mathbf{u},\mathbf{x}\rangle \text{ for all } \mathbf{x}'\in Q \text{ and } \mathbf{x}\in P\}.$$

Then

$$\Sigma_P=\{\sigma_Q\colon Q\ \text{ is a face in } P\}\cup\{\mathbf{0}\}$$

is a complete fan which is called the *normal fan* of the polytope P. If $\mathbf{0}$ is contained in the interior of P, then Σ_P may also be described as the set of cones over the faces of the polar polytope P^*.

The normal fan Σ_P is simplicial if and only if P is simple. In this case the cones in Σ_P are generated by those sets $\{\mathbf{a}_{i_1},\dots,\mathbf{a}_{i_k}\}$ for which the intersection $F_{i_1}\cap\dots\cap F_{i_k}$ is non-empty. The underlying simplicial complex $\mathscr{K}_{\Sigma_P}$ is the boundary of the polar simplicial polytope P^*.

The normal fan Σ_P of a polytope P contains information about the normals to the facets (the generators $\mathbf{a}_i$ of the edges of Σ_P) and the combinatorial structure of P (which sets of vectors $\mathbf{a}_i$ span a cone in Σ_P is determined by which facets intersect at a face), however, the scalars b_i in (2.1) are lost. Not every complete fan can be obtained by 'forgetting the numbers b_i' in a presentation of a polytope by inequalities, that is, not every complete fan is a normal fan. This fails even for regular fans in $\mathbb{R}^3$ (see [29], § 1.5 for an example). Moreover, complete simplicial fans and simplicial polytopes differ even as combinatorial objects: there are complete simplicial fans Σ whose underlying simplicial complex $\mathscr{K}_\Sigma$ cannot be obtained as the boundary of some simplicial polytope (although no regular examples of this sort are known).

6.2. Toric varieties. An *algebraic torus* is a commutative complex algebraic group isomorphic to a product $(\mathbb{C}^\times)^n$ of copies of the multiplicative group $\mathbb{C}^\times=\mathbb{C}\setminus\{0\}$. It contains a compact torus T^n as a Lie (but not algebraic) subgroup.

We shall often identify an algebraic torus with the standard model $(\mathbb{C}^\times)^n$.

A *toric variety* is a normal complex algebraic variety V containing an algebraic torus $(\mathbb{C}^\times)^n$ as a Zariski open subset in such a way that the natural action of $(\mathbb{C}^\times)^n$ on itself extends to an action on V.

It follows that $(\mathbb{C}^\times)^n$ acts on V with a dense orbit.

The algebraic geometry of toric varieties translates completely into the language of combinatorial and convex geometry. Namely, there is a bijective correspondence between rational fans in an n-dimensional space and complex n-dimensional toric varieties. Under this correspondence,

$$\begin{aligned}
\text{cones} &\longleftrightarrow \text{affine varieties,}\\
\text{complete fans} &\longleftrightarrow \text{compact (complete) varieties,}\\
\text{normal fans of polytopes} &\longleftrightarrow \text{projective varieties,}\\
\text{regular fans} &\longleftrightarrow \text{non-singular varieties,}\\
\text{simplicial fans} &\longleftrightarrow \text{orbifolds.}
\end{aligned}$$

The details of this classical correspondence can be found in any standard source on toric geometry, for instance, [20], [29], or [19]. Along with the classical construction, there is an alternative way to define a toric variety: as the quotient of an open subset in $\mathbb{C}^m$ (the complement of a coordinate subspace arrangement) by an action of a commutative algebraic group (a product of an algebraic torus and a finite group).

6.3. Quotients in algebraic geometry. Taking quotients of algebraic varieties by algebraic group actions is tricky for both topological and algebraic reasons. First, since algebraic groups are non-compact (as algebraic tori), their orbits may be not closed, and the quotients may be non-Hausdorff. Second, even if the quotient is Hausdorff as a topological space, it may fail to be an algebraic variety. This may be remedied to some extent by the notion of the categorical quotient.

Let X be an algebraic variety with an action of an affine algebraic group G. An algebraic variety Y is said to be a *categorical quotient* of X by the action of G if there exists a morphism $\pi\colon X \to Y$ which is constant on G-orbits of X and has the following universal property: for any morphism $\varphi\colon X \to Z$ which is constant on G-orbits, there is a unique morphism $\widehat{\varphi}\colon Y \to Z$ such that $\widehat{\varphi} \circ \pi = \varphi$. This is described by the diagram

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & Z \\
& Y \arrow[ur, dashed, "\widehat{\varphi}"'] &
\end{tikzcd}$$

The categorical quotient Y is unique up to isomorphism, and we shall denote it by $X/\!\!/G$ (although sometimes this notation is reserved for categorical quotients with additional nice properties).

Assume that $X = \operatorname{Spec} A$ is an affine variety, where $A = \mathbb{C}[X]$ is the algebra of regular functions on X and G is an algebraic torus (in fact, this construction works for any *reductive* affine algebraic group). Then the subalgebra $\mathbb{C}[X]^G$ of G-invariant functions (that is, functions f satisfying $f(gx) = f(x)$ for any $g \in G$ and $x \in X$) is finitely generated, and the corresponding affine variety $\operatorname{Spec}\mathbb{C}[X]^G$ is the categorical quotient $X/\!\!/G$. The quotient morphism $\pi\colon X \to X/\!\!/G$ is dual to the inclusion of algebras $\mathbb{C}[X]^G \to \mathbb{C}[X]$. The morphism π is surjective and induces a one-to-one correspondence between points in $X/\!\!/G$ and *closed* G-orbits in X (that is, $\pi^{-1}(x)$ contains a unique closed G-orbit for any $x \in X/\!\!/G$; see [19], Proposition 5.0.7).

Therefore, if all G-orbits of an affine variety X are closed, then the categorical quotient $X/\!\!/G$ is identified as a topological space with the ordinary 'topological' quotient X/G. In algebraic geometry quotients of this type are said to be *geometric* and also denoted by X/G.

Example 6.3. Let $\mathbb{C}^\times$ act on $\mathbb{C} = \operatorname{Spec}(\mathbb{C}[z])$ by scalar multiplication. There are two orbits: the closed orbit 0 and the open orbit $\mathbb{C}^\times$. The topological quotient $\mathbb{C}/\mathbb{C}^\times$ consists of two points, one of which is not closed, so the space is not Hausdorff.

On the other hand, the categorical quotient $\mathbb{C}/\!\!/\mathbb{C}^\times = \operatorname{Spec}(\mathbb{C}[z]^{\mathbb{C}^\times})$ is a single point, since any $\mathbb{C}^\times$-invariant polynomial is constant (and there is only one closed orbit).

Similarly, if $\mathbb{C}^\times$ acts on $\mathbb{C}^n = \operatorname{Spec}(\mathbb{C}[z_1,\dots,z_n])$ diagonally, then an invariant polynomial satisfies the condition $f(\lambda z_1,\dots,\lambda z_n) = f(z_1,\dots,z_n)$ for all $\lambda\in\mathbb{C}^\times$. Such a polynomial must be constant, so $\mathbb{C}^n/\!\!/\mathbb{C}^\times$ is again a point.

In good cases categorical quotients of more general (non-affine) varieties X may be constructed by 'gluing from pieces' as follows. Assume that G acts on X and $\pi\colon X\to Y$ is a morphism of varieties that is constant on G-orbits. If Y has an open affine cover $Y=\bigcup_\alpha V_\alpha$ such that $\pi^{-1}(V_\alpha)$ is affine and V_α is the categorical quotient (that is, $\pi|_{\pi^{-1}(V_\alpha)}\colon \pi^{-1}(V_\alpha)\to V_\alpha$ is the morphism dual to the inclusion of algebras $\mathbb{C}[\pi^{-1}(V_\alpha)]^G\to\mathbb{C}[\pi^{-1}(V_\alpha)]$) for all α, then Y is the categorical quotient $X/\!\!/G$.

Example 6.4. Let $\mathbb{C}^\times$ act on $\mathbb{C}^2\setminus\{0\}$ diagonally, where $\mathbb{C}^2=\operatorname{Spec}(\mathbb{C}[z_0,z_1])$. We have an open affine cover $\mathbb{C}^2\setminus\{0\}=U_0\cup U_1$, where

$$\begin{aligned}
U_0 &= \mathbb{C}^2\setminus\{z_0=0\} = \mathbb{C}^\times\times\mathbb{C} = \operatorname{Spec}(\mathbb{C}[z_0^{\pm1},z_1]),\\
U_1 &= \mathbb{C}^2\setminus\{z_1=0\} = \mathbb{C}\times\mathbb{C}^\times = \operatorname{Spec}(\mathbb{C}[z_0,z_1^{\pm1}]),\\
U_0\cap U_1 &= \mathbb{C}^2\setminus\{z_0z_1=0\} = \mathbb{C}^\times\times\mathbb{C}^\times = \operatorname{Spec}(\mathbb{C}[z_0^{\pm1},z_1^{\pm1}]).
\end{aligned}$$

The algebras of $\mathbb{C}^\times$-invariant functions are

$$\mathbb{C}[z_0^{\pm1},z_1]^{\mathbb{C}^\times}=\mathbb{C}[z_1/z_0],\quad \mathbb{C}[z_0,z_1^{\pm1}]^{\mathbb{C}^\times}=\mathbb{C}[z_0/z_1],$$
$$\mathbb{C}[z_0^{\pm1},z_1^{\pm1}]^{\mathbb{C}^\times}=\mathbb{C}[(z_1/z_0)^{\pm1}].$$

It follows that the varieties $V_i=U_i/\!\!/\mathbb{C}^\times=\mathbb{C}$ glue together along $V_0\cap V_1=(U_0\cap U_1)/\!\!/\mathbb{C}^\times=\mathbb{C}^\times$ in the standard way to produce the projective line $\mathbb{C}P^1$. We have that all $\mathbb{C}^\times$-orbits are closed in $\mathbb{C}^2\setminus\{0\}$, hence $\mathbb{C}P^1=(\mathbb{C}^2\setminus\{0\})/\mathbb{C}^\times$ is the geometric quotient.

Similarly, $\mathbb{C}P^n=(\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^\times$ is the geometric quotient for the diagonal action of $\mathbb{C}^\times$.

Example 6.5. Now we let $\mathbb{C}^\times$ act on $\mathbb{C}^2\setminus\{0\}$ by $\lambda\cdot(z_0,z_1)=(\lambda z_0,\lambda^{-1}z_1)$. Using the same affine cover of $\mathbb{C}^2\setminus\{0\}$ as in the previous example, we obtain the following algebras of $\mathbb{C}^\times$-invariant functions:

$$\mathbb{C}[z_0^{\pm1},z_1]^{\mathbb{C}^\times}=\mathbb{C}[z_0z_1],\quad \mathbb{C}[z_0,z_1^{\pm1}]^{\mathbb{C}^\times}=\mathbb{C}[z_0z_1],\quad \mathbb{C}[z_0^{\pm1},z_1^{\pm1}]^{\mathbb{C}^\times}=\mathbb{C}[(z_0z_1)^{\pm1}].$$

This time gluing together the varieties $V_i=U_i/\!\!/\mathbb{C}^\times=\mathbb{C}$ along $V_0\cap V_1=(U_0\cap U_1)/\!\!/\mathbb{C}^\times=\mathbb{C}^\times$ gives the space obtained from two copies of $\mathbb{C}$ by identifying all non-zero points. This space is not Hausdorff (the two zeros do not have disjoint neighbourhoods in the usual topology), and therefore it cannot be a categorical quotient, because algebraic varieties are Hausdorff spaces in the usual topology.

A toric variety V_Σ will be described as the categorical (or in good cases, geometric) quotient of the 'total space' $U(\Sigma)$ by an action of a commutative algebraic group G. We now proceed to describe G and $U(\Sigma)$.

6.4. Quotient construction of toric varieties. Following the algebraic tradition, we use the coordinate-free notation here. We fix a lattice N of rank n, and denote by $N_\mathbb{R}$ its ambient n-dimensional real vector space $N \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^n$. We also define the algebraic torus $\mathbb{C}_N^\times = N \otimes_\mathbb{Z} \mathbb{C}^\times \cong (\mathbb{C}^\times)^n$.

Let Σ be a rational fan in $N_\mathbb{R}$ with m edges generated by primitive vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ in N. We shall assume that the linear span of $\mathbf{a}_1, \dots, \mathbf{a}_m$ is the whole of $N_\mathbb{R}$.

We consider the map of lattices $A\colon \mathbb{Z}^m \to N$ sending the ith basis vector of $\mathbb{Z}^m$ to $\mathbf{a}_i \in N$. The corresponding map of algebraic tori

$$A \otimes_\mathbb{Z} \mathbb{C}^\times \colon (\mathbb{C}^\times)^m \to \mathbb{C}_N^\times$$

is surjective. We shall denote this map by $\exp A$.

Define the group $G = G_\Sigma$ as the kernel of the map $\exp A$. We therefore have an exact sequence of Abelian algebraic groups

$$1 \to G \to (\mathbb{C}^\times)^m \xrightarrow{\exp A} \mathbb{C}_N^\times \to 1. \tag{6.1}$$

Explicitly, G is given by

$$G = \biggl\{(z_1, \dots, z_m) \in (\mathbb{C}^\times)^m \colon \prod_{i=1}^m z_i^{\langle \mathbf{a}_i, \mathbf{u}\rangle} = 1 \text{ for all } \mathbf{u} \in N^*\biggr\}. \tag{6.2}$$

The group G is isomorphic to the product of $(\mathbb{C}^\times)^{m-n}$ and a finite Abelian group. If Σ is a regular fan with at least one n-dimensional cone, then $G \cong (\mathbb{C}^\times)^{m-n}$.

Given a cone $\sigma \in \Sigma$, we set $g(\sigma) = \{i_1, \dots, i_k\} \subset [m]$ if σ is spanned by $\mathbf{a}_{i_1} \dots, \mathbf{a}_{i_k}$. We define the simplicial complex $\mathscr{K}_\Sigma$ generated by all the subsets $g(\sigma) \subset [m]$:

$$\mathscr{K}_\Sigma = \{I \colon I \subset g(\sigma) \text{ for some } \sigma \in \Sigma\}.$$

If Σ is a simplicial fan, then each $I \subset g(\sigma)$ is $g(\tau)$ for some $\tau \in \Sigma$, and we obtain the 'underlying simplicial complex' of Σ defined in the beginning of this section. If Σ is the normal fan of a non-simple polytope P (that is, the fan over the faces of the polar polytope P^*), then $\mathscr{K}_\Sigma$ is obtained by replacing each face of ∂P^* by a simplex with the same set of vertices.

We now define the space $U(\Sigma)$ as the complement of the arrangement of coordinate subspaces in $\mathbb{C}^m$ determined by $\mathscr{K}_\Sigma$:

$$U(\Sigma) = \mathbb{C}^m \setminus \bigcup_{\{i_1, \dots, i_k\} \notin \mathscr{K}_\Sigma} \{\mathbf{z} \in \mathbb{C}^m \colon z_{i_1} = \dots = z_{i_k} = 0\}. \tag{6.3}$$

We observe that the subset $U(\Sigma) \subset \mathbb{C}^m$ depends only on the combinatorial structure of the fan Σ, while the subgroup $G \subset (\mathbb{C}^\times)^m$ depends on the geometric data, namely, the primitive generators of one-dimensional cones.

Since $U(\Sigma) \subset \mathbb{C}^m$ is invariant under the coordinatewise action of $(\mathbb{C}^\times)^m$, we obtain a G-action on $U(\Sigma)$ by restriction.

Theorem 6.6 (Cox [18], Theorem 2.1). *If the linear span of the one-dimensional cones in Σ is the whole space $N_{\mathbb{R}}$, then*

(a) *the toric variety V_Σ is naturally isomorphic to the categorical quotient $U(\Sigma)/\!\!/G$,*

(b) *V_Σ is the geometric quotient $U(\Sigma)/G$ if and only if the fan Σ is simplicial.*

The torus acting on $V_\Sigma = U(\Sigma)/\!\!/G$ is the quotient torus $\mathbb{C}^\times_N = (\mathbb{C}^\times)^m/G$.

Proposition 6.7. (a) *If Σ is a simplicial fan, then the G-action on $U(\Sigma)$ is almost free*;

(b) *If Σ is regular, then the G-action on $U(\Sigma)$ is free.*

Proof. The stabilizer of a point $\mathbf{z} \in \mathbb{C}^m$ under the action of $(\mathbb{C}^\times)^m$ is

$$(\mathbb{C}^\times)^{\omega(\mathbf{z})} = \{(t_1, \dots, t_m) \in (\mathbb{C}^\times)^m \colon t_i = 1 \text{ if } z_i \neq 0\},$$

where $\omega(\mathbf{z})$ is the set of zero coordinates of $\mathbf{z}$. The stabilizer of $\mathbf{z}$ under the G-action is $G_{\mathbf{z}} = (\mathbb{C}^\times)^{\omega(\mathbf{z})} \cap G$. Since G is the kernel of the map $\exp A \colon (\mathbb{C}^\times)^m \to \mathbb{C}^\times_N$ induced by the map of lattices $\mathbb{Z}^m \to N$, the subgroup $G_{\mathbf{z}}$ is the kernel of the composite map

$$(\mathbb{C}^\times)^{\omega(\mathbf{z})} \hookrightarrow (\mathbb{C}^\times)^m \xrightarrow{\exp A} \mathbb{C}^\times_N. \tag{6.4}$$

This homomorphism of tori is induced by the map of lattices $\mathbb{Z}^{\omega(\mathbf{z})} \to \mathbb{Z}^m \to N$, where $\mathbb{Z}^{\omega(\mathbf{z})} \to \mathbb{Z}^m$ is the inclusion of a coordinate sublattice.

Now let Σ be a simplicial fan and $\mathbf{z} \in U(\Sigma)$. Then $\omega(\mathbf{z}) = g(\sigma)$ for some cone $\sigma \in \Sigma$. Therefore, the set of primitive generators $\{\mathbf{a}_i \colon i \in \omega(\mathbf{z})\}$ is linearly independent. Hence, the map $\mathbb{Z}^{\omega(\mathbf{z})} \to \mathbb{Z}^m \to N$ taking $\mathbf{e}_i$ to $\mathbf{a}_i$ is a monomorphism, which implies that the kernel of (6.4) is a finite group.

If the fan Σ is regular, then $\{\mathbf{a}_i \colon i \in \omega(\mathbf{z})\}$ is part of a basis of N. In this case (6.4) is a monomorphism and $G_{\mathbf{z}} = \{1\}$. $\square$

The relationship between the algebraic quotient construction of V_Σ and the symplectic reduction construction of V_P (described in the previous section) is as follows. Let P be a Delzant polytope given by (2.1). Then the Delzant condition means exactly that the normal fan Σ_P is regular. The tori in the exact sequence (5.4) are maximal compact subgroups of the algebraic tori in (6.1). Also, it follows from Proposition 3.2 that the level set $\mu_\Gamma^{-1}(\Gamma\mathbf{b})$ (the moment-angle manifold $\mathscr{Z}_P$) is contained in $U(\Sigma_P)$.

Theorem 6.8. *Let P be a Delzant polytope with the normal fan Σ_P. Let V_P be the corresponding Hamiltonian toric manifold, and let V_{Σ_P} be the corresponding non-singular projective toric variety. The inclusion $\mathscr{Z}_P \subset U(\Sigma_P)$ induces a diffeomorphism*

$$V_P = \mathscr{Z}_P/T_\Gamma \xrightarrow{\cong} U(\Sigma_P)/G = V_{\Sigma_P}.$$

Therefore, any non-singular projective toric variety can be obtained as the symplectic quotient of $\mathbb{C}^m$ by an action of an $(m-n)$-torus.

A proof can be found in [3], Proposition VI.3.1.1 or in [33], Appendix 2; we shall also give a proof of a more general statement in § 10.

Remark. Projective embeddings of V_{Σ_P} correspond to *lattice* Delzant polytopes P, that is, Delzant polytopes with vertices in the lattice N. Any such embedding defines a symplectic structure on V_{Σ_P} by inducing the symplectic form from the projective space. It can be shown ([33], Appendix 2) that the diffeomorphism in Theorem 6.8 above preserves the cohomology class of the symplectic form, or equivalently, the two symplectic structures are T_N-equivariantly symplectomorphic.

Example 6.9. Let V_σ be the affine toric variety corresponding to an n-dimensional simplicial cone σ. We may write $V_\sigma = V_\Sigma$, where Σ is the simplicial fan consisting of all faces of σ. Then $m = n$, $U(\Sigma) = \mathbb{C}^n$, and $A\colon \mathbb{Z}^n \to N$ is the monomorphism onto the full-rank sublattice generated by $\mathbf{a}_1, \dots, \mathbf{a}_n$. Therefore, G is a finite group and $V_\sigma = \mathbb{C}^n/G = \operatorname{Spec}\mathbb{C}[z_1, \dots, z_n]^G$.

In particular, if we consider the cone σ generated by $2\mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{e}_2$ in $\mathbb{R}^2$, then G is $\mathbb{Z}_2$, embedded as $\{(1,1), (-1,-1)\}$ in $(\mathbb{C}^\times)^2$. The quotient construction realizes the quadratic cone

$$V_\sigma = \operatorname{Spec}\mathbb{C}[z_1, z_2]^G = \operatorname{Spec}\mathbb{C}[z_1^2, z_1z_2, z_2^2] = \{(u,v,w) \in \mathbb{C}^3 \colon v^2 = uw\}$$

as a quotient of $\mathbb{C}^2$ by $\mathbb{Z}_2$.

Example 6.10. Let Σ be the complete fan in $\mathbb{R}^2$ with the three maximal cones $\sigma_0 = \mathbb{R}_\geqslant(\mathbf{e}_1, \mathbf{e}_2)$, $\sigma_1 = \mathbb{R}_\geqslant(\mathbf{e}_2, -\mathbf{e}_1 - \mathbf{e}_2)$, and $\sigma_2 = \mathbb{R}_\geqslant(-\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_1)$. Then $\mathscr{K}_\Sigma$ is the boundary of a triangle, so the only non-simplex is $\{1,2,3\}$. Hence,

$$U(\Sigma) = \mathbb{C}^3 \setminus \{z_1 = z_2 = z_3 = 0\} = \mathbb{C}^3 \setminus \{\mathbf{0}\}.$$

The subgroup G defined by (6.2), is the diagonal $\mathbb{C}^\times$ in $(\mathbb{C}^\times)^3$. We therefore have $V_\Sigma = U(\Sigma)/G = \mathbb{C}P^2$. Since Σ is the normal fan of the standard 2-simplex, this agrees with the symplectic quotient $V_P = \mathscr{Z}_P/T_\Gamma$ in Example 5.7.

Example 6.11. Consider the fan Σ in $\mathbb{R}^2$ with three one-dimensional cones generated by the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $-\mathbf{e}_1 - \mathbf{e}_2$. This fan is not complete, but its one-dimensional cones span $\mathbb{R}^2$, so we may apply Theorem 6.6. The simplicial complex $\mathscr{K}_\Sigma$ consists of three separate points. The space $U(\Sigma)$ is the complement of the three coordinate lines in $\mathbb{C}^3$:

$$U(\Sigma) = \mathbb{C}^3 \setminus \bigl(\{z_1 = z_2 = 0\} \cup \{z_1 = z_3 = 0\} \cup \{z_2 = z_3 = 0\}\bigr).$$

The group G is the diagonal $\mathbb{C}^\times$ in $(\mathbb{C}^\times)^3$. Hence $V_\Sigma = U(\Sigma)/G$ is a quasi-projective variety obtained by removing three points from $\mathbb{C}P^2$.

