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Geometrical Considerations in the Measurement of the Volume of an Approximate Sphere

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Expressions are derived for the volume of an approximate sphere in terms of measured breadth, the distance between parallel planes tangent to opposite sides. The difference in volume of a ball and a true sphere of the same average breadth is shown to be of second order, and much smaller than the random and systematic errors in the measurements of the dimension. Thus, a ball commercially available at moderate cost can be used for absolute density measurements of high accuracy. Similar expressions are given for the area of an approximate circle.

Key words: Asphericity correction; density; spherical harmonics; volume of ball.

1. Introduction

The establishment of an absolute density standard, as is described by Bowman et al. in the companion article [1],[1] requires the determination of the volume of some object from its dimensions. The shapes that first come to mind are the cube or rectangular prism used by Cook [2] in the determination of the density of mercury, and the sphere. The latter is attractive because of its symmetry, and its relatively low cost in the form of ball bearings. Of course, no ball is a perfect sphere, and the question is immediately raised as to the importance of the topographic deviations from sphericity. In the following we will examine the determination of the volume from measurable quantities. The discussion will be restricted to objects which are everywhere convex, and free of sharp edges or corners.[2] Commercial ball bearings meet these requirements easily. We will be particularly concerned with features which occupy more than one percent of the surface. Irregularities smaller than a millimeter in extent would produce conspicuous distortions in the interference patterns. The method of measuring the interferograms averages over the irregularities associated with the texture of the surface. The observations with the profilometer and the homogeneous appearance of the ball as an optical surface make it improbable that a significant defect of intermediate size would escape detection. In the following we will show that the correction for asphericity is of the second order, given a good average for the diameter or the breadth of the object, and therefore smaller than first order uncertainties in the measurement. Then we will look at sampling schemes for obtaining that average.

1.1. Description of an Object in Terms of Spherical Coordinates

The volume V of an object is given by the integral,

$$V=\frac{1}{3}\int_s r^3 d\Omega \tag{1}$$

where r is the distance from the origin to the surface element, which subtends a solid angle $d\Omega$ at the origin, and the integration is performed over the full solid angle of 4π steradians. Let r_0 be the mean radius so that $r=r_0+\delta$. Then

$$V=4\pi r_0^3/3+r_0^2\int_s \delta d\Omega+r_0\int_s \delta^2 d\Omega+\int_s \delta^3 d\Omega/3.$$

Since the mean value of δ is zero, and that of δ^3 is very small, we have

$$V=4\pi[r_0^3/3+r_0\sigma_r^2] \tag{2}$$

where σ_r^2 is the mean square variation of the radius. Thus, the correction for irregularities is of the second order. If $\sigma_r/r_0=10^{-4}$ the correction is only 3 parts in 10^8.

But things are not this simple. Measurements with many metrological instruments, from the mechanic's micrometer to the Saunders interferometer [3], do not yield the diameter. Instead, they determine the breadth, that is, the separation of parallel planes

[1] Figures in brackets indicate the literature references at the end of this paper.
[2] This requires that the shape can be described by functions which are single valued and continuous, with single valued and finite first and second derivatives.

tangent to opposite sides of the object. Since the breadth favors the high spots, its average value will be somewhat greater than that of the diameter. In figure 1 let R be a point on a surface, and O be the origin. In general, the tangent plane PT which is normal to OR will not pass through R, but the point of tangency will be at the top of a nearby high spot. The tangential radius $p=\overline{OP}$ is given approximately by

$$p \simeq r + (\nabla r)^2/2R_L \tag{3}$$

where r is the distance OR; $R_L = CT$ the local radius of curvature in the region which includes R and the point of tangency; ∇r is the gradient, or the change of r with respect to the angle subtended at the origin in the direction of most rapid increase, i.e., towards the top of the hill. In the objects of present interest, $R_L \sim r_0$, and the gradient is so small that the difference between sine, angle, and tangent can be neglected. Let $p = p_0 + \Delta p$ where p_0 is the mean value of p,

$$\int_s \Delta p \, d\Omega = 0, \text{ and } 4\pi\sigma_p^2 = \int_s (\Delta p)^2 \, d\Omega.$$

Then

$$r \simeq p_0 + \Delta p - p_0^2 \, (\nabla r)^2/2R_L$$

$$r^3 \simeq p_0^3 + 3p_0^2 \, \Delta p + 3p_0 \, (\Delta p)^2 - 1.5 \, p_0^2 \, (\nabla r)^2/R_L.$$

Then the volume would be given by

$$V = \frac{1}{3} \int_s r^3 d\Omega = 4\pi p_0^3/3 + 4\pi p_0[\sigma_p^2 - \sigma_{\nabla r}^2/2] \tag{4}$$

where

$$4\pi\sigma_{\nabla r}^2 = p_0 \int_s (\nabla r^2/R_L) d\Omega.$$

As before, the topography correction to the volume is of the second order, this time involving the mean square of the gradient ∇r as well as that of the variation in breadth.

