GEOMETRIC, TOPOLOGICAL AND DIFFERENTIABLE RIGIDITY OF SUBMANIFOLDS IN SPACE FORMS *

HONG-WEI XU AND JUAN-RU GU

Abstract

Let M be an n-dimensional submanifold in the simply connected space form $F^{n+p}(c)$ with $c + H^2 > 0$, where H is the mean curvature of M. We verify that if $M^n(n \geq 3)$ is an oriented compact submanifold with parallel mean curvature and its Ricci curvature satisfies $Ric_M \geq (n-2)(c+H^2)$, then M is either a totally umbilic sphere, a Clifford hypersurface in an $(n+1)$-sphere with $n =$ even, or $\mathbb{C}P^2(\frac{4}{3}(c+H^2))$ in $S^7(\frac{1}{\sqrt{c+H^2}})$. In particular, if $Ric_M > (n-2)(c+H^2)$, then M is a totally umbilic sphere. We then prove that if $M^n(n \geq 4)$ is a compact submanifold in $F^{n+p}(c)$ with $c \geq 0$, and if $Ric_M > (n-2)(c+H^2)$, then M is homeomorphic to a sphere. It should be emphasized that our pinching conditions above are sharp. Finally, we obtain a differentiable sphere theorem for submanifolds with positive Ricci curvature.

1 Introduction

The investigation of curvature and topology of Riemannian manifolds and submanifolds is one of the main stream in global differential geometry. In 1951, Rauch first proved a topological sphere theorem for positive pinched compact manifolds. During the past sixty years, there are many progresses on sphere theorems for Riemannian manifolds and submanifolds [2, 8, 30, 37]. In 1960's, Berger and Klingenberg proved the famous topological sphere theorem for quarter-pinched compact manifolds. In 1966, Calabi and Gromoll initiated the differentiable pinching problem for positive pinched compact manifolds. In 1977, Grove and Shiohama [14] proved the celebrated diameter sphere theorem which is optimal for arbitrary n. In 1982, Hamilton [16] established the theory of Ricci flow and obtained the famous sphere theorem for 3-manifolds with positive Ricci curvature. Later some differentiable pinching theorems for Riemannian manifolds via Ricci flows were obtained by several authors [5, 8, 9, 17]. In 1988, Micallef and Moore [24] verified the topological sphere theorem for manifolds with pointwise 1/4-pinched curvatures via the techniques of minimal surface. In 1990's, Cheeger, Colding and Petersen [10, 27] obtained differentiable sphere theorems for manifolds with positive Ricci curvature. Recently Böhm and Wilking [3] proved that every manifold with 2-positive curvature operator must be diffeomorphic to a space form. More recently, Brendle and Schoen [6] proved the remarkable differentiable sphere theorem for manifolds with pointwise 1/4-pinched curvatures.

*2010 Mathematics Subject Classification. 53C20; 53C24; 53C40.

Keywords: Submanifolds, rigidity and sphere theorems, Ricci curvature, Ricci flow, stable currents.

Research supported by the NSFC, Grant No. 11071211; the Trans-Century Training Programme Foundation for Talents by the Ministry of Education of China.

Moreover, Brendle and Schoen [7] obtained a differentiable rigidity theorem for compact manifolds with weakly 1/4-pinched curvatures in the pointwise sense. Using Brendle and Schoen's result [7], Petersen and Tao [28] proved a classification theorem for compact and simply connected manifolds with almost 1/4-pinched sectional curvatures. The following important convergence result for the Ricci flow in higher dimensions, initiated by Brendle and Schoen [6] and finally verified by Brendle [4], cut open a new field in curvature and topology of manifolds [5, 8, 37].

Theorem A. *Let (M, g_0) be a compact Riemannian manifold of dimension $n(\geq 4)$. Assume that*

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} > 0$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$. Then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)} + \frac{2}{n} r_{g(t)} g(t)$$

exists for all time and converges to a constant curvature metric as $t \to \infty$. Here $r_{g(t)}$ denotes the mean value of the scalar curvature of $g(t)$.

Let M^n be an $n(\geq 2)$-dimensional submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. Denote by H and S the mean curvature and the squared length of the second fundamental form of M, respectively. After the pioneering rigidity theorem for minimal submanifolds in a sphere due to Simons [32], Lawson [18] and Chern-do Carmo-Kobayashi [11] obtained a famous rigidity theorem for oriented compact minimal submanifolds in S^{n+p} with $S \leq n/(2 - 1/p)$. It was partially extended to compact submanifolds with parallel mean curvature in a sphere by Okumura [25, 26], Yau [40] and others. In 1990, the first named author [35] proved the generalized Simons-Lawson-Chern-do Carmo-Kobayashi theorem for compact submanifolds with parallel mean curvature in a sphere.

Theorem B. *Let M be an n-dimensional oriented compact submanifold with parallel mean curvature in an $(n+p)$-dimensional unit sphere S^{n+p}. If $S \leq C(n, p, H)$, then M is either the totally umbilic sphere $S^n(\frac{1}{\sqrt{1+H^2}})$, a Clifford hypersurface in an $(n+1)$-sphere, or the Veronese surface in $S^4(\frac{1}{\sqrt{1+H^2}})$. Here the constant $C(n, p, H)$ is defined by*

$$C(n,p,H) = \begin{cases} \alpha(n,H), & for\ p = 1,\ or\ p = 2\ and\ H \neq 0, \\ \frac{n}{2-\frac{1}{p}}, & for\ p \geq 2\ and\ H = 0, \\ \min\left\{\alpha(n,H), \frac{n+nH^2}{2-\frac{1}{p-1}} + nH^2\right\}, & for\ p \geq 3\ and\ H \neq 0, \end{cases}$$

$$\alpha(n,H) = n + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}.$$

Later, the above pinching constant $C(n, p, H)$ was improved, by Li-Li [20] for $H = 0$ and by Xu [36] for $H \neq 0$, to

$$C'(n,p,H) = \begin{cases} \alpha(n,H), & \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\ \min\left\{\alpha(n,H), \frac{1}{3}(2n + 5nH^2)\right\}, & \text{otherwise.} \end{cases}$$

Using nonexistence for stable currents on compact submanifolds of a sphere and the generalized Poincaré conjecture in dimension $n(\geq 5)$ verified by Smale, Lawson and Simons [19] proved that if $M^n(n \geq 5)$ is an oriented compact submanifold in S^{n+p}, and if $S < 2\sqrt{n-1}$, then M is homeomorphic to a sphere. Let $F^{n+p}(c)$ be an $(n+p)$-dimensional simply connected space form with constant curvature c. Putting

$$\alpha(n, H, c) = nc + \frac{n^3}{2(n-1)}H^2 - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)cH^2},$$

we have $\min_H \alpha(n, H, c) = 2\sqrt{n-1}c$. Motivated by the rigidity theorem above, Shiohama and Xu [31] improved Lawson-Simons' result and proved the following optimal sphere theorem.

Theorem C. *Let $M^n(n \geq 4)$ be an oriented complete submanifold in $F^{n+p}(c)$ with $c \geq 0$. Suppose that $\sup_M(S - \alpha(n, H, c)) < 0$. Then M is homeomorphic to a sphere.*

Xu and Zhao [39] first investigated the differentiable pinching problem for submanifolds. Making use of the convergence results of Hamilton and Brendle for Ricci flow and the Lawson-Simons formula for the nonexistence of stable currents, Gu and Xu [15] proved the following differentiable sphere theorem for submanifolds in space forms.