7. Moment-angle complexes and polyhedral products

For any simple polytope $P = P(A, \mathbf{b})$ given by (2.1), we defined the moment-angle manifold $\mathscr{Z}_P = \mathscr{Z}_{A,\mathbf{b}}$ in the diagram (3.1), or equivalently, as the intersection of quadrics given by (3.2). Here, using a combinatorial decomposition of P into cubes, we represent $\mathscr{Z}_P$ as a union of products $(D^2)^I \times (S^1)^{[m]\setminus I}$ of disks and circles parametrized by simplices I in the associated simplicial complex $\mathscr{K}_P = \partial P^*$. This construction may be generalized to arbitrary simplicial complexes $\mathscr{K}_P$, leading to the notion of a *moment-angle complex* $\mathscr{Z}_\mathscr{K}$. We follow [13] (and the more detailed treatment given in [14]) in our description of moment-angle complexes.

The basic building block in the 'moment-angle' decomposition of $\mathscr{Z}_{\mathscr{K}}$ is the pair (D^2, S^1) consisting of a unit disk and circle, and the whole construction can be extended naturally to arbitrary pairs of spaces (X, A). The resulting complex $(X, A)^{\mathscr{K}}$ is now known as the 'polyhedral product space' over a simplicial complex $\mathscr{K}$; this terminology was suggested by William Browder (cf. [4]). Many spaces important for toric topology admit polyhedral product decompositions.

The construction of $\mathscr{Z}_{\mathscr{K}}$ and its generalization $(X, A)^{\mathscr{K}}$ is of a truly universal nature, and has remarkable functorial properties. The most basic of these is that the construction of $\mathscr{Z}_{\mathscr{K}}$ establishes a functor from simplicial complexes and simplicial maps to spaces with torus actions and equivariant maps. If $\mathscr{K}$ is a triangulated sphere, then $\mathscr{Z}_{\mathscr{K}}$ is a manifold, and most important geometric examples of $\mathscr{Z}_{\mathscr{K}}$ arise in this way.

Another important aspect of the theory of moment-angle complexes is their connection to coordinate subspace arrangements and their complements. These have appeared as the 'total spaces' $U(\Sigma)$ in the algebraic quotient construction of toric varieties reviewed in the previous section. Subspace arrangements and their complements have also played an important role in singularity theory, and, more recently, in the theory of linkages and robotic motion planning. Arrangements of coordinate subspaces in $\mathbb{C}^m$ correspond bijectively to simplicial complexes $\mathscr{K}$ on the set $[m]$, and the complement of such an arrangement is homotopy equivalent to the corresponding moment-angle complex $\mathscr{Z}_{\mathscr{K}}$ (see [13], Theorem 5.2.5 and Theorem 7.12 below).

7.1. Cubical decompositions.

Construction 7.1 (cubical subdivision of a simple polytope). Let P be a simple n-polytope with m facets $F_1, \dots, F_m$. We shall construct a piecewise linear embedding of P into the standard unit cube $\mathbb{I}^m \subset \mathbb{R}^m_{\geqslant}$, thereby inducing a cubical subdivision $\mathscr{C}(P)$ of P by the pre-images of faces of $\mathbb{I}^m$.

Denote by $\mathscr{S}$ the set of barycentres of all faces of P, including the vertices and the barycentre of the whole polytope. This will be the vertex set of $\mathscr{C}(P)$. Every $(n-k)$-face G of P is an intersection of k facets: $G = F_{i_1} \cap \dots \cap F_{i_k}$. We map the barycentre of G to $(\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{I}^m$, where $\varepsilon_i = 0$ if $i \in \{i_1, \dots, i_k\}$ and $\varepsilon_i = 1$ otherwise. The resulting map $\mathscr{S} \to \mathbb{I}^m$ can be extended linearly on the simplices of the barycentric subdivision of P to an embedding $c_P \colon P \to \mathbb{I}^m$. The case $n = 2$, $m = 3$ is shown in Fig. 7.1.

Any face of $\mathbb{I}^m$ has the form

$$C_{J \subset I} = \{(y_1, \dots, y_m) \in \mathbb{I}^m \colon y_j = 0 \text{ for } j \in J,\ y_j = 1 \text{ for } j \notin I\},$$

where $J \subset I$ is a pair of embedded (possibly empty) subsets of $[m]$. We also set

$$C_I = C_{\varnothing \subset I} = \{(y_1, \dots, y_m) \in \mathbb{I}^m \colon y_j = 1 \text{ for } j \notin I\}$$

to simplify the notation.

The image $c_P(P) \subset \mathbb{I}^m$ is the union of all faces $C_{J \subset I}$ such that $\bigcap_{i \in I} F_i \neq \varnothing$. For each such face $C_{J \subset I}$, the pre-image $c_P^{-1}(C_{J \subset I})$ is a face of the cubical complex $\mathscr{C}(P)$. The vertex set of $c_P^{-1}(C_{J \subset I})$ is the subset of $\mathscr{S}$ consisting of barycentres of all faces between the faces G and H of P, where $G = \bigcap_{j \in J} F_j$ and $H = \bigcap_{i \in I} F_i$. Therefore,

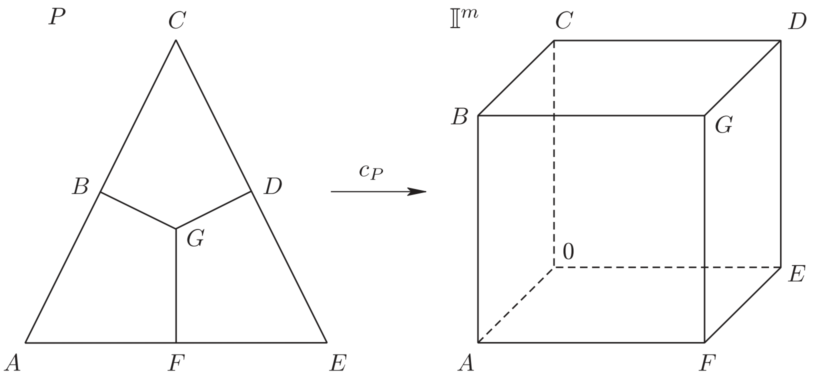


Figure 7.1. The embedding $c_P\colon P \to \mathbb{I}^m$ for $n = 2$, $m = 3$

faces in $\mathscr{C}(P)$ correspond to pairs of embedded faces $G \supset H$ of P, and we denote them by $C_{G\supset H}$. In particular, maximal (n-dimensional) faces in $\mathscr{C}(P)$ correspond to pairs $G = P$, $H = v$, where v is a vertex of P. For these maximal faces we use the abbreviated notation $C_v = C_{P\supset v}$.

For every vertex $v = F_{i_1} \cap \cdots \cap F_{i_n} \in P$ with $I_v = \{i_1, \dots, i_n\}$ we have

$$c_P(C_v) = C_{I_v} = \{(y_1, \dots, y_m) \in \mathbb{I}^m\colon y_j = 1 \text{ for } v \notin F_j\}. \tag{7.1}$$

We therefore obtain the following result.

Proposition 7.2. *A simple polytope P with m facets admits a cubical decomposition whose maximal faces C_v correspond to the vertices $v \in P$. The resulting cubical complex $\mathscr{C}(P)$ embeds canonically into $\mathbb{I}^m$ as described by* (7.1)*.*

7.2. Moment-angle complexes. The map $\mu\colon \mathbb{C}^m \to \mathbb{R}^m_{\geqslant}$ (see Example 5.1) identifies the unit cube $\mathbb{I}^m \subset \mathbb{R}^m_{\geqslant}$ with the quotient of the unit *polydisk*

$$\mathbb{D}^m = \bigl\{(z_1, \dots, z_m) \in \mathbb{C}^m\colon |z_i| \leqslant 1\bigr\}$$

by the coordinatewise action of $\mathbb{T}^m$.

We now define the space $\widetilde{\mathscr{Z}_P}$ from a diagram similar to (3.1) (which was used to define $\mathscr{Z}_P = \mathscr{Z}_{A,\mathbf{b}}$), in which the bottom map is replaced by $c_P\colon P \to \mathbb{I}^m$:

$$\begin{array}{ccc} \widetilde{\mathscr{Z}_P} & \xrightarrow{\ \widetilde{i}_Z\ } & \mathbb{D}^m \\ \Big\downarrow & & \Big\downarrow{\scriptstyle\mu} \\ P & \xrightarrow{\ c_P\ } & \mathbb{I}^m \end{array} \tag{7.2}$$

Proposition 7.3. *The space $\widetilde{\mathscr{Z}_P}$ is $\mathbb{T}^m$-equivariantly homeomorphic to the moment-angle manifold $\mathscr{Z}_P$.*

Proof. As we have seen in Proposition 4.2, $\mathscr{Z}_P$ is $\mathbb{T}^m$-equivariantly homeomorphic to the identification space

$$P \times \mathbb{T}^m/{\sim}, \quad \text{where } (\mathbf{x}, \mathbf{t}_1) \sim (\mathbf{x}, \mathbf{t}_2) \text{ if } \mathbf{t}_1^{-1}\mathbf{t}_2 \in \mathbb{T}^{I_{\mathbf{x}}}.$$

By restricting (4.1) to $\mathbb{D}^m \subset \mathbb{C}^m$ we get that

$$\mathbb{D}^m \cong \mathbb{I}^m \times \mathbb{T}^m/\sim, \qquad \text{where } (\mathbf{y}, \mathbf{t}_1) \sim (\mathbf{y}, \mathbf{t}_2) \text{ for } \mathbf{t}_1^{-1}\mathbf{t}_2 \in \mathbb{T}^{\omega(\mathbf{y})}.$$

As in the proof of Proposition 4.2, $\widetilde{\mathscr{Z}_P}$ is identified with $c_P(P) \times \mathbb{T}^m/\sim$. A point $\mathbf{x} \in P$ is mapped by c_P to the point $\mathbf{y} \in \mathbb{I}^m$ with $I_{\mathbf{x}} = \omega(\mathbf{y}) = \{i \in [m] \colon \mathbf{x} \in F_i\}$. We thus find that both $\mathscr{Z}_P$ and $\widetilde{\mathscr{Z}_P}$ are $\mathbb{T}^m$-equivariantly homeomorphic to $P \times \mathbb{T}^m/\sim$. $\square$

We shall therefore not distinguish between the spaces $\mathscr{Z}_P$ and $\widetilde{\mathscr{Z}_P}$, and we think of the maps i_Z and $\widetilde{i}_Z$ in the diagrams (3.1) and (7.2) as different embeddings of the same manifold $\mathscr{Z}_P$ in $\mathbb{C}^m$ (the first one is smooth but the second is not).

Given a vertex $v = F_{i_1} \cap \dots \cap F_{i_n} \in P$, we consider the restriction of the map $\widetilde{i}_Z \colon \mathscr{Z}_P \to \mathbb{D}^m$ to the subset $C_v \times \mathbb{T}^m/\sim \subset P \times \mathbb{T}^m/\sim = \mathscr{Z}_P$:

$$\begin{aligned}\widetilde{i}_Z(C_v \times \mathbb{T}^m/\sim) &= c_P(C_v) \times \mathbb{T}^m/\sim = C_{I_v} \times \mathbb{T}^m/\sim = \mu^{-1}(C_{I_v}) \\ &= \{(z_1, \dots, z_m) \in \mathbb{D}^m \colon |z_j|^2 = 1 \text{ for } v \notin F_j\}.\end{aligned}$$

Since $P = \bigcup_v C_v$, we get that

$$\widetilde{i}_Z(\mathscr{Z}_P) = \bigcup_v \mu^{-1}(C_{I_v}).$$

Note that $\mu^{-1}(C_{I_v})$ is a product of $|I_v| = n$ disks and $m-n$ circles. Since $\mu^{-1}(C_I) \cap \mu^{-1}(C_J) = \mu^{-1}(C_{I \cap J})$ for any $I, J \subset [m]$, we can rewrite the union above as

$$\widetilde{i}_Z(\mathscr{Z}_P) = \bigcup_{I \in \mathscr{K}_P} \mu^{-1}(C_I), \tag{7.3}$$

where

$$\mathscr{K}_P = \{I = \{i_1, \dots, i_k\} \subset [m] \colon F_{i_1} \cap \dots \cap F_{i_k} \neq \varnothing\}$$

is the boundary of the polar simplicial polytope P^*.

The decomposition (7.3) of $\mathscr{Z}_P$ into a union of products of disks and circles can now be generalized to an arbitrary simplicial complex.

Definition 7.4. Let $\mathscr{K}$ be a simplicial complex on the set $[m]$. We always assume that $\varnothing \in \mathscr{K}$. The *moment-angle complex* corresponding to $\mathscr{K}$ is defined as

$$\mathscr{Z}_{\mathscr{K}} = \bigcup_{I \in \mathscr{K}} B_I, \tag{7.4}$$

where

$$B_I = \mu^{-1}(C_I) = \{(z_1, \dots, z_m) \in \mathbb{D}^m \colon |z_j|^2 = 1 \text{ for } j \notin I\},$$

and the union in (7.4) is understood as a union of subsets inside the polydisk $\mathbb{D}^m$. Topologically, each B_I is a product of $|I|$ disks D^2 and $m - |I|$ circles S^1. We therefore may rewrite (7.4) as the following decomposition of $\mathscr{Z}_{\mathscr{K}}$ into a union of products of disks and circles:

$$\mathscr{Z}_{\mathscr{K}} = \bigcup_{I \in \mathscr{K}} \biggl(\prod_{i \in I} D^2 \times \prod_{i \notin I} S^1\biggr). \tag{7.5}$$

From now on we shall denote the space B_I by $(D^2, S^1)^I$.

We can rephrase (7.3) by saying that the map $\widetilde{i}_Z\colon \mathscr{Z}_P \to \mathbb{D}^m$ identifies the moment-angle manifold $\mathscr{Z}_P$ with the moment-angle complex $\mathscr{Z}_{\mathscr{K}_P}$ corresponding to $\mathscr{K}_P = \partial P^*$.

A *ghost vertex* of $\mathscr{K}$ is a one-element subset $\{i\} \in [m]$ which is not in $\mathscr{K}$ (that is, is not a vertex). Since facets of a simple polytope P correspond to vertices of $\mathscr{K}_P$, it is natural to add a ghost vertex to $\mathscr{K}_P$ for each redundant inequality in a generic presentation (2.1).

Example 7.5. 1. Let $\mathscr{K} = \Delta^{m-1}$ be the full simplex (the simplicial complex consisting of all subsets of $[m]$). Then $\mathscr{Z}_{\mathscr{K}} = \mathbb{D}^m$.

2. Let $\mathscr{K}$ be a simplicial complex on $[m]$, and let $\mathscr{K}^\circ$ be the complex on $[m+1]$ obtained by adding one ghost vertex $\circ = \{m+1\}$ to $\mathscr{K}$. Then in the decomposition (7.4) for $\mathscr{Z}_{\mathscr{K}^\circ}$ each B_I has the factor S^1 in the last coordinate, and

$$\mathscr{Z}_{\mathscr{K}^\circ} = \mathscr{Z}_{\mathscr{K}} \times S^1.$$

In the case $\mathscr{K} = \mathscr{K}_P$ this agrees with Proposition 4.3 (b).

In particular, if $\mathscr{K}$ is the 'empty' simplicial complex on $[m]$, consisting solely of the empty simplex $\varnothing$, then $\mathscr{Z}_{\mathscr{K}} = \mu^{-1}(1,\dots,1) = \mathbb{T}^m$ is the standard m-torus.

For an arbitrary complex $\mathscr{K}$ on $[m]$, the moment-angle complex $\mathscr{Z}_{\mathscr{K}}$ contains the m-torus $\mathbb{T}^m$ (corresponding to $\mathscr{K} = \varnothing$) and is contained in the polydisk $\mathbb{D}^m$ (corresponding to $\mathscr{K} = \Delta^{m-1}$).

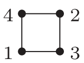
3. Let $\mathscr{K}$ be the complex consisting of two separate points. Then

$$\mathscr{Z}_{\mathscr{K}} = (D^2 \times S^1) \cup (S^1 \times D^2) = \partial(D^2 \times D^2) \cong S^3$$

is the standard decomposition of a 3-sphere into the union of two solid tori.

4. More generally, if $\mathscr{K} = \partial\Delta^{m-1}$ (the boundary of a simplex), then

$$\begin{aligned}\mathscr{Z}_{\mathscr{K}} &= (D^2 \times \dots \times D^2 \times S^1) \cup (D^2 \times \dots \times S^1 \times D^2) \cup \dots \cup (S^1 \times \dots \times D^2 \times D^2)\\ &= \partial\bigl((D^2)^m\bigr) \cong S^{2m-1}.\end{aligned}$$

5. Let 

be the boundary of a 4-gon. Then we have four maximal simplices $\{1,3\}$, $\{2,3\}$, $\{1,4\}$, $\{2,4\}$, and

$$\begin{aligned}\mathscr{Z}_{\mathscr{K}} &= (D^2 \times S^1 \times D^2 \times S^1) \cup (S^1 \times D^2 \times D^2 \times S^1)\\ &\qquad \cup (D^2 \times S^1 \times S^1 \times D^2) \cup (S^1 \times D^2 \times S^1 \times D^2)\\ &= \bigl((D^2 \times S^1) \cup (S^1 \times D^2)\bigr) \times D^2 \times S^1 \cup \bigl((D^2 \times S^1) \cup (S^1 \times D^2)\bigr) \times S^1 \times D^2\\ &= \bigl((D^2 \times S^1) \cup (S^1 \times D^2)\bigr) \times \bigl((D^2 \times S^1) \cup (S^1 \times D^2)\bigr) \cong S^3 \times S^3.\end{aligned}$$

The last example can be generalized as follows. Recall that the *join* of simplicial complexes $\mathscr{K}_1$ and $\mathscr{K}_2$ on respective sets $\mathscr{V}_1$ and $\mathscr{V}_2$ is the simplicial complex

$$\mathscr{K}_1 * \mathscr{K}_2 = \{I \subset \mathscr{V}_1 \sqcup \mathscr{V}_2 \colon I = I_1 \cup I_2,\ I_1 \in \mathscr{K}_1,\ I_2 \in \mathscr{K}_2\}$$

on the set $\mathscr{V}_1 \sqcup \mathscr{V}_2$.

Proposition 7.6. $\mathscr{Z}_{\mathscr{K}_1 * \mathscr{K}_2} = \mathscr{Z}_{\mathscr{K}_1} \times \mathscr{Z}_{\mathscr{K}_2}$.

Proof. Indeed,

$$\mathscr{Z}_{\mathscr{K}_1*\mathscr{K}_2}=\bigcup_{I_1\in\mathscr{K}_1,\,I_2\in\mathscr{K}_2}(D^2,S^1)^{I_1\sqcup I_2}=\bigcup_{I_1\in\mathscr{K}_1,\,I_2\in\mathscr{K}_2}(D^2,S^1)^{I_1}\times(D^2,S^1)^{I_2}$$
$$=\biggl(\bigcup_{I_1\in\mathscr{K}_1}(D^2,S^1)^{I_1}\biggr)\times\biggl(\bigcup_{I_2\in\mathscr{K}_2}(D^2,S^1)^{I_2}\biggr)=\mathscr{Z}_{\mathscr{K}_1}\times\mathscr{Z}_{\mathscr{K}_2}.$$

Corollary 7.7. *Let P and Q be two simple polytopes. Then $\mathscr{Z}_{P\times Q}\cong\mathscr{Z}_P\times\mathscr{Z}_Q$.*

Proof. Indeed, $\mathscr{K}_{P\times Q}=\mathscr{K}_P*\mathscr{K}_Q$.

Since $\mathscr{Z}_{\mathscr{K}_P}\cong\mathscr{Z}_P$, the moment-angle complex corresponding to the boundary of a simplicial polytope is a manifold. This is also true for the moment-angle complex corresponding to any triangulated sphere (although not every triangulation of a sphere is the boundary of a simplicial polytope; see for instance, [14], § 2.3).

Theorem 7.8 ([14], Lemma 7.13). *Let $\mathscr{K}$ be a triangulation of S^{n-1} with m indices. Then $\mathscr{Z}_{\mathscr{K}}$ is a* (*closed*) *topological manifold of dimension $m+n$.*

As we shall see in the next section, moment-angle complexes corresponding to complete simplicial fans are smooth manifolds. In general, it is not known whether a smooth structure exists on moment-angle manifolds corresponding to arbitrary triangulated spheres.

The topological structure of moment-angle complexes $\mathscr{Z}_{\mathscr{K}}$ is quite complicated in general. The cohomology ring of $\mathscr{Z}_{\mathscr{K}}$ was described in [13], § 4.2 (with field coefficients) and in [6] and [28] (with integer coefficients). It is known [31] that if $\mathscr{K}$ is the k-dimensional skeleton of the simplex Δ^{m-1} (for any k,m), then the corresponding moment-angle complex $\mathscr{Z}_{\mathscr{K}}$ is homotopy equivalent to a wedge of spheres. Also, it is known that if P is obtained from a simplex by successive truncation of vertices by hyperplanes (so that the polar polytope P^* is *stacked*), then $\mathscr{Z}_P$ is diffeomorphic to a connected sum of sphere products, with two spheres in each product (this result is due to McGavran, cf. [10], Theorem 6.3; see also [30]). Finding more series of polytopes or simplicial complexes for which the topology of $\mathscr{Z}_{\mathscr{K}}$ can be described explicitly is a challenging task. Many non-trivial topological phenomena occur already in the cohomology of $\mathscr{Z}_{\mathscr{K}}$. For instance, moment-angle manifolds are generally not *formal* (in the sense of rational homotopy theory); the first examples of $\mathscr{Z}_P$ with non-trivial Massey products in cohomology appear already for 3-dimensional polytopes P (see [5]).

7.3. Polyhedral products. The decomposition (7.5) of $\mathscr{Z}_{\mathscr{K}}$ using the disk and circle (D^2,S^1) is readily generalized to arbitrary pairs of spaces.

Construction 7.9 (polyhedral product). Let $\mathscr{K}$ be a simplicial product on $[m]$ and let

$$(\mathbf{X},\mathbf{A})=\{(X_1,A_1),\dots,(X_m,A_m)\}$$

be a set of m pairs of spaces with $A_i\subset X_i$. For each simplex $I\in\mathscr{K}$ we set

$$(\mathbf{X},\mathbf{A})^I=\biggl\{(x_1,\dots,x_m)\in\prod_{i=1}^m X_i\colon x_i\in A_i \text{ for } i\notin I\biggr\} \tag{7.6}$$

and define the *polyhedral product* of $(\mathbf{X}, \mathbf{A})$ corresponding to $\mathscr{K}$ by

$$(\mathbf{X}, \mathbf{A})^{\mathscr{K}} = \bigcup_{I \in \mathscr{K}} (\mathbf{X}, \mathbf{A})^I = \bigcup_{I \in \mathscr{K}} \biggl(\prod_{i \in I} X_i \times \prod_{i \notin I} A_i \biggr).$$

In the case when all the pairs (X_i, A_i) are the same, that is, $X_i = X$ and $A_i = A$ for $i = 1, \dots, m$, we use the notation $(X, A)^{\mathscr{K}}$ for $(\mathbf{X}, \mathbf{A})^{\mathscr{K}}$.

Example 7.10. 1. The moment-angle complex $\mathscr{Z}_{\mathscr{K}}$ is the polyhedral product $(D^2, S^1)^{\mathscr{K}}$ (when considered abstractly) or $(\mathbb{D}, \mathbb{S})^{\mathscr{K}}$ (when viewed as a subcomplex of $\mathbb{D}^m$).

2. The cubical subcomplex $c_P(P) \subset \mathbb{I}^m$ in Construction 7.1 is given by

$$c_P(P) = (\mathbb{I}, 1)^{\mathscr{K}_P},$$

where $\mathbb{I} = [0, 1]$ is the unit interval and 1 its endpoint. For general $\mathscr{K}$ the polyhedral product $(\mathbb{I}, 1)^{\mathscr{K}}$ is a cubical subcomplex of $\mathbb{I}^m$ that can be identified with the quotient of $\mathscr{Z}_{\mathscr{K}}$ by the action of $\mathbb{T}^m$. The space $(\mathbb{I}, 1)^{\mathscr{K}}$ is homeomorphic to the cone over $\mathscr{K}$ (see [14], Proposition 5.12). We use the notation $\operatorname{cc}(\mathscr{K}) = (\mathbb{I}, 1)^{\mathscr{K}}$.