In the above we have passed rapidly over several points which require closer examination. (a) It is not obvious that the expression for the volume is independent of the choice of the origin. (b) Equation (4) is in terms of the tangential radius, which cannot be measured since the center of a solid object is not accessible. Rather the measurements yield the breadth, which is the sum of two tangential radii. Therefore it is important to consider the important class of objects which exhibit odd lobing, with a valley opposite a hill. (c) The local radius near the hilltops is systematically less than the mean radius. We have assumed that they are approximately equal so that the difference leads to errors of a higher order than the second. (d) We can expect the gradient ∇p to approximate ∇r, but at this time have no good basis for estimating its magnitude. (e) In order for the terms of the first order to vanish, it is necessary to have a good average value of p experimentally as well as mathematically.

1.2. Description of an Object in Terms of Tangent Planes

We have reached the expression, eq (4), for the volume of an approximate sphere from a description in terms of the usual polar coordinates, r, θ, ϕ. Unfortunately the radius and its derivative with respect to the central angle are not directly measurable. To make the analysis correspond more closely to the measured quantities, we will look at a description of an object as the "envelope" of a set of tangent planes. The analytical description was developed by Meissner [4] in a study of objects of constant breadth. Equivalent graphical methods are used in the design of gears and cams, for example [5, 6]. More recent work has been published by Goldberg [7].

2. Area of an Approximate Circle

Before undertaking the more complex problem of the approximate sphere, let us examine that of the area of a plane figure bounded by the curve S in figure 1. The lines TP and T_1P_1 are two tangents to the curve, which intersect at the point Q between the points of tangency. Let OP and OP_1 be the perpendiculars from the origin to the two tangents. Let the tangential radius p be expressed as an analytic function of the polar angle θ. Then the distance $\overline{OP} = p(\theta)$, $\overline{OP_1} = p(\theta + \Delta\theta)$, where $POP_1 = PQP_1 = \Delta\theta$. From the

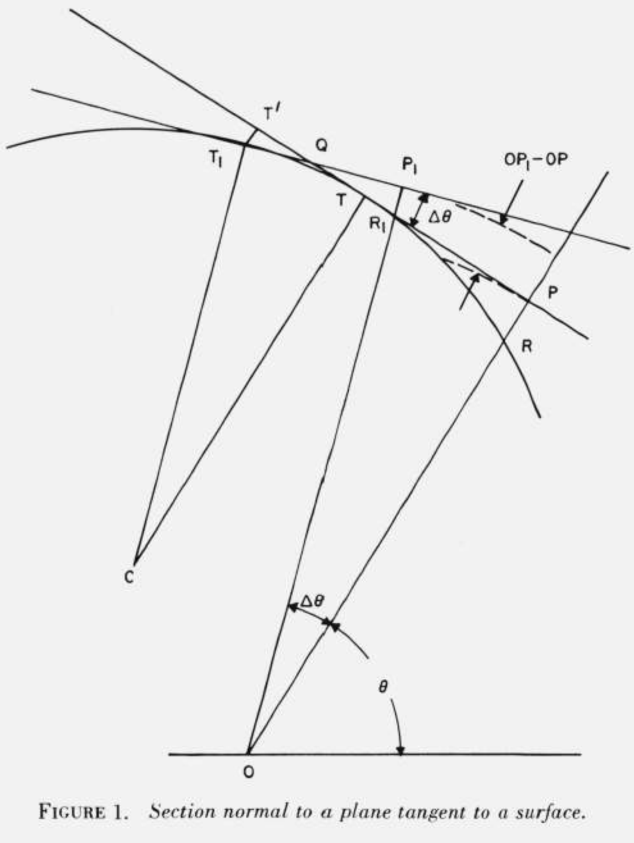

FIGURE 1. *Section normal to a plane tangent to a surface.*

figure it will be seen that the distances $\overline{P_1Q} \cdot \Delta\theta < \Delta p = \overline{OP_1} - \overline{OP} < \overline{PQ} \cdot \Delta\theta$. Divide through by $\Delta\theta$ and let $\Delta\theta \to 0$: $\overline{P_1Q} \to PQ \to \Delta p/\Delta\theta \to dp/d\theta \equiv p'$. Also the intersection Q of the two tangents becomes the point of tangency. Thus, the distance from the foot of the perpendicular to the point of tangency equals the derivative of p with respect to the angle at the origin. Returning to figure 1, we have two tangents PT and P_1T_1, with perpendiculars from the origin OP and OP_1. Let us project onto PT the vectors leading from O to T_1 by way of P and P_1. We have $\overline{OP_1} \sin \Delta\theta + \overline{P_1T_1} \cos \Delta\theta = \overline{PT'} = PT + TT'$. Let the length $\overline{OP} = p(\theta)$, $\overline{OP_1} = p(\theta + \Delta\theta)$; we have already shown that $\overline{PT} = p'(\theta)$, and $\overline{P_1T_1} = p'(\theta + \Delta\theta)$. Making these substitutions we obtain

