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Geometrical aspects of possibility measures on finite domain MV-clans

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Abstract In this paper we study generalized possibility and necessity measures on MV-algebras of $[0,1]$-valued functions (MV-clans) in the framework of idempotent mathematics, where the usual field of reals $\mathbb{R}$ is replaced by the *max-plus* semiring $\mathbb{R}_{\max}$. We prove results about extendability of partial assessments to possibility and necessity measures, and we characterize the geometrical properties of the space of homogeneous possibility measures. The aim of the present paper is also to support the idea that idempotent mathematics is the natural framework where to develop the theory of possibility and necessity measures, in the same way classical mathematics serves as a natural setting for probability theory.

Keywords Possibility measures, MV-algebras, Idempotent mathematics, Max-plus convexity.

1 Introduction and motivation

Possibility and necessity measures [9,31] on a Boolean algebra $\mathcal{B} = (B, \wedge, \vee, \neg, 1^B, 0^B)$ are $[0,1]$-valued, non-additive mappings $\mu : B \to [0,1]$ that can be framed in the context of *plausibility measures* [17] (also called *Sugeno* or *fuzzy measures* [29]). A plausibility measure $\mu : B \to [0,1]$ is just a normalized ($\mu(1^B) = 1$, and $\mu(0^B) = 0$) and monotone (for all $u, v \in B$ such that $u \leq v$, then $\mu(u) \leq \mu(v)$) mapping. Then,

- a *possibility measure* on $\mathcal{B}$ is a plausibility measure $\Pi : B \to [0,1]$ satisfying

$$\Pi(u \vee v) = \max\{\Pi(u), \Pi(v)\},$$

and

- a *necessity measure* on $\mathcal{B}$ is a plausibility measure $N : B \to [0,1]$ such that

$$N(u \wedge v) = \min\{N(u), N(v)\}.$$

In [10] (see also [11]), we consider possibility and necessity measures on (finite) MV-algebras [6,4] as a natural generalization of classical possibility and necessity measure on Boolean algebras. Indeed, any Boolean algebra is an MV-algebra, and moreover, in any MV-algebra $\mathcal{A}$, the set of idempotent elements is the domain of a Boolean subalgebra of $\mathcal{A}$. In this sense, MV-algebraic possibilities and necessities generalize the classical measures.

In Section 2, we recall the basic definition and the properties of possibility and necessity measures on MV-algebras. In particular, we concentrate on a representation theorem for *homogeneous* possibilities and necessities on MV-clans, in terms of a quasi Sugeno integral, as shown in [10, Theorem 3.3]. Sugeno, and generalized Sugeno integrals, can be regarded as the analogous versions of Riemann integral within the framework of *idempotent mathematics* (see [22] for a basic survey). Idempotent mathematics is based on the idea of replacing the usual arithmetic operations on reals by a new set of operations in such a way that the usual addition is substituted by an idempotent binary operation (usually max or min). The typical example, which is in fact crucial for the understanding of this paper, is the following: let $\mathbb{R}$ be the field of real numbers; then the *max-plus idempotent version* of $\mathbb{R}$ is the semiring $\mathbb{R}_{\max}$, whose domain is $\mathbb{R} \cup \{-\infty\}$, and where the operation of sum is replaced by max, and product between reals is replaced by the usual sum. The semiring $\mathbb{R}_{\max}$ can be regarded as the transformation of the real field $\mathbb{R}$ via the *Maslov dequantization*, which is related to the well-known logarithmic transformation that was used, e.g.,

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in the classical papers by Schrödinger [28] and Hopf [19]. It is worth noticing that the semiring $\mathbb{R}_{\min}$, obtained by replacing max by min in $\mathbb{R}_{\max}$, is the idempotent semiring which the rapidly growing discipline named *tropical mathematics* is built on [8,27][1].

In this paper we follow the intuition that the real idempotent semiring $\mathbb{R}_{\max}$ is the natural framework to study possibility measures, while $\mathbb{R}_{\min}$ is appropriate for necessity measures on MV-algebras; in an analogous form, the real field $\mathbb{R}$ is the natural setting to study states (probabilities) on MV-algebras [23]. Indeed we are going to prove that the algebraic and geometrical properties that hold for states, and that allow to extend de Finetti's coherence criterion to MV-algebras (cf. [21, 24]), naturally hold as well for possibility and necessity measures, when framed in the algebraic and geometrical setting of idempotent mathematics.

In fact, after this introduction and some preliminaries in next section, in Section 3 we characterize those partial and finite assessments that can be extended to possibility and necessity measures. Then, in Section 4, we focus on geometrical properties of the space $\boldsymbol{\pi}([0,1]^X)$ of homogeneous possibility measures defined over a finitely generated MV-clan $[0,1]^X$. In particular, we prove that $\boldsymbol{\pi}([0,1]^X)$ is max-plus convex (i.e. convex in the sense of $\mathbb{R}_{\max}$), and we show that the class of its extremal points coincides with the class $Hom([0,1]^X,[0,1])$ of real valued MV-homomorphisms of $[0,1]^X$, that, in turn, coincides with the class of *extremal states* over $[0,1]^X$ [23, Theorem 2.5]. Since any convex, max-plus convex, and min-plus convex set is fully characterized by its extremal elements, this last characterization indirectly proves that different uncertainty measures such as states, possibilities and necessities are characterized by the mathematical framework we choose to represent the nature of the uncertainty we are dealing with, rather than the way we choose to evaluate the information (i.e. the way we interpret the extremal measures).

This last consideration sheds a new light on uncertainty theories whose consequences we plan to investigate in our future work.

Finally, we would like to point out that the results we are going to present naturally apply to possibility and necessity measures defined on Boolean algebras as well. To the best of our knowledge the characterizations we provide are new, even for the classical case.

2 Preliminaries: MV-algebras, states and possibility measures

The language of Łukasiewicz logic Ł (cf. [6,16]), consists of a countable set of propositional variables $\{p_1,p_2,\ldots\}$, the binary connective $\to$, and the truth constant $\bot$. Formulas are defined by the usual inductive clauses. The following formulas provide an axiomatization for Ł:

(Ł1) $\varphi\to(\psi\to\varphi)$
(Ł2) $(\varphi\to\psi)\to((\psi\to\chi)\to(\varphi\to\chi))$
(Ł3) $((\varphi\to\bot)\to(\psi\to\bot))\to(\psi\to\varphi)$
(Ł4) $((\varphi\to\psi)\to\psi)\to((\psi\to\varphi)\to\varphi)$

The rule of inference of Ł is *modus ponens*: from φ and $\varphi\to\psi$, deduce ψ.

Further connectives in Ł are definable as follows: $\neg\varphi=\varphi\to\bot$; $\varphi\oplus\psi=\neg\varphi\to\psi$; $\varphi\odot\psi=\neg(\varphi\to\neg\psi)$; $\varphi\vee\psi=(\varphi\to\psi)\to\psi$; $\varphi\wedge\psi=\neg(\neg\varphi\vee\neg\psi)$; $\top=\neg\bot$.

Łukasiewicz logic is an algebraizable logic in the sense of Blok and Pigozzi [1], and its equivalent algebraic semantics is constituted by the class of MV-algebras [4,6]. In algebraic terms, an MV-algebra is a structure $\mathcal{A}=(A,\oplus,\neg,0^A)$ of type $(2,1,0)$ satisfying the following equations:

(MV1) $x\oplus(y\oplus z)=(x\oplus y)\oplus z$,
(MV2) $x\oplus y=y\oplus x$,
(MV3) $x\oplus 0^A=x$,
(MV4) $\neg\neg x=x$,
(MV5) $x\oplus\neg 0^A=\neg 0^A$,
(MV6) $\neg(\neg x\oplus y)\oplus y=\neg(\neg y\oplus x)\oplus x$.