3. If $\mathscr{K}$ consists of m separate points and $A_i = \mathrm{pt}$ (a point), then

$$(\mathbf{X}, \mathrm{pt})^{\mathscr{K}} = X_1 \vee X_2 \vee \dots \vee X_m$$

is the *wedge* (or *bouquet*) of the spaces X_i.

Remark. The decomposition of $\mathscr{Z}_{\mathscr{K}}$ into a union of products of disks and circles first appeared in [13], where the term 'moment-angle complex' for $\mathscr{Z}_{\mathscr{K}} = (D^2, S^1)^{\mathscr{K}}$ was also introduced. Several other examples of polyhedral products $(X, A)^{\mathscr{K}}$ (including those in Example 7.10) were also considered in [13]. The definition of $(X, A)^{\mathscr{K}}$ for an arbitrary pair of spaces (X, A) was suggested to the authors by N. Strickland (in a private communication, and also in an unpublished note) as a general framework for the constructions in [13]; this definition was also included in the final version of [13] and in [14]. Further generalizations of $(X, A)^{\mathscr{K}}$ to a set of pairs of spaces $(\mathbf{X}, \mathbf{A})$ were studied in the paper [31] of Grbić and Theriault, as well as the paper [4] of Bahri, Bendersky, Cohen, and Gitler, where the term 'polyhedral product' was introduced (following a suggestion of W. Browder). Since 2000, the terms 'generalized moment-angle complex', '$\mathscr{K}$-product', and 'partial product space' have also been used to refer to the spaces $(X, A)^{\mathscr{K}}$.

7.4. Complements of coordinate subspace arrangements, revisited. These spaces provide another important class of examples of polyhedral products. We can define the complement of a set of coordinate subspaces as in (6.3) for an arbitrary simplicial complex $\mathscr{K}$:

$$U(\mathscr{K}) = \mathbb{C}^m \setminus \bigcup_{\{i_1, \dots, i_k\} \notin \mathscr{K}} \{\mathbf{z} \in \mathbb{C}^m \colon z_{i_1} = \dots = z_{i_k} = 0\}. \tag{7.7}$$

It is easy to see that the complement of any set of coordinate subspaces of $\mathbb{C}^m$ has the form $U(\mathscr{K})$ for some simplicial complex $\mathscr{K}$ on $[m]$. If the arrangement of coordinate planes contains a hyperplane $\mathbf{z}_i = 0$, then $\{i\}$ is a ghost vertex of the corresponding simplicial complex $\mathscr{K}$.

Proposition 7.11. $U(\mathscr{K}) = (\mathbb{C}, \mathbb{C}^\times)^{\mathscr{K}}$.

Proof. For $\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{C}^m$ we wrote $\omega(\mathbf{z}) = \{i \in [m] \colon z_i = 0\} \subset [m]$, so

$$\begin{aligned} U(\mathscr{K}) &= \mathbb{C}^m \setminus \bigcup_{I \notin \mathscr{K}} L_I = \mathbb{C}^m \setminus \bigcup_{I \notin \mathscr{K}} \{\mathbf{z} \colon \omega(\mathbf{z}) \supset I\} = \mathbb{C}^m \setminus \bigcup_{I \notin \mathscr{K}} \{\mathbf{z} \colon \omega(\mathbf{z}) = I\} \\ &= \bigcup_{I \in \mathscr{K}} \{\mathbf{z} \colon \omega(\mathbf{z}) = I\} = \bigcup_{I \in \mathscr{K}} \{\mathbf{z} \colon \omega(\mathbf{z}) \subset I\} = \bigcup_{I \in \mathscr{K}} (\mathbb{C}, \mathbb{C}^\times)^I = (\mathbb{C}, \mathbb{C}^\times)^{\mathscr{K}}, \end{aligned}$$

where $L_I = \{\mathbf{z} \in \mathbb{C}^m \colon z_{i_1} = \dots = z_{i_k} = 0\}$ for $I = \{i_1, \dots, i_k\}$. $\square$

Since each coordinate subspace is invariant under the standard action of $\mathbb{T}^m$ on $\mathbb{C}^m$, the complement $U(\mathscr{K})$ is also a $\mathbb{T}^m$-invariant subset of $\mathbb{C}^m$.

Recall that a *deformation retraction* of a space X onto a subspace A is a continuous family of maps (a homotopy) $F_t \colon X \to X$, $t \in \mathbb{I}$, such that $F_0 = \mathrm{id}$ (the identity map), $F_1(X) = A$, and $F_t|_A = \mathrm{id}$ for all t. Often the term 'deformation retraction' refers only to the last map $f = F_1 \colon X \to A$ in the family. This map is a homotopy equivalence.

Theorem 7.12 [14]. *The moment-angle complex $\mathscr{Z}_{\mathscr{K}}$ is a $\mathbb{T}^m$-invariant subspace of $U(\mathscr{K})$, and there is a $\mathbb{T}^m$-equivariant deformation retraction*

$$\mathscr{Z}_{\mathscr{K}} \hookrightarrow U(\mathscr{K}) \to \mathscr{Z}_{\mathscr{K}}.$$

Proof. Since $\mathbb{D} \subset \mathbb{C}$ and $\mathbb{S} \subset \mathbb{C}^\times$, we have $\mathscr{Z}_{\mathscr{K}} = (\mathbb{D}, \mathbb{S})^{\mathscr{K}} \subset (\mathbb{C}, \mathbb{C}^\times)^{\mathscr{K}} = U(\mathscr{K})$, and the subset $\mathscr{Z}_{\mathscr{K}} \subset U(\mathscr{K})$ is obviously $\mathbb{T}^m$-invariant.

Any simplicial complex $\mathscr{K}$ can be obtained from Δ^{m-1} by successive removal of maximal simplices (so that we get a simplicial complex at each intermediate step), and we shall construct the deformation retraction $U(\mathscr{K}) \to \mathscr{Z}_{\mathscr{K}}$ by induction.

The base of induction is clear: if $\mathscr{K} = \Delta^{m-1}$, then $U(\mathscr{K}) = \mathbb{C}^m$, $\mathscr{Z}_{\mathscr{K}} = \mathbb{D}^m$, and the retraction $\mathbb{C}^m \to \mathbb{D}^m$ is evident.

The orbit space $\mathscr{Z}_{\mathscr{K}}/\mathbb{T}^m$ is the cubical complex $\mathrm{cc}(\mathscr{K}) = (\mathbb{I}, 1)^{\mathscr{K}}$ (see Example 7.10.2). The orbit space $U(\mathscr{K})/\mathbb{T}^m$ can be identified with

$$U(\mathscr{K})_{\geqslant} = U(\mathscr{K}) \cap \mathbb{R}^m_{\geqslant} = (\mathbb{R}_{\geqslant}, \mathbb{R}_{>})^{\mathscr{K}},$$

where $\mathbb{R}^m_{\geqslant}$ is viewed as a subset of $\mathbb{C}^m$.

We shall first construct a deformation retraction $r \colon U(\mathscr{K})_{\geqslant} \to \mathrm{cc}(\mathscr{K})$ of orbit spaces, and then cover it by a deformation retraction $\widetilde{r} \colon U(\mathscr{K}) \to \mathscr{Z}_{\mathscr{K}}$.

Assume now that $\mathscr{K}$ is obtained from a simplicial complex $\mathscr{K}'$ by removing one maximal simplex $J = \{j_1, \dots, j_k\}$, that is, $\mathscr{K} \cup J = \mathscr{K}'$. Then the cubical complex $\mathrm{cc}(\mathscr{K}')$ is obtained from $\mathrm{cc}(\mathscr{K})$ by adding a single k-dimensional face $C_J = (\mathbb{I}, 1)^J$. We also have $U(\mathscr{K}) = U(\mathscr{K}') \setminus L_J$, so that

$$U(\mathscr{K})_{\geqslant} = U(\mathscr{K}')_{\geqslant} \setminus \{\mathbf{y} \colon y_{j_1} = \dots = y_{j_k} = 0\}.$$

We may assume by induction that there is a deformation retraction $r' \colon U(\mathscr{K}')_{\geqslant} \to \mathrm{cc}(\mathscr{K}')$ such that $\omega(r'(\mathbf{y})) = \omega(\mathbf{y})$, where $\omega(\mathbf{y})$ is the set of zero coordinates of $\mathbf{y}$.

In particular, r' restricts to a deformation retraction

$$r'\colon U(\mathscr{K}')_{\geqslant} \setminus \{\mathbf{y}\colon y_{j_1}=\dots=y_{j_k}=0\} \to \mathrm{cc}(\mathscr{K}') \setminus \mathbf{y}_J,$$

where $\mathbf{y}_J$ is the point with coordinates $y_{j_1}=\dots=y_{j_k}=0$ and $y_j=1$ for $j\notin J$.

Since $J\notin\mathscr{K}$, we have $\mathbf{y}_J\notin\mathrm{cc}(\mathscr{K})$. On the other hand, $\mathbf{y}_J$ belongs to the extra face $C_J=(\mathbb{I},1)^J$ of $\mathrm{cc}(\mathscr{K}')$. We therefore may apply the deformation retraction r_J shown in Fig. 7.2 on the face C_J with centre at $\mathbf{y}_J$. In coordinates, a homotopy F_t between the identity map $\mathrm{cc}(\mathscr{K}')\setminus\mathbf{y}_J\to\mathrm{cc}(\mathscr{K}')\setminus\mathbf{y}_J$ (for $t=0$) and the retraction $r_J\colon \mathrm{cc}(\mathscr{K}')\setminus\mathbf{y}_J\to\mathrm{cc}(\mathscr{K})$ (for $t=1$) is given by

$$\begin{aligned}&F_t\colon \mathrm{cc}(\mathscr{K}')\setminus\mathbf{y}_J\to\mathrm{cc}(\mathscr{K}')\setminus\mathbf{y}_J,\\ &\quad (y_1,\dots,y_m,t)\mapsto(y_1+t\alpha_1y_1,\dots,y_m+t\alpha_my_m),\end{aligned}$$

where

$$\alpha_i=\begin{cases}\dfrac{1-\max_{j\in J}y_j}{\max_{j\in J}y_j} & \text{if } i\in J,\\ 0 & \text{if } i\notin J,\end{cases}\qquad 1\leqslant i\leqslant m.$$

We observe that $\omega(F_t(\mathbf{y}))=\omega(\mathbf{y})$ for any t and $\mathbf{y}\in\mathrm{cc}(\mathscr{K}')$. The composition

$$r\colon U(\mathscr{K})_{\geqslant}=U(\mathscr{K}')_{\geqslant}\setminus\{\mathbf{y}\colon y_{j_1}=\dots=y_{j_k}=0\}\xrightarrow{r'}\mathrm{cc}(\mathscr{K}')\setminus\mathbf{y}_J\xrightarrow{r_J}\mathrm{cc}(\mathscr{K}) \quad (7.8)$$

is a deformation retraction, and it satisfies $\omega(r(\mathbf{y}))=\omega(\mathbf{y})$, since this is true for r_J and r'. The inductive step is now complete. The required retraction $\widetilde{r}\colon U(\mathscr{K})\to\mathscr{Z}_{\mathscr{K}}$ covers r as shown in the following commutative diagram:

$$\begin{tikzcd}
\mathscr{Z}_{\mathscr{K}} \arrow[r, hook] \arrow[d, "\mu"] & U(\mathscr{K}) \arrow[r, "\widetilde{r}"] \arrow[d, "\mu"] & \mathscr{Z}_{\mathscr{K}} \arrow[d, "\mu"] \\
\mathrm{cc}(\mathscr{K}) \arrow[r, hook] & U_{\geqslant}(\mathscr{K}) \arrow[r, "r"] & \mathrm{cc}(\mathscr{K})
\end{tikzcd}$$

Explicitly, $\widetilde{r}$ is decomposed inductively in a way similar to (7.8),

$$\widetilde{r}\colon U(\mathscr{K})=U(\mathscr{K}')\setminus L_J\xrightarrow{\widetilde{r}'}\mathscr{Z}_{\mathscr{K}'}\setminus\mu^{-1}(\mathbf{y}_J)\xrightarrow{\widetilde{r}_J}\mathscr{Z}_{\mathscr{K}},$$

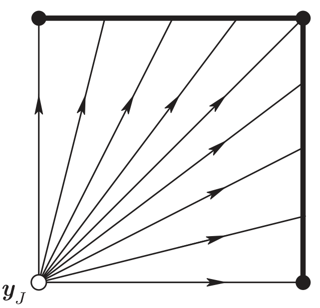


Figure 7.2. The retraction $r_J\colon \mathrm{cc}(\mathscr{K}')\setminus\mathbf{y}_J\to\mathrm{cc}(\mathscr{K})$

where $\mu^{-1}(\mathbf{y}_J) = \prod_{j\in J}\{0\} \times \prod_{j\notin J}\mathbb{S}$, and $\widetilde{r}_J$ is given by

$$\left(\sqrt{y_1}\,e^{i\varphi_1},\dots,\sqrt{y_m}\,e^{i\varphi_m}\right) \mapsto \left(\sqrt{y_1+\alpha_1 y_1}\,e^{i\varphi_1},\dots,\sqrt{y_m+\alpha_m y_m}\,e^{i\varphi_m}\right)$$

in the coordinates $(z_1,\dots,z_m) = \left(\sqrt{y_1}\,e^{i\varphi_1},\dots,\sqrt{y_m}\,e^{i\varphi_m}\right)$, with α_i as above. $\square$

As we shall see in § 9, in the case when $\mathscr{K} = \mathscr{K}_\Sigma$ is the underlying complex of a complete simplicial fan Σ, the deformation retraction $U(\mathscr{K}) \to \mathscr{Z}_{\mathscr{K}}$ can be realized as the quotient map for an action of $\mathbb{R}^{m-n}$ on $U(\mathscr{K})$.

In the remaining sections we shall concentrate on the geometric aspects of the theory of moment-angle complexes, and moment-angle manifolds corresponding to polytopes and complete simplicial fans will be our main objects of interest. Nevertheless, the homotopy theory of general moment-angle complexes has by now gained its own momentum, and we refer to [14], Chap. 6, [23], [31], [53], and [4] for the main stages of its development.

8. LVM-manifolds

Bosio and Meersseman [10] identified polytopal moment-angle manifolds $\mathscr{Z}_P$ with a class of non-Kähler complex-analytic manifolds introduced in the works of López de Medrano, Verjovsky, and Meersseman (LVM-manifolds). This was the starting point in the subsequent study of the complex geometry of moment-angle manifolds. Here we review the construction of LVM-manifolds and its connection to polytopal moment-angle manifolds.

The initial data of the construction of an LVM-manifold is a link of a homogeneous system of quadrics similar to (4.2), but with *complex* coefficients:

$$\mathscr{L} = \left\{ \begin{aligned} \mathbf{z}\in\mathbb{C}^m\colon \quad \sum_{k=1}^m |z_k|^2 &= 1, \\ \sum_{k=1}^m \zeta_k|z_k|^2 &= \mathbf{0} \end{aligned} \right\}, \tag{8.1}$$

where $\zeta_k \in \mathbb{C}^s$. We can obviously turn this link into the form (4.2) by identifying $\mathbb{C}^s$ with $\mathbb{R}^{2s}$ in the standard way, so that each ζ_k becomes a $\mathbf{g}_k \in \mathbb{R}^{m-n-1}$, where $n = m-2s-1$. We assume that the link is non-degenerate, that is, the system of complex vectors $(\zeta_1,\dots,\zeta_m)$ (or the corresponding system of real vectors $(\mathbf{g}_1,\dots,\mathbf{g}_m)$) satisfies the conditions (a) and (b) of Proposition 4.6.

Now define the manifold $\mathscr{N}$ as the projectivization of the intersection of homogeneous quadrics in (8.1):

$$\mathscr{N} = \{\mathbf{z}\in\mathbb{C}P^{m-1}\colon \zeta_1|z_1|^2+\dots+\zeta_m|z_m|^2 = \mathbf{0}\}, \qquad \zeta_k\in\mathbb{C}^s. \tag{8.2}$$

We therefore have a principal S^1-bundle $\mathscr{L}\to\mathscr{N}$.

Theorem 8.1 (Meersseman [43]). *The manifold $\mathscr{N}$ has a holomorphic atlas describing it as a compact complex manifold of complex dimension $m-1-s$.*

Sketch of proof. Consider the holomorphic action of $\mathbb{C}^s$ on $\mathbb{C}^m$ given by

$$\begin{gathered}\mathbb{C}^s \times \mathbb{C}^m \to \mathbb{C}^m,\\ (\mathbf{w}, \mathbf{z}) \mapsto \bigl(z_1 e^{\langle \zeta_1, \mathbf{w}\rangle}, \dots, z_m e^{\langle \zeta_m, \mathbf{w}\rangle}\bigr),\end{gathered} \tag{8.3}$$

where $\mathbf{w} = (w_1, \dots, w_s) \in \mathbb{C}^s$ and $\langle \zeta_k, \mathbf{w}\rangle = \zeta_{1k}w_1 + \dots + \zeta_{sk}w_s$.

Let $\mathscr{K}$ be the simplicial complex consisting of zero-sets of points of the link $\mathscr{L}$:

$$\mathscr{K} = \{\omega(\mathbf{z})\colon \mathbf{z} \in \mathscr{L}\}.$$

Observe that $\mathscr{K} = \mathscr{K}_P$, where P is the simple polytope associated with the link $\mathscr{L}$. Let $U = U(\mathscr{K})$ be the corresponding complement of the subspace arrangement given by (7.7), and note that Proposition 2.10 implies that U can also be defined as

$$U = \bigl\{(z_1, \dots, z_m) \in \mathbb{C}^m \colon \mathbf{0} \in \operatorname{conv}(\zeta_j \colon z_j \neq 0)\bigr\}.$$

An argument similar to that in the proof of Lemma 5.4 shows that the restriction of the action (8.3) to $U \subset \mathbb{C}^m$ is free. Also, this restricted action is proper (we shall prove this in more general context in Theorem 10.3), so the quotient $U/\mathbb{C}^s$ is Hausdorff. Using a holomorphic atlas transverse to the orbits of the free action of $\mathbb{C}^s$ on the complex manifold U, we get that the quotient $U/\mathbb{C}^s$ has the structure of a complex manifold.

On the other hand, it can be shown that the function $|z_1|^2 + \dots + |z_m|^2$ on $\mathbb{C}^m$ has a unique minimum when restricted to an orbit of the free action of $\mathbb{C}^s$ on U. The set of these minima can be described as

$$\mathscr{T} = \bigl\{\mathbf{z} \in \mathbb{C}^m \setminus \{\mathbf{0}\} \colon \zeta_1|z_1|^2 + \dots + \zeta_m|z_m|^2 = \mathbf{0}\bigr\}.$$

It follows that the quotient $U/\mathbb{C}^s$ can be identified with $\mathscr{T}$, and therefore $\mathscr{T}$ acquires the structure of a complex manifold of dimension $m - s$.

By projectivizing the construction we identify $\mathscr{N}$ with the quotient of the complement of a coordinate subspace arrangement in $\mathbb{C}P^{m-1}$ (the projectivization of U) by a holomorphic action of $\mathbb{C}^s$. In this way $\mathscr{N}$ becomes a compact complex manifold.

The manifold $\mathscr{N}$ with the complex structure in Theorem 8.1 is referred to as an *LVM-manifold*. These manifolds were described by Meersseman [43] as a generalization of the construction of López de Medrano and Verjovsky in [41].

Remark. The embedding of $\mathscr{T}$ in $\mathbb{C}^m$ and of $\mathscr{N}$ in $\mathbb{C}P^{m-1}$ given by (8.2) is not holomorphic.

A polytopal moment-angle manifold $\mathscr{Z}_P$ is diffeomorphic to a link (4.2), which can be turned into a complex link (8.1) whenever $m+n$ is odd. It follows that the quotient $\mathscr{Z}_P/S^1$ of an odd-dimensional moment-angle manifold has the complex-analytic structure of an LVM-manifold. By adding redundant inequalities and using the S^1-bundle $\mathscr{L} \to \mathscr{N}$, Bosio and Meersseman observed that $\mathscr{Z}_P$ or $\mathscr{Z}_P \times S^1$ also admits the structure of an LVM-manifold, depending on whether $m+n$ is even or odd.

We first summarize the effects that a redundant inequality in (2.1) has on the different spaces appearing above.

Proposition 8.2. *Assume that* (2.1) *is a generic presentation. Then the following conditions are equivalent*:

(a) $\langle \mathbf{a}_i, \mathbf{x}\rangle + b_i \geqslant 0$ *is a redundant inequality in* (2.1) (*that is*, $F_i = \varnothing$);

(b) $\mathcal{Z}_P \subset \{\mathbf{z} \in \mathbb{C}^m \colon z_i \neq 0\}$;

(c) $\{i\}$ *is a ghost vertex in* $\mathcal{K}_P$;

(d) $U(\mathcal{K}_P)$ *has the factor* $\mathbb{C}^\times$ *in the* i*th coordinate*;

(e) $\mathbf{0} \notin \operatorname{conv}(\mathbf{g}_k \colon k \neq i)$.

Proof. The equivalence of the first four conditions follows directly from the definitions. The equivalence (a) $\Leftrightarrow$ (e) follows from Proposition 2.10. $\square$

Theorem 8.3 [10]. *Let* $\mathcal{Z}_P$ *be the moment angle manifold corresponding to an* n*-dimensional simple polytope* (2.1) *defined by* m *inequalities*.

(a) *If* $m+n$ *is even, then* $\mathcal{Z}_P$ *has a complex structure as an LVM-manifold*.

(b) *If* $m+n$ *is odd, then* $\mathcal{Z}_P \times S^1$ *has a complex structure as an LVM-manifold*.

Proof. (a) We add one redundant inequality of the form $1 \geqslant 0$ to (2.1) and denote the resulting manifold in (3.1) by $\mathcal{Z}'_P$. Then $\mathcal{Z}'_P \cong \mathcal{Z}_P \times S^1$. By Proposition 4.5, $\mathcal{Z}_P$ is diffeomorphic to a link given by (4.2). Then $\mathcal{Z}'_P$ is given by the intersection of quadrics

$$\left\{\begin{array}{lccccccl} \mathbf{z} \in \mathbb{C}^{m+1}\colon & |z_1|^2 & + & \cdots & + & |z_m|^2 & = 1, \\ & \mathbf{g}_1|z_1|^2 & + & \cdots & + & \mathbf{g}_m|z_m|^2 & = \mathbf{0}, \\ & & & & & |z_{m+1}|^2 & = 1 \end{array}\right\},$$

which is diffeomorphic to the link given by

$$\left\{\begin{array}{lccccccccl} \mathbf{z} \in \mathbb{C}^{m+1}\colon & |z_1|^2 & + & \cdots & + & |z_m|^2 & + & |z_{m+1}|^2 & = 1, \\ & \mathbf{g}_1|z_1|^2 & + & \cdots & + & \mathbf{g}_m|z_m|^2 & & & = \mathbf{0}, \\ & |z_1|^2 & + & \cdots & + & |z_m|^2 & - & |z_{m+1}|^2 & = 0 \end{array}\right\}.$$

If we denote by $\varGamma^\star = (\mathbf{g}_1 \dots \mathbf{g}_m)$ the $(m-n-1)\times m$ matrix of coefficients of the homogeneous quadrics for $\mathcal{Z}_P$, then the corresponding matrix for $\mathcal{Z}'_P$ is

$$\varGamma^{\star\prime} = \begin{pmatrix} \mathbf{g}_1 & \cdots & \mathbf{g}_m & 0 \\ 1 & \cdots & 1 & -1 \end{pmatrix}.$$

Its height $m-n$ is even, so that we may think of its ith column as a complex vector ζ_i (by identifying $\mathbb{R}^{m-n}$ with $\mathbb{C}^{(m-n)/2}$) for $i = 1,\dots,m+1$. Now define

$$\mathcal{N}' = \{\mathbf{z} \in \mathbb{C}P^m \colon \zeta_1|z_1|^2 + \cdots + \zeta_{m+1}|z_{m+1}|^2 = \mathbf{0}\}. \tag{8.4}$$

Then $\mathcal{N}'$ has a complex structure as an LVM-manifold by Theorem 8.1. On the other hand,

$$\mathcal{N}' \cong \mathcal{Z}'_P/S^1 = (\mathcal{Z}_P \times S^1)/S^1 \cong \mathcal{Z}_P,$$

so that $\mathcal{Z}_P$ also acquires a complex structure.