$$\frac{\overline{TT'}}{\sin \Delta\theta} = p(\theta + \Delta\theta) + \frac{p'(\theta + \Delta\theta) \cos \Delta\theta - p'(\theta)}{\sin \Delta\theta},$$

and as $\Delta\theta \to 0$, $\overline{TT'} \to dS$, the element of arc, so that

$$dS/d\theta = p + p''.$$

The area A enclosed by the curve is given by

$$2A = \int_s p dS = \int_0^{2\pi} p^2 d\theta + \int_0^{2\pi} pp'' d\theta.$$

Substitute $p = p_0 + \delta p$, where p_0 is the mean value of p, and integrate the last term by parts.

$$2A = \int_0^{2\pi} [p^2 + 2p_0\delta p + (\delta p)^2 - (p')^2] d\theta + |pp'|_0^{2\pi}$$

or, since $\int_0^{2\pi} \delta p = 0$

$$A = \pi[p_0^2 + \sigma_p^2 - \sigma_{p'}^2] \tag{5}$$

where σ_p^2 and $\sigma_{p'}^2$ are mean square values of $p - p_0$ and p'.

There is no problem with the integration if the boundary curve is everywhere convex and free of steps and corners. With these restrictions it is possible to represent $p(\theta)$ by a Fourier series

$$p = p_0 + \sum_n C_n \sin (n\theta + \alpha_n)$$

whence

$$\sigma_p^2 = \sum C_n^2/2, \ \sigma_{p'}^2 = \sum n^2 C_n^2/2$$

and the area will be

$$A = \pi p_0^2 + \frac{\pi}{2} \sum (1 - n^2) C_n^2. \tag{6}$$

The "topography error" in the area is of the second order. Displacement of the origin changes the value of the constants for $n = 1$; because of the factor $(1 - n^2)$, the area is unchanged.

3. Volume of an Approximate Sphere

The problem of the approximate sphere is much more tedious than that of the approximate circle. The reader who is not interested in following the details of the analysis may wish to skip most of this section, stopping only to compare eqs (15) and (5) and eqs (20) and (6), and then to look at the calculation of volume.

The logic is similar to that of the approximate circle: (a) The shape of the object can be described in terms of a set of tangent planes. The position of each plane is defined by a vector $\mathbf{p}$ from the origin to the plane and normal to it. (b) The length p of the normal vector is a function of the polar coordinates θ and ϕ. Note that these coordinates do not refer directly to a point on or within the object but to the coordinates of a reference sphere with its center at the origin of $\mathbf{p}$. If the object has no sharp edges or vertices, each point on its surface defines a single tangent plane and therefore a single point (θ, ϕ), on the reference sphere. If the object is everywhere convex, each point on the reference sphere corresponds to a single point of tangency on the object. (c) We will first determine the vector from the origin to the point of contact between the tangent plane and the surface of the object. Then we will map out the element of the surface which corresponds to the increments $d\theta$ and $d\phi$, and write the expression for the element of volume. (d) In the analytical discussion we will use a series of spherical harmonics, which play the same part on the sphere as does the Fourier series on the circle. The volume integral will contain terms of the first, second, and higher orders in the spherical coordinates. As the first order terms integrate to zero, the main correction for topography will be found in the second order terms, which contain squares and products of the spherical harmonics. A sample calculation will show the magnitude of the second order terms.

3.1. Element of Volume

Figure 2 represents the plane tangent to the object at T and normal to the vector $\mathbf{p}(\theta, \phi)$ which terminates at P. The projections of all meridians of the reference sphere meet at the pole, $\theta = 0$. The intersection with the plane of the equator, $\theta = 90°$, forms the base of the triangle. The projection from the origin of the small circle of constant polar distance θ is a cone; its intersection with the tangent plane is symmetrical with respect to the meridian through P. The plane through P, orthogonal to the tangent and meridian planes, intersects the reference sphere in a great circle tangent to the small circle of constant θ. The angle ψ is measured along this circle.

Let figure 1 represent a meridian plane which contains the normals OP and OP_1 of two planes tangent to the object. T and T_1 are the perpendicular projections onto the plane of the figure of the two points of tangency. The argument proceeds as before, and the projection of PT on the tangent plane is equal to $\partial p/\partial\theta \equiv p_u$.