Further (definable) operations can be defined from $\oplus,\neg$ and 0 in a similar way as for the logical connectives above. In particular: $x\odot y=\neg(\neg x\oplus\neg y)$; $x\vee y=\neg(\neg x\oplus y)\oplus y$; $x\wedge y=\neg(\neg x\vee\neg y)$; $1^A=\neg 0^A$.

The class of MV-algebras forms a variety that we denote by $\mathbb{MV}$.

Example 1 The following are four relevant examples of MV-algebras:

(1) Every Boolean algebra is an MV-algebra, and moreover for every MV-algebra $\mathcal{A}$, the set $B(\mathcal{A})=\{x\in A : x\oplus x=x\}$ of its idempotent elements is the domain of the largest Boolean subalgebra of $\mathcal{A}$. The algebra having $B(\mathcal{A})$ as universe is usually called the *Boolean skeleton* of $\mathcal{A}$.

(2) Define on the real unit interval $[0,1]$ the operations $\oplus$ and $\neg$ as follows: for all $x,y\in[0,1]$,

[1] Papers on idempotent and tropical mathematics, usually adopt the following notation: the idempotent operation is denoted by $\oplus$, while $\odot$ denotes the usual sum. This notation is justified because the idempotent operation substitutes the sum, and the sum substitutes the product in the real field $\mathbb{R}$. In this paper, conversely, we will not adopt this notation because it would be misleading with respect to those used in many-valued logic (see Section 2), where $\oplus$ and $\odot$ represent respectively a t-conorm, and a t-norm. For this reason, we will keep the standard notation for max, min, $+$, and $\cdot$.

$$x \oplus y = \min\{1, x + y\}, \text{ and } \neg x = 1 - x.$$

Then the structure $[0,1]_{MV} = ([0,1], \oplus, \neg, 0)$ is an MV-algebra. The MV-algebra $[0,1]_{MV}$ is generic for the variety of MV-algebras (i.e. it generates the whole variety) and is usually called the *standard* MV-algebra. In equivalent terms, Łukasiewicz logic is complete with respect to the semantics defined by the standard MV-algebra.

(3) Fix a $k \in \mathbb{N}$, and let $F(k)$ be the set of all the McNaughton functions (cf. [6]) from the hypercube $[0,1]^k$ into $[0,1]$. In other words, $F(k)$ is the set of all those functions $f : [0,1]^k \to [0,1]$ which are continuous, piecewise linear and such that each piece has integer coefficients. The following pointwise operations defined on $F(k)$,

$$(f \oplus g)(x) = \min\{1, f(x) + g(x)\}, \text{ and}$$
$$(\neg f)(x) = 1 - f(x),$$

make the structure $\mathcal{F}(k) = (F(k), \oplus, \neg, \overline{0})$ into an MV-algebra, where $\overline{0}$ clearly denotes the function constantly equal to 0. Actually, $\mathcal{F}(k)$ is the free MV-algebra over k generators.

(4) Let X be a non-empty set, and let $[0,1]^X$ the set of all functions form X into $[0,1]$, endowed with operations defined by the pointwise application of those in $[0,1]_{MV}$. The structure $[0,1]^X$ is clearly MV-algebra. Every MV-subalgebra of $[0,1]^X$ is called an *MV-clan* (cf. [2,25]).

It is worth noticing that in $[0,1]_{MV}$, the standard interpretation of the lattice operations of $\wedge$ and $\vee$, is respectively in terms of min and max. Therefore, we will henceforth use both the notations $\wedge$ and min, and $\vee$ and max, without danger of confusion.

In this paper we will concentrate on MV-algebras which are MV-clans $[0,1]^X$ defined over a *finite* set X. These algebras can be identified with those being a finite direct product of $[0,1]_{MV}$. In what follows, to stress the fact that the elements in $[0,1]^X$ are functions defined over a finite domain, we will call those algebras *finite domain MV-clans.*

2.1 States on MV-algebras

By a state on an MV-algebra $\mathcal{A}$ (cf. [23]) we mean a map $s : A \to [0,1]$ satisfying the following:

(i) $s(1^A) = 1$,
(ii) for every $x, y \in A$ such that $x \odot y = 0^A$, $s(x \oplus y) = s(x) + s(y)$.

We denote by $\mathcal{S}(\mathcal{A})$ the subset of $[0,1]^A$ whose elements are the states on $\mathcal{A}$.

The restriction of every state s to $B(\mathcal{A})$, the Boolean skeleton of $\mathcal{A}$, is a finitely additive probability measure. Vice versa, states on MV-algebras can be represented by means of the Lebesgue integral of regular Borel probability measures. The following representation theorem was independently proved by Kroupa [20, Theorem 28] and Panti [26, Proposition 1.1][2].

Theorem 1 (Kroupa-Panti) *For every MV-algebra $\mathcal{A}$, there exists a canonical bijective correspondence between $\mathcal{S}(\mathcal{A})$ and the set $\mathcal{P}(Hom(\mathcal{A},[0,1]_{MV}))$ of regular Borel probability measures on the set $Hom(\mathcal{A},[0,1]_{MV})$ of homomorphims of $\mathcal{A}$ into $[0,1]_{MV}$.*

States on MV-algebras enjoy a similar characterization in terms of reversible betting schemes as in the case of finite additive probabilities on Boolean events by means of the so-called *de Finetti's coherence criterion.* According to de Finetti (cf. [7]), a $[0,1]$-valued assessment $\chi : \chi(\varphi_1) = \alpha_1, \ldots, \chi(\varphi_n) = \alpha_n$ of classical events $\varphi_1, \ldots, \varphi_n$ (represented as formulas of classical Boolean logic) is said to be *coherent* iff there is no system of reversible bets on the events which leads to a sure win independently of the truth status of the events $\varphi_1, \ldots, \varphi_n$. In other words, the assessment χ is coherent iff for every $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, there is a classical $\{0,1\}$-valued truth-evaluation V such that

$$\textstyle\sum_{i=1}^n \lambda_i(\alpha_i - V(\varphi_i)) \geq 0. \qquad \text{(C)}$$

The celebrated de Finetti's Theorem states that an assessment is coherent iff it can be *extended* to a finitely additive measure on the Boolean algebra of formulas. In other words, $\chi : \varphi_i \mapsto \alpha_i$ is coherent iff there exists a finitely additive measure P on formulas such that $P(\varphi_i) = \alpha_i$ for each i.

In [24], Mundici extends de Finetti's coherence criterion to formulas of the infinitely-valued Łukasiewicz calculus. In this setting, the notion of a coherent assignment is exactly as above, but requires the existence of a $[0,1]$-valued Łukasiewicz truth-evaluation V instead of the existence of a Boolean truth-evaluation V in condition (C) . In particular, Mundici proves the following theorem. [3]

[2] It is worth noticing that, while Kroupa proved the following Theorem 1 in the case of semisimple MV-algebras, Panti showed that the hypothesis on the semisimplicity of the MV-algebra can be relaxed, since, for every MV-algebra $\mathcal{A}$, there is a canonical bijection between the class $\mathcal{S}(\mathcal{A})$ of all the states on A, and the class $\mathcal{S}(\mathcal{A}/Rad(\mathcal{A}))$ of all the states on its most general semisimple quotient $\mathcal{A}/Rad(\mathcal{A})$

[3] It is worth noticing that Kühr and Mundici [21, Corollary 4.3] extend de Finetti's coherence criterion to any algebraizable (cf. [1]) logic $\mathcal{L}_\Omega$ whose equivalent algebraic semantics is given by the algebraic variety generated by the algebra $([0,1], \Omega)$, where Ω denotes a set of continuous operations on $[0,1]$. Therefore the following theorem can be reasonably seen as a particular case of [21, Corollary 4.3].