Theorem D. *Let M be an $n(\geq 4)$-dimensional oriented complete submanifold in $F^{n+p}(c)$ with $c \geq 0$. Assume that $S \leq 2c + \frac{n^2H^2}{n-1}$, where $c + H^2 > 0$. We have*
(i) If $c = 0$, then M is either diffeomorphic to S^n, $\mathbb{R}^n$, or locally isometric to $S^{n-1}(r) \times \mathbb{R}$.
(ii) If M is compact, then M is diffeomorphic to S^n .

Theorem D improves the differentiable pinching theorems due to Andrews-Baker and the authors [1, 38].

In 1979, Ejiri [12] obtained the following rigidity theorem for $n(\geq 4)$-dimensional oriented compact simply connected minimal submanifolds with pinched Ricci curvatures in a sphere.

Theorem E. *Let M be an $n(\geq 4)$-dimensional oriented compact simply connected minimal submanifold in S^{n+p}. If the Ricci curvature of M satisfies $Ric_M \geq n-2$, then M is either the totally geodesic submanifold S^n, the Clifford torus $S^m\big(\sqrt{\frac{1}{2}}\big) \times S^m\big(\sqrt{\frac{1}{2}}\big)$ in S^{n+1} with $n = 2m$, or $\mathbb{C}P^2(\frac{4}{3})$ in S^7. Here $\mathbb{C}P^2(\frac{4}{3})$ denotes the 2-dimensional complex projective space minimally immersed into S^7 with constant holomorphic sectional curvature $\frac{4}{3}$.*

The pinching constant above is the best possible in even dimensions. It's better than ones given by Simons [32] and Li-Li [20] in the sense of the average of Ricci curvatures. The following problem seems very attractive, which has been open for thirty years.

Open Problem A. *Is it possible to generalize Ejiri's rigidity theorem for minimal submanifolds to the cases of submanifolds with parallel mean curvature in a sphere?*

In 1987, Sun [33] showed that if M is an $n(\geq 4)$-dimensional compact oriented submanifold with parallel mean curvature in S^{n+p} and its Ricci curvature is not less than $\frac{n(n-2)}{n-1}(1+H^2)$, then M is a totally umbilic sphere. Afterward, Shen [29] and Li [21] proved that if M is a 3-dimensional oriented compact minimal submanifolds in S^{3+p} and $Ric_M \geq 1$, then M is totally geodesic.

The purposes of the present paper is to investigate rigidity of geometric, topological and differentiable structures of compact submanifolds in space forms. Our paper is organized as follows. Some notation and lemmas are prepared in Section 2. In Section 3, we generalize the Ejiri rigidity theorem for compact simply connected minimal submanifolds in a sphere to compact submanifolds with parallel mean curvature in space forms. More preciously, we prove that if M is an $n(\geq 3)$-dimensional oriented compact submanifold with parallel mean curvature in $F^{n+p}(c)$ with $c+H^2>0$, and if $Ric_M \geq (n-2)(c+H^2)$, then M is either a totally umbilic sphere, a Clifford hypersurface in an $(n+1)$-sphere with $n =$ even, or $\mathbb{C}P^2(\frac{4}{3}(c+H^2))$ in $S^7(\frac{1}{\sqrt{c+H^2}})$. In particular, we provide an affirmative answer to Open Problems A. In Section 4, we prove that if M is an $n(\geq 4)$-dimensional compact submanifold in $F^{n+p}(c)$ with $c \geq 0$, and if $Ric_M > (n-2)(c+H^2)$, then M is homeomorphic to a sphere. Moreover, we obtain a differentiable sphere theorem for compact submanifolds with positive Ricci curvature in a space form.

2 Notation and lemmas

Throughout this paper let M^n be an n-dimensional compact submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n+p,\ 1 \leq i, j, k, \cdots \leq n,\ n+1 \leq \alpha, \beta, \gamma, \cdots \leq n+p.$$

For an arbitrary fixed point $x \in M \subset N$, we choose an orthonormal local frame field $\{e_A\}$ in N^{n+p} such that e_i's are tangent to M. Denote by $\{\omega_A\}$ the dual frame field of $\{e_A\}$. Let Rm, h and ξ be the Riemannian curvature tensor, second fundamental form and mean curvature vector of M respectively, and $\overline{Rm}$ the Riemannian curvature tensor of N. Then

$$Rm = \sum_{i,j,k,l} R_{ijkl}\omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l,$$

$$\overline{Rm} = \sum_{A,B,C,D} \overline{R}_{ABCD}\omega_A \otimes \omega_B \otimes \omega_C \otimes \omega_D,$$

$$h = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha,\ \xi = \frac{1}{n}\sum_{\alpha,i} h^\alpha_{ii} e_\alpha,$$

$$R_{ijkl} = \overline{R}_{ijkl} + \sum_\alpha \left(h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}\right), \tag{2.1}$$

$$R_{\alpha\beta kl} = \overline{R}_{\alpha\beta kl} + \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}). \tag{2.2}$$

We define

$$S = |h|^2,\ H = |\xi|,\ H_\alpha = (h^\alpha_{ij})_{n\times n}.$$

Denote by $Ric(u)$ the Ricci curvature of M in direction of $u \in UM$. From the Gauss equation, we have

$$Ric(e_i) = \sum_j \overline{R}_{ijij} + \sum_{\alpha,j} \left[h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2\right]. \tag{2.3}$$

Set $Ric_{\min}(x) = \min_{u \in U_xM} Ric(u)$. Denote by $K(\pi)$ the sectional curvature of M for tangent 2-plane $\pi(\subset T_xM)$ at point $x \in M$, $\overline{K}(\pi)$ the sectional curvature of N for tangent 2-plane $\pi(\subset T_xN)$ at point $x \in N$. Set $\overline{K}_{\min} := \min_{\pi \subset T_xN} \overline{K}(\pi)$, $\overline{K}_{\max} := \max_{\pi \subset T_xN} \overline{K}(\pi)$. Then by Berger's inequality, we have

$$|\overline{R}_{ABCD}| \leq \frac{2}{3}(\overline{K}_{\max} - \overline{K}_{\min}) \tag{2.4}$$

for all distinct indices A, B, C, D.

When the ambient manifold N^{n+p} is the complete and simply connected space form $F^{n+p}(c)$ with constant curvature c, we have

$$\overline{R}_{ABCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2.5}$$

Then the scalar curvature of M is given by

$$R = n(n-1)c + n^2H^2 - S. \tag{2.6}$$

When M is a submanifold with parallel mean curvature vector ξ, we choose e_{n+1} such that it is parallel to ξ, and

$$trH_{n+1} = nH, \ trH_\alpha = 0, \ \text{for } \alpha \neq n+1. \tag{2.7}$$

Set

$$S_H = trH_{n+1}^2, \quad S_I = \sum_{\alpha \neq n+1} trH_\alpha^2.$$

The following lemma will be used in the proof of our rigidity theorem.