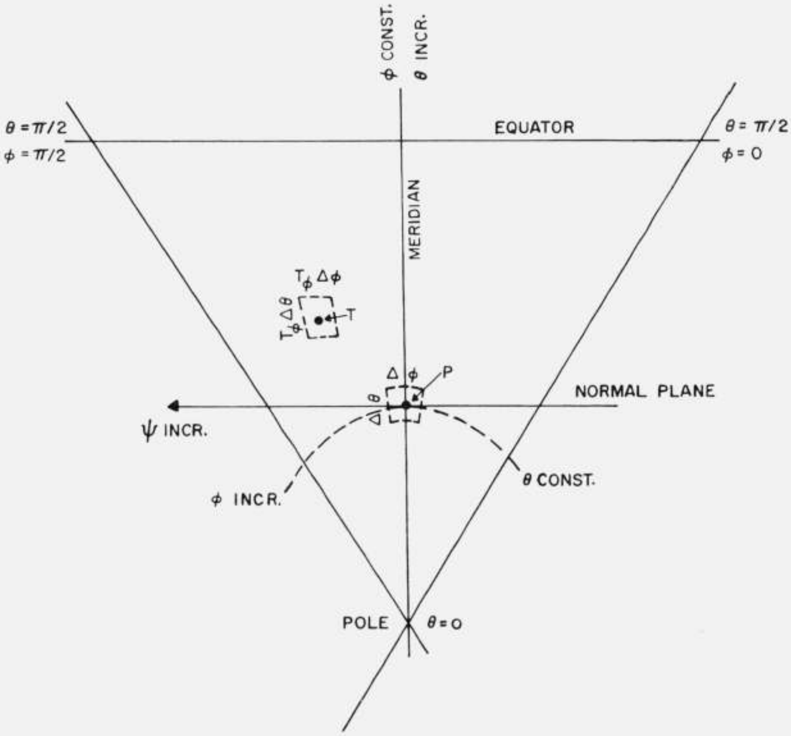

FIGURE 2. *Plane tangent to a surface.*

Now we relabel figure 1, replacing θ with ψ and let it represent projections on the plane normal to the meridian. The projection of PT on this plane is $dp/d\psi \equiv p_\psi$. Since the normal plane is tangent to the cone of constant polar angle θ, we can set $d\psi = \sin\theta d\phi$, and this component of PT, $p_\psi = p_\phi \csc\theta$.

Returning to figure 2 we have the sum of the vectors $\mathbf{p}_\theta$, parallel to the meridian, and $\mathbf{p}_\psi = \mathbf{p}_\phi \csc\theta$, perpendicular to it, connecting the point P to the point of tangency T between the plane and the surface of the object. Then, corresponding to an element $\Delta\theta\Delta\phi$ of the reference sphere, there is on the surface of the object an element whose edges are the vectors $\mathbf{T}_\theta\Delta\theta$ and $\mathbf{T}_\psi\Delta\phi$. The volume element will be a pyramid with this as the base and the origin as the vertex.

Now, let us define three orthogonal unit vectors, which move with the tangential radius vector. In terms of cartesian coordinates associated with the reference sphere [$\theta = 90°$, $\phi = 0$; $\theta = 90°$, $\phi = 90°$; $\theta = 0°$] the components of these are

$\mathbf{u} \equiv (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ along the radius vector,

$\mathbf{v} \equiv (\cos\theta \cos\phi, \cos\theta \sin\phi, -\sin\theta)$ in the meridian plane,

$\mathbf{w} \equiv (-\sin\phi, \cos\phi, 0)$ normal to the meridian plane.

By differentiation of the components the following can be verified:

$$\mathbf{u}_\theta = \mathbf{v} \qquad \mathbf{v}_\theta = -\mathbf{u} \qquad \mathbf{w}_\theta = 0$$

$$\mathbf{u}_\phi = \mathbf{w}\sin\theta \qquad \mathbf{v}_\phi = \mathbf{w}\cos\theta \qquad \mathbf{w}_\phi = -\mathbf{u}\sin\theta - \mathbf{v}\cos\theta$$

Then the tangential radius vector will be $\mathbf{p} = p\,\mathbf{u}$ and the vector to the corresponding point on the object will be:

$$\mathbf{t} = p\mathbf{u} + p_\theta\mathbf{v} + p_\phi \csc\theta\, \mathbf{w}.$$

In differentiating these vectors we must consider changes in both direction and magnitude:

$$\mathbf{t}_\theta = (p + p_{\theta\theta})\mathbf{v} + (p_{\phi\theta} - p_\phi \operatorname{ctn}\theta)\csc\theta\,\mathbf{w}$$

$$\mathbf{t}_\phi = (p_{\theta\phi} - p_\phi \operatorname{ctn}\theta)\mathbf{v} + (p\sin\theta + p_\theta\cos\theta + p_{\phi\phi}\csc\theta)\,\mathbf{w}.$$

Since no component in the $\mathbf{u}$ direction appears in the expression for $\mathbf{t}_\theta$ and $\mathbf{t}_\phi$, the derivative vectors in the surface of the object also lie in the tangent plane.