Theorem 2 *Let $\varphi_1, \ldots, \varphi_n$ be formulas in the language of Ł, and let $\chi : \varphi_i \mapsto \alpha_i$ be a $[0,1]$-valued assessment. Then the following are equivalent:*

(1) χ is coherent,
(2) χ extends to a state on $\mathcal{F}(k)$ (where k is the number of propositional variables occurring in the φ_i's),
(3) χ extends to a convex combination of at most $n+1$ homomorphisms of $\mathcal{F}(k)$ into $[0,1]_{MV}$

Notice that the equivalence (1)-(3) of above theorem establishes that, if $\varphi_1, \ldots, \varphi_n$ are formulas of Ł, then an assessment $\chi : \varphi_i \mapsto \alpha_i$ (for $i = 1, \ldots, n$) is coherent iff there exist at most $n+1$ homomorphisms $h_1, \ldots, h_t$ into $[0,1]_{MV}$ and t real numbers $\lambda_1, \ldots, \lambda_t$, such that

$$\sum_{i=1}^{t} \lambda_i = 1, \text{ and for all } j \in \{1, \ldots, n\},$$
$$\alpha_j = \sum_{i=1}^{t} \lambda_i \cdot h_i(\varphi_j).$$

2.2 Possibility and necessity measures on MV-algebras

Recall from Section 1 that a *possibility measure* on a Boolean algebra $\mathcal{B}$ is a plausibility measure $\Pi : B \to [0,1]$ such that the following $\vee$-decomposition property

$$\Pi(u_1 \vee u_2) = \max(\Pi(u_1), \Pi(u_2))$$

holds, while a *necessity measure* is a plausibility measure $N : B \to [0,1]$ satisfying the $\wedge$-decomposition property

$$N(u_1 \wedge u_2) = \min(N(u_1), N(u_2)).$$

Possibility and necessity are *dual* classes of measures, in the sense that if Π is a possibility measure, then the function

$$N(u) = 1 - \Pi(\neg u) \tag{1}$$

is a necessity measure, and vice versa. If $\mathcal{B}$ is the power set of a set X, then any dual pair of measures (Π, N) on $\mathcal{B}$ is induced by a *normalized possibility distribution*, i.e. a mapping $\pi : X \to [0,1]$ such that, $\sup_{x \in X} \pi(x) = 1$, and, for any $S \subseteq X$,

$$\Pi(S) = \sup\{\pi(x) \mid x \in S\} \text{ and}$$
$$N(S) = \inf\{1 - \pi(x) \mid x \notin S\}.$$

with the usual assumption that $\sup \emptyset = 0$ and $\inf \emptyset = 1$.

In [10], we considered natural extensions of the notion of possibility and necessity measures on MV-algebras[4].

[4] In [10], we actually introduced the slightly more general notion of L-valued possibility (necessity) measure on an MV-algebra, where L is in any MV-chain. In this paper, we concentrate on $[0,1]$-valued maps, and hence we will simply speak about "possibility" (resp. "necessity") measures, without specifying that $[0,1]_{MV}$ serves as range for the measures.

Definition 1 **([10])** Let $\mathcal{A} = (A, \oplus, \neg, 0^A)$ be an MV-algebra. A map $\mu : A \to [0,1]$ such that $\mu(1^A) = 1$, and $\mu(0^A) = 0$ is said to be:

(i) a *possibility measure* on $\mathcal{A}$ (and we will denote it by Π), provided that $\mu(u \vee v) = \max\{\mu(u), \mu(v)\}$ for any $u, v \in A$;
(ii) a *necessity measure* on $\mathcal{A}$ (and we will denote it by N), provided that $\mu(u \wedge v) = \min\{\mu(u), \mu(v)\}$ for any $u, v \in A$.

Clearly, the decomposition property for μ in terms of the max (resp. min) operator shows that possibility (necessity) measures are monotone maps. Moreover, it is worth noticing that possibility and necessity measures remain dual as in the classical case: if Π is a possibility measure on $\mathcal{A}$, then the map N, defined as $N(u) = 1 - \Pi(\neg u)$, is a necessity measure on $\mathcal{A}$ (and vice cersa).

Key examples of possibility and necessity measures over finite domain MV-clans are the following. Let X be a finite set, let $[0,1]^X$ be the corresponding finite domain MV-clan, and let $\pi : X \to [0,1]$ be a normalized possibility distribution over X (i.e. π is any mapping for which there exists at least an $x \in X$, with $\pi(x) = 1$). Consider the following maps from $[0,1]^X$ into $[0,1]$: for every $f \in [0,1]^X$,

$$\begin{aligned} \Pi(f) &= \max_{x \in X} \pi(x) \odot f(x), \text{ and} \\ N(f) &= \min_{x \in X} (1 - \pi(x)) \oplus f(x). \end{aligned} \tag{2}$$

In a similar way (probabilistic) states are related to Lebesgue integrals, this kind of possibility and necessity measures over MV-clans are related to a class of generalized Sugeno integrals [29]. Indeed, given a plausibility measure $\mu : 2^X \to [0,1]$, the $\odot$-*quasi Sugeno integral* of a function $f : X \to [0,1]$ with respect to μ [30,15] is defined as

$$\oint f \, d\mu = \max_{i=1,\ldots,n} f(x_{\sigma(i)}) \odot \mu(A_{\sigma(i)}) \tag{3}$$

where σ is a permutation of the indices such that

$$f(x_{\sigma(1)}) \geq f(x_{\sigma(2)}) \geq \ldots \geq f(x_{\sigma(n)}),$$

and $A_{\sigma(i)} = \{x_{\sigma(1)}, \ldots, x_{\sigma(i)}\}$. When μ is a (classical) possibility measure on 2^X induced by a (normalized) possibility distribution $\pi : X \to [0,1]$, i.e. when $\mu(A) = \max\{\pi(x) \mid x \in A\}$ for every $A \subseteq X$, then the above expression of the generalized Sugeno integral becomes (see e.g. [3])

$$\oint f \, d\pi = \max_{x \in X} \pi(x) \odot f(x) = \Pi(f). \tag{4}$$

Let $L \subseteq [0,1]$. Then, a necessity measure N on $[0,1]^X$ is said to be L-*homogenous* provided that for every $r \in L$, and for every $f \in [0,1]^X$, $N(\overline{r} \oplus f) = r \oplus N(f)$, where $\overline{r} \in [0,1]^X$ denotes the function that is constantly equal to r. A possibility measure $\Pi : A \to [0,1]$ is said to be *L-homogenous* provided that its dual necessity N defined as in (2) is L-homogenous, or in other words, such that $\Pi(\overline{r} \odot f) = r \odot \Pi(f)$. A possibility (necessity) measure which is $[0,1]$-homogenous is simply said to be *homogenous*. The class of all the homogenous possibility measures over $[0,1]^X$ will be henceforth denoted by $\boldsymbol{\pi}([0,1]^X)$.

The next theorem offers an axiomatic characterization of those measures for which there exists a possibility distribution that allows a representation in terms of a generalized Sugeno integral. The formulation we provide here is very general and makes only use of the structure of De Morgan triplets[5] over the real unit interval. The following result applies therefore to the specific De Morgan triplet $(\odot, \oplus, \neg)$ where $[0,1]_{MV} = ([0,1], \oplus, \neg, 0)$ is the standard MV-algbera.

