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# Geometrical First Principles in Proclus' *Commentary on the First Book of Euclid's Elements*

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## Abstract

In his commentary on Euclid, Proclus says both that the first principle of geometry are self-evident and that they are hypotheses received from the single, highest, unhypothetical science, which is probably dialectic. The implication of this seems to be that a geometer both does and does not know geometrical truths. This dilemma only exists if we assume that Proclus follows Aristotle in his understanding of these terms. This paper shows that this is not the case, and explains what Proclus himself means by definition, hypothesis, axiom, postulate, and the self-evident, and how geometry is a science that receives its principles from dialectic.

## Keywords

Neoplatonism; Proclus; Euclid; principles; mathematics; geometry

## 1 The Problem

### 1.1 *The Science of Mathematics Is a Hypothetical Science, but Also Makes Use of Principles That Are Self-evident*

The importance of Proclus' *Commentary on the First Book of Euclid's Elements* for the history and philosophy of mathematics is well known, but to date an accurate and thorough explanation of what he means by the hypothetical

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character of mathematics has not been given.[1] That he thinks arithmetic and geometry are hypothetical sciences is clear. In an important passage in which he discusses the status of geometrical principles,[2] Proclus begins by stating that the science of geometry 'is based ... on hypothesis and proves its later propositions from determinate first principles (*apo archôn hôrismenôn*)'.[3] The reason for this is that 'there is only one unhypothetical science, the other sciences receiving their first principles from it'.[4] Someone treating of the elements of geometry, therefore, should not give an argument for its first principles.[5] However, contrary to what we would expect, Proclus does not think that geometrical hypotheses are mere assumptions. Rather, they are known to the mathematician because they are self-evident:

> For no science demonstrates its own first principles or presents a reason for them; rather each holds them as self-evident (*autopistôs*), that is, as more evident than their consequences. The science knows them through themselves, and the later propositions through them.[6]

This is true for physics and medicine, for example, as for geometry, because all fall short of the one unhypothetical science, which is dialectic.[7]

Proclus' characterization of geometrical first principles in these two seemingly contradictory ways results from his combination of the Aristotelian idea of the intelligibility of first principles with his Platonic account of the hypothetical character of mathematics and all lower sciences, as compared with dialectic as the one unhypothetical science, drawn from his interpretation of

1) For Proclus on mathematics in general see O'Meara 1989; Cleary 2000; Lernould 2010; Nikulin 2002.

2) 75.6-26. All references to Proclus' commentary on Euclid are to page- and line-numbers in Friedlein 1873. Note that I shall also refer to Euclid's definitions, postulates, axioms, problems and theorems by their occurrence in the same edition of Proclus' text. Translations are generally based on Morrow 1970, often substantially revised.

3) 75.6-8: ἐξ ὑποθέσεως εἶναι φαμεν καὶ ἀπὸ ἀρχῶν ὡρισμένων τὰ ἐφεξῆς ἀποδεικνύναι. I shall discuss below the translation of ὁρισμένων as 'determinate'.

4) 75.9-10: μία γὰρ ἡ ἀνυπόθετος, αἱ δε ἄλλαι παρ᾽ ἐκείνης ὑποδέχονται τὰς ἀρχάς.

5) 75.13-14: τῶν μὲν ἀρχῶν μὴ διδόναι λόγον.

6) 75.14-19: οὐδεμία γὰρ ἐπιστήμη τὰς ἑαυτῆς ἀρχὰς ἀποδείκνυσιν, οὐδὲ ποιεῖται λόγον περὶ αὐτῶν, ἀλλ᾽ αὐτοπίστως ἔχει περὶ αὐτάς, καὶ μᾶλλόν εἰσιν αὐτῇ καταφανεῖς τῶν ἐφεξῆς. καὶ τὰς μὲν οἶδεν δι᾽ αὑτάς, τὰ δὲ μετὰ ταῦτα δι᾽ ἐκείνας. Cf. Aristotle, *An. Post.* 1.2, 72a31-3.

7) Proclus does not identify the one unhypothetical science with dialectic explicitly in the *in Eucl.*, but it is fairly certain that they are the same. See MacIsaac 2010, 130-2.

the Divided Line (10.16-12.2).[8] In this interpretation, the distinction between dialectic and mathematics lies in the status of their first principles. Both dialectic and the lower sciences move down from their principles, and mount back up to them, but dialectic takes the forms in *Nous* as the origin of its descent, and hence its correlative upward motion terminates in an unhypothetical *noêsis*, which itself aims towards *henôsis*. The mathematical sciences, on the other hand, begin from the *logoi* in the soul, and hence their return to principles terminates in the same *logoi*, and leaves the relation of these *logoi* to their own causes unexamined.[9]

### 1.2 *A Mathematician Without Dialectic Seems Ignorant of His Principles, but Still Seems to Arrive at Mathematical Knowledge*

But what exactly does Proclus mean by using the term 'hypotheses' to refer to these *logoi* from which mathematics begins and to which it returns? This question presses upon us because without an answer to it we cannot say that we understand clearly what is meant by mathematical knowledge in the most important text of ancient Platonic philosophy of mathematics. And this hypothetical character of mathematics initially seems problematic in Proclus, because it seems to imply that, according to him, a mathematician who is ignorant of dialectic would both know and not know the principles of his science, and hence both know and not know its conclusions.

In other words, we need to investigate why Proclus says two seemingly incompatible things about the principles of subordinate sciences. On the one hand they hold their principles as self-evident (*autopistôs*: 75.16) while on the other they receive their principles from dialectic, the one unhypothetical science: 'For genuine science is one, the science by which we are able to know all things, the science from which come the principles of all other sciences,

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8) In his commentary, Proclus is continually in conversation with Aristotle's *Posterior Analytics*. As I shall argue below, however, he does not always agree with Aristotle. And even when he does accept Aristotelian positions, they are understood from within a Neoplatonic framework.

9) See MacIsaac 2010, 136-7: 'Science première et sciences secondes effectuent le mouvement double descendant et ascendant, en usant des méthodes de la synthèse et de l'analyse, mais la science première, parce-qu'elle part des Formes premières, a vraiment pour objet ces mêmes Formes premières, et parce qu'elle a comme terme de sa remontée l'Un elle est *anhypothétique*; les mathématiques, parce-qu' elles prennent pour points de départ des Formes « secondes » . . . et parce que ces Formes secondes constituent le terme de leur remontée, sont *hypothétiques*.'

some immediately and some at further remove.'[10] Mathematics in particular is second in rank to dialectic, according to Plato, who 'says not that it does not know its principles, but that it takes them from the highest science [dialectic] and, holding them without demonstration, demonstrates their consequences'.[11] Insofar as the principles of mathematics are self-evident, the mathematician who is ignorant of dialectic can know them. But insofar as they are received from dialectic, he holds them only hypothetically, and we would be inclined to think that a hypothetical grasp falls short of knowledge.[12]

The counterpart to this question about how the same principles can be both hypothetical and self-evident is the following. In what way does dialectic furnish principles to lower sciences, and how do lower sciences like mathematics receive from dialectic their principles? If Proclus is not simply contradicting himself, he cannot mean that the lower sciences receive their principles from dialectic in the sense that they are simply the conclusions of that science.[13] Otherwise a mathematician who is ignorant of dialectic would not know his first principles, because as the conclusions of the science of dialectic they could only be known as following from the principles of that higher science. However, the same mathematician would know his first principles, because he holds them as self-evident. So both knowing and not knowing his principles, he both knows and does not know his conclusions, and both has and does not have mathematical knowledge. Of course, one could solve this dilemma by requiring all mathematicians to be dialecticians, but this seems an unreasonable requirement, as well as constituting a flat denial of Proclus' position that mathematical principles are self-evident. A better solution would be to discover what Proclus actually means when he says that lower sciences receive their principles from dialectic, and why he thinks that they can be both hypothetical and self-evident.

10) 31.22-32.2: μία γὰρ ἡ ὄντως ἐπιστήμη, καθ᾽ ἣν τὰ ὄντα πάντα γινώσκειν πεφύκαμεν, καὶ ἀφ᾽ ἧς πᾶσαι αἱ ἀρχαὶ ταῖς μὲν ἐγγυτέρω τεταγμέναις, ταῖς δὲ πορρωτέρω.

11) 32.5-7: μηδ᾽ ὅτι τὰς οἰκείας ἀρχὰς ἀγνοεῖν αὐτήν φησιν, ἀλλ᾽ ὅτι παρ᾽ ἐκείνης λαβοῦσαν καὶ ἀναποδείκτως ἔχουσαν ἐκ τούτων τὰ ἐφεξῆς ἀποδεικνύναι.

12) For hypotheses falling short of knowledge in Plato, see *Republic* 510b-d, 533b-d; *Meno* 86e-87b.

13) Even from a textual point of view, the hypotheses, axioms and postulates which Proclus finds in Euclid do not seem like the conclusions of the science of dialectic. The definition of a point as what has no parts (85.01) or of a line as length without breadth (96.16) are not found as the conclusions of, say, the *Elements of Theology* or the commentary on the *Parmenides*. Martijn's discussion of subalternate sciences (2010, 182-4) is relevant.