(b) The proof here is similar, but we have to add two redundant inequalities $1 \geqslant 0$ to (2.1). Then $\mathcal{Z}'_P \cong \mathcal{Z}_P \times S^1 \times S^1$ is given by the link

$$\left\{\begin{array}{lcccccccl} \mathbf{z} \in \mathbb{C}^{m+2}\colon & |z_1|^2 & +\cdots+ & |z_m|^2 & + & |z_{m+1}|^2 & + & |z_{m+2}|^2 & = 1, \\ & \mathbf{g}_1|z_1|^2 & +\cdots+ & \mathbf{g}_m|z_m|^2 & & & & & = \mathbf{0}, \\ & |z_1|^2 & +\cdots+ & |z_m|^2 & - & |z_{m+1}|^2 & & & = 0, \\ & |z_1|^2 & +\cdots+ & |z_m|^2 & & & - & |z_{m+2}|^2 & = 0 \end{array}\right\}.$$

The matrix of coefficients of the homogeneous quadrics is therefore

$$\Gamma^{\star\prime} = \begin{pmatrix} \mathbf{g}_1 & \cdots & \mathbf{g}_m & 0 & 0 \\ 1 & \cdots & 1 & -1 & 0 \\ 1 & \cdots & 1 & 0 & -1 \end{pmatrix}.$$

We think of its columns as a set of $m+2$ complex vectors $\zeta_1, \dots, \zeta_{m+2}$, and define

$$\mathscr{N}' = \bigl\{\mathbf{z} \in \mathbb{C}P^{m+1} \colon \zeta_1|z_1|^2 + \cdots + \zeta_{m+2}|z_{m+2}|^2 = \mathbf{0}\bigr\}. \qquad (8.5)$$

Then $\mathscr{N}'$ has a complex structure as an LVM-manifold. On the other hand,

$$\mathscr{N}' \cong \mathscr{Z}'_P/S^1 = (\mathscr{Z}_P \times S^1 \times S^1)/S^1 \cong \mathscr{Z}_P \times S^1,$$

and therefore $\mathscr{Z}_P \times S^1$ has a complex structure. □

In the next two sections we describe a more direct method of giving $\mathscr{Z}_P$ a complex structure, without referring to projectivized quadrics and LVM-manifolds. This approach, developed in [54], works not only in the polytopal case, but also for the moment-angle manifolds $\mathscr{Z}_{\mathscr{K}}$ corresponding to underlying complexes $\mathscr{K}$ of complete simplicial fans.

9. Moment-angle manifolds from simplicial fans

Let $\mathscr{K} = \mathscr{K}_\Sigma$ be the underlying complex of a complete simplicial fan Σ, and let $U(\mathscr{K})$ be the complement of the coordinate subspace arrangement (7.7) defined by $\mathscr{K}$. Here we shall identify the moment-angle manifold $\mathscr{Z}_{\mathscr{K}}$ with the quotient of $U(\mathscr{K})$ by a smooth action of a non-compact group isomorphic to $\mathbb{R}^{m-n}$, thereby defining a smooth structure on $\mathscr{Z}_{\mathscr{K}}$. A modification of this construction will be used in the next section to endow $\mathscr{Z}_{\mathscr{K}}$ with a complex structure. These results were obtained in the paper [54] of Ustinovsky and the author.

Recall from § 6.1 that a simplicial fan Σ can be defined by data $\{\mathscr{K}; \mathbf{a}_1, \dots, \mathbf{a}_m\}$, where

- $\mathscr{K}$ is a simplicial complex on $[m]$,
- $\mathbf{a}_1, \dots, \mathbf{a}_m$ is a configuration of vectors in $N_{\mathbb{R}} \cong \mathbb{R}^n$ such that the subset $\{\mathbf{a}_i \colon i \in I\}$ is linearly independent for any simplex $I \in \mathscr{K}$.

Here is an important point in which our approach to fans differs from the standard one adopted in toric geometry: since we allow ghost vertices in $\mathscr{K}$, we do not require that each vector $\mathbf{a}_i$ spans a one-dimensional cone in Σ. The vector $\mathbf{a}_i$ corresponding to a ghost vertex $\{i\} \in [m]$ may be zero. This formalism was also used in [7] under the name *triangulated vector configurations*.

Construction 9.1. For a set of vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ consider the linear map

$$A \colon \mathbb{R}^m \to N_{\mathbb{R}}, \qquad \mathbf{e}_i \mapsto \mathbf{a}_i, \qquad (9.1)$$

where $\mathbf{e}_1, \dots, \mathbf{e}_m$ is the standard basis of $\mathbb{R}^m$. Let

$$\mathbb{R}^m_> = \{(y_1, \dots, y_m) \in \mathbb{R}^m \colon y_i > 0\}$$

be the multiplicative group of m-tuples of positive real numbers, and define the subgroup

$$\begin{aligned} R = \exp(\operatorname{Ker} A) &= \{(e^{y_1},\dots,e^{y_m})\colon (y_1,\dots,y_m)\in \operatorname{Ker} A\} \\ &= \biggl\{(t_1,\dots,t_m)\in\mathbb{R}^m_>\colon \prod_{i=1}^m t_i^{\langle \mathbf{a}_i,\mathbf{u}\rangle}=1 \text{ for all } \mathbf{u}\in N^*_{\mathbb{R}}\biggr\}. \end{aligned} \tag{9.2}$$

We let $\mathbb{R}^m_>$ act on the complement $U(\mathscr{K})\subset\mathbb{C}^m$ by coordinatewise multiplications and consider the restricted action of the subgroup $R\subset\mathbb{R}^m_>$. Recall that an action of a topological group G on a space X is *proper* if the *group action map* $h\colon G\times X\to X\times X$, $(g,x)\mapsto(gx,x)$ is proper (the pre-image of a compact subset is compact).

Theorem 9.2 [54]. *Assume as given data $\{\mathscr{K};\mathbf{a}_1,\dots,\mathbf{a}_m\}$ satisfying the conditions above. Then:*

(a) *the group R given by* (9.2) *acts on $U(\mathscr{K})$ freely*;

(b) *if $\{\mathscr{K};\mathbf{a}_1,\dots,\mathbf{a}_m\}$ defines a simplicial fan Σ, then R acts on $U(\mathscr{K})$ properly, so the quotient $U(\mathscr{K})/R$ is a smooth Hausdorff $(m+n)$-dimensional manifold*;

(c) *if the fan Σ is complete, then $U(\mathscr{K})/R$ is homeomorphic to the moment-angle manifold $\mathscr{Z}_{\mathscr{K}}$.*

Therefore, $\mathscr{Z}_{\mathscr{K}}$ can be smoothed whenever $\mathscr{K}=\mathscr{K}_\Sigma$ for a complete simplicial fan Σ.

Proof. The statement (a) is proved in the same way as Proposition 6.7. Indeed, a point $\mathbf{z}\in U(\mathscr{K})$ has a non-trivial stabilizer with respect to the action of $\mathbb{R}^m_>$ only if some of its coordinates vanish. These $\mathbb{R}^m_>$-stabilizers are of the form $(\mathbb{R}_>,1)^I$ (see (7.6)) for some $I\in\mathscr{K}$. The restriction of $\exp A$ to any such $(\mathbb{R}_>,1)^I$ is an injection. Therefore, $R=\exp(\operatorname{Ker} A)$ intersects any $\mathbb{R}^m_>$-stabilizer only at the unit, which implies that the R-action on $U(\mathscr{K})$ is free.

Let us prove (b). Consider the map

$$h\colon R\times U(\mathscr{K})\to U(\mathscr{K})\times U(\mathscr{K}),\qquad (\mathbf{g},\mathbf{z})\mapsto(\mathbf{g}\mathbf{z},\mathbf{z})$$

for $\mathbf{g}\in R$, $\mathbf{z}\in U(\mathscr{K})$. Let $V\subset U(\mathscr{K})\times U(\mathscr{K})$ be a compact subset; we need to show that $h^{-1}(V)$ is compact. Since $R\times U(\mathscr{K})$ is metrizable, it suffices to check that any infinite sequence $\{(\mathbf{g}^{(k)},\mathbf{z}^{(k)})\colon k=1,2,\dots\}$ of points in $h^{-1}(V)$ contains a converging subsequence. Because $V\subset U(\mathscr{K})\times U(\mathscr{K})$ is compact, we may assume by passing to a subsequence that the sequence

$$\{h(\mathbf{g}^{(k)},\mathbf{z}^{(k)})\}=\{(\mathbf{g}^{(k)}\mathbf{z}^{(k)},\mathbf{z}^{(k)})\}$$

has a limit in $U(\mathscr{K})\times U(\mathscr{K})$. We set $\mathbf{w}^{(k)}=\mathbf{g}^{(k)}\mathbf{z}^{(k)}$ and assume that

$$\{\mathbf{w}^{(k)}\}\to\mathbf{w}=(w_1,\dots,w_m),\qquad \{\mathbf{z}^{(k)}\}\to\mathbf{z}=(z_1,\dots,z_m)$$

for some $\mathbf{w},\mathbf{z}\in U(\mathscr{K})$. We need to show that a subsequence of $\{\mathbf{g}^{(k)}\}$ has a limit in R. We write

$$\mathbf{g}^{(k)}=\bigl(g_1^{(k)},\dots,g_m^{(k)}\bigr)=\bigl(e^{\alpha_1^{(k)}},\dots,e^{\alpha_m^{(k)}}\bigr)\in R\subset\mathbb{R}^m_>,$$

$\alpha_j^{(k)} \in \mathbb{R}$. By passing to a subsequence we may assume that each sequence $\{\alpha_j^{(k)}\}$, $j = 1, \dots, m$, has a finite or infinite limit (including $\pm\infty$). Let

$$I_+ = \{j \colon \alpha_j^{(k)} \to +\infty\} \subset [m], \qquad I_- = \{j \colon \alpha_j^{(k)} \to -\infty\} \subset [m].$$

Since the sequences $\{\mathbf{z}^{(k)}\}$, $\{\mathbf{w}^{(k)} = \mathbf{g}^{(k)}\mathbf{z}^{(k)}\}$ are converging to $\mathbf{z}, \mathbf{w} \in U(\mathscr{K})$, respectively, we have $z_j = 0$ for $j \in I_+$ and $w_j = 0$ for $j \in I_-$. Then it follows from the decomposition $U(\mathscr{K}) = \bigcup_{I \in \mathscr{K}} (\mathbb{C}, \mathbb{C}^\times)^I$ that I_+ and I_- are simplices in $\mathscr{K}$. Let σ_+, σ_- be the corresponding cones of the simplicial fan Σ. Then $\sigma_+ \cap \sigma_- = \{\mathbf{0}\}$ by the definition of a fan. By Lemma 6.1 there exists a linear function $\mathbf{u} \in N_\mathbb{R}^*$ such that $\langle \mathbf{u}, \mathbf{a} \rangle > 0$ for any non-zero $\mathbf{a} \in \sigma_+$, and $\langle \mathbf{u}, \mathbf{a} \rangle < 0$ for any non-zero $\mathbf{a} \in \sigma_-$. Since $\mathbf{g}^{(k)} \in R$, it follows from (9.2) that

$$\sum_{j=1}^m \alpha_j^{(k)} \langle \mathbf{u}, \mathbf{a}_j \rangle = 0. \tag{9.3}$$

This implies that both I_+ and I_- are empty, since otherwise the latter sum tends to infinity. Thus, each sequence $\{\alpha_j^{(k)}\}$ has a finite limit α_j, and a subsequence of $\{\mathbf{g}^{(k)}\}$ converges to $(e^{\alpha_1}, \dots, e^{\alpha_m})$. Passing to the limit in (9.3), we find that $(e^{\alpha_1}, \dots, e^{\alpha_m}) \in R$. This proves the properness of the action. Since the Lie group $R(\Sigma)$ acts smoothly, freely, and properly on the smooth manifold $U(\mathscr{K})$, the orbit space $U(\mathscr{K})/R$ is Hausdorff and smooth by a standard result ([39], Theorem 9.16).

In the case of a complete fan it is possible to construct a smooth atlas on $U(\mathscr{K})/R$ explicitly. To do this, it is convenient to pre-factorize everything by the action of $\mathbb{T}^m$, as in the proof of Theorem 7.12. We have

$$U(\mathscr{K})/\mathbb{T}^m = (\mathbb{R}_\geqslant, \mathbb{R}_>)^{\mathscr{K}} = \bigcup_{I \in \mathscr{K}} (\mathbb{R}_\geqslant, \mathbb{R}_>)^I.$$

Since the fan Σ is complete, we may take the union above only over n-element simplices $I = \{i_1, \dots, i_n\} \in \mathscr{K}$. Consider one such simplex I; the generators of the corresponding n-dimensional cone $\sigma \in \Sigma$ are $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_n}$. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ denote the dual basis of $N_\mathbb{R}^*$, that is, $\langle \mathbf{a}_{i_k}, \mathbf{u}_j \rangle = \delta_{kj}$. Now consider the map

$$p_I \colon (\mathbb{R}_\geqslant, \mathbb{R}_>)^I \to \mathbb{R}^n_\geqslant,$$
$$(y_1, \dots, y_m) \mapsto \biggl(\prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u}_1 \rangle}, \dots, \prod_{i=1}^m y_i^{\langle \mathbf{a}_i, \mathbf{u}_n \rangle} \biggr),$$

where we set $0^0 = 1$. Note that zero cannot occur with a negative exponent in the right-hand side, hence p_I is well defined as a continuous map. Each subset $(\mathbb{R}_\geqslant, \mathbb{R}_>)^I$ is R-invariant, and it follows from (9.2) that p_I induces an injective map

$$q_I \colon (\mathbb{R}_\geqslant, \mathbb{R}_>)^I / R \to \mathbb{R}^n_\geqslant.$$

This map is also surjective since every $(x_1, \dots, x_n) \in \mathbb{R}^n_\geqslant$ is covered by $(y_1, \dots, y_m)$, where $y_{i_j} = x_j$ for $1 \leqslant j \leqslant n$ and $y_k = 1$ for $k \notin \{i_1, \dots, i_n\}$. Hence q_I is a homeomorphism. It is covered by a $\mathbb{T}^m$-equivariant homeomorphism

$$\overline{q}_I \colon (\mathbb{C}, \mathbb{C}^\times)^I / R \to \mathbb{C}^n \times \mathbb{T}^{m-n},$$

where $\mathbb{C}^n$ is identified with the quotient $\mathbb{R}^n_\geqslant \times \mathbb{T}^n/\sim$ (see (4.1)). Since $U(\mathscr{K})/R$ is covered by open subsets $(\mathbb{C},\mathbb{C}^\times)^I/R$ and $\mathbb{C}^n \times \mathbb{T}^{m-n}$ embeds as an open subset of $\mathbb{R}^{m+n}$, the set of homeomorphisms $\{\overline{q}_I \colon I \in \mathscr{K}\}$ provides an atlas for $(\mathscr{K})/R$. The changes of coordinates $\overline{q}_J\overline{q}_I^{-1} \colon \mathbb{C}^n \times \mathbb{T}^{m-n} \to \mathbb{C}^n \times \mathbb{T}^{m-n}$ are smooth by inspection; thus $U(\mathscr{K})/R$ is a smooth manifold.

Remark. The set of homeomorphisms $\{q_I \colon (\mathbb{R}_\geqslant,\mathbb{R}_>)^I/R \to \mathbb{R}^n_\geqslant\}$ defines an atlas for the smooth manifold with corners $\mathscr{Z}_\mathscr{K}/\mathbb{T}^m$. If $\mathscr{K} = \mathscr{K}_P$ for some simple polytope P, then this smooth structure with corners coincides with that of P.

It remains to prove the statement (c), that is, to identify $U(\mathscr{K})/R$ with $\mathscr{Z}_\mathscr{K}$. If X is a Hausdorff locally compact space with a proper G-action, and $Y \subset X$ a compact subspace which intersects every G-orbit at a single point, then Y is homeomorphic to the orbit space X/G. Therefore, we need to verify that each R-orbit intersects $\mathscr{Z}_\mathscr{K} \subset U(\mathscr{K})$ at a single point. We first prove that the R-orbit of any point $\mathbf{y} \in U(\mathscr{K})/\mathbb{T}^m = (\mathbb{R}_\geqslant,\mathbb{R}_>)^\mathscr{K}$ intersects $\mathscr{Z}_\mathscr{K}/\mathbb{T}^m$ at a single point. For this we use the cubical decomposition $\operatorname{cc}(\mathscr{K}) = (\mathbb{I},1)^\mathscr{K}$ of $\mathscr{Z}_\mathscr{K}/\mathbb{T}^m$ (see Example 7.10.2).

Assume first that $\mathbf{y} \in \mathbb{R}^m_>$. The R-action on $\mathbb{R}^m_>$ is obtained by exponentiating the linear action of $\operatorname{Ker} A$ on $\mathbb{R}^m$. Consider the subset $(\mathbb{R}_\leqslant,0)^\mathscr{K} \subset \mathbb{R}^m$, where $\mathbb{R}_\leqslant$ denotes the set of non-positive real numbers. It is taken by the exponential map $\exp \colon \mathbb{R}^m \to \mathbb{R}^m_>$ homeomorphically onto $\operatorname{cc}^\circ(\mathscr{K}) = ((0,1],1)^\mathscr{K} \subset \mathbb{R}^m_>$, where $(0,1]$ denotes the half-open interval $\{y \in \mathbb{R} \colon 0 < y \leqslant 1\}$. The map

$$A \colon (\mathbb{R}_\leqslant,0)^\mathscr{K} \to N_\mathbb{R} \tag{9.4}$$

takes every set $(\mathbb{R}_\leqslant,0)^I$ to $-\sigma$, where $\sigma \in \Sigma$ is the cone corresponding to $I \in \mathscr{K}$. Since Σ is complete, the map (9.4) is one-to-one.

The orbit of $\mathbf{y}$ under the action of R consists of points $\mathbf{w} \in \mathbb{R}^m_>$ such that $\exp A\mathbf{w} = \exp A\mathbf{y}$. Since $A\mathbf{y} \in N_\mathbb{R}$ and the map (9.4) is one-to-one, there is a unique point $\mathbf{y}' \in (\mathbb{R}_\leqslant,0)^\mathscr{K}$ such that $A\mathbf{y}' = A\mathbf{y}$. And since $\exp A\mathbf{y}' \subset \operatorname{cc}^\circ(\mathscr{K})$, the R-orbit of $\mathbf{y}$ intersects $\operatorname{cc}^\circ(\mathscr{K})$ and therefore $\operatorname{cc}(\mathscr{K})$ at a unique point.

Now let $\mathbf{y} \in (\mathbb{R}_\geqslant,\mathbb{R}_>)^\mathscr{K}$ be an arbitrary point. Let $\omega(\mathbf{y}) \in \mathscr{K}$ be the set of zero coordinates of $\mathbf{y}$, and let $\sigma \in \Sigma$ be the cone corresponding to $\omega(\mathbf{y})$. The cones containing σ constitute a fan $\operatorname{St}\sigma$ (called the *star* of σ) in the quotient space $N_\mathbb{R}/\mathbb{R}\langle \mathbf{a}_i \colon i \in \omega(\mathbf{y})\rangle$. The underlying simplicial complex of $\operatorname{St}\sigma$ is the *link* $\operatorname{lk}\omega(\mathbf{y})$ of $\omega(\mathbf{y})$ in $\mathscr{K}$. Observe now that the action of R on the set

$$\{(y_1,\dots,y_m) \in (\mathbb{R}_\geqslant,\mathbb{R}_>)^\mathscr{K} \colon y_i = 0 \text{ for } i \in \omega(\mathbf{y})\} \cong (\mathbb{R}_\geqslant,\mathbb{R}_>)^{\operatorname{lk}\omega(\mathbf{y})}$$

coincides with the action of the group $R_{\operatorname{St}\sigma}$ (defined by the fan $\operatorname{St}\sigma$). Here we can repeat the above arguments for the complete fan $\operatorname{St}\sigma$ and the action of $R_{\operatorname{St}\sigma}$ on $(\mathbb{R}_\geqslant,\mathbb{R}_>)^{\operatorname{lk}\omega(\mathbf{y})}$. As a result, we get that each R-orbit intersects $\operatorname{cc}(\mathscr{K})$ at a unique point.

To finish the proof of (c) we consider the commutative diagram

$$\begin{tikzcd}
\mathscr{Z}_\mathscr{K} \arrow[r] \arrow[d] & U(\mathscr{K}) \arrow[d, "\pi"] \\
\operatorname{cc}(\mathscr{K}) \arrow[r] & (\mathbb{R}_\geqslant,\mathbb{R}_>)^\mathscr{K}
\end{tikzcd}$$

where the horizontal arrows are embeddings and the vertical ones are projections onto the quotients of $\mathbb{T}^m$-actions. Note that the projection π commutes with the R-actions on $U(\mathscr{K})$ and $(\mathbb{R}_{\geqslant}, \mathbb{R}_{>})^{\mathscr{K}}$, and the subgroups R and $\mathbb{T}^m$ of $(\mathbb{C}^\times)^m$ intersect trivially. It follows that each R-orbit intersects the full pre-image $\pi^{-1}(\mathrm{cc}(\mathscr{K})) = \mathscr{Z}_{\mathscr{K}}$ at a unique point. Indeed, assume that $\mathbf{z}$ and $r\mathbf{z}$ are in $\mathscr{Z}_{\mathscr{K}}$ for some $\mathbf{z} \in U(\mathscr{K})$ and $r \in R$. Then $\pi(\mathbf{z})$ and $\pi(r\mathbf{z}) = r\pi(\mathbf{z})$ are in $\mathrm{cc}(\mathscr{K})$, which implies that $\pi(\mathbf{z}) = \pi(r\mathbf{z})$. Hence $\mathbf{z} = \mathbf{t}r\mathbf{z}$ for some $\mathbf{t} \in \mathbb{T}^m$. We may assume that $\mathbf{z} \in (\mathbb{C}^\times)^m$, so that the action of both R and $\mathbb{T}^m$ is free (otherwise consider the action on $U(\operatorname{lk}\omega(\mathbf{z}))$). It follows that $\mathbf{t}r = \mathbf{1}$, which implies that $r = \mathbf{1}$, since R and $\mathbb{T}^m$ intersect trivially. □

We do not know if Theorem 9.2 generalizes to other sphere triangulations.

Question 9.3. Describe the class of sphere triangulations $\mathscr{K}$ for which the moment-angle manifold $\mathscr{Z}_{\mathscr{K}}$ admits a smooth structure.

Remark. Even if $\mathscr{Z}_{\mathscr{K}}$ admits a smooth structure for some simplicial complex $\mathscr{K}$ not arising from a fan, such a structure does not come from a quotient $U(\mathscr{K})/R$ determined by data $\{\mathscr{K}; \mathbf{a}_1, \dots, \mathbf{a}_m\}$. In fact, the R-action on $U(\mathscr{K})$ is proper and the quotient $U(\mathscr{K})/R$ is Hausdorff *precisely when* $\{\mathscr{K}; \mathbf{a}_1, \dots, \mathbf{a}_m\}$ defines a fan, that is, the simplicial cones generated by any two subsets $\{\mathbf{a}_i \colon i \in I\}$ and $\{\mathbf{a}_j \colon j \in J\}$ with $I, J \in \mathscr{K}$ can be separated by a hyperplane. This observation is originally due to Bosio [9] (see also [2], § II.3 and [7]).