Theorem 3 ([11]) *Let X be a finite set, let $(\hat{\odot}, \hat{\oplus}, \neg)$ be a De Morgan triplet, and let $N, \Pi : [0,1]^X \to [0,1]$ be a necessity and a possibility measure (respectively) that are dual, i.e. they satisfy $N(f) = \neg\Pi(\neg f)$ for each $f \in [0,1]^X$. Then, Π and N are homogeneous if and only if there exists $\pi : X \to [0,1]$ such that $\Pi(f) = \max_{x\in X} \pi(x) \,\hat{\odot}\, f(x)$ and $N(f) = \min_{x\in X} \neg\pi(x) \,\hat{\oplus}\, f(x)$.*

Proof Suppose N is such that $N(\overline{r} \,\hat{\oplus}\, f) = r \,\hat{\oplus}\, N(f)$ for every $f \in [0,1]^X$ and $r \in [0,1]$. It is easy to check that every $f \in [0,1]^X$ can be written as

$$f = \bigwedge_{x\in X} \mathbf{x}^c \,\hat{\oplus}\, \overline{f(x)},$$

where $\mathbf{x}^c : X \to [0,1]$ is the characteristic function of the complement of the singleton $\{x\}$, i.e. $\mathbf{x}^c(y) = 1$ if $y \neq x$ and $\mathbf{x}^c(x) = 0$, and $\overline{f(x)}$ stands for the constant function of value $f(x)$.

Now, by applying the axioms of a necessity measure and the assumption that $N(\overline{r} \,\hat{\oplus}\, f) = r \,\hat{\oplus}\, N(f)$, we obtain that

$$\begin{aligned} N(f) &= N\left(\bigwedge_{x\in X} \mathbf{x}^c \,\hat{\oplus}\, \overline{f(x)}\right) \\ &= \min_{x\in X} N\left(\mathbf{x}^c \,\hat{\oplus}\, \overline{f(x)}\right) \\ &= \min_{x\in X} N(\mathbf{x}^c) \,\hat{\oplus}\, f(x). \end{aligned}$$

Finally, by putting $\pi(x) = 1 - N(\mathbf{x}^c)$, we finally get

$$N(f) = \min_{x\in X} (1 - \pi(x)) \,\hat{\oplus}\, f(x),$$

which, of course, by duality implies that

$$\Pi(f) = \max_{x\in X} \pi(x) \,\hat{\odot}\, f(x)$$

The converse is easy. □

It is worth noticing that in the above Theorem 3, when $\neg$ is the standard negation $\neg : x \in [0,1] \mapsto 1 - x$, then the possibility and the necessity measures Π and N are dual in the sense of (2).

3 Extension theorem for possibility measures via max-plus convex sets

In this section we will first introduce the necessary notions from max-plus convexity theory. We invite the reader to consult [13,32] and the references therein for a more complete treatment. The main structures we need for our investigation are defined as follows: expand the real line $\mathbb{R}$ by $-\infty$, and consider the structures $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \vee, +, -\infty, 0)$ and $\mathbb{R}_{\min} = (\mathbb{R} \cup \{-\infty\}, \wedge, +, -\infty, 0)$. $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$, where $\vee$ and $\wedge$ denote the max and min operators respectively, are respectively called the max-plus, and the min-plus semiring, and they serve as standard setting for the development of idempotent mathematics. Although it might be redundant, we will henceforth adopt the same notation of [13,32], and we will write $x \in \mathbb{R}_{\max}$ (or $x \in \mathbb{R}_{\min}$) to say that x is an element of the max-plus (min-plus) semiring.

We will limit our treatment to the case of $\mathbb{R}_{\max}$, and all the notions we are going to introduce (in particular: max-plus segment, and max-plus convex hull) can be easily rephrased in the framework of $\mathbb{R}_{\min}$ (and so obtaining the notions of min-plus segment, and min-plus convex-hull).

For every point $x = (x(1), \ldots, x(n)) \in \mathbb{R}^n_{\max}$, and every scalar $\lambda \in \mathbb{R}_{\max}$, we respectively denote by $\lambda + x$ and $\lambda \vee x$ the vectors in $\mathbb{R}^n_{\max}$ with entries $\lambda + x(j)$, and $\lambda \vee x(j)$ (for all $j = 1, \ldots, n$). For $x, y \in \mathbb{R}^n_{\max}$, the *max-plus segment* $S(x,y)$ joining x and y, is the set of all vectors of the form $(\alpha + x) \vee (\beta + y)$, where α and β are scalars satisfying $\alpha \vee \beta = 0$. Figure 1 shows how segments in $\mathbb{R}^2_{\max}$ and $\mathbb{R}^2_{\min}$ look like). A subset C of $\mathbb{R}^n_{\max}$ is said to be *max-plus convex* if C contains every max-plus segment joining two of its points.

Definition 2 Fix $x^1, \ldots, x^s \in \mathbb{R}^n_{\max}$. A point $x \in \mathbb{R}^n_{\max}$ is a *max-plus convex combination* of $x^1, \ldots, x^s$ if there exist $\alpha_1, \ldots \alpha_n \in \mathbb{R}_{\max}$ with $\bigvee_{i\le s} \alpha_i = 0$ and such that

$$x = \bigvee_{i\le s} (\alpha_i + x^i), \tag{5}$$

[5] A De Morgan triplet (see e.g. [12]) is a 3-tuple $(\hat{\odot}, \hat{\oplus}, \neg)$ where $\hat{\odot}$ is a t-norm, $\hat{\oplus}$ a t-conorm, $\neg$ a strong negation function such that $x \,\hat{\oplus}\, y = \neg(\neg x \,\hat{\odot}\, \neg y)$ for all $x, y \in [0,1]$.

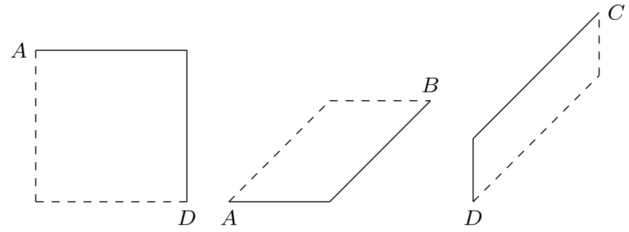

Fig. 1 Consider the points $A = (1/6, 1/2), B = (1/2, 2/3), C = (1, 1/2)$ and $D = (2/3, 0)$. The above lines represents the segments $S(A, D)$, $S(A, B)$, and $S(D, C)$ in the Max-Plus (continuous line), and Min-Plus (dashed line) semirings. Compare this picture with the first two pictures in Figure 2
.

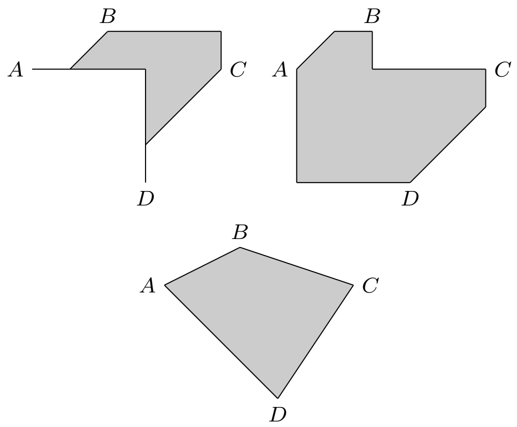

Fig. 2 Fixing the four points $A = (1/6, 1/2), B = (1/2, 2/3), C = (1, 1/2)$ and $D = (2/3, 0)$, the above pictures represent respectively, the subsets of $[0, 1]^2$ which are the max-plus (left), the min-plus (center), and the usual (right) convex hulls generated by $\{A, B, C, D\}$.