## 2 Axioms, Hypotheses, and Postulates according to Proclus' Report of Aristotle

In order to solve this dilemma, we need to look at Proclus' understanding of the first principles of geometry, and at what he means by the self-evident. Proclus says that Euclid distinguishes three sorts of principle: hypotheses, postulates, and axioms.[14] The most important of these for us are hypotheses, but in order to discover what Proclus means by a hypothesis we must look at all three, and we will also have to look at what he means by definitions.

After listing the three sorts of principle according to Euclid, Proclus continues with a passage that is generally taken to represent his own views on the distinction between them (76.6-77.2):

> Axiom, postulate, and hypothesis are not the same thing, as the inspired Aristotle somewhere says (*hôs pou phêsin ho daimonios Aristotelês*). When a proposition that is to be accepted into the rank of first principles is something both known (*gnôrimon*) to the learner and credible in itself (*kath' hauto piston*), such a proposition is an axiom: for example, that things equal to the same thing are equal to each other. When the student does not have a self-evident notion (*ennoian ... autopiston*) of the assertion proposed but nevertheless posits it and thus concedes the point to his teacher, such an assertion is a hypothesis. That a circle is a figure of such-and-such a sort we do not know by a common notion in advance of being taught, but upon hearing it we accept it without a demonstration. Whenever, on the other hand, the statement is unknown (*agnôston*) and nevertheless is taken as true without the student's conceding it, then, he says (*phêsin*), we call it a postulate: for example, that all right angles are equal. This characteristic of postulates is evidenced by the strenuous efforts that have been made to establish one of them,[15] as though nobody could concede it without more ado. In this way axiom, postulate, and hypothesis are distinguished according to Aristotle's teaching (*kata men tên Aristotelous huphêgêsin*).

In this passage and in his report of Euclid's taxonomy which precedes it, Proclus uses Aristotle's terms instead of Euclid's, substituting 'axioms' for common notions and 'hypotheses' for definitions. Despite this, it is clear that he thinks he is following Euclid's division of mathematical principles into definitions,

14) 76.4-6: καὶ αὐτὰς διαιρεῖ τὰς κοινὰς ἀρχὰς εἴς τε τὰς ὑποθέσεις καὶ τὰ αἰτήματα καὶ τὰ ἀξιώματα.

15) The fifth.

postulates and common notions,[16] and he seems to think that both 'hypothesis' and 'definition' are appropriate names for the first sort of principle, referring to the section of Euclid's text that contains them by both names.[17] So far as I know, all commentators have taken this passage to be more than a report of Aristotle's use of terms. They think it indicates that Proclus accepts generally the Aristotelian distinctions between axiom, hypothesis and postulate, as he reports them here. I will argue in what follows that this is mistaken.

There are very few treatments of the character of Proclus' geometrical first principles in themselves, and they generally focus on his substitution of the Aristotelian term 'hypothesis' for Euclid's 'definition'.[18] The general strategy of commentators has been to point out that Proclus is referring in the passage above to Aristotle's *Posterior Analytics*, and to assert that he agrees with this report of Aristotle's view.[19] They then focus on Aristotle's contention that a hypothesis involves an existence claim while a definition does not.[20] Earlier commentators then often concluded from Proclus' substitution that he confused definitions and hypotheses.[21] The few recent treatments have tried to defend Proclus on the grounds that, for him, definitions also involve existence claims and so the substitution is warranted (Giardina 2010, 178), or by arguing that the distinction between definition and hypothesis is not all that sharp

16) ὅροι, αἰτήματα, κοιναὶ ἔννοιαι. See Euclid, *Elements* book 1. He repeats his substitution of 'hypotheses' for 'definitions' at the beginning of his discussion of Euclid's postulates and axioms. See 178.7-8. For common notions as axioms, see 194.4-9, and cf. 181.24-182.20.

17) Definitions: 81.26; 178.7-8; 418.18. Hypotheses: 388.14.

18) While there are many studies on Proclus' conception of mathematics in general, most focusing on the Euclid commentary, very few attack this problem of geometrical principles itself. The only recent studies that are relevant are Giardina 2010; Martijn 2010; and Romano 2010. Breton 1969 has quite a long discussion (33-68) but, while quite interesting in its way, it is vitiated by his reliance on Aristotle's classification, a lack of precision about Proclus' system in general stemming from the early state of Proclus studies at the time, a preoccupation with contemporary problems in logic and philosophy of mathematics, and an almost complete absence of references to Proclus' text. However, his discussion of the parts of a geometrical argument and their relation to the soul's self-movement in Proclus is good (63-8).

19) For example, Romano 2010, 183-4 refers us to *An. Post.* 1.2, 72a1-24 and 1.10, 76b23 *ff.*, and says: 'Il est très evident, donc, que selon Proclus Euclide suit fidèlement la classification que donne Aristote des principes en mathématiques et qu'il reprend fidèlement la terminologie aristotélicienne . . . Proclus pense, donc, que la terminologie d'Aristote équivaut exactement à celle d'Euclide.' Since, according to Romano, Proclus thinks that Euclid and Aristotle say the same thing, it follows that Proclus agrees with both of them.

20) See Romano 2010, 184-5.

21) See references and discussion in Martijn 2010, 92.

in Aristotle (Romano 2010). One commentator (Martijn 2010, 95-6) recognises that neither in this passage nor anywhere else does Proclus say that a hypothesis must involve an existence claim, and so she rightly points out that Proclus does not subscribe to Aristotle's distinction between definitions and hypotheses. However, she does still pay attention to the question of existence claims, because she thinks that for Proclus definitions involve a certain type of implicit hypothesis, namely ones 'concerning the existence of the definienda'.

In my view, all of these treatments have missed the point. Because they all take Proclus to have had Aristotle's *Posterior Analytics* squarely in mind, they read Proclus' priorities off from Aristotle, for whom whether or not a hypothesis involves an existence claim is important. However, even though Proclus is continually engaging with the *Posterior Analytics* in his commentary, his main interest is not an exegesis of the Aristotelian text. His phrase 'as Aristotle says *somewhere*' (76.8: ὥς πού φησιν) indicates this, as does his bare-faced combination of Aristotelian and Platonic ideas with which this study began. In his corpus Proclus often refers to Aristotle, sometimes agreeing and sometimes disagreeing with him, but never simply accepting Aristotelian doctrines at face value. So an analysis of Proclus which takes the *Posterior Analytics* as its conceptual touchstone is likely to go astray. Moreover, as Martijn has pointed out, Proclus does not in fact subscribe to Aristotle's distinction between definition and hypothesis, so studies which try to defend him on the basis that he does are misguided. Her solution may be the correct one, but is addressing a point that is probably more important in the *Timaeus* commentary, which is her own main interest, than in the Euclid commentary.

Although Martijn has noticed that Proclus does not distinguish definitions and hypotheses in Aristotle's manner, and so is on much firmer ground than Romano, for example, who thinks that Proclus walks 'soigneusement dans les pas d'Aristote' (Romano 2010, 184), she does follow the universal opinion that at *in Eucl.* 76.6-77.2 Proclus is reporting a classification of axiom, hypothesis and postulate with which he agrees (Martijn 2010, 94). But does Proclus agree with this classification? I think he does not. First of all, at the beginning and at the end of the passage he attributes this characterization to Aristotle, so on a purely textual basis it seems to be the report of someone else's opinion rather than his own. Secondly, as I will discuss below, Proclus thinks that both axioms and postulates are easily known or easily constructed, and do not stand in need of proof. As first principles they are self-evident (*autopiston*: 178.9-179.14). Therefore, he clearly does not accept the characterization in this passage of a postulate as something that is unknown (*agnôston*), and the strenuous efforts taken to prove one of them (the fifth) as evidence of its character as a postulate. In fact he strikes the fifth postulate from the list because of its need for proof. Moreover, as his example of a hypothesis he gives the circle (*kuklon*), 'which we do not know by a common notion in advance of being taught'.[22] But later in the commentary he says himself that we do have an untaught grasp of the circular line (*peripherês*), and possess its notion from itself (*autothen*).[23] So if Proclus does not distinguish hypotheses and definitions in the way Aristotle does (i.e. with regard to existence claims), and he does not follow Aristotle's understanding of postulates, as reported here, and he gives an example of a hypothesis something whose characterization he himself does not agree with, we cannot simply take it for granted that he subscribes to this Aristotelian idea that a hypothesis is something posited without the possession of a self-evident notion (*ennoian . . . autopiston*).

The most obvious reason to reject this as Proclus' view are the passages which give rise to this study: first, his statement that geometrical principles are hypothetical, because there is only one unhypothetical science (75.6-9); secondly, his statement that geometry does know its first principles, even though it receives them from dialectic (31.22-32.7); and finally that no science demonstrates its own first principles, but holds them to be self-evident (*autopistôs*: 75.14-19; 179.12-14). In the face of this obvious textual evidence that Proclus thinks geometrical principles are both hypothetical and self-evident, it is fairly clear that he does not agree with the characterization of hypotheses that he ascribes to Aristotle in this passage.