Lemma 2.1([40]). *If M^n is a submanifold with parallel mean curvature in $F^{n+p}(c)$, then either $H \equiv 0$, or H is non-zero constant and $H_{n+1}H_\alpha = H_\alpha H_{n+1}$ for all α.*

We denote the first and the second covariant derivatives of h_{ij}^α by h_{ijk}^α and h_{ijkl}^α respectively. The Laplacian of h_{ij}^α is defined by $\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$. Following [40], we have

$$\Delta h_{ij}^{n+1} = \sum_{k,m}(h_{mk}^{n+1}R_{mijk} + h_{im}^{n+1}R_{mkjk}). \tag{2.8}$$

The nonexistence theorem for stable currents in a compact Riemannian manifold M isometrically immersed into $F^{n+p}(c)$ is employed to eliminate the homology groups $H_q(M;\mathbb{Z})$ for $0 < q < n$, which was initiated by Lawson-Simons [19] and extended by Xin [34].

Theorem 2.1. *Let M^n be a compact submanifold in $F^{n+p}(c)$ with $c \geq 0$. Assume that*

$$\sum_{k=q+1}^{n}\sum_{i=1}^{q}[2|h(e_i,e_k)|^2 - \langle h(e_i,e_i), h(e_k,e_k)\rangle] < q(n-q)c$$

holds for any orthonormal basis $\{e_i\}$ *of* T_xM *at any point* $x \in M$*, where* q *is an integer satisfying* $0 < q < n$*. Then there does not exist any stable* q*-currents. Moreover,* $H_q(M;\mathbb{Z}) = H_{n-q}(M;\mathbb{Z}) = 0$*, and* $\pi_1(M) = 0$ *when* $q = 1$*. Here* $H_i(M;\mathbb{Z})$ *is the* i*-th homology group of* M *with integer coefficients.*

To prove the rigidity and sphere theorems for submanifolds, we need to eliminate the fundamental group $\pi_1(M)$ under the Ricci curvature pinching condition, and get the following lemmas.

Lemma 2.2. *Let* M *be an* $n(\geq 4)$*-dimensional compact submanifold in* $F^{n+p}(c)$ *with* $c \geq 0$*. If the Ricci curvature of* M *satisfies*

$$Ric_M > \frac{n(n-1)}{n+2}(c + H^2),$$

then $H_1(M;\mathbb{Z}) = H_{n-1}(M;\mathbb{Z}) = 0$*, and* $\pi_1(M) = 0$*.*

Proof. From (2.6) and the assumption, we have

$$S - nH^2 < \frac{2n(n-1)}{n+2}(c + H^2).$$

This together with (2.3) implies that

$$\begin{aligned} & \sum_{k=2}^{n}[2|h(e_1, e_k)|^2 - \langle h(e_1, e_1), h(e_k, e_k)\rangle] \\ = \ & 2\sum_{\alpha}\sum_{k=2}^{n}(h_{1k}^{\alpha})^2 - \sum_{\alpha}\sum_{k=2}^{n}h_{11}^{\alpha}h_{kk}^{\alpha} \\ = \ & \sum_{\alpha}\sum_{k=2}^{n}(h_{1k}^{\alpha})^2 - Ric(e_1) + (n-1)c \\ \leq \ & \frac{1}{2}(S - nH^2) - Ric(e_1) + (n-1)c \\ < \ & \frac{n(n-1)}{n+2}(c + H^2) - \frac{n(n-1)}{n+2}(c + H^2) + (n-1)c \\ = \ & (n-1)c. \end{aligned} \tag{2.9}$$

This together with Theorem 2.1 implies that $H_1(M;\mathbb{Z}) = H_{n-1}(M;\mathbb{Z}) = 0$, and $\pi_1(M) = 0$. This proves Lemma 2.2. □

Lemma 2.3. *Let* M *be an* $n(\geq 4)$*-dimensional compact submanifold in* $F^{n+p}(c)$ *with* $c \geq 0$*. Assume that the Ricci curvature of* M *satisfies*

$$Ric_M \geq (n-2)(c + H^2).$$

We have the following possibilities:
(i) If $n = 4$ *and* $c > 0$*, then* $\pi_1(M) = 0$*.*
(ii) If $n \geq 5$*, then* $\pi_1(M) = 0$*.*

Proof. (i) If $n = 4$ and $c > 0$, then it follows from the assumption and (2.3) that

$$\begin{aligned}
&\sum_{k=2}^{4}[2|h(e_1, e_k)|^2 - \langle h(e_1, e_1), h(e_k, e_k)\rangle] \\
= \;& 2\sum_\alpha\sum_{k=2}^{4}(h^\alpha_{1k})^2 - \sum_\alpha\sum_{k=2}^{4}h^\alpha_{11}h^\alpha_{kk} \\
= \;& -2Ric(e_1) + 6c - \sum_\alpha[(h^\alpha_{11})^2 - trH_\alpha h^\alpha_{11}] \\
\le \;& -2Ric(e_1) + 6c + 4H^2 \\
< \;& 3c. \qquad (2.10)
\end{aligned}$$

This together with Theorem 2.1 implies that $\pi_1(M) = 0$.

(ii) If $n \ge 5$, then the assertion follows from Lemma 2.2.

This completes the proof of Lemma 2.3. □

3 Rigidity of submanifolds with parallel mean curvature

In this section, we generalize the Ejiri rigidity theorem to compact submanifolds with parallel mean curvature in space forms. To verify our rigidity theorem for submanifolds with parallel mean curvature in space forms, we need to prove the following theorem.

Theorem 3.1. *Let M be an $n(\ge 3)$-dimensional oriented compact submanifold with parallel mean curvature ($H \ne 0$) in $F^{n+p}(c)$. If*

$$Ric_M \ge (n-2)(c + H^2),$$

where $c + H^2 > 0$, then M is pseudo-umbilical.

Proof. The key ingredient of the proof is to derive a sharp estimate for ΔS_H. By the Gauss equation (2.1), (2.5) and (2.8), we have

$$\begin{aligned}
\frac{1}{2}\Delta S_H \;=\;& \sum_{i,j,k}(h^{n+1}_{ijk})^2 + \sum_{i,j}h^{n+1}_{ij}\Delta h^{n+1}_{ij} \\
=\;& \sum_{i,j,k}(h^{n+1}_{ijk})^2 + \sum_{i,j,k,m}h^{n+1}_{ij}h^{n+1}_{km}\Big[(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij})c \\
&+ \sum_\alpha(h^\alpha_{mj}h^\alpha_{ik} - h^\alpha_{mk}h^\alpha_{ij})\Big] \\
&+ \sum_{i,j,k,m}h^{n+1}_{ij}h^{n+1}_{im}\Big[(\delta_{mj}\delta_{kk} - \delta_{mk}\delta_{jk})c \\
&+ \sum_\alpha(h^\alpha_{mj}h^\alpha_{kk} - h^\alpha_{mk}h^\alpha_{jk})\Big]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + nc\sum_{i,j}(h_{ij}^{n+1})^2 - \Big[\sum_{i,j}(h_{ij}^{n+1})^2\Big]^2 \\
&\quad -n^2cH^2 + nH\sum_{i,j,k} h_{ij}^{n+1}h_{jk}^{n+1}h_{ki}^{n+1} \\
&\quad - \sum_{\alpha\neq n+1}\Big[\sum_{i,j}(h_{ij}^{n+1} - H\delta_{ij})h_{ij}^{\alpha}\Big]^2. \qquad (3.1)
\end{aligned}$$