The *mp-convex hull* of $\{x^1, \ldots, x^s\}$, that we denote by mp-co($\{x^1, \ldots, x^s\}$), is the set of all points in $\mathbb{R}^n_{\max}$ which are max-plus convex combinations of $x^1, \ldots, x^s$. Clearly any mp-convex subset $C \subseteq \mathbb{R}^n_{\max}$ is closed under finite mp-convex combinations, that is, for every choice of $x^1, \ldots, x^s \in C$, mp-co($\{x^1, \ldots, x^s\}$) $\subseteq C$.

In Figure 2, we show three examples of a max-plus, min-plus, and usual convex hull generated by four points of $\mathbb{R}^2_{\max}$.

The following presents a result that, although easy to prove, will be important for the rest of this paper.

Proposition 1 *For every $n \in \mathbb{N}$, the cube $[0, 1]^n$ is mp-convex. In particular $[0, 1]^n$ is closed under finite mp-convex combinations.*

Proof Let $s \in \mathbb{N}$ and let $i_0 \in \{1, \ldots, s\}$ be such that $\alpha_{i_0} = 0$. Then $\alpha_{i_0} + x^{i_0} = x^{i_0} \in [0, 1]^n$, i.e. for all $j = 1, \ldots, n$, $(\alpha_{i_0} + x^{i_0})(j) = x^{i_0}(j) \geq 0$. Hence, for all $j = 1, \ldots, n$, we have

$$\left(\bigvee_{i=1}^{s} \alpha_i + x^i\right)(j) \geq \alpha_{i_0} + x^{i_0}(j) = x^{i_0}(j) \geq 0. \quad (6)$$

Since $\alpha_i \leq 0$ for all i, and since $x^i(j) \leq 1$ for all i and for all j, then $\alpha_i + x^i(j) \leq 1$ for all i and for all j, in other words,

$$\left(\bigvee_{i=1}^{s} \alpha_i + x^i\right)(j) \leq 1. \quad (7)$$

Therefore, by (6) and (7), $\bigvee_{i=1}^{s} \alpha_i + x^i \in [0, 1]^n$, and the claim is proved. □

Consider now a finite set $\{f_1, \ldots, f_n\}$ of functions in the MV-algebra $[0, 1]^X$. The following theorem characterizes, in terms of max-plus convex sets, those mappings $B : f_i \mapsto \beta_i$ sending the f_i's into $[0, 1]$ that can be extended to a possibility measure Π on $[0, 1]^X$.

Theorem 4 *Let $f_1, \ldots, f_n \in [0, 1]^X$, consider the map $B : \{f_1, \ldots, f_n\} \to [0, 1]$, and write $B(f_i) = \beta_i$ for $i = 1, \ldots, n$. Then the following are equivalent:*

1. *There exists an homogeneous possibility measure $\Pi : [0, 1]^X \to [0, 1]$ that agrees with B over $\{f_1, \ldots, f_n\}$.*
2. *There exists a normalized possibility distribution $\pi : X \to [0, 1]$ such that, for every $i = 1 \ldots, n$,*

$$\beta_i = \oint_X f_i \, d\pi = \bigvee_{x \in X} \pi(x) \odot f_i(x).$$

3. *$(\beta_1, \ldots, \beta_n) \in$ mp-co $(\{(f_1(x), \ldots, f_n(x)) : x \in X\})$.*

Proof (1)⇔(2) is Theorem 3.

(2)⇒(3). Let $\pi : X \to [0, 1]$ satisfy $\bigvee_{x \in X} \pi(x) = 1$. Then by hypothesis, for every $i = 1, \ldots, n$,

$$\begin{aligned} \beta_i &= \oint_X f_i \, d\pi \\ &= \bigvee_{x \in X} \pi(x) \odot f_i(x) \\ &= \bigvee_{x \in X} \max\{0, \pi(x) + f_i(x) - 1\}. \end{aligned}$$

It is clear that, for each i,

$$\bigvee_{x \in X} \max\{0, \pi(x) + f_i(x) - 1\} =$$
$$= \max\{0, \bigvee_{x \in X} \pi(x) + f_i(x) - 1\},$$

and since $\bigvee_{x \in X}(\pi(x) - 1) = \left(\bigvee_{x \in X} \pi(x)\right) - 1 = 0$, and since for all $x \in X$, $f_i(x) \in [0, 1]$, from the above Proposition 1, $\bigvee_{x \in X}(\pi(x) - 1 + f_i(x)) \in [0, 1]$, and hence

$$\beta_i = \max\{0, \bigvee_{x \in X} \pi(x) + f_i(x) - 1\} = \bigvee_{x \in X} \pi(x) - 1 + f_i(x).$$

So $(\beta_1, \ldots, \beta_n) \in \text{mp-co}\left(\{(f_1(x), \ldots, f_n(x)) : x \in X\}\right)$.

(3)⇒(2). Assume that

$$(\beta_1, \ldots, \beta_n) \in \text{mp-co}\left(\{(f_1(x), \ldots, f_n(x)) : x \in X\}\right).$$

Therefore, for each $x \in X$ there is $\lambda_x \in \mathbb{R}_{\max}$ such that $\bigvee_{x\in X} \lambda_x = 0$, and

$$(\beta_1, \ldots, \beta_n) = \bigvee_{x\in X} \lambda_x + (f_1(x), \ldots, f_n(x))$$

that is, for every $i = 1, \ldots, n$, $\beta_i = \bigvee_{x\in X} \lambda_x + f_i(x)$. Since both $\beta_i, f_i(x) \in [0,1]$, we actually have

$$\beta_i = \bigvee_{x\in X} \max(\lambda_x, -1) + f_i(x).$$

Finally, for every $x \in X$, define $\pi(x) = \max(\lambda_x, -1) + 1 = \max(\lambda_x + 1, 0)$. Then, $\bigvee_{x\in X} \pi(x) = 1$, $0 \le \pi(x) \le 1$ for each $x \in X$, and

$$\begin{aligned}\beta_i &= \bigvee_{x\in X} \pi(x) - 1 + f_i(x)\\ &= \bigvee_{x\in X} \max\{0, \pi(x) + f_i(x) - 1\}\\ &= \oint_X f_i \, d\pi.\end{aligned}$$

Therefore the claim is proved. □

Remark 1 Theorem 4 establishes a one-to-one correspondence Ψ between normalized distributions over X and possibility measures over $[0,1]^X$. Notice that Ψ is *Max-Plus affine* in the sense that it preserves mp-convex combinations.

4 The space of possibility measures

In this section we are going to describe geometrical aspects of $\boldsymbol{\pi}([0,1]^X)$ as subset of $[0,1]^{[0,1]^X}$. Notice that the unit interval $[0,1]$ is endowed with the interval-topology inherited by $\mathbb{R}$ and, in turn, $[0,1]^{[0,1]^X}$ is endowed with the usual product topology defined from the one of $[0,1]$.