This is important because taking this passage as reporting Proclus' view leads one into serious trouble whenever he says that the same thing is both self-evident and hypothetical. In the face of such passages, Martijn was forced to assert that for Proclus the axioms of the philosophy of nature, which are held hypothetically, must be axioms only in a 'watered down sense'.[24] In other words, although she sees the dilemma in Proclus to which I am pointing, that something can both be an axiom and be hypothetical, she wants to solve it by discounting Proclus' notion of the self-evident. Moreover, she thinks this can be explained by the 'didactic' context of *phusiologia* in which hypotheses are only hypothetical for the student, but are known dialectically by the teacher.[25] However, this didactic suggestion of assigning the knowledge and the

22) 76.16: κοινὴν ἔννοιαν οὐ προειλήφαμεν ἀδιδάκτως.

23) 118.24-119.4: ἡ περιφερής . . . τούτων γὰρ καὶ οἱ πολλοὶ προλήψεις ἔχουσιν ἀδιδάκτους . . . αὐτόθεν τὰς ἐννοίας ἔχομεν.

24) Martijn 2010, 110-11. Her discussion is related to her treatment of the hypothetical character of *phusiologia* at Martijn 2010, 91-7.

25) Martijn 2010, 134-5. This solution is suggested by the didactic language at *An. Post.* 1.10, 76b28-9.

ignorance to two different sorts of person is not a very satisfying solution to our initial dilemma, in which Proclus really does seem to imply both that a mathematician who is ignorant of dialectic knows his first principles, because they are self-evident, and does not know them, because they are hypothetical. Not only does this solution require us to explain away the many places in which self-evidence seems to be a real notion for Proclus, it makes the unreasonable demand that only dialecticians know the first principles of subordinate sciences.[26] Mathematicians without dialectic would remain 'students', with only a hypothetical grasp of their own principles.

If the balance of evidence indicates that Proclus does not agree with Aristotle, why does this passage seem to say that he does? When Proclus says 'axiom, postulate, and hypothesis are not the same thing', and immediately remarks that Aristotle also distinguishes them, it seems natural to infer that he also agrees with the manner in which Aristotle distinguishes them, especially given the abrupt way in which he then launches into an explanation of it. However, all Proclus actually says is that (a) he (Proclus) thinks these three are distinct, (b) Aristotle thinks so too, and (c) this is the manner in which Aristotle distinguishes them. So accepting the evidence that Proclus generally disagrees with Aristotle's classification does not require us to think that Proclus means something different from what he says in this passage. Rather, we just have to avoid inferring something that he does not in fact say, namely that he agrees with Aristotle.[27]

So if this passage does not faithfully report Proclus' view, how does he in fact distinguish between axiom, postulate and hypothesis, and what does he really think is the relation between hypothesis and definition?

26) Martijn 2010, 94: 'The hypothetical nature of geometry, and, as we will see, of philosophy of nature, is no reason for Proclus to reject or criticise these sciences, as the good scientist would be able to present a justification for the starting points, but within a superordinate science.'

27) Note also that the passage reporting Aristotle's classification is followed immediately by a report of two other classifications: the Stoic classification of all such things as axioms, and an unnamed group that calls them all hypotheses. This in turn is followed by a long discussion of the difference between problems and theorems as reported by a variety of others, which Proclus variously agrees with, criticises, and nuances. See 75.26-81.6.

## 3 Proclus' Doctrine of Geometrical First Principles

### 3.1 *Analogy, Definition, and Hypothesis*

Proclus' understanding of the structure of the science of geometry is the following. First of all, he distinguishes first principles—definitions (*horoi*),[28] postulates (*aitêmata*), and axioms (*axiômata*)—from what follows from the first principles—problems (*problêmata*) and theorems (*theôrêmata*). A postulate constructs a geometrical figure while an axiom recognises the existence of an attribute. As first principles, both do so without demonstration. Problems and theorems are the demonstrated counterparts of these (178.9-179.12; 77.7-12).

Secondly, he holds that problems and theorems have six parts: enunciation (*protasis*), exposition (*ekthesis*), specification (*diorismos*), construction (*kataskeuê*), demonstration (*apodeixis*), and conclusion (*sumperasma*).[29] The enunciation lays out what is given (*to dedomenon*) and what is sought (*to zêtoumenon*), while the exposition treats separately what is given, and the specification does the same for what is sought. Not all problems have a separate exposition and specification, but all problems must have an enunciation. Moreover, the enunciation can sometimes state only what is sought, with the geometer supplying what is given from his understanding of the meaning of what is sought (204.8-17). In the construction the geometer draws the lines that are needed in order to discover what is sought. This step is often omitted in theorems (as opposed to problems), says Proclus, because often the exposition of what is given is sufficient to show the truth of the conclusion. The demonstration is the actual argument, which takes what is given and reasons to what is sought, and so it is always needed, as is the conclusion, which confirms the demonstration by referring it back to the enunciation. Therefore problems and theorems have six possible parts, of which three are essential: the enunciation, the demonstration and the conclusion. Finally, Proclus calls 'what is given' the hypothesis, whether it be in the enunciation or in a more detailed exposition. We will look at this use of hypothesis in more detail below.

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28) As mentioned above, Proclus uses the term hypothesis in his initial taxonomy of principles at 76.4-6, but he also refers to the section on definitions as such, and to each one of the definitions as a definition. So it is just as valid, and less confusing, to say that according to Proclus the first principles of geometry are definitions, postulates and axioms. Of course we cannot solve our initial dilemma in this simple way, because all three are hypothetical, as we shall discuss below.

29) For these terms see 203.4-6 and for his entire discussion of the structure of geometrical arguments see 200.22-207.25.

Our aim in this study is to understand how Proclus can call geometrical first principles both hypothetical and self-evident, and how this is related to his use of both 'hypothesis' and 'definition' to refer to the same sort of principle. In order to do this we shall begin by looking at what he means by definition and by hypothesis, which will require us first of all to examine his doctrine of analogy. After this we shall look at what he means by the self-evident, beginning with an examination of his doctrine of postulates and axioms, and then seeing how self-evidence relates to what we have discovered about definition and hypothesis. Finally, we shall answer the question of how, given the self-evident character of mathematical first principles, they can be said to be received from dialectic.

### 3.1.1 Analogy

Proclus' occasional substitution of the term 'hypothesis' for 'definition' in his delineation of principles seems to be deliberate. He is aware of the terminology of his source text, referring more than once to Euclid's section as dealing with 'definitions' (*horoi*: 81.26; 178.7-8; 418.18), and using the term *horos* himself many times. A clue to the relationship between the two terms in Proclus' mind can be gathered from the passage with which we began this study, which states that geometry: 'is based . . . on hypothesis and proves its later propositions from *determinate* first principles (*apo archôn horismenôn*)' (75.7-8; emphasis mine). *Hôrismenôn* and *horos* share a common root with *horizô*, so we shall need to investigate how Proclus thinks that being determinate in this way is related to Euclid's definitions. Moreover, this use of both terms in a single statement indicates that Proclus' occasional substitution of 'hypothesis' for Euclid's 'definition' does not seem to be a simple confusion of one term for the other, or even a simple distinction of one from the other. Rather, this passage indicates that there is a relationship between hypothesis and definition, such that geometry is a hypothetical science *because* it begins from principles which are determined, or which are defined in some way. If this is the case, then in order to understand what Proclus means by 'hypothesis' we first need to discover what he means by 'definition'.

Proclus' conception of definition, however, is imbedded within his idea of analogy. His general doctrine of *analogia* holds that in a certain way the same Forms exist both separate from sensible matter and in sensible matter,[30] and within those divisions in a number of ways. More precisely, each separate character (*idiotês*) which takes its origin from a henad at the summit of reality runs through the whole, all the way to the bottom, in what he calls a chain or

30) For Proclus there is also 'intelligible matter'. See 52.20-53.5 and MacIsaac 2001.

series (*seira*), taking on various ways of existing appropriate to each order of reality (*taxis*).[31]

Proclus discusses Euclid's three pages of definitions for ninety-three pages of Friedlein's text (85-177), and right from the beginning of his discussion he spends a good part of his time specifying the higher and lower analogical counterparts within his metaphysical hierarchy of each geometrical object. For example, he says that the point exists among the henads, in the intelligibility of *Nous*, in Soul's *dianoia* and *phantasia* (as understood by dialectic, as a unit in arithmetic, and as the point in geometry), as that which allows the celestial spheres to have axes and poles, and finally as dispersed in every position in a material body.[32] Proclus invokes his principle that all things are in all things but in a manner appropriate to where they appear, according to which we can think both that the form of Limit[33] exists variously—primarily as a point in *Nous*, a line in Soul, a plane in nature, and as all of these in body—and that the point exists in different ways in different orders (*taxeis*):

> Consequently all the limits are everywhere, and each appears in the mode of its own order, their appearances varying according to the power that prevails in them. As to the point, it is everywhere indivisible and distinguished by its simplicity from divisible things; but as it descends in the scale of being, even the point takes on the character distinctive of divisibles.[34]

All of Proclus' discussions of Euclid's definitions make explicit and extensive use of this notion of analogy. Particularly elaborate are his discussions of the straight line and the circle in the demiurgic *Nous*, of angles, triangles, squares, and especially the sections on figure and the circle.[35] Each reality is more genuinely what it is in the higher orders, but is manifested in more and more multiple ways as it descends through the cosmos, through what Proclus calls 'declension' (*huphesis*: 96.14-15).