Let $\{e_i\}$ be a frame diagonalizing the matrix H_{n+1} such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$, for all i, j. Set

$$\begin{aligned}
&f_k = \sum_i(\lambda_i^{n+1})^k, \\
&\mu_i^{n+1} = H - \lambda_i^{n+1},\ i = 1, 2, ..., n, \\
&B_k = \sum_i(\mu_i^{n+1})^k.
\end{aligned}$$

Then

$$\begin{aligned}
&B_1 = 0,\ B_2 = S_H - nH^2, \\
&B_3 = 3HS_H - 2nH^3 - f_3.
\end{aligned}$$

This together with (3.1) implies that

$$\begin{aligned}
\frac{1}{2}\Delta S_H &= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + ncS_H - S_H^2 - n^2cH^2 \\
&\quad +nHf_3 - \sum_{\alpha\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^{\alpha}\Big)^2 \\
&= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + ncS_H - S_H^2 - n^2cH^2 \\
&\quad +nH(3HS_H - 2nH^3 - B_3) - \sum_{\alpha\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^{\alpha}\Big)^2 \\
&= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + B_2[nc + 2nH^2 - S_H] \\
&\quad -nHB_3 - \sum_{\alpha\neq n+1}\Big(\sum_i \mu_i^{n+1}h_{ii}^{\alpha}\Big)^2. \qquad (3.2)
\end{aligned}$$

Let d be the infimum of the Ricci curvature of M. Then we have

$$Ric(e_i) = (n-1)c + nH\lambda_i^{n+1} - (\lambda_i^{n+1})^2 - \sum_{\alpha\neq n+1, j}(h_{ij}^{\alpha})^2 \geq d. \qquad (3.3)$$

This implies that

$$S - nH^2 \leq n[(n-1)(c + H^2) - d], \qquad (3.4)$$

and

$$
\begin{aligned}
&(n-2)H(\lambda_i^{n+1}-H)-(\lambda_i^{n+1}-H)^2\\
&\quad+(n-1)(c+H^2)-\sum_{\alpha\neq n+1,j}(h_{ij}^{\alpha})^2-d\geq 0.
\end{aligned} \tag{3.5}
$$

It follows from (3.5) that

$$
H(\lambda_i^{n+1}-H)\geq\frac{(\lambda_i^{n+1}-H)^2}{n-2}+\frac{\sum_{\alpha\neq n+1,j}(h_{ij}^{\alpha})^2}{n-2}+\frac{d}{n-2}-\frac{n-1}{n-2}(c+H^2).
$$

So,

$$
\begin{aligned}
-nHB_3 \quad \geq \quad & \frac{n}{n-2}\sum_i(\mu_i^{n+1})^4+\frac{n}{n-2}\sum_{\alpha\neq n+1}\sum_{i,j}(h_{ij}^{\alpha})^2(\mu_i^{n+1})^2\\
&+\frac{n}{n-2}[d-(n-1)(c+H^2)]B_2.
\end{aligned} \tag{3.6}
$$

From (3.2) and (3.6), we get

$$
\begin{aligned}
\frac{1}{2}\Delta S_H \quad \geq \quad & \sum_{i,j,k}(h_{ijk}^{n+1})^2+B_2\Big\{nc+2nH^2-S_H\\
&+\frac{n}{n-2}[d-(n-1)(c+H^2)]\Big\}+\frac{n}{n-2}\sum_i(\mu_i^{n+1})^4\\
&+\sum_{\alpha\neq n+1}\Big[\frac{n}{n-2}\sum_i(h_{ii}^{\alpha})^2(\mu_i^{n+1})^2-\Big(\sum_i\mu_i^{n+1}h_{ii}^{\alpha}\Big)^2\Big]\\
\geq \quad & \sum_{i,j,k}(h_{ijk}^{n+1})^2+B_2\Big\{nc+2nH^2-S_H\\
&+\frac{n}{n-2}[d-(n-1)(c+H^2)]\Big\}\\
&+\frac{B_2^2}{n-2}-\frac{n-3}{n-2}\sum_{\alpha\neq n+1}\Big(\sum_i\mu_i^{n+1}h_{ii}^{\alpha}\Big)^2\\
\geq \quad & \sum_{i,j,k}(h_{ijk}^{n+1})^2+B_2\Big\{nc+nH^2-\frac{n-3}{n-2}(S-nH^2)\\
&+\frac{n}{n-2}[d-(n-1)(c+H^2)]\Big\}.
\end{aligned} \tag{3.7}
$$

This together with (3.4) implies that

$$
\begin{aligned}
\frac{1}{2}\Delta S_H \quad \geq \quad & \sum_{i,j,k}(h_{ijk}^{n+1})^2+\frac{n}{n-2}B_2\{(n-2)(c+H^2)\\
&-(n-3)[(n-1)(c+H^2)-d]+[d-(n-1)(c+H^2)]\}\\
= \quad & \sum_{i,j,k}(h_{ijk}^{n+1})^2+nB_2[d-(n-2)(c+H^2)].
\end{aligned} \tag{3.8}
$$

By the assumption, we have $d \geq (n-2)(c+H^2)$. This together with (3.8) and the maximum principal implies that S_H is a constant, and

$$(S_H - nH^2)[d - (n-2)(c+H^2)] = 0. \tag{3.9}$$

Suppose that $S_H \neq nH^2$. Then $d = (n-2)(c+H^2)$. We consider the following two cases:

(i) If $n = 3$, then the inequalities in (3.7) and (3.8) become equalities. Thus, we have

$$h_{ij}^{\alpha} = 0, \ \text{ for } \alpha \neq n+1, \ \ i \neq j,$$
$$|\mu_i^{n+1}| = |\mu_j^{n+1}|, \ \ \mu_i^{n+1} = h_{ii}^{\alpha}, \ \text{ for } \alpha \neq n+1, \ \ 1 \leq i, j \leq n. \tag{3.10}$$

This implies $\mu_i^{n+1} = 0$, $i = 1, 2, \cdots, n$. It follows from Gauss equation that $c + H^2 = 0$. This contradicts with assumption.

(ii) If $n \geq 4$, then the inequalities in (3.7) and (3.8) become equalities and we have

$$Ric_M \equiv (n-2)(c+H^2),$$
$$h_{ij}^{\alpha} = 0, \ \text{ for } \alpha \neq n+1, \ \ i \neq j,$$
$$|\mu_i^{n+1}| = |\mu_j^{n+1}|, \ \ \mu_i^{n+1} = h_{ii}^{\alpha}, \ \text{ for } \alpha \neq n+1, \ \ 1 \leq i, j \leq n. \tag{3.11}$$

It follows from Gauss equation that $\mu_i^{n+1} = 0$ and $c + H^2 = 0$. This contradicts with assumption.

Therefore, $S_H = nH^2$, i.e., M is a pseudo-umbilical submanifold. This completes the proof of Theorem 3.1. □

The following lemma due to Yau [40] will be used in the proof of our geometric rigidity theorem, i.e., Theorem 3.3.