Lemma 1 *For $\Pi_1, \Pi_2 \in \boldsymbol{\pi}([0,1]^X)$, the mp-segment $S(\Pi_1, \Pi_2)$ is contained in $\boldsymbol{\pi}([0,1]^X)$, i.e. $\boldsymbol{\pi}([0,1]^X)$ is a mp-convex subset of $[0,1]^{[0,1]^X}$.*

Proof Let $\Pi_1, \Pi_2 \in \boldsymbol{\pi}([0,1]^X)$, let $\alpha, \beta \in \mathbb{R}$ such that $\max(\alpha, \beta) = 0$, and let $\Pi : [0,1]^X \to [0,1]$ be defined as

$$\Pi = (\alpha + \Pi_1) \vee (\beta + \Pi_2). \tag{8}$$

Then, for every $f, g \in [0,1]^X$,

$$\begin{aligned}\Pi(f \vee g) &= (\alpha + \Pi_1(f \vee g)) \vee (\beta + \Pi_2(f \vee g))\\ &= (\alpha + \max\{\Pi_1(f), \Pi_1(g)\}) \vee\\ &\quad (\beta + \max\{\Pi_2(f), \Pi_2(g)\})\\ &= \max\{\alpha + \Pi_1(f), \alpha + \Pi_1(g)\} \vee\\ &\quad \max\{\beta + \Pi_2(f), \beta + \Pi_2(g)\}\\ &= \max\{\alpha + \Pi_1(f), \beta + \Pi_2(f)\} \vee\\ &\quad \max\{\alpha + \Pi_1(g), \beta + \Pi_2(g)\}\\ &= \Pi(f) \vee \Pi(g).\end{aligned}$$

Let now r be a real number in $[0,1]$, and let $f \in [0,1]^X$. Then, by (8), we get

$$\begin{aligned}r \odot \Pi(f) = \max\{0, \max\{&\alpha + r - 1 + \Pi_1(f),\\ &\beta + r - 1 + \Pi_2(f)\}\}.\end{aligned} \tag{9}$$

On the other hand, $\Pi(r \odot f) = (\alpha + r \odot \Pi_1(f)) \vee (\beta + r \odot \Pi_2(f)) = (\alpha + \max\{0, r + \Pi_1(f) - 1\}) \vee (\beta + \max\{0, r + \Pi_2(f) - 1\})$. Now we distinguish the following cases:

1. Assume $r \odot \Pi_1(f) > 0$ and $r \odot \Pi_2(f) > 0$. Then, for $i = 1, 2$, $r \odot \Pi_i(f) = r + \Pi_i(f) - 1$, and hence $\Pi(r \odot f) = (\alpha + r + \Pi_1(f) - 1) \vee (\beta + r + \Pi_2(f) - 1) = [r - 1 + (\alpha + \Pi_1(f))] \vee [r - 1 + (\beta + \Pi_2(f))] = r - 1 + [(\alpha + \Pi_1(f)) \vee (\beta + \Pi_2(f))] = r - 1 + \Pi(f) = r \odot \Pi(f)$.
2. If $r \odot \Pi_1(f) = 0$ and $r \odot \Pi_2(f) > 0$, then $\Pi(r \odot f) = \alpha \vee (\beta + r - 1 + \Pi_2(f))$, where $r - 1 + \Pi_2(f) > 0$. Now we distinguish:
   (a) If $\alpha \le 0$, and $\beta = 0$, then $\Pi(r \odot f) = r - 1 + \Pi_2(f)$, and from (9), the claim follows.
   (b) If $\alpha = 0$, and $0 \le \beta \le -(r - 1 + \Pi_2(f))$, then $\beta + r - 1 + \Pi_2(f) \le 0$, and hence $\Pi(r \odot f) = \alpha = 0$. From (9), $r \odot \Pi(f) = 0$ as well.
   (c) Finally if $\alpha = 0$, and $-\beta \le r - 1 + \Pi_2(f)$, then $\beta + r - 1 + \Pi_2(f) \ge 0$, and hence $\Pi(r \odot f) = \beta + r - 1 + \Pi_2(f)$. Again, (9) ensures $r \odot \Pi(f) = \max\{0, \max\{0, \beta + r - 1 + \Pi_2(f)\}\} = \beta + r - 1 + \Pi_2(f)$.
3. The case $r \odot \Pi_1(f) > 0$ and $r \odot \Pi_2(f) = 0$ is dual to the above one, and omitted.
4. If $r \odot \Pi_1(f) = r \odot \Pi_2(f) = 0$, then for $i = 1, 2$, $\Pi_i(f) \le 1 - r$. Moreover, since $\alpha \vee \beta = 0$, $\Pi(r \odot f) = 0$. By (8), $\Pi(f) = (\alpha + \Pi_1(f)) \vee (\beta + \Pi_2(f))$. Since $\alpha, \beta \le 0$, $\Pi(f) \le \max\{\Pi_1(f), \Pi_2(f)\} \le 1 - r$, and consequently $r \odot \Pi(f) = \Pi(r \odot f) = 0$.

Therefore $\Pi \in \boldsymbol{\pi}([0,1]^X)$, and the claim is proved. □

Theorem 5 *The set $\boldsymbol{\pi}([0,1]^X)$ of homogeneous possibility measures on the MV-algebra $[0,1]^X$ is closed by finite mp-convex combinations.*

Proof As we already noticed in Definition 2, every mp-convex set C is closed under finite mp-convex combination. This remark and Lemma 1 prove the claim. In what follows we are going to provide a direct proof.

The case with two elements $\mathit{\Pi}_1, \mathit{\Pi}_2 \in \boldsymbol{\pi}([0,1]^X)$ is solved in Lemma 1. Now, we proceed by induction on the (finite) number of the elements in $\boldsymbol{\pi}([0,1]^X)$ involved in the definition of the finite mp-convex combination. Take $\mathit{\Pi}_1, \ldots, \mathit{\Pi}_n \in \boldsymbol{\pi}([0,1]^X)$ (with $n \geq 3$), and let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_{\max}$ such that $\bigvee_i \alpha_i = 0$. Without loss of generality, assume that $\alpha_1 = 0$. Then,

$$\begin{aligned}\mathit{\Pi} &= \textstyle\bigvee_{i=1}^n (\alpha_i + \mathit{\Pi}_i) \\ &= [\delta + ((\alpha_1 + \mathit{\Pi}_1) \vee (\alpha_2 + \mathit{\Pi}_2))] \vee [\textstyle\bigvee_{i=3}^n (\alpha_i + \mathit{\Pi}_i)],\end{aligned}$$

where $\delta = 0$. Therefore from Lemma 1, and since $\alpha_1 = 0$, $\mathit{\Pi}' = (\alpha_1 + \mathit{\Pi}_1) \vee (\alpha_2 + \mathit{\Pi}_2) \in \boldsymbol{\pi}([0,1]^X)$, and hence

$$\mathit{\Pi} = \bigvee_{i=1}^n (\alpha_i + \mathit{\Pi}_i) = (\delta + \mathit{\Pi}') \vee \left(\bigvee_{i=3}^n (\alpha_i + \mathit{\Pi}_i) \right).$$

Since $\delta \vee \bigvee_{i=3}^n \alpha_i = 0$, the inductive hypothesis ensures that $\mathit{\Pi} \in \boldsymbol{\pi}([0,1]^X)$. □

Definition 3 (Extreme point) Let C be a max-plus convex subset of $\mathbb{R}^n_{\max}$. An element $c \in C$ is an *extreme point* of C if for all $y, z \in C$ and for all $\alpha, \beta \in \mathbb{R}_{\max}$ such that $\max\{\alpha, \beta\} = 0$, the condition

$$\text{if } c = (y + \alpha) \vee (z + \beta), \text{ then } c = y \text{ or } c = z, \tag{10}$$

is satisfied. We denote by $ext(C)$ the set of extreme points of C.

Now we concentrate on providing a characterization for the set of extreme points of $\boldsymbol{\pi}([0,1]^X)$ in terms of MV-homomorphisms of $[0,1]^X$ into the standard MV-algebra $[0,1]$. Indeed, notice that, from [23, Theorem 2.5], a map $h : [0,1]^X \to [0,1]$ is an MV-homomorphism iff $\{f \in [0,1]^X : h(f) = 1\}$ is a maximal MV-filter of $[0,1]^X$ iff, by Chang-Belluce representation theorem for semisimple MV-algebras (cf. [5,6], see also [23, Theorem 1.2]), there exists a $y \in X$ such that, for every $f \in [0,1]^X$, $h(f) = f(y)$. This general result implies the following proposition that, nevertheless, we are going to prove to make the paper as self-contained as possible.