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31) For *analogia* in Proclus, see Martijn 2010, 202-3; MacIsaac 2007; Gersh 1978; Gersh 1973 83-90; Charles 1969. Although it can be misleading, it is convenient to think of 'vertical' series that run down through the 'horizontal' orders of *Nous*, Soul, etc.

32) 91.11-93.5. For the point as dispersed in body: 88.18-20.

33) I.e. limit as boundary, rather than the principle Limit from the pair Limit/Unlimited.

34) 92.16-21. Note the verbal echo in the first line here, πάντα ἄρα πανταχοῦ καὶ ἕκαστα κατὰ τὴν οἰκεῖαν τάξιν ἐκφαίνεται, of Prop. 103 of the *Elements of Theology*: πάντα ἐν πᾶσιν, οἰκείως δὲ ἐν ἑκάστῳ.

35) Straight line and circle in *Nous*: 107.11-109.4; angles: 128.26 *ff.*; triangles: 166.14-168.2; squares: 173.2-174.16; figure: 136.20-146.17; the circle: 146.24-156.5.

For Proclus, only dialectic has as its purview how things exist in all orders of reality. A special science like geometry has an eye only to geometrical being, even though points, lines and figures exist analogically above and below this sort of being. The basic meaning of 'definition', for Proclus, is a kind of boundary. As we shall see in the next section, the definitions proper to geometry give a kind of narrowing of psychic focus, and are what allow the geometer to attend accurately and exclusively to geometrical being. So analogy is the background of definition, because in beginning from 'determinate / defined first principles' (*archôn horismenôn*) each science attends to a kind of being that is bounded or marked off from what lies analogically above and below it.

### 3.1.2 The Non-Propositional Character of Philosophy in Proclus

Before we examine Proclus' conception of definition, however, we need to avoid a possible source of confusion. Proclus does not really think of geometry, or of philosophy in general, in terms of propositions. He does think that philosophy is discursive, as opposed to the non-discursive thinking of *Nous*. But this discursivity does not consist in articulating propositions which are in some way *about* other things in the world.[36] Rather, Proclus' idea is that the soul already possesses all of its *logoi*, and its discursivity consists in moving from the examination of one to the examination of another. He also thinks that the *logoi* which are resident in your soul or my soul are the same *logoi* which belong to the hypostasis of Soul as a whole, not just specifically but numerically. In other words, what you or I think are not just our own conceptions of geometrical objects, but the actual point, line, circle, etc. as they exist on the psychic level. Otherwise, he would say, our thought does not reach the realities it seeks.

Further, because of his doctrine of analogy, the *logoi* that the Soul possesses are the whole of reality on the psychic level. They are the Forms in *Nous* as they have descended to Soul; they are the paradigms of all the *logoi* in nature and body as a whole. For Proclus, the soul's thoughts are not *about* something else, some things that exist 'in the world', because in a way for him there is no world for discursive thought outside of the soul. Of course, there is a sense in which the soul's thinking is *about* something else, namely through the idea of analogy itself, but even in these cases the soul has discursive knowledge of its higher and lower counterparts through itself. The soul can put aside discursivity in a way to rise to *noêsis*, or it can attend to the sensations

36) I do not wish to discuss the ontological status of propositions. The only important point is that a proposition is generally taken to be distinct from the thing that it is about.

and passions that come to it from the body. But when it thinks discursively it always has itself as its object. What comes closest to propositions in Proclus, spoken or written arguments, are only images of the real arguments, which are the discursive activities of soul.[37]

This picture needs to be corrected somewhat, because in a way for Proclus geometrical reasoning *is about* something else. There are distinctions within the soul's discursivity, and the figures which the soul constructs in the geometrical matter of its imagination (*phantasia*) are about the geometrical realities that exist in its discursive reason proper (*dianoia*). Also, in making use in geometry of its imagination, which lies next below discursive reason, the soul begins to introduce a numerical multiplicity into the objects of its own thinking that was not present before. Nevertheless, whether in discursive reason or imagination, the soul actually *is* the objects it thinks about:[38]

> When, therefore, geometry says something about the circle or its diameter, or about its accidental characteristics, such as tangents to it or segments of it and the like, let us not say that it is instructing us either about the circles in the sense world, for it attempts to separate [itself] from them, or about the forms in its discursive reason (*dianoia*). For the circle [in discursive reason] is one, yet geometry speaks of many circles, setting them forth individually and studying the identical features in all of them; and that circle [in discursive reason] is indivisible, yet the circle in geometry is divisible. Nevertheless we must grant the geometer that he is investigating the universal, only that this universal is obviously the universal present in the imagined circles. Thus while he sees one circle [the circle in imagination], he is studying another, the circle in discursive reason, yet he makes his demonstrations about the former. For discursive reason contains the [geometrical] *logoi* but, being unable to see them when they are wrapped up, unfolds and exposes them and presents them to the imagination (*phantasia*) sitting in the vestibule; and in imagination, or with its aid, it unfolds its knowledge of them, happy in their separation from sensible things and finding in the matter of imagination a medium apt for receiving its forms (*tôn heautês eidôn*).[39]

37) For Proclus' account of language and its relation to thought in his *Cratylus* commentary, see MacIsaac 2013, 97-102.

38) See MacIsaac 2001 for a discussion of imagination and geometry in Proclus.

39) 54.14-55.6. See also 45.6-46.3.

This general understanding of the soul's thinking has important implications for his account of geometrical first principles. On the one hand, he does refer to them in a way that sounds as if they are propositional. Euclid's *Elements* is a book, after all, and Proclus is discussing the contents of this book, which is *about* geometrical reality. But on the other hand, he always has in mind the idea that definitions, hypotheses, axioms, postulates and the demonstrations that make use of them are primarily the soul's own activities with regard to the actual realities that lie inside it.

Another way of thinking about this is that the principles of geometry are really the point, the line, the circle, the angle, equality, etc. As we shall see in the following sections, definitions, hypotheses, axioms and postulates gain their character as principles from these, insofar as they either make their boundaries clear, or use them within geometrical arguments.

### 3.1.3 Definition

Proclus sometimes uses the term *logos* for definition (e.g. 93.20), but most often he uses *horos* or *horismos* for 'a definition', and *horizô* or *aphorizô* for the act of defining, with their participles used to describe something as being bounded or determined, in the manner which we will look at below.

Proclus often refers to definitions in what we would think of as the ordinary sense, as the answer to the question 'what is it?'[40] But this sense is based on a more fundamental idea of definition as the kind of limit (*peras*) that bounds or encloses something. He makes this clear in his discussion of Euclid's definition of *horos* itself (136.1-12):

> 'Def. XIII. A boundary (*horos*) is what is the limit (*peras*) of something.'
>
> The term boundary (*horon*) should not be applied to every magnitude—for there is also a boundary (*horos*) and limit (*peras*) of a line—but to the areas (*choria*) within surfaces and solids. Now he calls a boundary (*horon*) the enclosure (*periochên*) that marks off (*aphorizousan*) each

40) See 201.15-202.8. In addition to 'what is it? (*ti esti*), geometry asks 'does the object exist as defined?' (*to ei estin auto kath' hauto*) and 'what sort of thing is it?' (*to hopoion ti estin*). The following is a relatively complete survey of Proclus' use of *horos* and its cognates in the 'ordinary' sense of definition: *horos*: 117.12; 135.16; 178.8; 189.4; 191.25; 201.22; 281.5; 418.18; *horismos*: 109.6; 110.16; 110.27; 113.12; 128.17; 134.8; 136.23; 144.9; 158.26; 206.13; 209.19; 215.17; *horizô* (finite and infinitive): 110.11; 114.15; 120.15; 120.26; 135.19; 165.11; 176.5; *horizô* (participle): 144.8; 218.5; *aphorizô*: 80.21; 95.21; 109.21; 117.1; 123.18; 125.2; 127.18; 128.10; 134.20; 135.18; 136.6; 136.14; 143.9; 143.23; 155.4; 175.5. This 'ordinary' sense comprises only about a quarter of Proclus' uses of these terms.

> area, and it is defined as a limit (*peras aphorizetai*) in this sense, not as the point is said to be the limit of a line (*peras grammês*), but as that which encloses and encompasses [an area] apart from what lies around it. The name has been at home in geometry from the beginning, [as the art] by which men were accustomed to measure lands (*ta chôria*) and to keep their boundaries (*horous*) from getting mixed up; and it is from this [activity] that they came to the idea of this science.

In saying that a *horos* is a limit that encloses an area, Proclus is making explicit what is taken for granted in Euclid's own definition, which simply says that a *horos* is a limit of something.[41] Proclus emphasises this idea of enclosing because he sees the geometrical boundary as an analogue of boundary in general, which gives definition and determination to intelligible or sensible matter. In his discussion of the definition of figure (*schêma*),[42] he says (142.15-24):

> For everything that has matter, whether intelligible or sensible, has a boundary (*horon*) coming from outside itself, and is not itself a limit (*peras*), but is limited (*peperasmenon*). It is not its own boundary (*horos*), because in it what bounds (*to horizon*) is other than what is bounded (*to horizomenon*), and [this boundary] is not in it, but it is contained by it [the boundary]. Since it [figure] is born with quantity and subsists with it, quantity is its substratum, and the figure is the quantity's definition (*logos*), shape (*morphê*), and form (*eidos*). For figure limits it [quantity], giving it a character and such-and-such a boundary (*horon*), either simple or composite.