The volume element can be expressed in terms of the determinant of the components of $\mathbf{t}$, $\mathbf{t}_\theta$ and $\mathbf{t}_\phi$

$$dV = \begin{vmatrix} p & p_\theta & p_\phi \csc\theta \\ 0 & p + p_{\theta\theta} & (p_{\phi\theta} - p_\phi \operatorname{ctn}\theta)\csc\theta \\ 0 & p_{\phi\theta} - p_\phi \operatorname{ctn}\theta & (p\sin\theta + p_\theta\cos\theta + p_{\phi\phi}\csc\theta) \end{vmatrix} \frac{d\theta d\phi}{3} \tag{10}$$

3.2. Volume Integral

Let p_0 be the average value of p $(4\pi p_0 = \int_s p d\Omega)$, and let Y be the relatively small deviation from the average value, so that $p = p_0 + Y$. Later we will express Y as a linear combination of spherical harmonics. Now let $p = p_0 + Y$ where Y is a linear combination of spherical harmonics. Now we expand the determinant, sort out terms, and start the integration:

$$V = \left\{ \begin{array}{l} 4\pi p_0^3/3 + (p_0^2/3) \iint_s [Y_{\theta\theta} + Y_\theta \text{ctn}\theta + Y_{\phi\phi}\csc^2\theta + 3Y] \sin\theta d\theta d\phi \\ + \dfrac{p_0}{3} \left[\begin{array}{l} \iint_s [2Y_{\theta\theta} + 2Y_\theta \text{ctn}\theta + 2Y_{\phi\phi}\csc^2\theta + 3Y] Y \sin\theta d\theta d\phi \\ + \iint_s \cos\theta (Y_{\theta\theta} Y_\theta d\theta) d\phi + \iint_s (Y_{\theta\theta}\csc\theta)(Y_{\phi\phi} d\phi) d\theta \\ - \iint_s Y_{\theta\phi} Y_{\phi\theta} \csc\theta d\theta d\phi + \iint_s \dot{Y}_{\theta\phi} Y_\phi \text{ctn}\theta \csc\theta d\theta d\phi \\ + \iint_s \text{ctn}\theta \csc\theta (Y_{\phi\theta} T_\phi d\theta) d\phi - \iint_s Y_\phi^2 \text{ctn}^2\theta \csc\theta d\theta d\phi \end{array} \right] \\ + \iint_s [Y^3 \text{ etc}] \sin\theta d\theta d\phi/3 \end{array} \right\} \quad (11)$$

From the outcome of the problem of the approximate circle we expect that the integral of the first order terms in Y will vanish. If the maximum value of Y is small compared to p_0 the terms of the third order in Y will be small compared to those of the second, and may be neglected. In order to get the second order terms into more manageable form we will make a number of integrations by parts, starting with the first term in the second line. With $U = \cos\theta$, $dV = Y_{\theta\theta} Y_\theta d\theta$, integrate by parts, obtaining for the integrand $+ (1/2)(Y_\theta \sin\theta)(Y_\theta d\theta) d\phi$. Again integrate by parts, obtaining $-(1/2)(Y_{\theta\theta} + Y_\theta \text{ctn}\theta) Y \sin\theta d\theta d\phi$. Integrate the next term by parts, obtaining as integrand, $-Y_{\theta\theta\phi} Y_\phi \csc\theta d\phi d\theta$. This combines with the terms in the third line, to form $-d(Y_{\theta\phi} Y_\phi \csc\theta) d\phi$. Integrate by parts the first term of the last line and combine with the final term to obtain $(1/2)(Y_\phi \csc\theta) \times (Y_\phi d\phi) d\theta$. Integrate this by parts to obtain $-(1/2) \times (Y_{\phi\phi} Y \csc\theta d\theta d\phi$. Equation (11) now boils down to

$$V = 4\pi p_0^3/3 + (p_0^2/3) \iint_s [Y_{\theta\theta} + Y_\theta \text{ctn}\theta + Y_{\phi\phi}\csc^2\theta + 3Y] \sin\theta d\theta d\phi + (p_0/2) \iint_s [Y_{\theta\theta} + Y_\Theta \text{ctn}\theta + Y_{\phi\phi}\csc^2\theta) + 2Y] Y \sin\theta d\theta d\phi \quad (12)$$

together with six "UV" terms. Later, when Y is expanded in spherical harmonics, we will be able to show that the terms of the first order in Y and the "UV" terms reduce to zero.