Lemma 3.2. *Let N^{n+p} be a conformally flat manifold. Let N_1 be a subbundle of the normal bundle of M^n with fiber dimension k. Suppose M is umbilical with respect to N_1 and N_1 is parallel in the normal bundle. Then M lies in an $(n+p-k)$-dimensional umbilical submanifold N' of N such that the fiber of N_1 is everywhere perpendicular to N'.*

We are now in a position to give an affirmative answer to Open Problems A. More generally, we prove the following rigidity theorem for compact submanifolds with parallel mean curvature in space forms.

Theorem 3.3. *Let M be an $n(\geq 3)$-dimensional oriented compact submanifold with parallel mean curvature in $F^{n+p}(c)$ with $c + H^2 > 0$. If*

$$Ric_M \geq (n-2)(c+H^2),$$

then M is either the totally umbilic sphere $S^n(\frac{1}{\sqrt{c+H^2}})$, the Clifford hypersurface $S^m\Big(\frac{1}{\sqrt{2(c+H^2)}}\Big) \times S^m\Big(\frac{1}{\sqrt{2(c+H^2)}}\Big)$ in the totally umbilic sphere $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ with $n = 2m$, or $\mathbb{C}P^2(\frac{4}{3}(c+H^2))$ in $S^7(\frac{1}{\sqrt{c+H^2}})$. Here $\mathbb{C}P^2(\frac{4}{3}(c+H^2))$ denotes the 2-dimensional complex projective space minimally immersed in $S^7(\frac{1}{\sqrt{c+H^2}})$ with constant holomorphic sectional *curvature* $\frac{4}{3}(c+H^2)$.

Proof. Case I. $H=0$. If $n=3$, then the assertion follows from Shen and Li's results [29, 21].

If $n\geq 4$, then it follows from Lemma 2.3 that M is simply connected. Hence the assertion follows from Theorem E.

Case II. $H\neq 0$. When $p=1$, we get the conclusion from Theorem 3.1.

When $p\geq 2$, we know from the assumption and Theorem 3.1 that M is pseudo-umbilical. It is seen from Lemma 3.2 that M lies in an $(n+p-1)$-dimensional totally umbilic submanifold $F^{n+p-1}(\tilde{c})$ of $F^{n+p}(c)$, i.e., the isometric immersion from M into $F^{n+p}(c)$ is given by

$$i\circ\varphi: M\rightarrow F^{n+p-1}(\tilde{c})\rightarrow F^{n+p}(c),$$

where $\varphi: M^n\rightarrow F^{n+p-1}(\tilde{c})$ is an isometric immersion with mean curvature vector ξ_1, and $i: F^{n+p-1}(\tilde{c})\rightarrow F^{n+p}(c)$ is a totally umbilic submanifold with mean curvature vector ξ_2. Denote by h_2 the second fundamental form of isometric immersion i. Set

$$H_1=|\xi_1|,\quad H_2=|\xi_2|. \tag{3.12}$$

We know that $\xi=\xi_1+\eta$, where $\eta=\frac{1}{n}\sum_i h_2(e_i,e_i)$ and $\{e_i\}$ is a local orthonormal frame field in M. Since $\xi_1\perp\xi$, and $\eta\parallel\xi$, we obtain $\xi_1=0$, and $\eta=\xi$. Noting that $F^{n+p-1}(\tilde{c})$ is a totally umbilic submanifold in $F^{n+p}(c)$, we have $|\eta|=H_2$. Thus,

$$H^2=H_1^2+|\eta|^2=H_2^2. \tag{3.13}$$

This together with the Gauss equation implies that

$$\tilde{c}=c+H^2. \tag{3.14}$$

Hence, M is an oriented compact minimal submanifold in $S^{n+p-1}(\frac{1}{\sqrt{c+H^2}})$. Now we consider the following two cases:

(i) $n=3$. It follows from Shen and Li's results [29, 21] that M is the totally umbilic sphere $S^3(\frac{1}{\sqrt{c+H^2}})$.

(ii) $n\geq 4$. From the assumption and Lemma 2.3, we know that M is simply connected. Therefore, it follows from Theorem E that M is either the totally umbilic sphere $S^n(\frac{1}{\sqrt{c+H^2}})$, the Clifford hypersurface $S^m\big(\frac{1}{\sqrt{2(c+H^2)}}\big)\times S^m\big(\frac{1}{\sqrt{2(c+H^2)}}\big)$ in the totally umbilic sphere $S^{n+1}(\frac{1}{\sqrt{c+H^2}})$ with $n=2m$, or $\mathbb{C}P^2(\frac{4}{3}(c+H^2))$ in $S^7(\frac{1}{\sqrt{c+H^2}})$.

Combing (i) and (ii), we complete the proof of Theorem 3.3. □

Remark 3.1. It's obvious that the pinching condition in Theorem 3.3 is sharp.

As a consequence of Theorem 3.3, we get the following:

Corollary 3.4. *Let M^n be an $n(\geq 3)$-dimensional oriented compact submanifold with parallel mean curvature in $F^{n+p}(c)$ with $c+H^2>0$. If*

$$Ric_M>(n-2)(c+H^2),$$

then M is the totally umbilic sphere $S^n(\frac{1}{\sqrt{c+H^2}})$.

4 Sphere theorems for submanifolds

In this section, we investigate rigidity of topological and differentiable structures of compact submanifolds in space forms. Motivated by Theorem 3.3, we first prove the following topological sphere theorem for compact submanifolds in space forms.

Theorem 4.1. *Let M be an $n(\geq 4)$-dimensional compact submanifold in $F^{n+p}(c)$ with $c \geq 0$. If*

$$Ric_M > (n-2)(c+H^2),$$

then M is homeomorphic to a sphere.

Proof. Assume that $2 \leq q \leq \frac{n}{2}$. Setting

$$T_\alpha := \frac{trH_\alpha}{n},$$

we have $\sum_\alpha T_\alpha^2 = H^2$, and

$$Ric(e_i) = (n-1)c + \sum_\alpha \Big[nT_\alpha h_{ii}^\alpha - (h_{ii}^\alpha)^2 - \sum_{j\neq i}(h_{ij}^\alpha)^2\Big]. \tag{4.1}$$

Then we get

$$\begin{aligned}
&\sum_{k=q+1}^{n}\sum_{i=1}^{q}[2|h(e_i,e_k)|^2 - \langle h(e_i,e_i), h(e_k,e_k)\rangle] \\
= \;& 2\sum_\alpha\sum_{k=q+1}^{n}\sum_{i=1}^{q}(h_{ik}^\alpha)^2 - \sum_\alpha\sum_{k=q+1}^{n}\sum_{i=1}^{q}h_{ii}^\alpha h_{kk}^\alpha \\
= \;& \sum_\alpha\Big[2\sum_{k=q+1}^{n}\sum_{i=1}^{q}(h_{ik}^\alpha)^2 - \Big(\sum_{i=1}^{q}h_{ii}^\alpha\Big)\Big(trH_\alpha - \sum_{i=1}^{q}h_{ii}^\alpha\Big)\Big] \\
\leq \;& \sum_\alpha\Big[2\sum_{k=q+1}^{n}\sum_{i=1}^{q}(h_{ik}^\alpha)^2 - nT_\alpha\sum_{i=1}^{q}h_{ii}^\alpha + q\sum_{i=1}^{q}(h_{ii}^\alpha)^2\Big] \\
\leq \;& q\sum_{i=1}^{q}[(n-1)c - Ric(e_i)] + n(q-1)\sum_\alpha\sum_{i=1}^{q}T_\alpha h_{ii}^\alpha \\
\leq \;& q^2[(n-1)(c+H^2) - Ric_{\min}] \\
& -q(n-q)H^2 + n(q-1)\sum_\alpha\sum_{i=1}^{q}T_\alpha(h_{ii}^\alpha - T_\alpha) \\
\leq \;& q(n-q)[(n-1)(c+H^2) - Ric_{\min}] \\
& -q(n-q)H^2 + n(q-1)\sum_\alpha\sum_{i=1}^{q}T_\alpha(h_{ii}^\alpha - T_\alpha).
\end{aligned} \tag{4.2}$$