Figure here is described as a particular case of the limitation of matter by its boundaries. The matter of figure is quantity, and this quantity is given form or definition by the lines (or planes) which are its boundary (*horos*). An important part of his conception is that without boundaries things would 'slip away into boundlessness (*eis aoristian . . . proelthon*)' (86.19-20), so that something's boundaries are what allows it to be some particular thing.

41) Proclus himself once refers to the limit of a line as a *horos* (115.10-116.3), and very occasionally uses verbal forms to say that the line is bounded by points (86.21; 101.11; 277.20). Euclid's definition of *horos* comes just before his definition of figure as that which is contained by one or more *horoi*, so he also has in mind *horos* as that which encloses an area.

42) 136.18-19: 'Def. XIV. A figure is that which is contained by any boundary or boundaries (*tinos ê tinôn horôn*).'

Proclus' idea of analogy, which we have already seen holds that geometrical objects exist in different ways at higher and lower places in the hierarchy, is in play here in a wider sense—what it means to be a *horos* is understood analogically as well. The primal boundaries come from above for things that lie below them, beginning from the One itself (115.10-19):

> If we take these propositions as likenesses, we can understand that every being simpler than what immediately follows it supplies the boundary and limit (*ton horon epagei kai to peras*) to its successor. Soul bounds (*aphorizei*) and perfects the activity of nature, nature does likewise for the motion in bodies, and prior to both of them *Nous* measures the revolutions of Soul and the One measures the life of *Nous* itself, for the One is the measure of all things. So also in geometry the solid is bounded (*horizetai*) by the surface, the surface in turn is bounded (*horizetai*) by the line, and the line by the point, for the point is the limit (*peras*) of them all.

The reception of a boundary or determination is connected with the third moment in the triad remaining, procession, and reversion. In that sense, higher things give themselves their own boundaries through their reversion upon their causes, while the boundaries of lower things are an image of this self-related activity.

> Every compound gets its boundary (*horon*) from the simple and every divisible thing from the indivisible . . . Now in imagined and perceived objects the very points that are in the line limit it, but in the region of immaterial forms the partless idea of the point has prior existence. As it proceeds from that region, this very first of all ideas expands itself, moves, and flows towards infinity and, imitating the indefinite dyad (*tên aoriston duada*),[43] is mastered by its own principle, unified by it, and constrained on all sides. Thus it is at once unlimited and limited—in its own procession unlimited, but limited by virtue of its participation in its limit-like cause. For as it goes forth, it is held by itself within the compass of that cause and is bounded (*horizetai*) by its unifying power. Hence also in [sensible] likenesses the points that constitute the extremity and the

43) I.e. which is bounded by the monad (101.8-9).

> beginning (*to peras kai tên archên*) of a line are said to bound it (*horizein autên legetai*).[44]

This idea of higher things supplying boundaries for the lower is understood to take place within the 'series', the central concept of analogy:

> For the *logos* of the point presides over this entire series (*tês seiras*) [i.e. point, line, plane, body], and unites and contains its divisions, and bounds (*horizei*) its processions, produces them all and comprehends them from all sides.[45]

The provision of boundaries extends down to the heavenly bodies, the sublunary elements, and even the semi-regular occurrence of such things as droughts and floods.[46] In these metaphysical senses—the higher as supplying boundaries for the lower, the connection of boundary and reversion, and the bounding of a series as a whole—a thing's boundary allows it to exist as the sort of thing that it is. This metaphysical sense of boundary is the analogue of the boundaries that make a portion of a plane surface into a triangle, or make a particular configuration of land about one's house into one's property.

If we look at *horos* and its cognates specifically in mathematics, Proclus mentions it a number of times as one of the methods of dialectic employed by mathematics (i.e. definition), although in each case he does not give much detail about how it works.[47] More importantly, the fundamental meaning of *horos* as boundary also runs all through Proclus' discussion of mathematics, even apart from the examples given above. So, for example, Proclus says that arithmetic provides boundaries to geometry, presumably by providing rules regarding such things as the addition or subtraction of equals.[48] He mentions boundaries of judgment (*horous . . . tês kriseôs*) that seem to mark off both

44) 101.2-102.2. Proclus uses the term 'procession' in this passage, but not 'reversion', although it is clear the limitation is through reversion. Cf. 153.22-154.24, where this is explicit: 'the circumference is like a separate center converging upon it, striving to be the centre and become one with it and to bring the reversion back to the point from which the procession began.'

45) 89.10-14. For other examples of the definition of the lower by the higher, see 13.8-26; 26.23-27.10; 187.7; 290.23-291.18.

46) 89.28-90.6; 137.8-138.10; 149.8-150.12.

47) 42.15-43.10; 57.18-58.3; 69.8-19.

48) 48.9-15. See, for example, Proclus statement that Euclid's axioms are true not only in geometry, but in mathematics in general: 195.23-196.14.

what a mathematician knows from what someone who is 'generally educated' (*haplôs pepaideumenos*) knows, and to mark off the various sorts of demonstrations a mathematician makes, whether general or specific (32.23-33.2).

His principal use within mathematics of *horos* and its cognates in the sense of boundary is to speak generally about boundaries giving determinate existence to what would otherwise be boundless or indeterminate. So, for example, mathematics concerns itself with finite (*horismenon*) quantities and magnitudes, because it is impossible to grasp infinite or unlimited (*aperia*) quantities or magnitudes in thought.[49] The infinite is only present in the imagination, according to Proclus, by a sort of negation. It is unable to encompass the infinite in thought (*aoristainousês peri to nooumenon*), and only knows it by bringing its power of proceeding to a halt, in the way that sight experiences darkness by not seeing (see 284.22-286.4).

Particularly clear is Proclus' discussion of Proposition XIII, where he sets up the right angle as the *horos* of all angles (293.15-23):[50]

> From this we can see how equality is a measure and a boundary (*horos*) of inequality as well. For even though the diminution and increase of the obtuse and acute angles is indefinite (*aoristos*) and unlimited (*apeiros*), yet this increase and diminution are said to be limited and bounded (*perainesthai kai aphorizesthai*) by the right angle. And though each of them departs in a different direction from likeness to the right angle, yet both of them by a certain unity of nature refer back to the boundary (*horon*) of the right angle.

In the obvious way, the right angle is a boundary between the acute and the obtuse angles, and two right angles are the upper boundary of the obtuse. But more importantly, it is its equality, i.e. its stability and constancy, that makes the right angle a boundary. Proclus contrasts this equality with the indeterminate increase and decrease of the other angles, which is due to the infinite divisibility of the space that the right angle bounds (see 188.13).

From his other discussions, it is clear that what he means by their indeterminacy is that an acute or an obtuse angle can be any of a possible infinite number of angles. These terms do not name some one thing that exists, because they name many things at the same time. A right angle, on the other

49) 36.3-7. This restriction is attributed to the Pythagoreans, but Proclus seems to agree with it.

50) 166.20-2: 'Prop. XIII, Theor. VI. If a straight line set up on a straight line makes angles, it will make either two right angles or angles equal to two right angles.'

hand, is always one and the same thing, and so is determinate enough to exist without any further specification. Even though one can extend to any length the lines that meet to form a right angle, the right angle *qua* angle is always the same thing.

This idea of *horos* and its cognates within mathematics as a boundary of what would otherwise be indeterminate, necessary for something to exist or to be constructed, is indicated by passages in which he speaks of 'fixing the point (*horisai to sêmeion*) at which intersection occurs' (369.23), of 'angles BAC and DCA [being] defined (*hôrismenês*) by points F and G' (370.4), of rejecting a definition of the angle that makes use of the idea of a 'first interval' because we cannot 'determine (*aphorisômen*) the first interval', as every interval is infinitely divisible (125.23), and of Euclid making 'determinate (*hôrismenên*) the kind of angle to be constructed' by specifying that it be constructed on a straight line (333.18).

Proclus also uses *horos* and its cognates in two epistemological senses that are related to the ontological sense that we have just examined, determinacy as a necessary condition of existence. Some problems have more than one solution. The fewer the solutions a given problem admits of, the more determinate or bounded it is, while the more it admits of, the more indeterminate it is:

> In general we shall see that some problems have a unique solution, others more than one, and some an indefinite number. We call 'ordered', to use Amphinomus' term, those that have only one solution, 'intermediate' those that have more than one but a finite number (*kata arithmon hôrismenon*) and 'unordered' those having an indefinite variety (*apeirakôs poikillomena*) of solutions.[51]

Further, some problems are stated in a manner in which not enough is given for someone to know exactly what is being asked:

> A problem is deficient and is called 'less than a problem' when it needs to have something added to bring it from indeterminacy into order and scientific determinateness (*ek tês aoristias eis taxin kai horon epistêmonikon*), such as 'to construct an isosceles triangle'. This is insufficiently determinate (*aoristôdes*) and requires an addition specifying what sort of

51) 220.7-12. See also 253.14; 395.13-18.