10. Complex geometry of moment-angle manifolds

Here we show that the even-dimensional moment-angle manifold $\mathscr{Z}_{\mathscr{K}}$ corresponding to a complete simplicial fan Σ admits the structure of a complex manifold. The idea is to replace the action of $R \cong \mathbb{R}^{m-n}_{>}$ on $U(\mathscr{K})$ (whose quotient is $\mathscr{Z}_{\mathscr{K}}$) by a holomorphic action of $\mathbb{C}^{\frac{m-n}{2}}$ on the same space.

In this section we assume that $m - n$ is even. We can always achieve this by adding a ghost vertex with any corresponding vector to our data $\{\mathscr{K}; \mathbf{a}_1, \dots, \mathbf{a}_m\}$; topologically, this amounts to multiplying $\mathscr{Z}_{\mathscr{K}}$ by a circle. Let $\ell = (m - n)/2$.

We identify $\mathbb{C}^m$ (as a real vector space) with $\mathbb{R}^{2m}$ using the map

$$(z_1, \dots, z_m) \mapsto (x_1, y_1, \dots, x_m, y_m),$$

where $z_k = x_k + iy_k$, and we consider the $\mathbb{R}$-linear map

$$\operatorname{Re} \colon \mathbb{C}^m \to \mathbb{R}^m, \qquad (z_1, \dots, z_m) \mapsto (x_1, \dots, x_m).$$

In order to obtain a complex structure on the quotient $\mathscr{Z}_{\mathscr{K}} \cong U(\mathscr{K})/R$, we replace the action of R by the action of a holomorphic subgroup $C \subset (\mathbb{C}^\times)^m$ by means of the following construction.

Construction 10.1. Let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be a configuration of vectors that span $N_{\mathbb{R}} \cong \mathbb{R}^n$. Assume further that $m - n = 2\ell$ is even. Some of the vectors $\mathbf{a}_i$ may be zero. Recall the map $A \colon \mathbb{R}^m \to N_{\mathbb{R}}$ with $\mathbf{e}_i \mapsto \mathbf{a}_i$.

We choose a complex ℓ-dimensional subspace of $\mathbb{C}^m$ which projects isomorphically onto the real $(m - n)$-dimensional subspace $\operatorname{Ker} A \subset \mathbb{R}^m$. More precisely, let $\mathfrak{c} \cong \mathbb{C}^\ell$, and choose a linear map $\Psi \colon \mathfrak{c} \to \mathbb{C}^m$ satisfying the two conditions:

(a) the composite map $\mathfrak{c} \xrightarrow{\Psi} \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m$ is a monomorphism;

(b) the composite map $\mathfrak{c} \xrightarrow{\Psi} \mathbb{C}^m \xrightarrow{\mathrm{Re}} \mathbb{R}^m \xrightarrow{A} N_{\mathbb{R}}$ is zero.

These two conditions are equivalent to the following:

(a′) $\Psi(\mathfrak{c}) \cap \overline{\Psi(\mathfrak{c})} = \{\mathbf{0}\}$;

(b′) $\Psi(\mathfrak{c}) \subset \mathrm{Ker}(A_{\mathbb{C}} \colon \mathbb{C}^m \to N_{\mathbb{C}})$,

where $\overline{\Psi(\mathfrak{c})}$ is the complex conjugate space and $A_{\mathbb{C}} \colon \mathbb{C}^m \to N_{\mathbb{C}}$ is the complexification of the real map $A \colon \mathbb{R}^m \to N_{\mathbb{R}}$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} \mathfrak{c} & \xrightarrow{\Psi} & \mathbb{C}^m & \xrightarrow{\mathrm{Re}} & \mathbb{R}^m & \xrightarrow{A} & N_{\mathbb{R}} \\ & & \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \exp} & & \\ & & (\mathbb{C}^\times)^m & \xrightarrow{|\cdot|} & \mathbb{R}^m_> & & \end{array} \tag{10.1}$$

where the vertical arrows are the coordinatewise exponential maps, and $|\cdot|$ denotes the map $(z_1, \dots, z_m) \mapsto (|z_1|, \dots, |z_m|)$. Now let

$$C_\Psi = \exp \Psi(\mathfrak{c}) = \bigl\{ \bigl(e^{\langle \psi_1, \mathbf{w} \rangle}, \dots, e^{\langle \psi_m, \mathbf{w} \rangle} \bigr) \in (\mathbb{C}^\times)^m \bigr\}, \tag{10.2}$$

where $\mathbf{w} \in \mathfrak{c}$ and $\psi_i \in \mathfrak{c}^*$ is given by the ith coordinate projection $\mathfrak{c} \xrightarrow{\Psi} \mathbb{C}^m \to \mathbb{C}$. Then $C_\Psi \cong \mathbb{C}^\ell$ is a complex analytic (but not algebraic) subgroup of $(\mathbb{C}^\times)^m$, and therefore there is a holomorphic action of C_Ψ on $\mathbb{C}^m$ and on $U(\mathscr{K})$ obtained by restriction.

Example 10.2. Let $\mathbf{a}_1, \dots, \mathbf{a}_m$ be the configuration of $m = 2\ell$ zero vectors. We supplement it by the empty simplicial complex $\mathscr{K}$ on $[m]$ (with m ghost vertices), so that the data $\{\mathscr{K}; \mathbf{a}_1, \dots, \mathbf{a}_m\}$ define a complete fan in 0-dimensional space. Then $A \colon \mathbb{R}^m \to \mathbb{R}^0$ is a zero map and the condition (b) of Construction 10.1 is void. The condition (a) means that $\mathfrak{c} \xrightarrow{\Psi} \mathbb{C}^{2\ell} \xrightarrow{\mathrm{Re}} \mathbb{R}^{2\ell}$ is an isomorphism of real spaces.

Consider the quotient $(\mathbb{C}^\times)^m / C_\Psi$ (note that $U(\mathscr{K}) = (\mathbb{C}^\times)^m$ in our case). The exponential map $\mathbb{C}^m \to (\mathbb{C}^\times)^m$ identifies $(\mathbb{C}^\times)^m$ with the quotient of $\mathbb{C}^m$ by the imaginary lattice $\Gamma = \mathbb{Z}\langle 2\pi i \mathbf{e}_1, \dots, 2\pi i \mathbf{e}_m \rangle$. The condition (a) implies that the projection $p \colon \mathbb{C}^m \to \mathbb{C}^m / \Psi(\mathfrak{c})$ is non-singular on the imaginary subspace of $\mathbb{C}^m$. In particular, $p(\Gamma)$ is a lattice of rank $m = 2\ell$ in $\mathbb{C}^m / \Psi(\mathfrak{c}) \cong \mathbb{C}^\ell$. Therefore,

$$(\mathbb{C}^\times)^m / C_\Psi \cong (\mathbb{C}^m / \Gamma) / \Psi(\mathfrak{c}) = \bigl(\mathbb{C}^m / \Psi(\mathfrak{c}) \bigr) / p(\Gamma) \cong \mathbb{C}^\ell / \mathbb{Z}^{2\ell}$$

is a complex compact ℓ-dimensional torus.

Any complex torus can be obtained in this way. Indeed, let $\Psi \colon \mathfrak{c} \to \mathbb{C}^m$ be given by a $(2\ell) \times \ell$ matrix $\begin{pmatrix} -B \\ E \end{pmatrix}$, where E is the identity matrix and B is a square matrix of size ℓ. Then $p \colon \mathbb{C}^m \to \mathbb{C}^m / \Psi(\mathfrak{c})$ is given by the matrix $(E \ B)$ in appropriate bases, and $(\mathbb{C}^\times)^m / C_\Psi$ is isomorphic to the quotient of $\mathbb{C}^\ell$ by the lattice $\mathbb{Z}\langle \mathbf{e}_1, \dots, \mathbf{e}_\ell, \mathbf{b}_1, \dots, \mathbf{b}_\ell \rangle$, where $\mathbf{b}_k$ is the kth column of B. (The condition (b) implies that the imaginary part of B is non-singular.)

For example, if $\ell = 1$, then $\Psi \colon \mathbb{C} \to \mathbb{C}^2$ is given by $w \mapsto (\beta w, w)$ for some $\beta \in \mathbb{C}$, so that the subgroup (10.2) is

$$C_\Psi = \{ (e^{\beta w}, e^w) \} \subset (\mathbb{C}^\times)^2.$$

The condition (a) implies that $\beta \notin \mathbb{R}$. Then $\exp\Psi\colon \mathbb{C} \to (\mathbb{C}^\times)^2$ is an embedding, and

$$(\mathbb{C}^\times)^2/C_\Psi \cong \mathbb{C}/(\mathbb{Z} \oplus \beta\mathbb{Z}) = T^1_{\mathbb{C}}(\beta)$$

is a complex 1-dimensional torus with lattice parameter $\beta \in \mathbb{C}$.

Theorem 10.3 [54]. *Assume that the data $\{\mathscr{K}; \mathbf{a}_1, \dots, \mathbf{a}_m\}$ define a complete fan Σ in $N_{\mathbb{R}} \cong \mathbb{R}^n$ and $m - n = 2\ell$. Let $C_\Psi \cong \mathbb{C}^\ell$ be given by* (10.2). *Then:*

(a) *the holomorphic action of C_Ψ on $U(\mathscr{K})$ is free and proper, and the quotient $U(\mathscr{K})/C_\Psi$ is a compact complex manifold*;

(b) *$U(\mathscr{K})/C_\Psi$ is diffeomorphic to the moment-angle manifold $\mathscr{Z}_{\mathscr{K}}$.*

Therefore, $\mathscr{Z}_{\mathscr{K}}$ has a complex structure, in which each element of $\mathbb{T}^m$ acts by a holomorphic transformation.

Remark. A result similar to Theorem 10.3 was obtained by Tambour [57], but his approach was somewhat different: he constructed complex structures on manifolds $\mathscr{Z}_{\mathscr{K}}$ arising from *rationally* starshaped spheres $\mathscr{K}$ (underlying complexes of complete rational simplicial fans) by relating them to a class of generalized LVM-manifolds described by Bosio in [9].

Proof of Theorem 10.3. We first prove the statement (a). The stabilizer subgroups of the $(\mathbb{C}^\times)^m$-action on $U(\mathscr{K})$ are of the form $(\mathbb{C}^\times, 1)^I$ for $I \in \mathscr{K}$. In order to show that $C_\Psi \subset (\mathbb{C}^\times)^m$ acts freely we need to check that C_Ψ has trivial intersection with any stabilizer subgroup of $(\mathbb{C}^\times)^m$. Since C_Ψ embeds into $\mathbb{R}^m_>$ by (10.1), it is enough to check that the image of C_Ψ in $\mathbb{R}^m_>$ intersects the image of $(\mathbb{C}^\times, 1)^I$ in $\mathbb{R}^m_>$ trivially. The former image is R and the latter image is $(\mathbb{R}_>, 1)^I$; the triviality of their intersection follows from Theorem 9.2 (a).

We now prove the properness of this action. Consider the projection $\pi\colon U(\mathscr{K}) \to (\mathbb{R}_\geqslant, \mathbb{R}_>)^{\mathscr{K}}$ onto the quotient of the $\mathbb{T}^m$-action and the commutative square

$$\begin{array}{ccc}
C_\Psi \times U(\mathscr{K}) & \xrightarrow{h_{\mathbb{C}}} & U(\mathscr{K}) \times U(\mathscr{K}) \\
\Big\downarrow{\scriptstyle f\times\pi} & & \Big\downarrow{\scriptstyle \pi\times\pi} \\
R \times (\mathbb{R}_\geqslant, \mathbb{R}_>)^{\mathscr{K}} & \xrightarrow{h_{\mathbb{R}}} & (\mathbb{R}_\geqslant, \mathbb{R}_>)^{\mathscr{K}} \times (\mathbb{R}_\geqslant, \mathbb{R}_>)^{\mathscr{K}}
\end{array}$$

where $h_{\mathbb{C}}$ and $h_{\mathbb{R}}$ denote the group action maps and $f\colon C_\Psi \to R$ is the isomorphism given by the restriction of $|\cdot|\colon (\mathbb{C}^\times)^m \to \mathbb{R}^m_>$. The pre-image $h_{\mathbb{C}}^{-1}(V)$ of a compact subset $V \in U(\mathscr{K}) \times U(\mathscr{K})$ is a closed subset of $W = (f \times \pi)^{-1} \circ h_{\mathbb{R}}^{-1} \circ (\pi \times \pi)(V)$. The image $(\pi \times \pi)(V)$ is compact, the action of R on $(\mathbb{R}_\geqslant, \mathbb{R}_>)^{\mathscr{K}}$ is proper by Theorem 9.2 (a), and the map $f \times \pi$ is proper as the quotient projection for a compact group action. Hence W is a compact subset of $C_\Psi \times U(\mathscr{K})$, and $h_{\mathbb{C}}^{-1}(V)$ is compact as a closed subset of W.

The group $C_\Psi \cong \mathbb{C}^l$ acts holomorphically, freely, and properly on the complex manifold $U(\mathscr{K})$, and therefore the quotient manifold $U(\mathscr{K})/C_\Psi$ has a complex structure.

As in the proof of Theorem 9.2, it is possible to describe a holomorphic atlas of $U(\mathscr{K})/C_\Psi$. Since the action of C_Ψ on the quotient $U(\mathscr{K})/\mathbb{T}^m = (\mathbb{R}_\geqslant, \mathbb{R}_>)^{\mathscr{K}}$ coincides with the action of R on the same space, the quotient of $U(\mathscr{K})/C_\Psi$ by the action of $\mathbb{T}^m$ has exactly the same structure of a smooth manifold with corners as the quotient of $U(\mathscr{K})/R$ by $\mathbb{T}^m$ (see the proof of Theorem 9.2). This structure is determined by the atlas $\{q_I\colon (\mathbb{R}_\geqslant,\mathbb{R}_>)^I/R\to\mathbb{R}^n_\geqslant\}$, which lifts to a covering of $U(\mathscr{K})/C_\Psi$ by the open subsets $(\mathbb{C},\mathbb{C}^\times)^I/C_\Psi$. For any $I\in\mathscr{K}$ the subset $(\mathbb{C},\mathbb{T})^I\subset(\mathbb{C},\mathbb{C}^\times)^I$ intersects each orbit of the C_Ψ-action on $(\mathbb{C},\mathbb{C}^\times)^I$ transversely at a single point. Therefore, every $(\mathbb{C},\mathbb{C}^\times)^I/C_\Psi\cong(\mathbb{C},\mathbb{T})^I$ acquires the structure of a complex manifold. Since $(\mathbb{C},\mathbb{C}^\times)^I\cong\mathbb{C}^n\times(\mathbb{C}^\times)^{m-n}$ and the action of C_Ψ on the $(\mathbb{C}^\times)^{m-n}$ factor is free, the complex manifold $(\mathbb{C},\mathbb{C}^\times)^I/C_\Psi$ is the total space of a holomorphic $\mathbb{C}^n$-bundle over the complex torus $(\mathbb{C}^\times)^{m-n}/C_\Psi$ (see Example 10.2). Writing trivializations of these $\mathbb{C}^n$-bundles for every I, we obtain a holomorphic atlas for $U(\mathscr{K})/C_\Psi$.

The proof of the statement (b) follows the lines of the proof of Theorem 9.2 (b). We need to show that each C_Ψ-orbit intersects $\mathscr{Z}_\mathscr{K}\subset U(\mathscr{K})$ at a single point. First we show that the C_Ψ-orbit of any point in $U(\mathscr{K})/\mathbb{T}^m$ intersects $\mathscr{Z}_\mathscr{K}/\mathbb{T}^m=\operatorname{cc}(\mathscr{K})$ at a single point; this follows from the fact that the actions of C_Ψ and R coincide on $U(\mathscr{K})/\mathbb{T}^m$. Then we show that each C_Ψ-orbit intersects the pre-image $\pi^{-1}(\operatorname{cc}(\mathscr{K}))$ at a single point by using the fact that C_Ψ and $\mathbb{T}^m$ have trivial intersection in $(\mathbb{C}^\times)^m$. $\square$

Example 10.4 (Hopf manifold). Let $\mathbf{a}_1,\dots,\mathbf{a}_{n+1}$ be a set of vectors which span $N_\mathbb{R}\cong\mathbb{R}^n$ and satisfy a linear relation $\lambda_1\mathbf{a}_1+\dots+\lambda_{n+1}\mathbf{a}_{n+1}=\mathbf{0}$ with all $\lambda_k>0$. Let Σ be the complete simplicial fan in $N_\mathbb{R}$ whose cones are generated by all the proper subsets of $\mathbf{a}_1,\dots,\mathbf{a}_{n+1}$. To make $m-n$ even we add one more ghost vector $\mathbf{a}_{n+2}$. Hence $m=n+2$, $\ell=1$, and we have one more linear relation $\mu_1\mathbf{a}_1+\dots+\mu_{n+1}\mathbf{a}_{n+1}+\mathbf{a}_{n+2}=\mathbf{0}$ with $\mu_k\in\mathbb{R}$. The subspace $\operatorname{Ker}A\subset\mathbb{R}^{n+2}$ is spanned by $(\lambda_1,\dots,\lambda_{n+1},0)$ and $(\mu_1,\dots,\mu_{n+1},1)$.

Then $\mathscr{K}=\mathscr{K}_\Sigma$ is the boundary of an n-dimensional simplex with $n+1$ vertices and one ghost vertex, $\mathscr{Z}_\mathscr{K}\cong S^{2n+1}\times S^1$, and $U(\mathscr{K})=(\mathbb{C}^{n+1}\setminus\{\mathbf{0}\})\times\mathbb{C}^\times$.

The conditions (a) and (b) of Construction 10.1 imply that C_Ψ is a 1-dimensional subgroup of $(\mathbb{C}^\times)^m$ given in appropriate coordinates by

$$C_\Psi=\bigl\{(e^{\zeta_1 w},\dots,e^{\zeta_{n+1}w},e^w)\colon w\in\mathbb{C}\bigr\}\subset(\mathbb{C}^\times)^m,$$

where $\zeta_k=\mu_k+\alpha\lambda_k$ for some $\alpha\in\mathbb{C}\setminus\mathbb{R}$. By changing the basis of $\operatorname{Ker}A$ if necessary we may assume that $\alpha=i$. The moment-angle manifold $\mathscr{Z}_\mathscr{K}\cong S^{2n+1}\times S^1$ acquires a complex structure as the quotient $U(\mathscr{K})/C_\Psi$:

$$\begin{aligned}\bigl(\mathbb{C}^{n+1}\setminus\{\mathbf{0}\}\bigr)\times\mathbb{C}^\times/\bigl\{(z_1,\dots,z_{n+1},t)\sim(e^{\zeta_1 w}z_1,\dots,e^{\zeta_{n+1}w}z_{n+1},e^w t)\bigr\}\\ \cong\bigl(\mathbb{C}^{n+1}\setminus\{\mathbf{0}\}\bigr)/\bigl\{(z_1,\dots,z_{n+1})\sim(e^{2\pi i\zeta_1}z_1,\dots,e^{2\pi i\zeta_{n+1}}z_{n+1})\bigr\},\end{aligned}$$

where $\mathbf{z}\in\mathbb{C}^{n+1}\setminus\{\mathbf{0}\}$ and $t\in\mathbb{C}^\times$. The right-hand side is the quotient of $\mathbb{C}^{n+1}\setminus\{\mathbf{0}\}$ by a diagonalizable action of $\mathbb{Z}$. It is known as a *Hopf manifold*. For $n=0$ we obtain the complex torus (elliptic curve) in Example 10.2.

Theorem 10.3 can be generalized to the quotients of $\mathscr{Z}_\mathscr{K}$ by freely acting subgroups $H\subset\mathbb{T}^m$ (see [14], § 8.5). These include both toric quotients and LVM-manifolds.

Construction 10.5. Let Σ be a complete simplicial fan in $N_{\mathbb{R}}$ defined by the data $\{\mathscr{K};\mathbf{a}_1,\dots,\mathbf{a}_m\}$, and let $H\subset\mathbb{T}^m$ be a subgroup which acts freely on the corresponding moment-angle manifold $\mathscr{Z}_{\mathscr{K}}$. Then H is a product of a torus and a finite group, and $h=\dim H\leqslant m-n$ by Proposition 6.7 (H must intersect trivially with an n-dimensional coordinate subtorus in $\mathbb{T}^m$). Under an additional assumption on H, we shall define a holomorphic subgroup D of $(\mathbb{C}^\times)^m$ and introduce a complex structure on $\mathscr{Z}_{\mathscr{K}}/H$ by identifying it with the quotient $U(\mathscr{K})/D$.

The additional assumption is compatibility with the fan data. We recall the map $A_{\mathbb{R}}\colon\mathbb{R}^m\to N_{\mathbb{R}}$, $\mathbf{e}_i\mapsto\mathbf{a}_i$, and let $\mathfrak{h}\subset\mathbb{R}^m$ be the Lie algebra of the subgroup $H\subset\mathbb{T}^m$. Assume that $\mathfrak{h}\subset\operatorname{Ker}A_{\mathbb{R}}$. We also assume that $m-n-h=2\ell$ is even (this can be satisfied by adding a zero vector to $\mathbf{a}_1,\dots,\mathbf{a}_m$). Let $T=\mathbb{T}^m/H$ be the quotient torus, $\mathfrak{t}$ its Lie algebra, and $\rho\colon\mathbb{R}^m\to\mathfrak{t}$ the map of Lie algebras corresponding to the quotient projection $\mathbb{T}^m\to T$.

Let $\mathfrak{c}\cong\mathbb{C}^\ell$ and choose a linear map $\Omega\colon\mathfrak{c}\to\mathbb{C}^m$ satisfying the two conditions:

(a) the composite map $\mathfrak{c}\xrightarrow{\Omega}\mathbb{C}^m\xrightarrow{\operatorname{Re}}\mathbb{R}^m\xrightarrow{\rho}\mathfrak{t}$ is a monomorphism;

(b) the composite map $\mathfrak{c}\xrightarrow{\Omega}\mathbb{C}^m\xrightarrow{\operatorname{Re}}\mathbb{R}^m\xrightarrow{A}N_{\mathbb{R}}$ is zero.

Equivalently, choose a complex subspace $\mathfrak{c}\subset\mathfrak{t}_{\mathbb{C}}$ such that the composite map $\mathfrak{c}\to\mathfrak{t}_{\mathbb{C}}\xrightarrow{\operatorname{Re}}\mathfrak{t}$ is a monomorphism.

As in Construction 10.1, $\exp\Omega(\mathfrak{c})\subset(\mathbb{C}^\times)^m$ is a holomorphic subgroup isomorphic to $\mathbb{C}^\ell$. Let $H_{\mathbb{C}}\subset(\mathbb{C}^\times)^m$ be the complexification of H (it is a product of an algebraic torus of dimension h and a finite group). It follows from (a) that the subgroups $H_{\mathbb{C}}$ and $\exp\Omega(\mathfrak{c})$ intersect trivially in $(\mathbb{C}^\times)^m$. We therefore define a complex $(h+\ell)$-dimensional subgroup

$$D_{H,\Omega}=H_{\mathbb{C}}\times\exp\Omega(\mathfrak{c})\subset(\mathbb{C}^\times)^m. \tag{10.3}$$

Theorem 10.6 ([54], Theorem 3.7). *Let Σ, $\mathscr{K}$, and $D_{H,\Omega}$ be as above. Then*:

(a) *the holomorphic action of the group $D_{H,\Omega}$ on $U(\mathscr{K})$ is free and proper, and the quotient $U(\mathscr{K})/D_{H,\Omega}$ has the structure of a compact complex manifold of complex dimension $m-h-\ell$;*

(b) *there is a diffeomorphism between $U(\mathscr{K})/D_{H,\Omega}$ and $\mathscr{Z}_{\mathscr{K}}/H$ defining a complex structure on the quotient $\mathscr{Z}_{\mathscr{K}}/H$ in which each element of $T=\mathbb{T}^m/H$ acts by a holomorphic transformation.*

The proof is similar to that of Theorem 10.3 and is omitted.