Proposition 2 *Consider a $\mathit{\Pi} \in \boldsymbol{\pi}([0,1]^X)$. Then $\mathit{\Pi} \in Hom([0,1]^X, [0,1])$ iff there exists $x_0 \in X$ such that, for every $f \in [0,1]^X$, $\mathit{\Pi}(f) = f(x_0)$.*

Proof The left-to-right direction is clear and omitted. Conversely, if $\mathit{\Pi} \in \boldsymbol{\pi}([0,1]^X)$, by Theorem 3, there exists a possibility distribution $\pi : X \to [0,1]$ such that, for every $f \in [0,1]^X$, $\mathit{\Pi}(f) = \bigvee_{x \in X} \pi(x) \odot f(x)$. Let $x_0 \in X$ be such that $\pi(x_0) = 1$, and let $f_{x_0} \in [0,1]^X$ the function defined by putting $f_{x_0}(x) = 1$ if $x = x_0$ and $f_{x_0}(x) = 0$ otherwise. Then it is clear that $\mathit{\Pi}(f_{x_0}) = \pi(x_0)$ and $\mathit{\Pi}(1 - f_{x_0}) = \max\{\pi(x) \mid x \neq x_0\}$. If $\mathit{\Pi}$ is an MV-homomorphism, in particular, it must verify that $\mathit{\Pi}(1 - f_{x_0}) = 1 - \mathit{\Pi}(f_{x_0})$, hence $0 = 1 - \pi(x_0) = \max\{\pi(x) \mid x \neq x_0\}$, that is, it must be $\pi(x) = 0$ for all $x \neq x_0$. Therefore, $\mathit{\Pi}(f) = \bigvee_{x \in X} \pi(x) \odot f(x) = \pi(x_0) \odot f(x_0) = f(x_0)$. □

Proposition 3 *Consider a $\mathit{\Pi} \in \boldsymbol{\pi}([0,1]^X)$. If $\mathit{\Pi} \in Hom([0,1]^X, [0,1])$ then $\mathit{\Pi} \in \mathrm{ext}(\boldsymbol{\pi}([0,1]^X))$.*

Proof Let $\mathit{\Pi}$ be an homogeneous possibility measure on $[0,1]^X$. Assume $\mathit{\Pi}$ is a mp-convex combination of two homogeneous possibility measures, that is, there exist distinct $\mathit{\Pi}_1, \mathit{\Pi}_2 \in \boldsymbol{\pi}([0,1]^X)$ and $\alpha, \beta \in \mathbb{R}$ such that $\mathit{\Pi} = (\alpha + \mathit{\Pi}_1) \vee (\beta + \mathit{\Pi}_2)$ with $\max(\alpha, \beta) = 0$. By Proposition 2 there exists $x_0 \in X$ such that, for any f, $\mathit{\Pi}(f) = f(x_0)$. For any $x \in X$, consider the functions f_x defined as $f_x(y) = 1$ if $y = x_0$ and $f_x(y) = 0$ otherwise. Then we have:

$$\begin{aligned}1 &= f_{x_0}(x_0) = \mathit{\Pi}(f_{x_0}) \\ &= \max(\alpha + \pi_1(x_0), \beta + \pi_2(x_0))\end{aligned} \tag{11}$$

and, if $y \neq x_0$,

$$\begin{aligned}0 &= f_y(x_0) = \mathit{\Pi}(f_y) \\ &= \max(\alpha + \pi_1(y), \beta + \pi_2(y)),\end{aligned} \tag{12}$$

where π_1 and π_2 are the corresponding possibility distributions of $\mathit{\Pi}_1$ and $\mathit{\Pi}_2$ respectively. Since $\max(\alpha, \beta) = 0$, we can assume without loss of generality that $\alpha = 0$, and hence:

1. If $\beta < 0$, then $\pi_1(x_0) = 1$ by (11), and (12) implies $\pi_1(y) = 0$ for all $y \neq x_0$. In this case, for any f, $\mathit{\Pi}_1(f) = \bigvee_{x \in X} \pi_1(x) \odot f(x) = f(x_0) = \mathit{\Pi}(f)$, i.e. $\mathit{\Pi}_1 = \mathit{\Pi}$.
2. If also $\beta = 0$, then since by (12) $\pi_1(y) = \pi_2(y) = 0$ for all $y \neq x_0$, then either $\pi_1(x_0) = 1$, and hence $\mathit{\Pi}(f) = \mathit{\Pi}_1(f)$ for all f, or $\pi_2(x_0) = 1$, and hence for all f, $\mathit{\Pi}(f) = \mathit{\Pi}_2(f)$. Also notice that, since we are assuming $\mathit{\Pi}_1 \neq \mathit{\Pi}_2$, then it cannot happen that both $\pi_1(x_0) = \pi_2(x_0) = 1$.

Therefore, in any case, either $\mathit{\Pi} = \mathit{\Pi}_1$ or $\mathit{\Pi} = \mathit{\Pi}_2$. Consequently, $\mathit{\Pi}$ is extremal. □

Lemma 2 *Let $\mathit{\Pi} \in \boldsymbol{\pi}([0,1]^X)$ and let π be its corresponding possibility distribution. If the set $\{x \in X \mid \pi(x) > 0\}$ has at least two elements, then $\mathit{\Pi}$ is not extremal.*

Proof Since π is normalized, there exists x_0 such that $\pi(x_0) = 1$, and by hypothesis there exists another element y_0 such that $\pi(y_0) > 0$. Consider two possibility distributions π_1 and π_2 defined as follows:

$$\pi_1(x) = \begin{cases} 1, \text{if } x = y_0 \\ 0, \text{otherwise} \end{cases} \qquad \pi_2(x) = \begin{cases} 0, & \text{if } x = y_0 \\ \pi(x), & \text{otherwise} \end{cases}.$$

Then, if Π_1 and Π_2 are the corresponding possibility measures defined from π_1 and π_2 and letting $\alpha = \pi(y_0) - 1$, it is easy to check that

$$\Pi = (\alpha + \Pi_1) \vee \Pi_2$$

and since $\Pi \neq \Pi_1$ and $\Pi \neq \Pi_2$, Π is not extremal. □

As a consequence, the only candidate $\Pi \in \boldsymbol{\pi}([0,1]^X)$ to be extremal are those defined as Sugeno integrals from a possibility distribution π such that, for some $x_0 \in X$, $\pi(x_0) = 1$ and $\pi(x) = 0$ for $x \neq x_0$, and hence $\Pi(f) = f(x_0)$ for any $f \in [0,1]^X$. By Proposition 2, these Π are precisely those $\Pi \in \boldsymbol{\pi}([0,1]^X)$ that are MV-homomorphisms. Therefore we have the following corollary.

Theorem 6 *$\Pi \in \mathrm{ext}(\boldsymbol{\pi}([0,1]^X)$ if and only if $\Pi \in Hom([0,1]^X, [0,1])$.*

Consequently, a map Π in $[0,1]^{[0,1]^X}$ is an extremal homogeneous possibility measure iff Π is a homomorphism of $[0,1]^X$ into the standard MV-algebra $[0,1]$ iff, from [23, Theorem 2.5] (also see [14, Theorem 12.18, Corollary 12.20]) the set $\{f \in [0,1]^X : \Pi(f) = 1\}$ is a maximal MV-filter of $[0,1]^X$. Even more precisely, Theorem 6, together with [23, Theorem 2.5], implies that the space of extremal homogeneous possibility measures on $[0,1]^X$, endowed with the product topology inherited by $[0,1]^{[0,1]^X}$ is a non-empty compact Hausdorff space which is homeomorphic to the space of maximal MV-filters of $[0,1]^X$, endowed with the spectral topology, that, in turn, is homeomorphic to X with the topology having for basis (of clopen) the class of its subsets.