> isosceles is wanted, whether one having its base greater or less than each of the equal sides.[52]

These more epistemological senses of *horos* and its cognates are obviously related to the senses specified above. There, a boundary makes something determinate enough to exist, while here the idea is that a problem with a finite number of solutions is more bounded than the infinite and so (presumably) more intelligible, and that a construction can only be made if its boundaries are laid out clearly enough. There the idea is that determinacy is needed for existence, here determinacy is needed for clarity, but in both cases it is a boundary that gives the needed determinacy.

Finally, all of these related senses of *horos* as a boundary that gives determination are in the background of his understanding of definition in the manner in which we would most naturally think of it, as the formula that answers the question 'what is it?' (201.15-202.8). In discussing the definition of the diameter of a circle,[53] Proclus begins by saying this (156.12-19):

> The author of the *Elements* himself makes clear that he is defining (*horizetai*) not every diameter, but the diameter of the circle. The square also has a diameter, and so does the parallelogram in general, and among solid figures the sphere. But in these cases such a line is also called a 'diagonal', and in the case of spheres an 'axis' also, 'diameter' alone being used for the circle. Even for the ellipse, the cylinder, and the cone we are accustomed to say 'axis', 'diameter' being peculiar to the circle.

Presumably, had Euclid simply defined *diametros* as 'a straight line drawn through the centre and terminated in both directions by the line enclosing the figure', rather than specifying that this line is to be drawn in a circle, it would have been unclear which sort of *diametros* he was speaking about. Further, although there are many straight lines and many points in a circle, Proclus says that Euclid makes clear he means the diameter by specifying that it is a straight line that passes through the centre point and does not go past the circumference as its boundary (*horon*) (156.19-157.3).

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52) 221.23-222.5. Note that if the sides are all equal it is an equilateral triangle, so the point is that one needs to specify which of the two species of isosceles triangle is sought. See also 377.22-378.2.

53) 156.6-11: 'Def. XVII. A diameter of the circle is a straight line drawn through the centre and terminated in both directions by the circumference of the circle; and such a straight line also bisects the circle.'

Proclus seems to understand Euclid's geometrical *horos* as the analogue of *horos* in general, as a boundary that encloses and that makes what is enclosed something definite. In order to be correct, each definition—point, line, surface, right angle, circle, triangle, etc.—has to be detailed enough to make clear what is being defined. It has to mark the definiendum off unambiguously from all other things, and so be neither too wide nor too narrow. For example, in his definition of an angle, Proclus observes that Euclid makes it clear that he is speaking of the intersection of lines on a plane surface, not on a sphere or within a solid (127.17-128.2). Proclus' approval or disapproval of Euclid's definitions generally are of this character, showing how Euclid either reached or failed to reach the appropriate level of detail, and how without this level of detail they would be ambiguous, i.e. undetermined (*aoristos*).[54] We can see this, for example, in his approval of Euclid's inclusion of 'indefinite extension' in his definition of parallel lines, because 'absence of intersection does not always make lines parallel, for the circumferences of concentric circles do not intersect; the lines must also be extended indefinitely' (176.18-21).

It is this understanding of definition as the analogue of *horos* in general, the 'enclosure that marks off each area' (136.5-6), that explains Proclus' use of the term *hôrismenôn* when he says that hypothetical sciences begin from determinate first principles (*apo archôn hôrismenôn*) (75.6-8). Proclus does not simply say that they begin from *apo horôn*, from definitions, because the principles of a science like geometry include also hypotheses, axioms, and postulates. So his phrase indicates rather that these other principles are bounded in some way. What the definitions of a science do is make its boundaries clear. The special scientist has already to have some sense of what sort of reality he is studying, and so in that sense he has already to be working within assumed boundaries. But without definitions his understanding of the limits and character of his science's proper objects remains vague. A science's definitions allow the scientist to focus on his science's proper objects with sufficient clarity that he can use them as the hypotheses of scientific demonstration.[55]

We can see this in his discussion of the point, when Proclus asks whether Euclid's definition is adequate (93.6-7): 'But is the point the only thing that is without parts, or is not this a characteristic also of the instant in time and of the monad among numbers?' His answer is that the geometer, the arithmetician, the physician, and the physicist each look at a particular bounded part of reality, and so this definition is adequate within geometry. So Euclid's

54) See 93.6-94.7; 102.22-103.18; 113.6-25; 114.3-19; 116.4-14; 116.25-117.6; 127.17-128.22; 134.8-135.23; 142.8-144.5; 151.13-153.9; 156.11-157.9; 165.19-166.13; 175.5-177.25.

55) We shall show this relation between definition and hypothesis in the next section.

definition is not sufficient for a 'philosopher, whose field of inquiry is the entire field of beings,'[56] but it is sufficient for a geometer, because (93.11-94.4):

> The scientist in a special area—conducting his inquiry from certain determinate first principles (*apo tinôn hôrismenôn archôn*) to which alone he refers his results, without attending to the procession of beings in the cosmos (*tas de proodous tôn ontôn*)—has the responsibility of examining and expounding only that indivisible nature which is appropriate to his first principles. It is his responsibility to see that simplicity which is primary in the object he studies. In geometrical matter, then, the point alone is without parts, and in arithmetic the monad; and the definition of the point (*ho tou sêmeiou logos*), though it may be imperfect from another point of view, is perfect as far as the science before us is concerned. The physician (*iatros*) says that the elements of bodies are fire, water, and the like, and he carries his analysis of bodies only thus far; but the physicist (*phusikos*) proceeds to other elements simpler than these. The former defines (*horizetai*) as element what is simple to sense-perception, the other what is simple in thought; and each of them is right with regard to his own science. We must not therefore consider the definition (*horon*) of point mistaken, nor judge that it is imperfect; for with respect to geometrical matter (*geômetrikên hulên*) and the starting points of this science, it is adequately given.

This case is analogous to what we saw above, where the definition of the diameter of a circle was deemed adequate precisely because its scope was limited to the circle. Had Euclid simply referred to a line that ran through the centre of a figure, the definition of the diameter would apply ambiguously to more than one thing. Here 'that which has no parts' is ambiguous if understood generally, but when understood from within the various sciences, as applying to various kinds of being, it means unit, point, one of the four elements, or something 'simpler' than these latter.

We can also see from this passage the manner in which for Proclus definitions are not primarily verbal formulae, but have to do instead with the soul's attention to its object. The formula 'the only thing that is without parts' is the same for the monad and for the point, but the arithmetician and the geometer each have a 'responsibility to see that simplicity which is primary in the object he studies'. It is through his definitions that the basic outlines of the part of being to which a scientist attends become clear. At this point one might ask

56) 93.7-9: τῷ μὲν φιλοσόφῳ περὶ πάντων ποιουμένῳ τῶν ὄντων λόγους.

whether *horoi* are 'definitions' possessed by the soul or whether instead they are the 'boundaries' of the geometrical objects to which these definitions refer. The answer is that they are both, because in a way these are two aspects of the same thing. We should remember that for Proclus the objects of philosophical science are the soul's own content. Geometrical objects, in particular, exist in the soul's imagination. The soul is what it understands, so the definition of the circle is primarily a thought of the soul whose object is itself. The formula is merely a linguistic image of this: 'Deff. XV and XVI. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another. And the point is called the centre of the circle' (lemma at 146.18-23). So the *horos* of the circle allows the soul to focus its attention on just that portion of its own geometrical matter whose *horos* marks it off from everything that is not a circle.

So when Proclus says that a hypothetical science begins from determinate first principles, *horismenôn* really has two related senses, both of which can be derived from Proclus' understanding of a boundary as the formal element that gives identity to a certain sort of matter. (1) On the one hand the principles are bounded in the sense of being defined—a point is that which has no parts, a line is length without breadth, etc. These formulae are 'boundaries' which give each thing a definite character, marking off a certain portion of geometrical matter from all other portions, in a manner analogous to the boundaries that mark off a piece of land. Points, lines and circles, so defined, can then serve as the first principles of geometrical arguments within the special sort of hypotheses which are axioms and postulates,[57] and as the beginning points of theorems and problems as well. (2) On the other hand, the principles are bounded by being understood to apply only within a particular sort of being—in this case geometrical being. The geometer's arguments which begin from the point, the line and the circle, etc., do not allow him to surpass the beginning points of his science and become an arithmetician, nor do they allow him to descend beyond its ambit and become a physicist. They make clear the outer boundaries of the kind being that he studies.

So our conclusion is that Proclus treats Euclid's definitions as boundaries (1) that severally delineate the formal characteristics of objects within geometrical being, marking them off from each other, and (2) as applying only to geometrical matter, that mark out collectively the boundaries of this particular region of being.

57) As we shall see below, although geometrical first principles are hypothetical, not all hypotheses are made up of first principles.

### 3.1.4 Hypothesis

We looked at Proclus' conception of definition in order that we might see how it is related to his understanding of hypothesis, so this is what we shall now turn to.