The record of the radial profilometer is a polar plot of the variations in radius. The amplification, which may be as high as 20000, can be chosen to provide a vivid display of the rate of change q of the tangential radius p as a function of the angular coordinates. A given profile is likely to sample some regions of relatively high relief and other regions of low relief. Therefore one or more profiles, in various orientations, will provide an estimate of the mean square value of the rate of change. If α is the azimuth at a point (θ, ϕ), the rate of change q will be

$$q = p_\theta \cos\alpha + \csc\theta p_\phi \sin\alpha,$$

and the value of q in the direction of its maximum increase is that of the gradient,

$$\nabla p = \mathbf{v} p_\theta + \mathbf{w} \csc\theta p_\phi.$$

At any point on the sphere the value of q^2 averaged with respect to azimuth is

$$\overline{q^2} = \frac{1}{2}\,\overline{[p_\theta^2 + \csc^2\theta p_\phi^2]} = \frac{1}{2}\,\overline{(\nabla p)^2}$$

The average value over the sphere is given by

$$4\pi\sigma_q^2 = \int_s \overline{q^2} d\Omega = (1/2) \iint_s Y_\theta \sin\theta (Y_\theta d\theta) d\phi + (1/2) \iint_s Y_\phi \csc\theta (Y_\phi d\phi) d\theta \quad (13)$$

Integration by parts, with the integrands divided as shown, gives

$$4\pi\sigma^2_{\bar{q}} = -(1/2) \iint_s (Y_{\theta\theta} + Y_\theta \text{ctn}\theta + Y_{\phi\phi}\csc^2\theta)\ Y \sin\theta d\theta d\phi + \ldots. \quad (14)$$

Two "UV" terms will be evaluated later. If we define the mean square variation of p by

$$4\pi\sigma_p^2 = \iint Y^2 \sin\theta d\theta d\phi,$$

we get for the approximate sphere

$$V = 4\pi p_0^3/3 + 4\pi p_0 (\sigma_p^2 - \sigma_q^2). \quad (15)$$

Comparison with eq (2) for the volume based on deviations from the mean radius shows that the mean breadth is slightly larger than the radius,

$$r_0 \simeq p_0\,(1-\sigma_q^2/p_0^2)^{1/2}.$$

3.3. Representation in Spherical Harmonics

At this point we will express Y as the sum of spherical harmonics Y_n of degree n. It will then be possible to compute the volume for specific shapes, to evaluate the first order terms in eq (12) and to show that the "UV" terms of eqs (12) and (14) reduce to zero. An object as smooth as a ball bearing easily meets the requirements for convergence of a series of spherical harmonics. Summaries of the theory of spherical harmonics will be found in [8] or [9]. We will make use of the following properties:

Each spherical harmonic can be expressed as a linear combination of Laplace functions which are products of Legendre polynomials in $\cos\theta$ and sines or cosines of $m\phi$.

$$Y=\sum_n Y_n=\sum_n\sum_m [A_{n,m}\cos m\phi + B_{n,m}\sin m\phi]\,P_n^m(\cos\theta). \qquad (16)$$

The Y_n satisfy Legendre's differential equation:

$$(Y_n)_{\theta\theta}+(Y_n)_\theta\,\mathrm{ctn}_\theta+(Y_n)_{\phi\phi}\csc_\theta^2=-n(n+1)Y_n. \qquad (17)$$

Also,

$$\iint_s Y_n \sin\theta d\theta d\phi=0,\ \iint_s Y_nY_k\sin\theta d\theta d\phi=0 \text{ if } n\neq k.$$

Letting $x=\cos_\theta$, we can define the Legendre polynomials of degree n and order m:

$$P_n=P_n^0=(2^n\,n!)^{-1}(d/dx)^n(x^2-1)^n$$

$$P_m^m=(1-x^2)^{m/2}(d/dx)^mP_n.$$

If the shape of an object can be represented by a single spherical harmonic Y_n, the radial profile on the section of greatest relief will show n lobes. The zonal harmonics P_n describe n lobed figures of revolution symmetrical about the polar axis. The sectoral harmonics $P_n^n\cos n\phi$ describe figures with maximum relief along the equator, and with nodal lines along meridians. If n is odd, a valley will be found at the antipodes of every hill, while if n is even a hill will be opposite a hill and a valley opposite a valley.

Any distribution of lobes over the sphere can be described with spherical harmonics. A few examples will illustrate the possibilities:

$$P_3^2\cos 2\phi=(3.75)\,(\cos\theta-\cos 3\theta)\,\cos 2\phi$$

has lobes in the directions of the vertices of a tetrahedron oriented so that the polar axis passes through the midpoint of two opposite sides.

$$P_4+(1/168)P_4^4\cos 4\phi$$

has octahedral symmetry with a lobe at each pole and four on the equator.

$$P_6+(1/3960)P_6^5\cos 5\phi$$

has lobes in the direction of the normals to faces of a dodecahedron.