Example 10.7. 1. If H is trivial ($h=0$) then we obtain Theorem 10.3.

2. Let H be the diagonal circle in $\mathbb{T}^m$. The condition $\mathfrak{h}\subset\operatorname{Ker}A_{\mathbb{R}}$ implies that the sum of the vectors $\mathbf{a}_1,\dots,\mathbf{a}_m$ is zero; this can always be achieved by rescaling them (since Σ is a complete fan). As a result, we obtain a complex structure on the quotient $\mathscr{Z}_{\mathscr{K}}/S^1$ by the diagonal circle in $\mathbb{T}^m$, provided that $m-n$ is odd. In the polytopal case $\mathscr{K}=\mathscr{K}_P$, the quotient $\mathscr{Z}_{\mathscr{K}}/S^1$ embeds into $\mathbb{C}^m\setminus\{\mathbf{0}\}/\mathbb{C}^\times=\mathbb{C}P^{m-1}$ as an intersection of homogeneous quadrics (8.2), and the complex structure on $\mathscr{Z}_{\mathscr{K}}/S^1$ is that of an *LVM-manifold* (see § 8).

3. Let $h=\dim H=m-n$. Then $\mathfrak{h}=\operatorname{Ker}A$. Since $\mathfrak{h}$ is the Lie algebra of a torus, the $(m-n)$-dimensional subspace $\operatorname{Ker}A\subset\mathbb{R}^m$ is rational. By Gale duality, this implies that the fan Σ is also rational. We have $\ell=0$, $D_{H,\Omega}=H_{\mathbb{C}}\cong(\mathbb{C}^\times)^{m-n}$, and $U(\mathscr{K})/H_{\mathbb{C}}=\mathscr{Z}_{\mathscr{K}}/H$ is the toric variety corresponding to Σ.

As was shown by Ishida [34], any compact complex manifold with a *maximal* effective holomorphic action of a torus is biholomorphic to a quotient $\mathcal{Z}_{\mathcal{K}}/H$ of a moment-angle manifold with a complex structure described by Theorem 10.6. (An effective action of T^k on an m-dimensional manifold M is said to be *maximal* if there exists a point $x \in M$ whose stabilizer has dimension $m-k$; the two extreme cases are the free action of a torus on itself and the half-dimensional torus action on a toric manifold.) The argument in [36] for recovering a fan Σ from a maximal holomorphic torus action builds on the papers [37] and [38], where this result was proved in particular cases. The main result in [38] provides the following purely complex-analytic description of toric manifolds V_Σ.

Theorem 10.8 ([38], Theorem 1). *Let M be a compact connected complex manifold of complex dimension n, equipped with an effective action of T^n by holomorphic transformations. If the action has fixed points, then there exist a complete regular fan Σ and a T^n-equivariant biholomorphism between V_Σ and M.*

11. Holomorphic principal bundles over toric varieties and Dolbeault cohomology

In the case of rational simplicial normal fans Σ_P a construction of Meersseman and Verjovsky [44] identifies the corresponding projective toric variety V_P as the base of a holomorphic principal *Seifert fibration* whose total space is the moment-angle manifold $\mathcal{Z}_P$ with the complex structure of an LVM-manifold, and whose fibre is a compact complex torus of complex dimension $\ell = \frac{m-n}{2}$. (Seifert fibrations are generalizations of holomorphic fibre bundles to the case when the base is an orbifold.) If V_P is a projective toric manifold, then there is a holomorphic free action of a complex ℓ-dimensional torus $T^\ell_{\mathbb{C}}$ on $\mathcal{Z}_P$ with quotient V_P.

Holomorphic (Seifert) fibrations with total space $\mathcal{Z}_{\mathcal{K}}$ were defined in [54] for arbitrary complete rational simplicial fans Σ by using the construction of a complex structure on $\mathcal{Z}_{\mathcal{K}}$ described in the previous section. By an application of the Borel spectral sequence to the holomorphic fibration $\mathcal{Z}_{\mathcal{K}} \to V_\Sigma$, the Dolbeault cohomology of $\mathcal{Z}_{\mathcal{K}}$ can be described and some Hodge numbers can be calculated explicitly.

Here we make the additional assumption that the set of integral linear combinations of the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ is a full-rank lattice (a discrete subgroup isomorphic to $\mathbb{Z}^n$) in $N_{\mathbb{R}} \cong \mathbb{R}^n$. We denote this lattice by $N_{\mathbb{Z}}$ or simply N. This assumption implies that the complete simplicial fan Σ defined by the data $\{\mathcal{K}; \mathbf{a}_1, \dots, \mathbf{a}_m\}$ is *rational*. We also continue assuming that $m-n$ is even and setting $\ell = \frac{m-n}{2}$.

Because of our rationality assumption, the algebraic group G is defined by (6.2). Furthermore, since we defined N as the lattice generated by $\mathbf{a}_1, \dots, \mathbf{a}_m$, the group G is isomorphic to $(\mathbb{C}^\times)^{2\ell}$ (that is, there are no finite factors). We also observe that C_Ψ lies in G as an ℓ-dimensional complex subgroup. This follows from the condition (b′) in Construction 10.1.

The quotient construction (§ 6.4) identifies the toric variety V_Σ with $U(\mathcal{K})/G$, provided that the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ are *primitive* generators of the edges of Σ. In our data $\{\mathcal{K}; \mathbf{a}_1, \dots, \mathbf{a}_m\}$ the vectors $\mathbf{a}_1, \dots, \mathbf{a}_m$ are not necessarily primitive in the lattice N generated by them. Nevertheless, the quotient $U(\mathcal{K})/G$ is still isomorphic to V_Σ (see [2], Proposition II.3.1.7). Indeed, let $\mathbf{a}'_i \in N$ be the primitive generator along $\mathbf{a}_i$, so that $\mathbf{a}_i = r_i\mathbf{a}'_i$ for some positive integer r_i. Then we have a finite branched covering

$$U(\mathscr{K}) \to U(\mathscr{K}), \qquad (z_1,\dots,z_m) \mapsto (z_1^{r_1},\dots,z_m^{r_m}),$$

which maps the group G defined by $\mathbf{a}_1,\dots,\mathbf{a}_m$ to the group G' defined by $\mathbf{a}'_1,\dots,\mathbf{a}'_m$ (see (6.2)). We therefore obtain a covering $U(\mathscr{K})/G \to U(\mathscr{K})/G'$ of the toric variety $V_\Sigma \cong U(\mathscr{K})/G \cong U(\mathscr{K})/G'$ over itself. Having this in mind, we can relate the quotients $V_\Sigma \cong U(\mathscr{K})/G$ and $\mathscr{Z}_{\mathscr{K}} \cong U(\mathscr{K})/C_\Psi$ as follows.

Proposition 11.1. *Assume that the data $\{\mathscr{K};\mathbf{a}_1,\dots,\mathbf{a}_m\}$ define a complete simplicial rational fan Σ, and let G and C_Ψ be the groups defined by* (6.2) *and* (10.2).

(a) *The toric variety V_Σ is identified, as a topological space, with the quotient of $\mathscr{Z}_{\mathscr{K}}$ by the holomorphic action of the complex compact torus G/C_Ψ.*

(b) *If the fan Σ is regular, then V_Σ is the base of a holomorphic principal bundle with total space $\mathscr{Z}_{\mathscr{K}}$ and fibre the complex compact torus G/C_Ψ.*

Proof. To prove (a) we just observe that

$$V_\Sigma = U(\mathscr{K})/G = \bigl(U(\mathscr{K})/C_\Psi\bigr)/(G/C_\Psi) \cong \mathscr{Z}_{\mathscr{K}}/(G/C_\Psi),$$

where we have used Theorem 10.3. The quotient G/C_Ψ is a compact complex ℓ-torus by Example 10.2. To prove (b) we observe that the holomorphic action of G on $U(\mathscr{K})$ is free by Proposition 6.7, and the same is true for the action of G/C_Ψ on $\mathscr{Z}_{\mathscr{K}}$. A holomorphic free action of the torus G/C_Ψ gives rise to a principal bundle. $\square$

Remark. As in the projective situation of [44], if the fan Σ is not regular, then the quotient projection $\mathscr{Z}_{\mathscr{K}} \to V_\Sigma$ in Proposition 11.1 (a) is a holomorphic principal *Seifert fibration* for an appropriate orbifold structure on V_Σ.

For a complex n-dimensional manifold M the space $\Omega^*_{\mathbb{C}}(M)$ of complex differential forms on M decomposes into the direct sum $\bigoplus_{0\leqslant p,q\leqslant n}\Omega^{p,q}(M)$ of the subspaces of (p,q)-*forms*, and the *Dolbeault differential* $\bar\partial\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ is defined. The dimensions $h^{p,q}(M)$ of the Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(M)$ are known as the *Hodge numbers* of M. They are important invariants of the complex structure on M.

The Dolbeault cohomology of a compact complex ℓ-torus $T^\ell_{\mathbb{C}}$ is isomorphic to an exterior algebra on 2ℓ generators:

$$H^{*,*}_{\bar\partial}(T^\ell_{\mathbb{C}}) \cong \Lambda[\xi_1,\dots,\xi_\ell,\eta_1,\dots,\eta_\ell], \tag{11.1}$$

with $\xi_1,\dots,\xi_\ell \in H^{1,0}_{\bar\partial}(T^\ell_{\mathbb{C}})$ the classes of basis holomorphic 1-forms and $\eta_1,\dots,\eta_\ell \in H^{0,1}_{\bar\partial}(T^\ell_{\mathbb{C}})$ the classes of basis antiholomorphic 1-forms. In particular, the Hodge numbers are given by $h^{p,q}(T^\ell_{\mathbb{C}}) = \binom{\ell}{p}\binom{\ell}{q}$.

The de Rham cohomology of a complete non-singular toric variety V_Σ admits a Hodge decomposition with only non-trivial components of bidegree (p,p), $0 \leqslant p \leqslant n$ ([20], § 12). This, together with the cohomology calculation of Danilov and Jurkiewicz ([20], § 10), gives the following description of the Dolbeault cohomology:

$$H^{*,*}_{\bar\partial}(V_\Sigma) \cong \mathbb{C}[v_1,\dots,v_m]/(\mathscr{I}_{\mathscr{K}} + \mathscr{J}_\Sigma), \tag{11.2}$$

where $v_i \in H^{1,1}_{\bar\partial}(V_\Sigma)$ are the cohomology classes corresponding to torus-invariant divisors (one for each one-dimensional cone in Σ), the ideal $\mathscr{I}_{\mathscr{K}}$ is generated by the monomials $v_{i_1}\cdots v_{i_k}$ for which $\mathbf{a}_{i_1},\dots,\mathbf{a}_{i_k}$ do not span a cone in Σ (the *Stanley–Reisner ideal* of $\mathscr{K}$), and the ideal $\mathscr{J}_\Sigma$ is generated by the linear forms $\sum_{j=1}^m\langle\mathbf{a}_j,\mathbf{u}\rangle v_j$ with $\mathbf{u}\in N^*$. We have $h^{p,p}(V_\Sigma)=h_p$, where $(h_0,h_1,\dots,h_n)$ is the *h-vector* of $\mathscr{K}$ ([14], § 2.1) and $h^{p,q}(V_\Sigma)=0$ for $p\neq q$.

Theorem 11.2 [54]. *Assume that the data $\{\mathscr{K};\mathbf{a}_1,\dots,\mathbf{a}_m\}$ define a complete regular fan Σ in $N_{\mathbb{R}}\cong\mathbb{R}^n$, $m-n=2\ell$, and let $\mathscr{Z}_{\mathscr{K}}$ be the corresponding moment-angle manifold with a complex structure defined in Theorem 10.3. Then the Dolbeault cohomology algebra $H^{*,*}_{\bar\partial}(\mathscr{Z}_{\mathscr{K}})$ is isomorphic to the cohomology of the differential bigraded algebra*

$$\bigl[\Lambda[\xi_1,\dots,\xi_\ell,\eta_1,\dots,\eta_\ell]\otimes H^{*,*}_{\bar\partial}(V_\Sigma),d\bigr] \tag{11.3}$$

with differential d of bidegree $(0,1)$ defined on the generators as follows:

$$dv_i=d\eta_j=0,\quad d\xi_j=c(\xi_j),\qquad 1\leqslant i\leqslant m,\quad 1\leqslant j\leqslant \ell,$$

where $c\colon H^{1,0}_{\bar\partial}(T^\ell_{\mathbb{C}})\to H^2(V_\Sigma,\mathbb{C})=H^{1,1}_{\bar\partial}(V_\Sigma)$ is the first Chern class map of the principal $T^\ell_{\mathbb{C}}$-bundle $\mathscr{Z}_{\mathscr{K}}\to V_\Sigma$.

Proof. We use the notion of a minimal Dolbeault model of a complex manifold [27], § 4.3. Let $[B,d_B]$ be such a model for V_Σ, that is, $[B,d_B]$ is a minimal commutative bigraded differential algebra together with a quasi-isomorphism $f\colon B^{*,*}\to\Omega^{*,*}(V_\Sigma)$ (that is, f commutes with the differentials d_B and $\bar\partial$, and induces an isomorphism in cohomology). Consider the differential bigraded algebra

$$\begin{gathered}\bigl[\Lambda[\xi_1,\dots,\xi_\ell,\eta_1,\dots,\eta_\ell]\otimes B,d\bigr],\\ \text{where}\quad d\big|_B=d_B,\quad d(\xi_i)=c(\xi_i)\in B^{1,1}=H^{1,1}_{\bar\partial}(V_\Sigma),\quad d(\eta_i)=0.\end{gathered} \tag{11.4}$$

By [27], Corollary 4.66, this algebra gives a model for the Dolbeault cohomology algebra of the total space $\mathscr{Z}_{\mathscr{K}}$ of the principal $T^\ell_{\mathbb{C}}$-bundle $\mathscr{Z}_{\mathscr{K}}\to V_\Sigma$, provided that V_Σ is strictly formal. We recall from [27], Definition 4.58 that a complex manifold M is *strictly formal* if there exists a differential bigraded algebra $[Z,\delta]$ together with quasi-isomorphisms

$$\begin{tikzcd}
{[\Omega^{*,*},\bar\partial]} & {[Z,\delta]} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, "\simeq"] & {[\Omega^*,d_{\mathrm{DR}}]} \\
 & {[H^{*,*}_{\bar\partial}(M),0]} &
\end{tikzcd}$$

linking together the de Rham algebra, the Dolbeault algebra, and the Dolbeault cohomology.

The toric manifold V_Σ is formal in the usual (de Rham) sense by [53], Corollary 7.2. The Hodge decomposition in [20], § 12 implies that V_Σ satisfies the $\partial\bar\partial$-lemma ([27], Lemma 4.24). Therefore, V_Σ is strictly formal by the same argument as used for Theorem 4.59 in [27], and (11.4) is a model for its Dolbeault cohomology.

The usual formality of V_Σ implies the existence of a quasi-isomorphism $\phi_B\colon B \to H^{*,*}_{\bar\partial}(V_\Sigma)$, which extends to a quasi-isomorphism

$$\mathrm{id}\otimes\phi_B\colon \bigl[\Lambda[\xi_1,\dots,\xi_\ell,\eta_1,\dots,\eta_\ell]\otimes B, d\bigr] \to \bigl[\Lambda[\xi_1,\dots,\xi_\ell,\eta_1,\dots,\eta_\ell]\otimes H^{*,*}_{\bar\partial}(V_\Sigma), d\bigr]$$

by Lemma 14.2 in [26]. Thus, the differential algebra $\bigl[\Lambda[\xi_1,\dots,\xi_\ell,\eta_1,\dots,\eta_\ell]\otimes H^{*,*}_{\bar\partial}(V_\Sigma), d\bigr]$ provides a model for the Dolbeault cohomology of $\mathscr{Z}_{\mathscr{K}}$, as claimed. □

Remark. If V_Σ is projective, then it is Kähler; in this case the model in Theorem 11.2 coincides with the model for the Dolbeault cohomology of the total space of a holomorphic torus principal bundle over a Kähler manifold ([27], Theorem 4.65).

The first Chern class map c in Theorem 11.2 can be described explicitly in terms of the map $\varPsi$ defining the complex structure on $\mathscr{Z}_{\mathscr{K}}$. We recall the map $A_{\mathbb{C}}\colon \mathbb{C}^m \to N_{\mathbb{C}}$ with $\mathbf{e}_i \mapsto \mathbf{a}_i$ and the Gale dual $(m-n)\times m$ matrix $\varGamma = (\gamma_{jk})$ whose rows form a basis of linear relations between the vectors $\mathbf{a}_1,\dots,\mathbf{a}_m$. By Construction 10.1, $\Im\varPsi \subset \operatorname{Ker} A_{\mathbb{C}}$. Denote by $\operatorname{Ann} U$ the annihilator of a linear subspace $U\subset\mathbb{C}^m$, that is, the subspace of linear functions on $\mathbb{C}^m$ vanishing on U.

Lemma 11.3. *Let k be the number of zero vectors among $\mathbf{a}_1,\dots,\mathbf{a}_m$. The first Chern class map*

$$c\colon H^{1,0}_{\bar\partial}(T^\ell_{\mathbb{C}}) \to H^2(V_\Sigma,\mathbb{C}) = H^{1,1}_{\bar\partial}(V_\Sigma)$$

of the principal $T^\ell_{\mathbb{C}}$-bundle $\mathscr{Z}_{\mathscr{K}} \to V_\Sigma$ is given by the composition

$$\operatorname{Ann}\operatorname{Im}\varPsi/\operatorname{Ann}\operatorname{Ker}A_{\mathbb{C}} \xrightarrow{i} \mathbb{C}^m/\operatorname{Ann}\operatorname{Ker}A_{\mathbb{C}} \xrightarrow{p} \mathbb{C}^{m-k}/\operatorname{Ann}\operatorname{Ker}A_{\mathbb{C}},$$

where i is the inclusion and p is the projection forgetting the coordinates in $\mathbb{C}^m$ corresponding to zero vectors.

Explicitly, the map c is given on the generators of $H^{1,0}_{\bar\partial}(T^\ell_{\mathbb{C}})$ by

$$c(\xi_j) = \mu_{j1}v_1 + \dots + \mu_{jm}v_m, \qquad 1 \leqslant j \leqslant \ell,$$

where $M = (\mu_{jk})$ is an $\ell\times m$ matrix satisfying the two conditions:

(a) *$\varGamma M^t\colon \mathbb{C}^\ell \to \mathbb{C}^{2\ell}$ is a monomorphism*;

(b) *$M\varPsi = 0$*.

Proof. Let $A^t_{\mathbb{C}}\colon N^*_{\mathbb{C}} \to \mathbb{C}^m$, $\mathbf{u}\mapsto(\langle\mathbf{a}_1,\mathbf{u}\rangle,\dots,\langle\mathbf{a}_m,\mathbf{u}\rangle)$, be the dual map. We have

$$H^1(T^\ell_{\mathbb{C}};\mathbb{C}) = \mathbb{C}^m/\operatorname{Im}A^t_{\mathbb{C}} = (\operatorname{Ker}A_{\mathbb{C}})^*, \qquad H^2(V_\Sigma;\mathbb{C}) = \mathbb{C}^{m-k}/\operatorname{Im}A^t_{\mathbb{C}}.$$

The first Chern class map $c\colon H^1(T^\ell_{\mathbb{C}};\mathbb{C}) \to H^2(V_\Sigma;\mathbb{C})$ (the transgression) is then given by $p\colon \mathbb{C}^m/\operatorname{Im}A^t_{\mathbb{C}} \to \mathbb{C}^{m-k}/\operatorname{Im}A^t_{\mathbb{C}}$. To separate out the holomorphic part of c we need to identify the subspace of holomorphic differentials $H^{1,0}_{\bar\partial}(T^\ell_{\mathbb{C}}) \cong \mathbb{C}^\ell$ inside the space of all 1-forms $H^1(T^\ell_{\mathbb{C}};\mathbb{C}) \cong \mathbb{C}^{2\ell}$. Since

$$T^\ell_{\mathbb{C}} = G/C_\varPsi = (\operatorname{Ker}\exp A_{\mathbb{C}})/(\exp\operatorname{Im}\varPsi),$$

holomorphic differentials on $T^\ell_{\mathbb{C}}$ correspond to $\mathbb{C}$-linear functions on $\operatorname{Ker}A_{\mathbb{C}}$ which vanish on $\operatorname{Im}\varPsi$. The space of functions on $\operatorname{Ker}A_{\mathbb{C}}$ is $\mathbb{C}^m/\operatorname{Im}A^t_{\mathbb{C}} = \mathbb{C}^m/\operatorname{Ann}\operatorname{Ker}A_{\mathbb{C}}$, and the functions vanishing on $\operatorname{Im}\varPsi$ form the subspace $\operatorname{Ann}\operatorname{Im}\varPsi/\operatorname{Ann}\operatorname{Ker}A_{\mathbb{C}}$. The condition (b) says exactly that the linear functions on $\mathbb{C}^m$ corresponding to the rows of M vanish on $\operatorname{Im}\varPsi$. The condition (a) says that the rows of M constitute a basis in the complement of $\operatorname{Ann}\operatorname{Ker}A_{\mathbb{C}}$ in $\operatorname{Ann}\operatorname{Im}\varPsi$. □

It is interesting to compare Theorem 11.2 with the following description of the de Rham cohomology of $\mathscr{Z}_{\mathscr{K}}$.

Theorem 11.4 ([14], Theorem 7.36). *Let $\mathscr{Z}_{\mathscr{K}}$ and V_Σ be as in Theorem 11.2. The de Rham cohomology $H^*(\mathscr{Z}_{\mathscr{K}})$ is isomorphic to the cohomology of the differential graded algebra*

$$\bigl[\Lambda[u_1,\dots,u_{m-n}]\otimes H^*(V_\Sigma),d\bigr],$$

where $\deg u_j=1$, $\deg v_i=2$, *and the differential d is defined on the generators as*

$$dv_i=0,\qquad du_j=\gamma_{j1}v_1+\dots+\gamma_{jm}v_m,\quad 1\leqslant j\leqslant m-n.$$

This theorem follows from the more general result ([14], Theorem 7.7) describing the cohomology of $\mathscr{Z}_{\mathscr{K}}$. For more information about $H^*(\mathscr{Z}_{\mathscr{K}})$ see [14], Chap. 8, and [52], § 4.

There are two classical spectral sequences for the Dolbeault cohomology. First, the *Borel spectral sequence* [8] of a holomorphic bundle $E\to B$ with a compact Kähler fibre F, which has $E_2=H_{\bar\partial}(B)\otimes H_{\bar\partial}(F)$ and converges to $H_{\bar\partial}(E)$. Second, the *Frölicher spectral sequence* ([32], § 3.5), whose E_1-term is the Dolbeault cohomology of a complex manifold M and which converges to the de Rham cohomology of M. Theorem 11.2 implies the following collapses for these spectral sequences.

Corollary 11.5. (a) *The Borel spectral sequence of the holomorphic principal bundle $\mathscr{Z}_{\mathscr{K}}\to V_\Sigma$ collapses at the E_3-term, that is, $E_3=E_\infty$.*

(b) *The Frölicher spectral sequence of $\mathscr{Z}_{\mathscr{K}}$ collapses at the E_2-term.*

Proof. To prove (a) we just observe that the differential algebra (11.3) is the E_2-term of the Borel spectral sequence, and its cohomology is the E_3-term.