In addition to his occasional substitution of the term 'hypothesis' for Euclid's 'definition',[58] Proclus uses the term 'hypothesis' and its cognates in three related ways: (1) in the first way, he uses the term to distinguish all the other sciences from dialectic, sciences which begin from 'determinate first principles' and so which begin from and return to principles which lie lower down than the intelligibles in *Nous*;[59] (2) secondly, Proclus uses verbal forms (e.g. *hupothemenos, hupothômetha, hupotheteon*) to indicate the action of positing or assuming something, usually (but not always) as the beginning of a demonstration;[60] (3) finally, he generally calls the premises of a geometrical demonstration the hypotheses, in the sense of the 'what is given' in an enunciation (*protasis*) or in a separate exposition (*ekthesis*).[61] It is clear how these three senses are related. We could think of a hypothesis as something which is (2) *posited* or assumed, serving as (3) the *beginning points of a demonstration*, a demonstration (1) whose point of departure and whose aim in the returning ascent *falls short of Nous*.

What this points to is that for Proclus the basic meaning of hypothesis is something that is *used* as the beginning point of an argument, with the strong sense of 'setting up' or 'placing' derived from *tithêmi*. So something becomes a hypothesis—the 'what is given' from which an inquiry which falls short of *Nous* begins—if it is used as such. This idea is confirmed by a passage on the principle of conversion (252.5-14; cf. 244.10-11):

> Conversion among geometers has two meanings. In the strict and primary sense it occurs when two theorems interchange their conclusions and their hypotheses with each other, that is, when the conclusion of the first becomes the hypothesis of the second and the hypothesis of the first is adduced as conclusion of the second. For example [Prop. V]: 'In an isosceles triangle the angles at the base are equal' (here the hypothesis is 'isosceles triangle', and the equality of the angles at the base is the

58) 76.5; 76.8; 76.15; 178.2; 178.7.

59) 9.26; 11.8; 11.22; 19.9; 27.1; 31.13; 31.20; 57.19; 75.7.

60) 13.7; 17.11; 75.20; 83.14; 131.10; 254.8; 255.9.

61) 240.13; 244.15-245.12; 252.7-255.9; 265.10; 349.19-350.5. Proclus says that 'what is given' is given in one or more of four ways: in position (*thesei*), ratio (*logôi*), magnitude (*megethei*), or species (*eidei*) (205.14).

conclusion); and [Prop. VI]: 'Triangles having equal angles at the base are isosceles.'

We should notice that what Proclus has given us here, strictly speaking, are the enunciations of Propositions V and VI, in which 'what is given' and 'what is sought' are exchanged (see 246.13-20). He can refer to 'what is given' as the hypothesis and 'what is sought' within the enunciation as the conclusion because although 'what is sought' is part of the enunciation, in another way it is the same as the conclusion. The difference between them is that in the enunciation 'what is sought' only needs to be understood clearly, whereas the conclusion is drawn only after the construction and/or demonstration that proves 'what is sought' to be the case.[62] So, strictly speaking, in converse propositions the hypothesis of one, the 'what is given' in the enunciation, is used in the other both as 'what is sought' in the enunciation and as the conclusion. So both through the exchange of 'what is given' in the enunciation of one with the conclusion of the other, and through the transposition of 'what is given' and 'what is sought' within the enunciations, it is clear that the hypothesis is whatever is *used* as the 'what is given' in the enunciation of the argument.[63]

If this is the meaning of hypothesis in Proclus—something used as the beginning point of a demonstration that falls short of *Nous*—we can see two related senses in which hypothesis is related to definition. These correspond to the two senses of *horismenos* which we discussed at the end of the section on definition, in relation to his statement that geometry 'is based . . . on hypothesis and proves its later propositions from determinate first principles' (75.6-8). In the first sense,[64] the soul takes up the hypotheses of geometry from within a bounded field of inquiry (31.14-32.2):

62) Construction *and* demonstration is needed for problems, but often demonstration *without* construction is sufficient for theorems.

63) It is possible that Proclus uses the term 'hypothesis' to refer to the entire enunciation, including 'what is sought', because this is also taken up by the soul and used beforehand to set the parameters of the argument. Proclus' usage is not unambiguous, but I base my interpretation on this passage on conversion, and the passage at 244.15-17 on simple theorems, in which 'hypotheses and conclusions are indivisible, having one thing given and one thing to be proved' (ὅσα κατὰ τὰς ὑποθέσεις καὶ κατὰ τὰ συμπεράσματα ἀδιαίρετά ἐστιν, ἓν ἔχοντα τὸ δεδομένον καὶ τὸ ζητούμενον ἕν). Here there is a parallel between hypothesis and 'what is given', and conclusion and 'what is sought'. But see 204.13-19, and my discussion in the notes below.

64) The second one discussed above.

> The unhypothetical science of the whole of things mounts upwards to the Good, to the cause high above all else, making the Good the goal of its ascent, but that which shows what follows from previously determined starting-points (*hôrismenas archas prostêsamenên*) moves not towards a principle, but to a conclusion. In this sense, then, he says, because mathematics uses hypotheses, it falls below the unhypothetical and perfect science. For genuine science is one, the science by which we are able to know all things, the science from which come the principles of all other sciences, some immediately and some at further remove.

As the first principle of the whole of reality, the Good is the universal beginning point of dialectic. The inquirer in a special science, on the other hand, chooses only one part of reality as his particular beginning point, and it is only beings of this particular sort that he 'uses' or takes up as hypotheses. Proclus' use of the phrase *hôrismenas archas prostêsamenên* indicates the idea of a previous choice and ties it to the cognate of *horos*. Therefore, it seems that it is this fundamental choice of what sort of being to attend to that gives a science its initial boundaries. Following on this, the scientist's totality of definitions mark out clearly for him the boundaries of this certain type of being within the analogical hierarchy—geometrical being, physical being, etc. So this sense of 'hypothetical', as indicating a fundamental choice of a certain region of being to restrict one's attention to, seems to be prior to the sense of *hôrismenos* as indicating the bounded nature of a subordinate sciences as a whole. The prior choice of a certain sort of being sets boundaries that are made sharper and clearer by definition.

In the second sense,[65] this priority of hypothesis over definition seems to be reversed. In this sense they are related because the hypotheses of arguments in the subordinate sciences make use of things that have previously been defined. As we saw above, definitions are formal elements—the boundaries that make geometrical matter into geometrical objects—and as such they allow the materials out of which the geometer frames his axioms, postulates, problems and theorems to be unambiguous.[66] Strictly speaking, a geometer does not use definitions as the premises of arguments, because 'what is given' and 'what is sought' are the geometrical objects themselves.[67] So, for example, what is

65) The first sense discussed above.

66) They are also of use with the 'demonstration' (*apodeixis*). See note below.

67) *Pace* Martijn 2010, 93: 'Proclus has two more interesting reasons for calling the definitions hypotheses. First of all, the definitions, as well as the axioms and the theorems, are hypotheses in that they function as premises in geometrical demonstrations.'

sought in Euclid's third postulate is a circle,[68] but what allows any circle to be what it is and what allows the geometer to understand what is sought is the definition of a circle.[69] Likewise, 'what is given' and 'what is sought' in the first problem are a straight line and an equilateral triangle, not their definitions.[70] He would be a poor geometer who, when asked to construct a circle, simply wrote out its definition.[71]

Although definitions are not used in hypotheses, they are what allow the geometer to delineate the geometrical object with enough clarity that he may use it in a hypothesis. Proclus says that without definitions all problems would be deficient problems, in which not enough is given for the geometer to know what is being asked (221.23-222.5). So Euclid's definitions lend his hypotheses sufficient 'scientific determination' (*horon epistêmonikon*: 221.25-222.1), by delineating unambiguously the components of the enunciation. A particularly interesting example of this idea is Proclus' observation that sometimes the enunciation (*protasis*) of a problem gives only what is sought, as when we are asked 'to construct an isosceles triangle having each of its base angles double the other angle' (204.1-2). In such a case (204.13-19):

68) 185.4-5: 'Post. III . . . and to describe a circle with any centre and distance.'

69) 146.18-23: 'Deff. XV and XVI. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another. And the point is called the centre of the circle.' But see the discussion below of the manner in which the construction of a circle is self-evident and how this relates to its definition. Further, in a postulate there is only 'what is sought' in a weak sense, because of the simplicity of the construction. See also Harari 2010.

70) 200.19-21: 'Prop. I. On a given finite straight line to construct an equilateral triangle.'

71) More properly, simply thinking of the definition is not the same as constructing a circumference in one's geometrical imagination. What is the relation between these two and the verbal formula, all of which can be called the *horos* of the circle? There are really three things here. (1) First is the *logos* in *dianoia* (and prior to that the partless *logos* in the soul's *nous*, etc.); (2) second is this same *logos* projected into the *geometrikên hulên* of imagination where it becomes an actual circumference equidistant from its centre, making this matter into a certain sort of figure; (3) third comes the formula written down in Euclid's book. The written formula serves to remind (in the sense of *anamnêsis*) anyone who reads it of the geometrical circle (2) and of the dianoetic *logos* (1) that precedes both. Whenever Proclus refers to Euclid's definitions in what I have been calling the 'ordinary' sense, he means this written formula (3) which refers back to the dianoetic *logos* (1). The premises of geometrical arguments are not definitions but they are definite, because they are geometrical objects which have boundaries (2), and which are intelligible through their definitions (1). See 54.27-55.4. See MacIsaac 2013, 97-102.