Let $AP\cos m\phi$ be a sample term in the series expansion of Y [if $m=0$, $P\equiv P_n$; if $m\neq 0$, $P\equiv P_n^m$]. Since the variables are separated, the "UV" terms from eq (11) can be written:

$$\begin{aligned}\frac{P_0A^2}{3}\Big\{ & (1/2)\Big|P_\theta^2\cos\theta\Big|_0^\pi\int_0^{2\pi}\cos^2 m\phi d\phi \\ & +(1/2)\Big|P_\theta P\sin\theta\Big|_0^\pi\int_0^{2\pi}\cos^2 m\phi d\phi \\ & +m\Big|\sin m\phi\cos m\phi\Big|_0^{2\pi}\int_0^\pi P_{\theta\theta}P\csc\theta d\theta \\ & -m^2\Big|P_\theta P\csc\theta\Big|_0^\pi\int_0^{2\pi}\sin^2 m\phi d\phi \\ & (m^2/2)\Big|P^2\,\mathrm{ctn}\,\theta\csc\theta\Big|_0^\pi\int_0^{2\pi}\sin^2 m\phi d\phi \\ & +(m/2)\Big|\sin m\phi\cos m\phi\Big|_0^{2\pi}\int_0^\pi P^2\csc\theta d\theta\Big\}.\end{aligned}$$

If $m=0$ only the first two terms appear. Since $P\equiv P_n$ is a polynomial in $\cos\theta$, $P_\theta=-\sin\theta\,P_n'$. Therefore, both the terms contain the factor $\sin^2\theta$ which is zero at the limits $\theta=0$ and $\theta=\pi$.

If $m=1$, $P\equiv P_n$, $\sin\theta P_\theta=P_n'\cos\theta-P_n''\sin^2\theta$, and $P_{\theta\theta}=-P_n'\sin\theta-3P_n''\sin\theta\cos\theta+P_n''\sin^3\theta$. For the two terms which contain $\Big|\sin m\phi\cos m\phi\Big|_0^{2\pi}=0$, it is sufficient to note that both $P_{\theta\theta}$ and P contain $\sin\theta$ as a factor, so that the integrals with respect to θ are finite. For the other four terms the ϕ integrals equal π. For these, retaining only the part which is significant at the limits, we have:

$$\begin{aligned}\frac{\pi p_0A^2}{3}\Big\{ & (1/2)\Big|(P')^2\cos^3\theta\Big|_0^\pi+1/2\Big|(P')^2\sin^2\theta\cos\theta\Big|_0^\pi \\ & -\Big|(P')^2\cos\theta\Big|_0^\pi+1/2\Big|(P')^2\cos\theta\Big|_0^\pi\Big\}=0.\end{aligned}$$

For $m>1$, all terms contain $\sin\theta$ in the numerator and reduce to zero at both limits.

The "UV" terms from the integration of eq (13) are

$$1/2\int_0^{2\pi}\left|Y_\theta Y\sin\theta\right|_0^\pi d\phi-1/2\int_0^{2\pi}\left|Y_\phi Y\csc\theta\right|_0^{2\pi}d\theta$$

both of which occurred in the integration of eq (11) and were shown to vanish for all values of m.

Having justified the omission of the "UV" terms from eq (12), let us expand Y in a series of spherical harmonics, then use Legendre's differential equation (17) to eliminate the derivatives of Y_n. The result is

$$V=4\pi p_0^3/3$$

$$+(p_0^2/3)\sum_n(3-n-n^2)\int_0^{2\pi}\int_0^\pi Y_n\sin\theta d\theta d\phi$$

$$+(p_0/2)\sum_n\sum_k(2-n-n^2)\int_0^{2\pi}\int_0^\pi Y_nY_k\sin\theta d\theta d\phi. \tag{18}$$

The integrals of Y_n and of the cross products Y_nY_k, $n\neq k$, vanish and we have

$$V=4\pi p_0^3/3-(p_0/2)\sum(n-1)(n+2)\int_0^{2\pi}\int_0^\pi Y_n^2\sin\theta d\theta d\phi. \tag{19}$$

If the standard deviation σ_n of the spherical harmonic of degree n is given by

$$4\pi\sigma_n^2=\int_0^{2\pi}\int_0^\pi Y_n^2\sin\theta d\theta d\phi,$$

$$V=4\pi p_0^3/3-2\pi p_0\sum(n-1)(n+2)\sigma_n^2. \tag{20}$$

3.4. Calculation of Volume

As an example, let us calculate the topography correction for a ball bearing of a type which is produced in quantity by centerless grinding. The radial profiles usually show three or five lobes. The balls are graded on the basis of the tolerance for the difference in diameter of circles inscribed and circumscribed on the lobe pattern. 2.5-inch balls with a 100 microinch tolerance can be bought in small quantities for about $25. The shape of one of these balls might be represented by the formula:

$$p=p_0+A(\cos\theta-\cos^3\theta)\cos 2\phi$$

$$=p_0+(A/15)P_3^2\cos 2\phi$$

where A is the tolerance and p_0 is the half width. This ball would have tetrahedral symmetry, and the radial profile would have three lobes. The standard deviation of p is given by

$$4\pi\sigma_p^2=A^2\int_0^{2\pi}\cos^2 2\phi d\phi\int_0^\pi(\cos\theta-\cos^3\theta)^2\sin\theta d\theta$$

or

$$\sigma_p^2=4A^2/105.$$

The volume is

$$V=4\pi p_0^3/3-2\pi p_0(3-1)(3+2)\sigma_p^2$$

$$=(4\pi p_0^3/3)(1-4A^2/7p_0^2)$$

For $A=10^{-4}$ inch and $p_0=1.25$ inch the correction to the volume is about 4 parts in 10^9.