On the other hand, we obtain

$$\begin{aligned}
&\sum_{k=q+1}^{n}\sum_{i=1}^{q}[2|h(e_i,e_k)|^2-\langle h(e_i,e_i),h(e_k,e_k)\rangle]\\
=\;&\sum_{\alpha}\Big[2\sum_{k=q+1}^{n}\sum_{i=1}^{q}(h_{ik}^{\alpha})^2-\frac{n-q}{n}\Big(\sum_{i=1}^{q}h_{ii}^{\alpha}\Big)\Big(trH_{\alpha}-\sum_{i=1}^{q}h_{ii}^{\alpha}\Big)\\
&-\frac{q}{n}\Big(\sum_{k=q+1}^{n}h_{kk}^{\alpha}\Big)\Big(trH_{\alpha}-\sum_{k=q+1}^{n}h_{kk}^{\alpha}\Big)\Big]\\
\leq\;&\sum_{\alpha}\Big[2\sum_{k=q+1}^{n}\sum_{i=1}^{q}(h_{ik}^{\alpha})^2-(n-q)T_{\alpha}\sum_{i=1}^{q}h_{ii}^{\alpha}+\frac{q(n-q)}{n}\sum_{i=1}^{q}(h_{ii}^{\alpha})^2\\
&-qT_{\alpha}\sum_{k=q+1}^{n}h_{kk}^{\alpha}+\frac{q(n-q)}{n}\sum_{k=q+1}^{n}(h_{kk}^{\alpha})^2\Big]\\
\leq\;&\frac{q(n-q)}{n}S-\sum_{\alpha}\Big[qnT_{\alpha}^2+(n-2q)T_{\alpha}\sum_{i=1}^{q}h_{ii}^{\alpha}\Big]\\
\leq\;&q(n-q)[(n-1)(c+H^2)-Ric_{\min}]\\
&-q(n-q)H^2-(n-2q)\sum_{\alpha}\sum_{i=1}^{q}T_{\alpha}(h_{ii}^{\alpha}-T_{\alpha}).
\end{aligned}\tag{4.3}$$

It follows from (4.2), (4.3) and the assumption that

$$\begin{aligned}
&\sum_{k=q+1}^{n}\sum_{i=1}^{q}[2|h(e_i,e_k)|^2-\langle h(e_i,e_i),h(e_k,e_k)\rangle]\\
\leq\;&\frac{n-2q}{q(n-2)}\Big\{q(n-q)[(n-1)(c+H^2)-Ric_{\min}]\\
&-q(n-q)H^2+n(q-1)\sum_{\alpha}\sum_{i=1}^{q}T_{\alpha}(h_{ii}^{\alpha}-T_{\alpha})\Big\}\\
&+\frac{n(q-1)}{q(n-2)}\Big\{q(n-q)[(n-1)(c+H^2)-Ric_{\min}]\\
&-q(n-q)H^2-(n-2q)\sum_{\alpha}\sum_{i=1}^{q}T_{\alpha}(h_{ii}^{\alpha}-T_{\alpha})\Big\}\\
=\;&q(n-q)[(n-1)(c+H^2)-Ric_{\min}]-q(n-q)H^2\\
<\;&q(n-q)c.
\end{aligned}\tag{4.4}$$

This together with Theorem 2.1 implies that

$$H_q(M;\mathbb{Z})=H_{n-q}(M;\mathbb{Z})=0,$$

for all $2\leq q\leq\frac{n}{2}$.

Since $(n-2)(c+H^2)\geq\frac{n(n-1)}{n+2}(c+H^2)$, we get from the assumption and Lemma 2.2 that

$$H_1(M;\mathbb{Z})=H_{n-1}(M;\mathbb{Z})=0,$$

and M is simply connected.

From above discussion, we know that M is a homotopy sphere. This together with the generalized Paincaré conjecture implies that M is a topological sphere. This completes the proof of Theorem 4.1. □

Remark 4.1. It's seen from Theorem 3.3 that the pinching condition in Theorem 4.1 is sharp.

In the next, we investigate differentiable pinching problem on compact submanifolds in a Riemannian manifold, and obtain the following theorem.

Theorem 4.2. *Let (M, g_0) be an $n(\geq 4)$-dimensional compact submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. If the Ricci curvature of M satisfies*

$$Ric_M > \Big[\frac{3n^2-9n+8}{3(n-2)}\overline{K}_{\max} - \frac{8}{3(n-2)}\overline{K}_{\min}\Big] + \frac{n(n-3)}{n-2}H^2,$$

then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t}g(t) = -2Ric_{g(t)} + \frac{2}{n}r_{g(t)}g(t),$$

exists for all time and converges to a constant curvature metric as $t \to \infty$. Moreover, M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n.

Proof. Set $T_\alpha = \frac{1}{n}trH_\alpha$. Then $\sum_\alpha T_\alpha^2 = H^2$, and

$$\begin{aligned} h_{ii}^\alpha h_{jj}^\alpha &= \frac{1}{2}[(h_{ii}^\alpha + h_{jj}^\alpha - 2T_\alpha)^2 - (h_{ii}^\alpha - T_\alpha)^2 - (h_{jj}^\alpha - T_\alpha)^2] \\ &\quad + T_\alpha(h_{ii}^\alpha - T_\alpha) + T_\alpha(h_{jj}^\alpha - T_\alpha) + T_\alpha^2. \end{aligned} \tag{4.5}$$

We rewrite (2.3) as

$$\begin{aligned} Ric(e_i) &= \sum_j \overline{R}_{ijij} + (n-1)H^2 + (n-2)\sum_\alpha T_\alpha(h_{ii}^\alpha - T_\alpha) \\ &\quad - \sum_\alpha (h_{ii}^\alpha - T_\alpha)^2 - \sum_{\alpha, j\neq i}(h_{ij}^\alpha)^2. \end{aligned} \tag{4.6}$$

This implies that

$$\begin{aligned} -\sum_\alpha (h_{ii}^\alpha - T_\alpha)^2 &\geq Ric_{\min} - (n-1)(\overline{K}_{\max} + H^2) \\ &\quad - (n-2)\sum_\alpha T_\alpha(h_{ii}^\alpha - T_\alpha) + \sum_{\alpha, j\neq i}(h_{ij}^\alpha)^2, \end{aligned} \tag{4.7}$$

and

$$\sum_\alpha T_\alpha(h_{ii}^\alpha - T_\alpha) \geq \frac{1}{n-2}[Ric_{\min} - (n-1)(\overline{K}_{\max} + H^2)]. \tag{4.8}$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame and $\lambda \in \mathbb{R}$.