Now, we concentrate on finitely generated mp-convex sets. Since by Theorem 4, an assessment B on a finite subset A' of $[0,1]^X$ is extendable to possibility measure on $[0,1]^X$ iff B belongs to the mp-convex hull generated by the values of the functions in A', the results we are going to show will offer a refinement of the extension theorem proved in Section 3.

Lemma 3 *For every $f_1, \dots, f_n \in [0,1]^X$, we have:*

$$\mathrm{ext}(\text{mp-co}\{(f_1(x), \dots, f_n(x)) : x \in X\}) \subseteq \subseteq \{(f_1(x), \dots, f_n(x)) : x \in X\}).$$

Proof Assume $b \in \mathrm{ext}(\text{mp-co}\{(f_1(x), \dots, f_n(x)) : x \in X\})$. Therefore there exist $\alpha_1, \dots, \alpha_m \in \mathbb{R}_{\max}$ with $\bigvee_{i=1}^m \alpha_i = 0$, such that

$$b = \bigvee_{i=1}^m \alpha_i + \mathcal{F}_i.$$

where we denote by $\mathcal{F}_i$ the vector $(f_1(x_i), \dots, f_n(x_i))$ (with $i = 1, \dots, m$). We now prove that, for some i, $b = \mathcal{F}_i$. We proceed by induction on m. If $m = 2$ the claim is clearly proved by definition of extreme point. Conversely, assume $m > 2$. Then

$$b = (\alpha_1 + \mathcal{F}_1) \vee \left(\bigvee_{i=2}^m \alpha_i + \mathcal{F}_i \right).$$

We can assume (after renaming the indexes) that $\bigvee_{i=2}^m \alpha_i = 0$, and hence

$$\mathcal{F}' = \bigvee_{i=2}^m \alpha_i + \mathcal{F}_i \in \text{mp-co}\{(f_1(x), \dots, f_n(x)) : x \in X\}.$$

Letting $\delta' = 0$, $b = (\alpha_1 + \mathcal{F}_1) \vee (\delta' + \mathcal{F}')$ with $\alpha_1 \vee \delta' = 0$. Then, since b is extremal, either $b = \mathcal{F}_1$, or $b = \mathcal{F}'$. If $b = \mathcal{F}_1$, we are done. Conversely, if $b = \mathcal{F}' = \bigvee_{i=2}^m \alpha_i + \mathcal{F}_i$, the inductive hypothesis ensures that $b = \mathcal{F}_i$ for some $i = 2, \dots, m$. □

Since the operations max and $\odot$ are continuous with respect to the usual topology over $[0,1]$, it is clear that the class $\boldsymbol{\pi}([0,1]^X)$ is a closed subset of $[0,1]^{[0,1]^X}$. In fact, let $\Pi \in [0,1]^{[0,1]^X}$ such that $\Pi = \lim_i \Pi_i$ where $\Pi_1, \Pi_2, \Pi_3 \dots \in \boldsymbol{\pi}([0,1]^X)$. Then, for every $f_1, f_2 \in [0,1]^X$ and for every $r \in [0,1]$, one has:

(a) $\lim_i \Pi_i(f_1 \vee f_2) = \lim_i \max(\Pi_i(f_1), \Pi_i(f_2)) = \max(\lim_i \Pi_i(f_1), \lim_i \Pi_i(f_2))$;
(b) $\lim_i \Pi_i(f_1 \odot \overline{r}) = \lim_i (\Pi_i(f_1) \odot r) = (\lim_i \Pi_i(f_1)) \odot r$;
(c) $\lim_i \Pi_i(\overline{1}) = \lim_i 1 = 1$.

Hence $\Pi = \lim_i \Pi_i \in \boldsymbol{\pi}([0,1]^X)$.

Therefore, being $[0,1]$ a compact space, and since the product topology preserves compactness, the following lemma immediately holds.

Lemma 4 *The class of homogeneous possibility measures $\boldsymbol{\pi}([0,1]^X)$ is compact. Therefore, in particular, for $f_1, \dots, f_n \in [0,1]^X$, the set $\text{mp-co}(\{(f_1(x), \dots, f_n(x)) : x \in X\})$ is a compact subset of $[0,1]^n$.*

Finally, the following result provides a refinement of Theorem 4.

Theorem 7 *For every $f_1, \dots, f_n \in [0,1]^X$, every $[0,1]$-valued assessment $B : f_i \mapsto \beta_i$ extends to a homogeneous possibility measure on $[0,1]^X$ iff there are $m \leq n+1$ elements $x_1, \dots, x_m \in X$ such that*

$$(\beta_1, \dots, \beta_n) \in \text{mp-co}(\{(f_1(x_i), \dots, f_n(x_i)) : i = 1, \dots, m\}).$$

Proof From Theorem 4, B extends to a $\Pi \in \boldsymbol{\pi}([0,1]^X)$ iff $(\beta_1, \dots, \beta_n) \in \text{mp-co}(\{(f_1(x), \dots, f_n(x)) : x \in X\})$. From Lemma 4, $\text{mp-co}(\{(f_1(x), \dots, f_n(x)) : x \in X\})$ is a compact subset of $\mathbb{R}^n_{\max}$ and therefore from the mp-version of Minkowski Theorem [13, Theorem 3.2], there are $m \leq n+1$ points $e_1, \ldots, e_m$ belonging to $\text{ext}(\text{mp-co}(\{(f_1(x), \ldots, f_n(x)) : x \in X\}))$ such that

$$\text{mp-co}(\{(f_1(x), \ldots, f_n(x)) : x \in X\}) = \\ = \text{mp-co}(\{e_1, \ldots, e_m\}).$$

Then the claim follows from Lemma 3. □

For every $x \in X$, the map $h : [0,1]^X \to [0,1]$ defined as $h(f) = f(x)$ is an MV-homomorphism of $[0,1]^X$ into $[0,1]_{MV}$. Therefore, whenever we fix $f_1, \ldots, f_n \in [0,1]^X$, and $x_1, \ldots, x_m \in X$, there are $h_1, \ldots, h_m \in Hom([0,1]^X, [0,1])$ such that, for every $i = 1, \ldots, n$, and every $j = 1, \ldots, m$, $h_j(f_i) = f_i(x_j)$. Therefore the following result is an immediate consequence of Theorem 7 (compare it with Theorem 2).

Corollary 1 *Let $f_1, \ldots, f_n \in [0,1]^X$. A $[0,1]$-valued assessment $B : f_i \mapsto \beta_i \in [0,1]$ extends to a possibility measure $\Pi \in \boldsymbol{\pi}([0,1]^X)$ iff B coincides with the restriction to $f_1, \ldots, f_n$ of an mp-convex combination of at most $n+1$ homomorphisms of $[0,1]^X$ in $[0,1]$.*

5 Concluding remarks

In this paper we have been concerned with geometrical aspects of a class of generalized possibility and necessity measures on MV-algebras of functions $[0,1]^X$, where X is a finite set, in the setting of idempotent mathematics. We have shown that suitable counterparts of known results for the case of finite-additive measures on MV-algebras also hold for possibility and necessity measures when the real field $\mathbb{R}$ is appropriately replaced by the well-known max-plus and min-plus semiring structures $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$. Further research will focus on generalizing the results to MV-clans $[0,1]^X$ when X is infinite, and, in particular, to semisimple MV-algebras, which can be represented as MV-algebras of continuous and real-valued functions defined on a compact and Hausdorff space X.

Our future work will also concern with generalizations of possibility and necessity measures in the context of Gödel and product algebras (cf. [16]). Although we do not know yet if those measures could have a geometrical characterization like the one we have presented in this paper, we believe that, at least for the case of possibility (necessity) measures on Gödel algebras, a similar treatment could be provided.

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