4. Measurement of Mean Breadth

In measuring the object it is desirable to have as large a number of independent observations as practicable, to provide estimates of precision and of topographic variations. There are some advantages in having the directions of measurement distributed systematically so that each can be associated with the same solid angle and therefore have equal weight. It is possible to do this by measuring in directions normal to the faces of a regular polygon. Three orthogonal directions are normal to the faces of a cube, four directions are normal to the faces of an octahedron, six are determined by a dodecahedron and ten by an icosahedron.

Mapped on a sphere, latitudes and longitudes of these sets of directions can be as follows:

	Set of three			
N. Latitude... Longitude....	90° —	0° 0°	0° 90°	
	Set of four			
N. Latitude... Longitude....	35.264° ±45°	35.264° = $\tan^{-1}(\sqrt{2}/2)$ ±135°		
	Set of six			
N. Latitude... Longitude....	90° —	26.565° ±36°	26.565° ±108°	26.565° = $\tan^{-1}(1/2)$ 180°
	Set of ten			
Longitude.... N. Latitude... N. Latitude...	0 52.623° 10.812°	±72° 52.623° 10.812°	±144° 52.623° = $\tan^{-1}(3+\sqrt{5})/4$ 10.812° = $\tan^{-1}(3-\sqrt{5})/4$	

These orientations are chosen so that each direction of the set of four directions is towards the centroid of a spherical triangle (an octant) determined by the set of three. Similarly, each direction of the set of 10 is toward the centroid of three adjacent directions of the set of six.

The set of six or the set of 10 directions each corresponds to the locations of the lobes of one of the spherical harmonics of degree six. If the lobe pattern of the ball is represented by spherical harmonics of degree less than six, the observations of either set are distributed over hills and valleys so as to average out the lobe pattern *exactly*, and thus give a correct mean width.

If the ball had lobes distributed like the vertices of a dodecahedron and is oriented just right, it would be possible for the observations of the set of six to fall in the low spots or for those of the set of 10 to fall on the high spots. Either set would yield an erroneous result.

What is more likely is for the ball to have a single feature, such as a dent or abraded patch, covering one or two percent of its surface. Then there would be a chance that this feature would be missed in a set of observations. In this experiment the chance of missing an isolated feature was reduced by making four series of observations in ten directions with the ball relocated at random between series.

An estimate of the magnitude of the even spherical harmonics can be made on the basis of the interferometric measurements. However, this information is incomplete on two counts. First, the breadth is not affected by the odd spherical harmonics, but the volume is. Second, the expression for the volume contains the square of the degree n, and the interferometer provides no information on the number of lobes.

Observations with the radial profilometer are essential in estimating the error due to topography. The profiles show the odd as well as the even harmonics. If a single harmonic dominates the topography the number n of lobes can be counted. If the pattern is irregular, a radial profile provides a sample from which to estimate a mean square rate of change of radius (σ_q^2 in eq (15)). The angular resolution of the trace is of the order of a degree, so that the chance of finding a small defect is improved.

5. Summary

We have evaluated the topography correction for the difference between the volume of an approximate sphere and that of a true sphere of the same average breadth. It is of second order in the variations in tangential radius, and therefore much smaller than the random and systematic errors of the measurements which can be made on the object. For example, a commercially available ball bearing of moderate cost can be expected to have a topography correction of a few parts in 10^9. Its mean breadth can be measured to a part in 10^6 corresponding to a few parts in a million in its volume.

In order to have no question as to the integral part of the order of interference, Bowman [1] used balls of much better quality. The manufacturer's nominal tolerances were one or two microinches. The radial profiles appeared to be elliptical ($n=2$) with a difference between the major and minor axes less than one part in a million. Harmonics of degree greater than two were too small to detect. In sixteen sets of interferometric measurements, the greatest and least breadths of a ball in a set of ten differed by less than two parts in a million. If all of this difference resulted from a spherical harmonic of degree two, the correction to the volume would be a few parts in 10^{12}. The effect of harmonics of higher degree and smaller amplitude might be comparable. In any case the topography corrections are several orders of magnitude smaller than the uncertainties inherent in the interferometer.

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