By comparing the Dolbeault and de Rham cohomology algebras of $\mathscr{Z}_{\mathscr{K}}$ given by Theorems 11.2 and 11.4 we observe that the elements $\eta_1,\dots,\eta_\ell\in E_1^{0,1}$ cannot survive in the E_∞-term of the Frölicher spectral sequence. The only possible non-trivial differential on these elements is $d_1\colon E_1^{0,1}\to E_1^{1,1}$. By Theorem 11.4, the cohomology algebra of $[E_1,d_1]$ is exactly the de Rham cohomology of $\mathscr{Z}_{\mathscr{K}}$, thus proving (b). $\square$

Theorem 11.4 can also be interpreted as a collapse result for the Leray–Serre spectral sequence of the principal T^{m-n}-bundle $\mathscr{Z}_{\mathscr{K}}\to V_\Sigma$.

In order to proceed with our calculation of Hodge numbers, we need the following bounds for the dimension of $\operatorname{Ker}c$ in Lemma 11.3.

Lemma 11.6. *Let k be the number of zero vectors among $\mathbf{a}_1,\dots,\mathbf{a}_m$. Then*

$$k-\ell\leqslant\dim_{\mathbb{C}}\operatorname{Ker}\bigl(c\colon H^{1,0}_{\bar\partial}(T^\ell_{\mathbb{C}})\to H^{1,1}_{\bar\partial}(V_\Sigma)\bigr)\leqslant\frac{k}{2}\,.$$

In particular, c is a monomorphism if $k\leqslant 1$.

Proof. Consider the diagram

$$\begin{tikzcd}
\operatorname{Ann}\operatorname{Im}\varPsi/\operatorname{Ann}\operatorname{Ker}A_{\mathbb{C}} \arrow[r, "i"] \arrow[d, "\cong"] & \mathbb{C}^m/\operatorname{Ann}\operatorname{Ker}A_{\mathbb{C}} \arrow[r, "p"] \arrow[d, "\mathrm{Re}"] & \mathbb{C}^{m-k}/\operatorname{Ann}\operatorname{Ker}A_{\mathbb{C}} \arrow[d, "\mathrm{Re}"] \\
\mathbb{R}^{m-n} \arrow[r, equal] & \mathbb{R}^{m-n} \arrow[r, "p'"] & \mathbb{R}^{m-n-k}
\end{tikzcd}$$

The left vertical arrow has the form $H^{1,0}_{\bar\partial}(T^\ell_{\mathbb C})\to H^1(T^\ell_{\mathbb C},\mathbb C)\to H^1(T^\ell_{\mathbb C},\mathbb R)$, so is an ($\mathbb R$-linear) isomorphism, and any real-valued function on the lattice Γ defining the torus $T^\ell_{\mathbb C}=\mathbb C^\ell/\Gamma$ is the real part of the restriction to Γ of some $\mathbb C$-linear function on $\mathbb C^\ell$.

Since the diagram above is commutative, the kernel of $c=p\circ i$ has real dimension at most k, which implies the upper bound on its complex dimension. For the lower bound, $\dim_{\mathbb C}\operatorname{Ker} c\geqslant \dim H^{1,0}_{\bar\partial}(T^\ell_{\mathbb C})-\dim H^{1,1}_{\bar\partial}(V_\Sigma)=\ell-(2\ell-k)=k-\ell$. □

Theorem 11.7. *Let $\mathscr Z_{\mathscr K}$ be as in Theorem 11.2, and let k be the number of zero vectors among $\mathbf a_1,\dots,\mathbf a_m$. Then the following hold for the Hodge numbers $h^{p,q}=h^{p,q}(\mathscr Z_{\mathscr K})$:*

(a) $\binom{k-\ell}{p}\leqslant h^{p,0}\leqslant\binom{[k/2]}{p}$ *for $p\geqslant 0$, and in particular, $h^{p,0}=0$ for $p>0$ if $k\leqslant 1$;*

(b) $h^{0,q}=\binom{\ell}{q}$ *for $q\geqslant 0$;*

(c) $h^{1,q}=(\ell-k)\binom{\ell}{q-1}+h^{1,0}\binom{\ell+1}{q}$ *for $q\geqslant 1$;*

(d) $\ell(3\ell+1)/2-h_2(\mathscr K)-\ell k+(\ell+1)h^{2,0}\leqslant h^{2,1}\leqslant \ell(3\ell+1)/2-\ell k+(\ell+1)h^{2,0}$.

Proof. Let $A^{p,q}$ denote the bidegree (p,q) component of the differential algebra in Theorem 11.2, and let $Z^{p,q}\subset A^{p,q}$ denote the subspace of d-cocycles. Then $d^{1,0}\colon A^{1,0}\to Z^{1,1}$ coincides with the map c, and the required bounds for $h^{1,0}=\operatorname{Ker} d^{1,0}$ were already established in Lemma 11.6. Since $h^{p,0}=\dim\operatorname{Ker} d^{p,0}$ and $\operatorname{Ker} d^{p,0}$ is the pth exterior power of the space $\operatorname{Ker} d^{1,0}$, the statement (a) follows.

The differential is trivial on $A^{0,q}$, hence $h^{0,q}=\dim A^{0,q}$, thus proving (b).

The space $Z^{1,1}$ is spanned by the cocycles v_i and $\xi_i\eta_j$ with $\xi_i\in\operatorname{Ker} d^{1,0}$. Hence $\dim Z^{1,1}=2\ell-k+h^{1,0}\ell$. Also, $\dim d(A^{1,0})=\ell-h^{1,0}$, hence $h^{1,1}=\ell-k+h^{1,0}(\ell+1)$. Similarly, $\dim Z^{1,q}=(2\ell-k)\binom{\ell}{q-1}+h^{1,0}\binom{\ell}{q}$ (with basis of $v_i\eta_{j_1}\cdots\eta_{j_{q-1}}$ and $\xi_i\eta_{j_1}\cdots\eta_{j_q}$, where $\xi_i\in\operatorname{Ker} d^{1,0}$, $j_1<\dots<j_q$), and the image of $d\colon A^{1,q-1}\to Z^{1,q}$ is a subspace of dimension $(\ell-h^{1,0})\binom{\ell}{q-1}$. This proves (c).

We have $A^{2,1}=U\oplus W$, where U has a basis of monomials ξ_iv_j and W has a basis of monomials $\xi_i\xi_j\eta_k$. Therefore,

$$h^{2,1}=\dim U-\dim dU+\dim W-\dim dW-\dim dA^{2,0}. \tag{11.5}$$

Now $\dim U=\ell(2\ell-k)$, $0\leqslant\dim dU\leqslant h_2(\mathscr K)$ (since $dU\subset H^{2,2}_{\bar\partial}(V_\Sigma)$), $\dim W-\dim dW=\dim\operatorname{Ker} d|_W=\ell h^{2,0}$, and $\dim dA^{2,0}=\binom{\ell}{2}-h^{2,0}$. Substituting all this into (11.5), we obtain the inequalities in (d). □

Remark. At most one ghost vertex needs to be added to $\mathscr K$ to make $\dim\mathscr Z_{\mathscr K}=m+n$ even. Since $h^{p,0}(\mathscr Z_{\mathscr K})=0$ when $k\leqslant 1$, the manifold $\mathscr Z_{\mathscr K}$ does not have holomorphic forms of arbitrary degree in this case.

If $\mathscr Z_{\mathscr K}$ is a torus (so that $\mathscr K$ is empty), then $m=k=2\ell$ and $h^{1,0}(\mathscr Z_{\mathscr K})=h^{0,1}(\mathscr Z_{\mathscr K})=\ell$. Otherwise Theorem 11.7 implies that $h^{1,0}(\mathscr Z_{\mathscr K})<h^{0,1}(\mathscr Z_{\mathscr K})$, and

therefore the moment-angle manifold $\mathscr{Z}_{\mathscr{K}}$ is not Kähler (in the polytopal case this was observed in [43], Theorem 3).

Example 11.8. Let $\mathscr{Z}_{\mathscr{K}} \cong S^1 \times S^{2n+1}$ be the Hopf manifold in Example 10.4. Our rationality assumption is that $\mathbf{a}_1 \dots, \mathbf{a}_{n+2}$ span an n-dimensional lattice N in $N_{\mathbb{R}} \cong \mathbb{R}^n$; in particular, the fan Σ defined by the proper subsets of the set of $\mathbf{a}_1, \dots, \mathbf{a}_{n+1}$ is rational. We assume further that Σ is regular (this is equivalent to the condition $\mathbf{a}_1 + \dots + \mathbf{a}_{n+1} = \mathbf{0}$), so that Σ is the normal fan of a Delzant n-dimensional simplex Δ^n. We have $V_\Sigma = \mathbb{C}P^n$, and (11.2) describes its cohomology as the quotient of $\mathbb{C}[v_1, \dots, v_{n+2}]$ by the sum of two ideals: the ideal $\mathscr{I}$ generated by $v_1 \cdots v_{n+1}$ and v_{n+2}, and the ideal $\mathscr{J}$ generated by $v_1 - v_{n+1}, \dots, v_n - v_{n+1}$. The differential algebra in Theorem 11.2 is therefore given by $\bigl[\Lambda[\xi, \eta] \otimes \mathbb{C}[t]/t^{n+1}, d\bigr]$, with $dt = d\eta = 0$ and $d\xi = t$ for a suitable choice of t. The non-trivial cohomology classes are represented by the cocycles 1, η, ξt^n, and $\xi\eta t^n$, and this gives the following non-zero Hodge numbers of $\mathscr{Z}_{\mathscr{K}}$: $h^{0,0} = h^{0,1} = h^{n+1,n} = h^{n+1,n+1} = 1$. We note that the Dolbeault cohomology and the Hodge numbers do not depend on the choice of complex structure (the map $\varPsi$).

Example 11.9 (Calabi–Eckmann manifold). Let $\{\mathscr{K}; \mathbf{a}_1, \dots, \mathbf{a}_{n+2}\}$ be the data defining the normal fan of the product $P = \Delta^p \times \Delta^q$ of two Delzant simplices with $p + q = n$, $1 \leqslant p \leqslant q \leqslant n-1$. That is, $\mathbf{a}_1, \dots, \mathbf{a}_p, \mathbf{a}_{p+2}, \dots, \mathbf{a}_{n+1}$ is a basis of the lattice N and there are two relations $\mathbf{a}_1 + \dots + \mathbf{a}_{p+1} = \mathbf{0}$ and $\mathbf{a}_{p+2} + \dots + \mathbf{a}_{n+2} = \mathbf{0}$. The corresponding toric variety V_Σ is $\mathbb{C}P^p \times \mathbb{C}P^q$, and its cohomology ring is isomorphic to $\mathbb{C}[x, y]/(x^{p+1}, y^{q+1})$. Consider the map

$$\varPsi \colon \mathbb{C} \to \mathbb{C}^{n+2}, \qquad w \mapsto (1, \dots, 1, \alpha w, \dots, \alpha w),$$

where $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and αw appears $q+1$ times. This map satisfies the conditions of Construction 10.1. The resulting complex structure on $\mathscr{Z}_P \cong S^{2p+1} \times S^{2q+1}$ is that of a *Calabi–Eckmann manifold*. We denote complex manifolds obtained in this way by $CE(p, q)$ (the complex structure depends on the choice of $\varPsi$, but we do not reflect this in the notation). Each manifold $CE(p, q)$ is the total space of a holomorphic principal bundle over $\mathbb{C}P^p \times \mathbb{C}P^q$ with fibre the complex 1-torus $\mathbb{C}/(\mathbb{Z} \oplus \alpha\mathbb{Z})$.

Theorem 11.2 and Lemma 11.3 provide the following description of the Dolbeault cohomology of $CE(p, q)$:

$$H^{*,*}_{\bar\partial}\bigl(CE(p, q)\bigr) \cong H\bigl[\Lambda[\xi, \eta] \otimes \mathbb{C}[x, y]/(x^{p+1}, y^{q+1}), d\bigr],$$

where $dx = dy = d\eta = 0$ and $d\xi = x - y$ for an appropriate choice of the generators x, y. We therefore get that

$$H^{*,*}_{\bar\partial}\bigl(CE(p, q)\bigr) \cong \Lambda[\omega, \eta] \otimes \mathbb{C}[x]/(x^{p+1}), \tag{11.6}$$

where $\omega \in H^{q+1,q}_{\bar\partial}\bigl(CE(p, q)\bigr)$ is the cohomology class of the cocycle $\xi(x^{q+1} - y^{q+1})/(x - y)$. This calculation is originally from [8], § 9. We note that the Dolbeault cohomology of a Calabi–Eckmann manifold depends only on p, q and not on the complex parameter α (nor the map $\varPsi$).

Example 11.10. Let $P = \Delta^1 \times \Delta^1 \times \Delta^2 \times \Delta^2$. Then the moment-angle manifold $\mathscr{Z}_P$ has two different structures of a product of Calabi–Eckmann manifolds, namely, $CE(1,1) \times CE(2,2)$ and $CE(1,2) \times CE(1,2)$. Using the isomorphism (11.6), we observe that these two complex manifolds have different Hodge numbers: $h^{2,1} = 1$ in the first case, and $h^{2,1} = 0$ in the second. This shows that the choice of the map $\varPsi$ affects not only the complex structure of $\mathscr{Z}_{\mathscr{K}}$ but also its Hodge numbers, unlike in the previous examples of complex tori, Hopf manifolds, and Calabi–Eckmann manifolds. Of course, this is not very surprising from the complex-analytic point of view.

12. Hamiltonian-minimal Lagrangian submanifolds

In this last section we apply the accumulated knowledge of the topology of moment-angle manifolds in a somewhat different area: Lagrangian geometry. The systems of real quadrics which we used in §§ 3 and 4 to define moment-angle manifolds also give rise to a family of Hamiltonian-minimal Lagrangian submanifolds in a complex space or in more general toric varieties.

Hamiltonian minimality (H-minimality for short) for Lagrangian submanifolds is a symplectic analogue of minimality in Riemannian geometry. A Lagrangian immersion is said to be H-minimal if the variations of its volume along all Hamiltonian vector fields are zero. This notion was introduced in the paper [51] of Oh in connection with the celebrated *Arnold conjecture* on the number of fixed points of a Hamiltonian symplectomorphism. The simplest example of an H-minimal Lagrangian submanifold is the coordinate torus [51] $S^1_{r_1} \times \dots \times S^1_{r_m} \subset \mathbb{C}^m$, where $S^1_{r_k}$ denotes the circle of radius $r_k > 0$ in the kth coordinate subspace of $\mathbb{C}^m$. More examples of H-minimal Lagrangian submanifolds in a complex space were constructed in the papers [17], [34], [1], among others.

In [46] Mironov proposed a general construction of H-minimal Lagrangian immersions $N \looparrowright \mathbb{C}^m$ arising from intersections of real quadrics. These systems of quadrics are the same as those we used to define moment-angle manifolds, and therefore one can apply toric methods for analysing the topological structure of N. In [47] an effective criterion was obtained for $N \looparrowright \mathbb{C}^m$ to be an embedding: the polytope corresponding to the intersection of quadrics must be Delzant. As a consequence, any Delzant polytope gives rise to an H-minimal Lagrangian submanifold $N \subset \mathbb{C}^m$. As in the case of moment-angle manifolds, the topology of N is quite complicated even for low-dimensional polytopes: for example, a Delzant 5-gon gives rise to a manifold N which is the total space of a bundle over a 3-torus with fibre a surface of genus 5. Furthermore, by combining Mironov's construction with symplectic reduction, a new family of H-minimal Lagrangian submanifolds of toric varieties was defined in [48]. This family includes many previously constructed explicit examples in $\mathbb{C}^m$ and $\mathbb{C}P^{m-1}$.

12.1. Preliminaries. Let (M,ω) be a symplectic manifold of dimension $2n$. An immersion $i\colon N \looparrowright M$ of an n-dimensional manifold N is said to be *Lagrangian* if $i^*(\omega) = 0$. If i is an embedding, then $i(N)$ is a *Lagrangian submanifold* of M. A vector field X on M is *Hamiltonian* if the 1-form $\omega(X,\,\cdot\,)$ is exact.

Assume now that M is Kähler, so that it has a Riemannian metric and a symplectic structure that are compatible. A Lagrangian immersion $i\colon N \looparrowright M$ is said to be *Hamiltonian-minimal* (*H-minimal*) if the variations of the volume of $i(N)$ along all Hamiltonian vector fields with compact support are zero, that is,

$$\frac{d}{dt}\operatorname{vol}\bigl(i_t(N)\bigr)\Big|_{t=0}=0,$$

where $i_t(N)$ is a deformation of $i(N)$ along a Hamiltonian vector field, $i_0(N) = i(N)$, and $\operatorname{vol}(i_t(N))$ is the volume of the deformed part of $i_t(N)$. An immersion i is *minimal* if the variations of the volume of $i(N)$ along *all* vector fields are zero.

Our basic example is $M=\mathbb{C}^m$ with the Hermitian metric $2\sum_{k=1}^m d\overline{z}_k\otimes dz_k$. Its imaginary part is the symplectic form in Example 5.1. In the end we consider the more general case when M is a toric manifold.

12.2. Construction. We consider an intersection of quadrics similar to (3.4), but in the real space:

$$\mathscr{R}=\biggl\{\mathbf{u}=(u_1,\dots,u_m)\in\mathbb{R}^m\colon\ \sum_{k=1}^m\gamma_{jk}u_k^2=\delta_j\ \text{for}\ 1\leqslant j\leqslant m-n\biggr\}. \qquad (12.1)$$

We assume the non-degeneracy and rationality conditions on the coefficient vectors $\gamma_i=(\gamma_{1i},\dots,\gamma_{m-n,i})^t\in\mathbb{R}^{m-n}$, $i=1,\dots,m$:

(a) $\delta\in\mathbb{R}_\geqslant\langle\gamma_1,\dots,\gamma_m\rangle$;

(b) if $\delta\in\mathbb{R}_\geqslant\langle\gamma_{i_1},\dots,\gamma_{i_k}\rangle$, then $k\geqslant m-n$;

(c) the vectors $\gamma_1,\dots,\gamma_m$ generate a lattice $L\cong\mathbb{Z}^{m-n}$ in $\mathbb{R}^{m-n}$.

These conditions guarantee that $\mathscr{R}$ is a smooth n-dimensional submanifold of $\mathbb{R}^m$ (by the argument for Proposition 3.4) and that

$$T_\Gamma=\bigl\{\bigl(e^{2\pi i\langle\gamma_1,\varphi\rangle},\dots,e^{2\pi i\langle\gamma_m,\varphi\rangle}\bigr)\in\mathbb{T}^m\bigr\}$$

is an $(m-n)$-dimensional torus subgroup of $\mathbb{T}^m$. We identify the torus T_Γ with $\mathbb{R}^{m-n}/L^*$ and represent its elements by vectors $\varphi\in\mathbb{R}^{m-n}$. We also define the group

$$D_\Gamma=\frac12L^*/L^*\cong(\mathbb{Z}_2)^{m-n}.$$

Note that D_Γ embeds canonically as a subgroup of T_Γ.

We view the intersection $\mathscr{R}$ as a subset of the intersection $\mathscr{Z}$ of Hermitian quadrics or as a subset of the whole complex space $\mathbb{C}^m$. We 'spread' $\mathscr{R}$ by the action of T_Γ, that is, we consider the set of T_Γ-orbits going through points in $\mathscr{R}$. More precisely, we consider the map

$$j\colon\mathscr{R}\times T_\Gamma\to\mathbb{C}^m,$$
$$(\mathbf{u},\varphi)\mapsto\mathbf{u}\cdot\varphi=\bigl(u_1e^{2\pi i\langle\gamma_1,\varphi\rangle},\dots,u_me^{2\pi i\langle\gamma_m,\varphi\rangle}\bigr)$$

and observe that $j(\mathscr{R}\times T_\Gamma)\subset\mathscr{Z}$. We let D_Γ act on $\mathscr{R}_\Gamma\times T_\Gamma$ diagonally; this action is free since it is free on the second factor. The quotient

$$N=\mathscr{R}\times_{D_\Gamma}T_\Gamma$$

is an m-dimensional manifold.

For any point $\mathbf{u}=(u_1,\dots,u_m)\in\mathscr{R}$ we have the sublattice

$$L_{\mathbf{u}}=\mathbb{Z}\langle\gamma_k\colon u_k\neq 0\rangle\subset L=\mathbb{Z}\langle\gamma_1,\dots,\gamma_m\rangle.$$

The next result says that the set of T_Γ-orbits through points in $\mathscr{R}$ is an immersion of N.

Lemma 12.1. (a) *The map $j\colon\mathscr{R}\times T_\Gamma\to\mathbb{C}^m$ induces an immersion $i\colon N\looparrowright\mathbb{C}^m$.*
(b) *The immersion i is an embedding if and only if $L_{\mathbf{u}}=L$ for any $\mathbf{u}\in\mathscr{R}$.*

Proof. Take $\mathbf{u}\in\mathscr{R}$, $\varphi\in T_\Gamma$, and $g\in D_\Gamma$. We have $\mathbf{u}\cdot g\in\mathscr{R}$ and $j(\mathbf{u}\cdot g,g\varphi)=\mathbf{u}\cdot g^2\varphi=\mathbf{u}\cdot\varphi=j(\mathbf{u},\varphi)$. Hence the map j is constant on D_Γ-orbits, and therefore it induces a map of the quotient $N=(\mathscr{R}\times T_\Gamma)/D_\Gamma$, which we denote by i.

Assume that $j(\mathbf{u},\varphi)=j(\mathbf{u}',\varphi')$. Then $L_{\mathbf{u}}=L_{\mathbf{u}'}$ and

$$u_ke^{2\pi i\langle\gamma_k,\varphi\rangle}=u'_ke^{2\pi i\langle\gamma_k,\varphi'\rangle}\quad\text{for } k=1,\dots,m. \tag{12.2}$$

Since both u_k and u'_k are real, this implies that $e^{2\pi i\langle\gamma_k,\varphi-\varphi'\rangle}=\pm1$ whenever $u_k\neq0$, or equivalently, $\varphi-\varphi'\in\frac12L^*_{\mathbf{u}}/L^*$. In other words, (12.2) implies that $\mathbf{u}'=\mathbf{u}\cdot g$ and $\varphi'=g\varphi$ for some $g\in\frac12L^*_{\mathbf{u}}/L^*$. The latter is a finite group by Lemma 5.4, hence the pre-image of any point of $\mathbb{C}^m$ under j consists of a finite number of points. If $L_{\mathbf{u}}=L$, then $\frac12L^*_{\mathbf{u}}/L^*=\frac12L^*/L^*=D_\Gamma$. Therefore, $(\mathbf{u},\varphi)$ and $(\mathbf{u}',\varphi')$ represent the same point in N. The statement (b) follows. To prove (a), it remains to observe that $L_{\mathbf{u}}=L$ for generic $\mathbf{u}$ (with all coordinates non-zero). □

Theorem 12.2 ([46], Theorem 1). *The immersion $i\colon N\looparrowright\mathbb{C}^m$ is H-minimal Lagrangian. Moreover, if $\sum_{k=1}^m\gamma_k=0$, then i is a minimal Lagrangian immersion.*

Proof. Here we only prove that i is a Lagrangian immersion. Let

$$(\mathbf{x},\varphi)\mapsto\mathbf{z}(\mathbf{x},\varphi)=\bigl(u_1(\mathbf{x})e^{2\pi i\langle\gamma_1,\varphi\rangle},\dots,u_m(\mathbf{x})e^{2\pi i\langle\gamma_m,\varphi\rangle}\bigr)$$

be a local coordinate system on $N=\mathscr{R}\times_{D_\Gamma}T_\Gamma$, where $\mathbf{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $\varphi=(\varphi_1,\dots,\varphi_{m-n})\in\mathbb{R}^{m-n}$. Let $\langle\xi,\eta\rangle_{\mathbb{C}}=\sum_{i=1}^m\overline{\xi}_i\eta_i=\langle\xi,\eta\rangle+i\omega(\xi,\eta)$ be the Hermitian scalar product of vectors $\xi,\eta\in\mathbb{C}^m$. Then

$$\biggl\langle\frac{\partial\mathbf{z}}{\partial x_k},\frac{\partial\mathbf{z}}{\partial\varphi_j}\biggr\rangle_{\mathbb{C}}=2\pi i\biggl(\gamma_{j1}u_1\frac{\partial u_1}{\partial x_k}+\dots+\gamma_{jm}u_m\frac{\partial u_m}{\partial x_k}\biggr)=0,$$

where the last equality follows by differentiating the equations of quadrics in (12.1). Moreover, $\biggl\langle\dfrac{\partial\mathbf{z}}{\partial x_k},\dfrac{\partial\mathbf{z}}{\partial x_j}\biggr\rangle_{\mathbb{C}}\in\mathbb{R}$ and $\biggl\langle\dfrac{\partial\mathbf{z}}{\partial\varphi_k},\dfrac{\partial\mathbf{z}}{\partial\varphi_j}\biggr\rangle_{\mathbb{C}}\in\mathbb{R}$. It follows that

$$\omega\biggl(\frac{\partial\mathbf{z}}{\partial x_k},\frac{\partial\mathbf{z}}{\partial\varphi_j}\biggr)=\omega\biggl(\frac{\partial\mathbf{z}}{\partial x_k},\frac{\partial\mathbf{z}}{\partial x_j}\biggr)=\omega\biggl(\frac{\partial\mathbf{z}}{\partial\varphi_k},\frac{\partial\mathbf{z}}{\partial\varphi_j}\biggr)=0,$$

that is, the restriction of the symplectic form to the tangent space of N is zero. □

Remark. The equality $\sum_{k=1}^m\gamma_k=0$ cannot hold for a compact $\mathscr{R}$ (or N).