From (2.1), (2.4), (4.5), (4.7) and (4.8), we have

$$\begin{aligned}
& R_{1313} + R_{2323} - |R_{1234}| \\
= \;& \overline{R}_{1313} + \overline{R}_{2323} + \sum_\alpha \Big[h_{11}^\alpha h_{33}^\alpha - (h_{13}^\alpha)^2 + h_{22}^\alpha h_{33}^\alpha - (h_{23}^\alpha)^2\Big] \\
& -|\overline{R}_{1234} + \sum_\alpha (h_{13}^\alpha h_{24}^\alpha - h_{14}^\alpha h_{23}^\alpha)| \\
\geq \;& 2\overline{K}_{\min} - \frac{2}{3}(\overline{K}_{\max} - \overline{K}_{\min}) \\
& -\frac{1}{2}\sum_\alpha \Big[3(h_{13}^\alpha)^2 + 3(h_{23}^\alpha)^2 + (h_{14}^\alpha)^2 + (h_{24}^\alpha)^2\Big] \\
& +\sum_\alpha \Big[-\frac{(h_{11}^\alpha - T_\alpha)^2}{2} - \frac{(h_{33}^\alpha - T_\alpha)^2}{2} \\
& +T_\alpha(h_{11}^\alpha - T_\alpha) + T_\alpha(h_{33}^\alpha - T_\alpha) + T_\alpha^2\Big] \\
& +\sum_\alpha \Big[-\frac{(h_{22}^\alpha - T_\alpha)^2}{2} - \frac{(h_{33}^\alpha - T_\alpha)^2}{2} \\
& +T_\alpha(h_{22}^\alpha - T_\alpha) + T_\alpha(h_{33}^\alpha - T_\alpha) + T_\alpha^2\Big] \\
\geq \;& \frac{8}{3}\Big(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\Big) - \frac{1}{2}\sum_\alpha \Big[3(h_{13}^\alpha)^2 + 3(h_{23}^\alpha)^2 + (h_{14}^\alpha)^2 + (h_{24}^\alpha)^2\Big] \\
& +2[Ric_{\min} - (n-1)(\overline{K}_{\max} + H^2)] + 2H^2 \\
& +\frac{1}{2}\sum_{\alpha, j\neq 1}(h_{1j}^\alpha)^2 + \frac{1}{2}\sum_{\alpha, j\neq 2}(h_{2j}^\alpha)^2 + \sum_{\alpha, j\neq 3}(h_{3j}^\alpha)^2 \\
& +\frac{n-4}{2}\sum_{\alpha, i\neq 1,3} T_\alpha(h_{ii}^\alpha - T_\alpha) + \frac{n-4}{2}\sum_{\alpha, i\neq 2,3} T_\alpha(h_{ii}^\alpha - T_\alpha) \\
\geq \;& \frac{8}{3}\Big(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\Big) + 2H^2 \\
& +(n-2)[Ric_{\min} - (n-1)(\overline{K}_{\max} + H^2)]. \qquad (4.9)
\end{aligned}$$

Same argument implies that

$$\begin{aligned}
& R_{1414} + R_{2424} - |R_{1234}| \\
\geq \;& \frac{8}{3}\Big(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\Big) + 2H^2 \\
& +(n-2)[Ric_{\min} - (n-1)(\overline{K}_{\max} + H^2)]. \qquad (4.10)
\end{aligned}$$

This together with (4.9) and the assumption implies

$$\begin{aligned}
& R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \\
\geq \;& R_{1313} + R_{2323} - |R_{1234}| + \lambda^2(R_{1414} + R_{2424} - |R_{1234}|) \\
> \;& 0. \qquad (4.11)
\end{aligned}$$

It follows from Theorem A that M is diffeomorphic to a spherical space form. In particular, if M is simply connected, then M is diffeomorphic to S^n. This completes the proof of Theorem 4.2. □

Theorem 4.3. *Let M be an $n(\geq 4)$-dimensional compact submanifold in $F^{n+p}(c)$ with $c \geq 0$. If*

$$Ric_M > (n-2)(1+\varepsilon_n)(c+H^2),$$

then M is diffeomorphic to S^n. Here

$$\varepsilon_n = \begin{cases} 0, & \text{for } 4 \leq n \leq 6, \\ \frac{n-4}{(n-2)^2}, & \text{for } n \geq 7. \end{cases}$$

Proof. When $n = 5, 6$, it is well known that there is only one differentiable structure on S^n. This together with Theorem 4.1 implies M is diffeomorphic to S^n. When $n \neq 5, 6$, it follows from Theorem 4.2 that M is diffeomorphic to a spherical space form. On the other hand, it follows from Lemma 2.2 that M is simply connected. Therefore, M is diffeomorphic to S^n. This completes the proof of Theorem 4.3. □

Remark 4.2. When $4 \leq n \leq 6$, the pinching condition in Theorem 4.3 is sharp. When $n \geq 7$, we have $0 \leq \varepsilon_n < \frac{1}{n}$ and $\lim_{n\to\infty} \varepsilon_n = 0$. Therefore, the pinching condition in Theorem 4.3 is close to the best possible.

Motivated by our rigidity and sphere theorems, we would like to propose the following conjecture.

Conjecture A. *Let M be an $n(\geq 3)$-dimensional compact oriented submanifold in the space form $F^{n+p}(c)$ with $c + H^2 > 0$. If*

$$Ric_M \geq (n-2)(c+H^2),$$

then M is diffeomorphic to either the standard n-sphere S^n, the Clifford hypersurface $S^m\big(\frac{1}{\sqrt{2}}\big)\times S^m\big(\frac{1}{\sqrt{2}}\big)$ in S^{n+1} with $n = 2m$, or $\mathbb{C}P^2$. In particular, if $Ric_M > (n-2)(c+H^2)$, then M is diffeomorphic to S^n.

To verify Conjecture A, we hope to prove the following conjecture on the normalized Ricci flow.

Conjecture B. *Let (M, g_0) be an $n(\geq 4)$-dimensional compact submanifold in an $(n+p)$-dimensional space form $F^{n+p}(c)$ with $c + H^2 > 0$. If the Ricci curvature of M satisfies*

$$Ric_M > (n-2)(c+H^2),$$

then the normalized Ricci flow with initial metric g_0

$$\frac{\partial}{\partial t}g(t) = -2Ric_{g(t)} + \frac{2}{n}r_{g(t)}g(t),$$

exists for all time and converges to a constant curvature metric as $t \to \infty$. Moreover, M is diffeomorphic to a spherical space form.

Theorems 4.2 and 4.3 provide partial affirmative answers to Conjectures A and B. Motivated by our rigidity and sphere theorems, we would like to propose the following conjecture on the mean curvature flow in higher codimensions.

Conjecture C. *Let $F_0 : M \to F^{n+p}(c)$ be an n-dimensional compact submanifold in an $(n+p)$-dimensional space form $F^{n+p}(c)$ with $c + H^2 > 0$. If the Ricci curvature of M satisfies*

$$Ric_M > (n-2)(c+H^2),$$

then the mean curvature flow

$$\begin{cases} \frac{\partial}{\partial t}F(x,t) = n\xi(x,t), \ x \in M, \, t \geq 0, \\ F(\cdot,0) = F_0(\cdot), \end{cases}$$

exists smooth solution $F_t(\cdot)$, and $F_t(\cdot)$ converges to a round point in finite time, or $c > 0$ and $F_t(\cdot)$ converges to a totally geodesic sphere as $t \to \infty$. In particular, M is diffeomorphic to S^n.