Recall from Theorem 3.5 that a non-degenerate intersection of quadrics (3.4) or (12.1) defines a simple polyhedron (2.1), and $\mathscr{Z}$ is identified with the moment-angle manifold $\mathscr{Z}_P$. We can now summarize the results of the previous sections in the following criterion for $i\colon N\to\mathbb{C}^m$ to be an embedding.

Theorem 12.3. *Let $\mathscr{Z}$ and $\mathscr{R}$ be the intersections of Hermitian and real quadrics defined by* (3.4) *and* (12.1), *respectively, and satisfying the conditions* (a)–(c) *at the beginning of* § 12.2. *Let P be the associated simple polyhedron and $N=\mathscr{R}\times_{D_\Gamma}T_\Gamma$. The following conditions are equivalent*:

(a) *$i\colon N\to\mathbb{C}^m$ is an embedding of an H-minimal Lagrangian submanifold*;

(b) *$L_{\mathbf{u}}=L$ for any $\mathbf{u}\in\mathscr{R}$*;

(c) *the torus T_Γ acts freely on the moment-angle manifold $\mathscr{Z}=\mathscr{Z}_P$*;

(d) *P is a Delzant polyhedron.*

Proof. The equivalence (a) $\Leftrightarrow$ (b) follows from Lemma 12.1 and Theorem 12.2. The equivalence (b) $\Leftrightarrow$ (c) is Lemma 5.4, and the equivalence (c) $\Leftrightarrow$ (d) is Theorem 5.3 (c). □

Toric topology provides large families of explicitly constructed Delzant polytopes. Basic examples include simplices and cubes in all dimensions. It is easy to see that the Delzant condition is preserved under several operations on polytopes, such as taking products or cutting vertices or faces by specially chosen hyperplanes. This is sufficient to show that many important families of polytopes such as *associahedra* (Stasheff polytopes), *permutahedra*, and general *nestohedra* admit Delzant realizations (see, for example, [55] and [12]).

12.3. Topology of Lagrangian submanifolds N. We start by reviewing three simple properties linking the topological structure of N to that of the intersections of quadrics $\mathscr{Z}$ and $\mathscr{R}$.

Proposition 12.4. (a) *The immersion of N in $\mathbb{C}^m$ factors as $N\looparrowright\mathscr{Z}\hookrightarrow\mathbb{C}^m$.*

(b) *N is the total space of a bundle over the torus T^{m-n} with fibre $\mathscr{R}$.*

(c) *If $N\to\mathbb{C}^m$ is an embedding, then N is the total space of a principal T^{m-n}-bundle over the n-dimensional manifold $\mathscr{R}/D_\Gamma$.*

Proof. The statement (a) is clear. Since D_Γ acts freely on T_Γ, the projection $N=\mathscr{R}\times_{D_\Gamma}T_\Gamma\to T_\Gamma/D_\Gamma$ onto the second factor is a fibre bundle with fibre $\mathscr{R}$. Then (b) follows from the fact that $T_\Gamma/D_\Gamma\cong T^{m-n}$.

If $N\to\mathbb{C}^m$ is an embedding, then T_Γ acts freely on $\mathscr{Z}$ by Theorem 12.3, and the action of D_Γ on $\mathscr{R}$ is also free. Therefore, the projection $N=\mathscr{R}\times_{D_\Gamma}T_\Gamma\to\mathscr{R}/D_\Gamma$ onto the first factor is a principal T_Γ-bundle, which proves (c). □

Remark. The quotient $\mathscr{R}/D_\Gamma$ is a *real toric variety*, or a *small cover*, over the corresponding polytope P (see [21] and [14]).

Example 12.5 (one quadric). Suppose that $\mathscr{R}$ is given by a single equation

$$\gamma_1u_1^2+\cdots+\gamma_mu_m^2=\delta \tag{12.3}$$

in $\mathbb{R}^m$. We assume that $\mathscr{R}$ is compact, so that γ_i and δ are positive real numbers, $\mathscr{R}\cong S^{m-1}$, and the associated polytope P is an n-simplex $\varDelta^n$. In this case

$N \cong S^{m-1} \times_{\mathbb{Z}_2} S^1$, where the generator of $\mathbb{Z}_2$ acts by the standard free involution on S^1 and by a certain involution τ on S^{m-1}. The topological type of N depends on τ. Namely,

$$N \cong \begin{cases} S^{m-1} \times S^1 & \text{if } \tau \text{ preserves the orientation of } S^{m-1}, \\ \mathscr{K}^m & \text{if } \tau \text{ reverses the orientation of } S^{m-1}, \end{cases}$$

where $\mathscr{K}^m$ is the *m-dimensional Klein bottle.*

Proposition 12.6. *Let $m - n = 1$ (one quadric). An H-minimal Lagrangian embedding of $N \cong S^{m-1} \times_{\mathbb{Z}/2} S^1$ in $\mathbb{C}^m$ is obtained if and only if $\gamma_1 = \dots = \gamma_m$ in (12.3). In this case the topological type of $N = N(m)$ depends only on the parity of m and is given by*

$$\begin{aligned} &N(m) \cong S^{m-1} \times S^1 && \textit{if } m \textit{ is even}, \\ &N(m) \cong \mathscr{K}^m && \textit{if } m \textit{ is odd}. \end{aligned}$$

Proof. Since there exists a $\mathbf{u} \in \mathscr{R}$ with only one non-zero coordinate, Theorem 12.3 implies that N embeds in $\mathbb{C}^m$ if and only if γ_i generates the same lattice as the whole set $\gamma_1, \dots, \gamma_m$ for each i. Therefore, $\gamma_1 = \dots = \gamma_m$. In this case $D_\Gamma \cong \mathbb{Z}_2$ acts by the standard antipodal involution on S^{m-1}, which preserves orientation if m is even and reverses orientation otherwise. $\square$

Both examples of H-minimal Lagrangian embeddings given by Proposition 12.6 are well known. The Klein bottle $\mathscr{K}^m$ with even m does not admit Lagrangian embeddings in $\mathbb{C}^m$ (see [50] and [56]).

Example 12.7 (two quadrics). In the case $m-n=2$ the topology of $\mathscr{R}$ and N can be completely described by analysing the action of the two commuting involutions on the intersection of quadrics. We consider only the compact case here.

Using Proposition 2.8, we write $\mathscr{R}$ in the form

$$\begin{aligned} \gamma_{11} u_1^2 + \dots + \gamma_{1m} u_m^2 &= c, \\ \gamma_{21} u_1^2 + \dots + \gamma_{2m} u_m^2 &= 0, \end{aligned} \tag{12.4}$$

where $c > 0$ and $\gamma_{1i} > 0$ for all i.

Proposition 12.8. *There is a number p with $0 < p < m$ such that $\gamma_{2i} > 0$ for $i = 1, \dots, p$ and $\gamma_{2i} < 0$ for $i = p+1, \dots, m$ in (12.4), possibly after reordering the coordinates $u_1, \dots, u_m$. The corresponding manifold $\mathscr{R} = \mathscr{R}(p,q)$, where $q = m-p$, is diffeomorphic to $S^{p-1} \times S^{q-1}$. Its associated polytope P either coincides with Δ^{m-2} (if one of the inequalities in (2.1) is redundant) or is combinatorially equivalent to the product $\Delta^{p-1} \times \Delta^{q-1}$ (if there are no redundant inequalities).*

Proof. We observe that $\gamma_{2i} \neq 0$ for all i in (12.4), since $\gamma_{2i} = 0$ implies that the vector $\delta = (c\ 0)^t$ is in the cone generated by γ_i, which contradicts Proposition 3.4 (b). By reordering the coordinates, we can make the first p of the numbers γ_{2i} be positive and the rest negative. Then $1 < p < m$, because otherwise (12.4) is empty. Further, (12.4) is the intersection of the cone over the product of two ellipsoids of dimensions $p-1$ and $q-1$ (given by the second quadric) with an $(m-1)$-dimensional ellipsoid (given by the first quadric). Therefore, $\mathcal{R}(p,q) \cong S^{p-1} \times S^{p-1}$. The statement about the polytope follows from the combinatorial fact that a simple n-polytope with at most $n+2$ facets is combinatorially equivalent to a product of simplices; the case of one redundant inequality corresponds to $p=1$ or $q=1$. $\square$

An element $\varphi \in D_\Gamma = \frac{1}{2}L^*/L^* \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $\mathcal{R}(p,q)$ by

$$(u_1,\dots,u_m) \mapsto (\varepsilon_1(\varphi)u_1,\dots,\varepsilon_m(\varphi)u_m),$$

where $\varepsilon_k(\varphi) = e^{2\pi i\langle\gamma_k,\varphi\rangle} = \pm 1$ for $1 \leqslant k \leqslant m$.

Lemma 12.9. *Suppose that D_Γ acts on $\mathcal{R}(p,q)$ freely and $\varepsilon_i(\varphi) = 1$ for some i with $1 \leqslant i \leqslant p$ and $\varphi \in D_\Gamma$. Then $\varepsilon_l(\varphi) = -1$ for all l with $p+1 \leqslant l \leqslant m$.*

Proof. Assume the opposite, that is, that $\varepsilon_i(\varphi) = 1$ for some $1 \leqslant i \leqslant p$ and $\varepsilon_j(\varphi) = 1$ for some $p+1 \leqslant j \leqslant m$. Then $\gamma_{2i} > 0$ and $\gamma_{2j} < 0$ in (12.4), so we can choose $\mathbf{u} \in \mathcal{R}(p,q)$ whose only non-zero coordinates are u_i and u_j. The element $\varphi \in D_\Gamma$ fixes this $\mathbf{u}$, leading to a contradiction. $\square$

Lemma 12.10. *Suppose that D_Γ acts on $\mathcal{R}(p,q)$ freely. Then there exist two generating involutions $\varphi_1,\varphi_2 \in D_\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ whose action on $\mathcal{R}(p,q)$ is described by either* (a) *or* (b) *below, possibly after reordering the coordinates*:

$$\begin{aligned}\varphi_1 &\colon (u_1,\dots,u_m) \mapsto (u_1,\dots,u_k,-u_{k+1},\dots,-u_p,-u_{p+1},\dots,-u_m),\\ \varphi_2 &\colon (u_1,\dots,u_m) \mapsto (-u_1,\dots,-u_k,u_{k+1},\dots,u_p,-u_{p+1},\dots,-u_m);\end{aligned} \tag{a}$$

$$\begin{aligned}\varphi_1 &\colon (u_1,\dots,u_m) \mapsto (-u_1,\dots,-u_p,u_{p+1},\dots,u_{p+l},-u_{p+l+1},\dots,-u_m),\\ \varphi_2 &\colon (u_1,\dots,u_m) \mapsto (-u_1,\dots,-u_p,-u_{p+1},\dots,-u_{p+l},u_{p+l+1},\dots,u_m);\end{aligned} \tag{b}$$

here $0 \leqslant k \leqslant p$ and $0 \leqslant l \leqslant q$.

Proof. By Lemma 12.9 for each of the three non-zero elements $\varphi \in D_\Gamma$, we have either $\varepsilon_i(\varphi) = -1$ for $1 \leqslant i \leqslant p$ or $\varepsilon_i(\varphi) = -1$ for $p+1 \leqslant i \leqslant m$. Therefore, we can choose two different non-zero elements $\varphi_1,\varphi_2 \in D_\Gamma$ such that either $\varepsilon_i(\varphi_j) = -1$ for $j=1,2$ and $p+1 \leqslant i \leqslant m$, or $\varepsilon_i(\varphi_j) = -1$ for $j=1,2$ and $1 \leqslant i \leqslant p$. This corresponds to the cases (a) and (b) above, respectively. In the first case we may assume after reordering the coordinates that φ_1 acts as in (a). Then φ_2 also acts as in (a) since otherwise the composition $\varphi_1 \cdot \varphi_2$ cannot act freely by Lemma 12.9. The second case is treated similarly. $\square$

Each of the actions of D_Γ described in Lemma 12.10 can be realized by a particular intersection of quadrics (12.4). For example, the system of quadrics

$$\begin{aligned}2u_1^2+\dots+2u_k^2+u_{k+1}^2+\dots+u_p^2+u_{p+1}^2+\dots+u_m^2 &= 3,\\ u_1^2+\dots+u_k^2+2u_{k+1}^2+\dots+2u_p^2-u_{p+1}^2-\dots-u_m^2 &= 0\end{aligned} \tag{12.5}$$

gives the first action of Lemma 12.10; the second action is realized similarly. Note that the lattice L corresponding to (12.5) is a sublattice of index 3 in $\mathbb{Z}^2$. We can rewrite (12.5) as

$$\begin{aligned}u_1^2+\dots+u_k^2+u_{k+1}^2+\dots+u_p^2 \qquad\qquad\qquad\qquad &= 1,\\ u_1^2+\dots+u_k^2 \qquad\qquad\qquad\qquad +u_{p+1}^2+\dots+u_m^2 &= 2,\end{aligned} \tag{12.6}$$

in which case $L = \mathbb{Z}^2$. The action of the two involutions $\psi_1, \psi_2 \in D_\Gamma = \frac{1}{2}\mathbb{Z}^2/\mathbb{Z}^2$ corresponding to the standard basis vectors of $\frac{1}{2}\mathbb{Z}^2$ is given by

$$\begin{aligned} \psi_1 &\colon (u_1, \dots, u_m) \mapsto (-u_1, \dots, -u_k, -u_{k+1}, \dots, -u_p, u_{p+1}, \dots, u_m), \\ \psi_2 &\colon (u_1, \dots, u_m) \mapsto (-u_1, \dots, -u_k, u_{k+1}, \dots, u_p, -u_{p+1}, \dots, -u_m). \end{aligned} \tag{12.7}$$

We denote the manifold N corresponding to (12.6) by $N_k(p,q)$. Then

$$N_k(p,q) \cong (S^{p-1} \times S^{q-1}) \times_{\mathbb{Z}/2\times\mathbb{Z}/2} (S^1 \times S^1), \tag{12.8}$$

where the action of the two involutions on $S^{p-1} \times S^{q-1}$ is given by (12.7). Note that ψ_1 acts trivially on S^{q-1} and antipodally on S^{p-1}. Therefore,

$$N_k(p,q) \cong N(p) \times_{\mathbb{Z}/2} (S^{q-1} \times S^1),$$

where $N(p)$ is the manifold in Proposition 12.6. If $k = 0$, then the second involution ψ_2 acts trivially on $N(p)$, and $N_0(p,q)$ coincides with the product $N(p) \times N(q)$ of the two manifolds in Example 12.5. In general, the projection

$$N_k(p,q) \to S^{q-1} \times_{\mathbb{Z}/2} S^1 = N(q)$$

describes $N_k(p,q)$ as the total space of a fibration over $N(q)$ with fibre $N(p)$.

We summarize the above facts and observations in the following topological classification result for compact H-minimal Lagrangian submanifolds $N \subset \mathbb{C}^m$ obtained from intersections of two quadrics.

Theorem 12.11. *Let $N \to \mathbb{C}^m$ be the embedding of the H-minimal Lagrangian submanifold corresponding to a compact intersection of two quadrics. Then N is diffeomorphic to some manifold $N_k(p,q)$ given by* (12.8)*, where $p+q = m$, $0 < p < m$, and $0 \leqslant k \leqslant p$. Moreover, any such triple (k,p,q) can be realized by N.*

In the case of at most two quadrics considered above, the topology of $\mathscr{R}$ is relatively simple, and in order to analyse the topology of N, one only needs to describe the action of involutions on $\mathscr{R}$. When the number of quadrics is more than two, the topology of $\mathscr{R}$ becomes an issue as well.

Example 12.12 (three quadrics). In the case $m - n = 3$ the topology of the compact manifolds $\mathscr{R}$ and $\mathscr{Z}$ was fully described in [40], Theorem 2. Each of these manifolds is diffeomorphic to a product of three spheres or to a connected sum of products of spheres with two spheres in each product.

We note that for $m - n = 3$ the manifold $\mathscr{R}$ (or $\mathscr{Z}$) can be distinguished topologically by looking at the planar Gale diagram of the associated simple polytope P (see § 2). This agrees with the classification of n-dimensional simple polytopes with $n+3$ facets which is well-known in combinatorial geometry.

The smallest polytope with $m - n = 3$ is a pentagon. It has many Delzant realizations, for instance,

$$P = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \geqslant 0,\ x_2 \geqslant 0,\ -x_1 + 2 \geqslant 0,\ -x_2 + 2 \geqslant 0,\ -x_1 - x_2 + 3 \geqslant 0\}.$$

In this case, $\mathscr{R}$ is an oriented surface of genus 5 (see [14], Example 6.40), and the moment-angle manifold $\mathscr{Z}$ is diffeomorphic to a connected sum of five copies of $S^3 \times S^4$.

We therefore obtain an H-minimal Lagrangian submanifold $N \subset \mathbb{C}^5$ which is the total space of a bundle over T^3 with fibre a surface of genus 5.

Now let the polytope P associated with the intersection of quadrics (12.1) be a polygon (that is, $n = 2$). If there are no redundant inequalities, then P is an m-gon and $\mathscr{R}$ is an orientable surface S_g of genus $g = 1+2^{m-3}(m-4)$ by [14], Example 6.40. If there are k redundant inequalities, then P is an $(m-k)$-gon. In this case $\mathscr{R} \cong \mathscr{R}' \times (S^0)^k$, where $\mathscr{R}'$ corresponds to an $(m-k)$-gon without redundant inequalities. That is, $\mathscr{R}$ is a disjoint union of 2^k surfaces of genus $1 + 2^{m-k-3}(m-k-4)$.

The corresponding H-minimal Lagrangian submanifold $N \subset \mathbb{C}^m$ is the total space of a bundle over T^{m-2} with fibre S_g. This is an aspherical manifold for $m \geqslant 4$.

12.4. Generalization to toric manifolds. Consider two sets of quadrics:

$$\mathscr{Z}_\Gamma = \biggl\{\mathbf{z} \in \mathbb{C}^m \colon \sum_{k=1}^m \gamma_k |z_k|^2 = \mathbf{c}\biggr\}, \qquad \gamma_k, \mathbf{c} \in \mathbb{R}^{m-n},$$

$$\mathscr{Z}_\Delta = \biggl\{\mathbf{z} \in \mathbb{C}^m \colon \sum_{k=1}^m \delta_k |z_k|^2 = \mathbf{d}\biggr\}, \qquad \delta_k, \mathbf{d} \in \mathbb{R}^{m-\ell},$$

such that $\mathscr{Z}_\Gamma$, $\mathscr{Z}_\Delta$, and $\mathscr{Z}_\Gamma \cap \mathscr{Z}_\Delta$ satisfy the non-degeneracy and rationality conditions (a)–(c) in § 12.2. Assume also that the polyhedra associated with $\mathscr{Z}_\Gamma$, $\mathscr{Z}_\Delta$, and $\mathscr{Z}_\Gamma \cap \mathscr{Z}_\Delta$ are Delzant.

The idea is to use the first set of quadrics to produce a toric manifold V via symplectic reduction (as described in § 5), and then to use the second set of quadrics to define an H-minimal Lagrangian submanifold of V.

Construction 12.13. Let the real intersections of quadrics $\mathscr{R}_\Gamma$, $\mathscr{R}_\Delta$, the tori $T_\Gamma \cong \mathbb{T}^{m-n}$, $T_\Delta \cong \mathbb{T}^{m-\ell}$, and the groups $D_\Gamma \cong \mathbb{Z}_2^{m-n}$, $D_\Delta \cong \mathbb{Z}_2^{m-\ell}$ be as before.

We consider the toric variety V obtained as the symplectic quotient of $\mathbb{C}^m$ by the action of the torus corresponding to the first set of quadrics: $V = \mathscr{Z}_\Gamma / T_\Gamma$. It is a Kähler manifold of real dimension $2n$. The quotient $\mathscr{R}_\Gamma / D_\Gamma$ is the set of real points of V (the fixed point set of the complex conjugation, or the real toric manifold); it has dimension n. Consider the subset of $\mathscr{R}_\Gamma / D_\Gamma$ defined by the second set of quadrics:

$$\mathscr{S} = (\mathscr{R}_\Gamma \cap \mathscr{R}_\Delta)/D_\Gamma,$$

for which $\dim \mathscr{S} = n + \ell - m$. Finally, define the following n-dimensional submanifold of V:

$$N = \mathscr{S} \times_{D_\Delta} T_\Delta.$$

Theorem 12.14. *N is an H-minimal Lagrangian submanifold of the toric manifold V.*

Proof. Let $\widehat{V}$ be the symplectic quotient of V by the action of the torus corresponding to the second set of quadrics, that is, $\widehat{V} = (V \cap \mathscr{Z}_\Delta)/T_\Delta = (\mathscr{Z}_\Gamma \cap \mathscr{Z}_\Delta)/(T_\Gamma \times T_\Delta)$. It is a toric manifold of real dimension $2(n+\ell-m)$. The submanifold of real points

$$\widehat{N} = N/T_\Delta = (\mathscr{R}_\Gamma \cap \mathscr{R}_\Delta)/(D_\Gamma \times D_\Delta) \hookrightarrow (\mathscr{Z}_\Gamma \cap \mathscr{Z}_\Delta)/(T_\Gamma \times T_\Delta) = \widehat{V}$$

is the fixed point set of the complex conjugation, hence it is a totally geodesic submanifold. In particular, $\widehat{N}$ is a minimal submanifold of $\widehat{V}$. According to Corollary 2.7 in [24], N is an H-minimal submanifold of V. $\square$

Example 12.15. 1. If $m-\ell=0$, that is, $\mathscr{Z}_\Delta=\varnothing$, then $V=\mathbb{C}^m$, and we obtain the original construction of H-minimal Lagrangian submanifolds N of $\mathbb{C}^m$.

2. If $m-n=0$, that is, $\mathscr{Z}_\Gamma=\varnothing$, then N is the set of real points of V. It is minimal (totally geodesic).

3. If $m-\ell=1$, that is, $\mathscr{Z}_\Delta\cong S^{2m-1}$, then we get H-minimal Lagrangian submanifolds of $V=\mathbb{C}P^{m-1}$. This includes the families of projective examples in [45], [42], and [49].

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