Recently, Andrews and Baker [1], Liu, Xu, Ye and Zhao [22, 23] obtained some convergence theorems for the mean curvature flow of higher codimension under certain pinching conditions on the second fundamental form of M.

References

[1] B. Andrews and C. Baker, Mean curvature flow of pinched submanifolds to spheres, *J. Differential Geom.*, **85**(2010), 357-396.

[2] M. Berger, Riemannian geometry during the second half of the twentieth century, University Lecture Series, Vol.**17**, American Mathematical Society, Providence, RI, 2000.

[3] C. Böhm and B. Wilking, Manifolds with positive curvature operator are space forms, *Ann. of Math.*, **167**(2008), 1079-1097.

[4] S. Brendle, A general convergence result for the Ricci flow in higher dimensions, *Duke Math. J.*, **145**(2008), 585-601.

[5] S. Brendle, Ricci Flow and the Sphere Theorem, Graduate Studies in Mathematics, Vol.**111**, Americam Mathematical Society, 2010.

[6] S. Brendle and R. Schoen, Manifolds with 1/4-pinched curvature are space forms, *J. Amer. Math. Soc.*, **22**(2009), 287-307.

[7] S. Brendle and R. Schoen, Classification of manifolds with weakly 1/4-pinched curvatures, *Acta Math.*, **200**(2008), 1-13.

[8] S. Brendle and R. Schoen, Sphere theorems in geometry, Surveys in Differential Geometry, Vol.**13**, 2009, 49-84.

[9] H. D. Cao, B. L. Chen and X. P. Zhu, Recent developments on Hamilton's Ricci flow, Surveys in Differential Geometry, Vol.**12**, 2008, 47-112.

[10] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below. I, *J. Differential Geom.*, **46**(1997), 406-480.

[11] S. S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, in Functional Analysis and Related Fields, Springer-Verlag, New York(1970).

[12] N. Ejiri, Compact minimal submanifolds of a sphere with positive Ricci curvature, *J. Math. Soc. Japan*, **31**(1979), 251-256.

[13] K. Grove and P. Petersen, A pinching theorem for homotopy spheres, *J. Amer. Math. Soc.*, **3**(1990), 671-677.

[14] K. Grove and K. Shiohama, A generalized sphere theorem, *Ann. of Math.*, **106**(1977), 201-211.

[15] J. R. Gu and H. W. Xu, The sphere theorem for manifolds with positive scalar curvatue, *J. Differential Geom.*, **92**(2012), 507-545.

[16] R. Hamilton, Three manifolds with positive Ricci curvature, *J. Differential Geom.*, **17**(1982), 255-306.

[17] G. Huisken, Ricci deformation of the metric on a Riemannian manifold, *J. Differential Geom.*, **21**(1985), 47-62.

[18] B. Lawson, Local rigidity theorems for minimal hyperfaces, *Ann. of Math.*, **89**(1969), 187-197.

[19] B. Lawson and J. Simons, On stable currents and their application to global problems in real and complex geometry, *Ann. of Math.*, **98**(1973), 427-450.

[20] A. M. Li and J. M. Li, An intrinsic rigidity theorem for minimal submanifolds in a sphere, *Arch. Math.*, **58**(1992), 582-594.

[21] H. Z. Li, Curvature pinching for odd-dimensional minimal submanifolds in a sphere, *Publ. Inst. Math.* (*Beograd*), **53**(1993), 122-132.

[22] K. F. Liu, H. W. Xu, F. Ye and E. T. Zhao, The extension and convergence of mean curvature flow in higher codimension, arXiv:math.DG/1104.0971.

[23] K. F. Liu, H. W. Xu, F. Ye and E. T. Zhao, Mean curvature flow of higher codimension in hyperbolic spaces, arXiv:math.DG/1105.5686, to appear in *Comm. Anal. Geom.*

[24] M. Micallef and J. D. Moore, Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes, *Ann. of Math.*, **127**(1988), 199-227.

[25] M. Okumura, Submanifolds and a pinching problem on the second fundamental tensor, *Trans. Amer. Math. Soc.*, **178**(1973), 285-291.

[26] M. Okumura, Hypersurfaces and a pinching problem on the second fundamental tensor, *Amer. J. Math.*, **96**(1974), 207-213.

[27] P. Petersen, On eigenvalue pinching in positive Ricci curvature, *Invent. Math.*, **138**(1999), 1-21.

[28] P. Petersen and T. Tao, Classification of almost quarter-pinched manifolds, *Proc. Amer. Math. Soc.*, **137**(2009), 2437-2440.

[29] Y. B. Shen, Curvature pinching for three-dimensional minimal submanifolds in a sphere, *Proc. Amer. Math. Soc.*, **115**(1992), 791-795.

[30] K. Shiohama, Sphere theorems, Handbook of Differential Geometry, Vol. **1**, F.J.E. Dillen and L.C.A. Verstraelen (eds.), Elsevier Science B.V., Amsterdam, 2000.

[31] K. Shiohama and H. W. Xu, The topological sphere theorem for complete submanifolds, *Compositio Math.*, **107**(1997), 221-232.

[32] J. Simons, Minimal varieties in Riemannian manifolds, *Ann. of Math.*, **88**(1968), 62-105.

[33] Z. Q. Sun, Submanifolds with constant mean curvature in spheres, *Adv. Math.* (*China*), **16**(1987), 91-96.

[34] Y. L. Xin, Application of integral currents to vanishing theorems, *Scient. Sinica*(A), **27**(1984), 233-241.

[35] H. W. Xu, Pinching theorems, global pinching theorems, and eigenvalues for Riemannian submanifolds, *Ph.D. dissertation, Fudan University*, 1990.

[36] H. W. Xu, A rigidity theorem for submanifolds with parallel mean curvature in a sphere, *Arch. Math.*, **61**(1993), 489-496.

[37] H. W. Xu, Recent developments in differentiable sphere theorems, Fifth International Congress of Chinese Mathematicians, part 1, 2, 415-430, AMS/IP Stud. Adv. Math., 51, pt. 1, 2, Amer. Math. Soc., Providence, RI, 2012.

[38] H. W. Xu and J. R. Gu, An optimal differentiable sphere theorem for complete manifolds, *Math. Res. Lett.*, **17**(2010), 1111-1124.

[39] H. W. Xu and E. T. Zhao, Topological and differentiable sphere theorems for complete submanifolds, *Comm. Anal. Geom.*, **17**(2009), 565-585.

[40] S. T. Yau, Submanifolds with constant mean curvature I, II, *Amer. J. Math.*, **96, 97**(1974, 1975), 346-366, 76-100.

Hong-Wei Xu
Center of Mathematical Sciences
Zhejiang University
Hangzhou 310027
China
E-mail address: xuhw@cms.zju.edu.cn

Juan-Ru Gu
Center of Mathematical Sciences
Zhejiang University
Hangzhou 310027
China
E-mail address: gujr@cms.zju.edu.cn