> we understand the proposal on the basis of preexisting knowledge (*ek proginôskomenôn*), for as it happens we know (*eidotes tunchanomen*) the meaning of 'isosceles' (*ti to isoskeles*), of 'equality', and of 'double'; and such preexisting knowledge, Aristotle says, is the characteristic feature of all discursive learning. Nevertheless nothing lies before us (*hupokeitai . . . ouden hêmin*), as in other problems.

It is because we happen to understand the definition of the terms in question that we know what is being sought. But simply knowing the meaning of these terms does not mean that we have 'something given', like a finite straight line or some other such thing. Further, 'what is sought' is not merely the meaning of 'isosceles triangle' of this sort. Rather, the actual triangle is taken up into the enunciation as what will be sought as a conclusion, through the appropriate construction. The 'what is sought' is something that the soul proposes to draw in itself, rather than something whose meaning it simply proposes to rest in.[72] The situation is the same for theorems. As we saw above, it is one thing to understand the meaning of something like the equality of angles at the base of an isosceles triangle, and another to hypothesise this as the 'what is sought', whose truth comes to light in the conclusion, through the demonstration.

This is also how we have to interpret even Proclus' statement that the most perfect form of demonstration (*apodeixis*) establishes 'what is sought by means of definitions as middle terms' (206.12-16). So, for example, one constructs the equilateral triangle by describing circles at either end of a finite line. This means, Proclus says, is 'suggested to us by the definition of the circle, which says that all the lines drawn from its centre are equal' (209.18-20). But one does not construct or demonstrate the *definition* of the circle. Rather, one is able to construct lines equal to the given line and to demonstrate that the resulting figure is an equilateral triangle on the basis of one's understanding of what a circle is, which we gain through its definition.[73]

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72) Morrow translates *hupokeitai . . . ouden* as 'no specific hypothesis', most likely because he also thinks that the hypothesis is the 'what is given' in the enunciation. However, one could use this passage as evidence for the position that a hypothesis includes 'what is sought' as well as 'what is given', if one thinks that it does not make sense to say a problem does not have a hypothesis. See my discussion in the notes, above. I think Proclus' theory is clear, even if his use of this term is not.

73) Does this mean that, although definitions are not themselves used in hypotheses, they are used in demonstrations? Proclus' theory of demonstrative arguments holds that the conclusions are established by a causal relation between the being of the premises and the conclusion (see Harari 2010). So if what is given is an isosceles triangle, and what is sought is the equality of angles at its base, insofar as the *apodeixis* proper is the rational articula-

On the basis of these complementary relations of definitions and hypotheses, we can see why Proclus occasionally substitutes 'hypothesis' for 'definition' in his taxonomy of geometrical principles. Unsurprisingly, Proclus distinguishes things with a finer-toothed comb than does Euclid. So he refers to the actual section of Euclid's text as containing definitions because it is there that Euclid defines geometrical being in both of the senses which we have been discussing, as marking geometrical being off from all other sorts of being, and as articulating in an unambiguous manner the formal element of various geometrical objects. But Proclus distinguishes this formal aspect of geometrical objects—what makes them exist and makes them intelligible—from their actual use as the beginning points of geometrical arguments. That an equilateral triangle has three equal sides is a definition, but the *construction* of one uses a hypothesis, i.e. it takes up an actual finite line as 'what is given' in the enunciation.

At this point it is clear that Proclus does not confuse definitions and hypotheses and that he does not think they are the same, as is asserted by some interpreters. This impression comes from thinking that he agrees with the Aristotelian taxonomy at 76.6-77.2. But obviously there is much more going on in Proclus' account of definition and hypothesis than simple agreement or disagreement with Aristotle. Proclus is comfortable calling the same geometrical first principles both definitions and hypotheses precisely because he distinguishes them clearly from each other, and relates them to each other, as two activities of the soul concerning the same things, the geometrical objects themselves. He is well aware that the section of Euclid's text in question is where he explains clearly what a point, circle, or triangle are, and so he generally refers to it as the section of definitions. But he can substitute 'hypothesis' for 'definition' in his taxonomy of principles, because points, circles and triangles are used as the hypotheses of geometrical arguments. Moreover, Proclus can move back and forth between the two terms because each brings out a different aspect of the same general character of geometry as a subordinate science. Geometry's principles are hypothetical because its point of departure

tion of the causal relation between the two, it probably makes use of the knowledge that 'an isosceles triangle is such and such a figure', and so likely does make use of definitions. The reason for this is probably that in its hypothesis the soul proposes to move from a geometrical object that is given to something that is sought (an object or a property). The means by which it moves from the one to the other is sometimes by drawing lines, but sometimes simply by recognising the properties of what it has drawn. In this latter case, the articulation of an object's definition allows the soul to see that it has some property that will lead to what is sought.

and aim in the returning ascent falls short of dialectic's ascent to *Nous*; its principles are determinate because the science remains within the boundaries that they articulate clearly.

### 3.2 *The Self-evident*

It should be clear now what Proclus means when he says that geometry is a hypothetical science. But how are its first principles self-evident? If the basic meaning of hypothesis is 'used as the premise of an argument', then the primary hypotheses of geometry are its axioms and postulates. These are self-evident, according to Proclus, and its theorems and problems are derived from them.

Hypotheses would seem to fall into three sorts: things that are unknown; things that are known as the result of previous arguments; things that are self-evident. As we saw above, Proclus says that for Aristotle a hypothesis is something for which we do not have a self-evident notion. The passage has an overtone of the hearer not only lacking a self-evident notion, but accepting something as true on the authority of the person who proposes it without actually knowing that it is the case,[74] in line with our common understanding of the 'hypothetical' as something that is merely assumed. Although one could use something that is unknown in this way as a premise, for Proclus this is not what would make the argument hypothetical. And of course, such an argument would not constitute a demonstration.

An argument that takes as its premises things that are known as the conclusions of previous arguments is also hypothetical, and can be a demonstration, but the hypotheses from which it begins are not geometrical principles. In order to be a geometrical principle, the hypothesis must be self-evident and indemonstrable, which is the case only for axioms and postulates, and in a certain way for definitions, as we will see below.

At this point we are in fact able to give a preliminary answer to our initial dilemma. Geometrical principles can be both hypothetical and self-evident because 'hypothetical' does not mean 'merely assumed', i.e. unknown, but refers instead to what is used as the beginning point of an argument that falls short of *Nous*. But exactly how such a beginning point can be self-evident, and how it is received by geometry from dialectic remains to be investigated.

74) 76.12-14: ὅταν δὲ μὴ ἔχῃ μὲν ἔννοιαν ὁ ἀκούων τοῦ λεγομένου τὴν αὐτόπιστον, τίθεται δὲ ὅμως καὶ συγχωρεῖ τῷ λαμβάνοντι. Proclus has in mind *An. Post.* 1.10, 76b28-30: ὅσα μὲν οὖν δεικτὰ ὄντα λαμβάνει αὐτὸς μὴ δείξας, ταῦτ', ἐὰν μὲν δοκοῦντα λαμβάνῃ τῷ μανθάνοντι, ὑποτίθεται, καὶ ἔστιν οὐχ ἁπλῶς ὑπόθεσις ἀλλὰ πρὸς ἐκεῖνον μόνον.

### 3.2.1 Axioms and Postulates

Proclus asserts that the first principles of mathematics are self-evident (*autopiston*) and indemonstrable (*anapodeiktos*) in the Prologues.[75] But he discusses these concepts extensively in only two places, in his introduction to Euclid's postulates and axioms, and in his discussion of the axioms themselves (178.9-184.10; 193.8-194.14). We shall begin in this section by looking at what he thinks postulates and axioms are. In the next section we shall look at what he means by saying that the self-evident is simple and easy to grasp. In the third section we shall look at the postulates and axioms themselves to see how this actually plays out. In the fourth we shall examine briefly to what extent he thinks Euclid's definitions are also self-evident. And finally, in Section 3.2.5 we shall draw our conclusions about self- evidence and hypothesis.

Proclus describes axioms and postulates as follows (178.9-179.12):

> It is a common character of axioms and postulates alike that they do not require demonstration (*apodeixeôs*) or geometrical evidence but are taken as known (*gnôrima*) and used as starting-points (*archas*) for what follows. They differ from one another in the way in which theorems have been distinguished from problems. Just as in a theorem we put forward something to be seen and known as a consequence of our hypotheses but in a problem are required to procure or construct something, so in the same way axioms take for granted things that are immediately evident to our knowledge (*autothen eis gnôsin esti kataphanê*) and easily grasped by our untaught understandings, whereas in a postulate we ask leave to assume something that can easily be brought about or devised (*euporista kai eumêchana*), not requiring any labor of thought for its acceptance nor any complex construction (*poikilias . . . kataskeuês*). Hence clear knowledge without demonstration (*anapodeiktos*) and assumption without construction (*akataskeuos*)[76] distinguish axioms and postulates, just as knowing from demonstration and accepting conclusions by the aid of constructions differentiate theorems from problems.

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75) Self-evident: 75.15; indemonstrable: 32.6. See also 255.17 (where he identifies the self-evident with 'common notions', *koinai ennoiai*); 322.9. The occurrence of self-evident at 76.13 is within his explanation of Aristotle's taxonomy of geometrical first principles.

76) Proclus uses the term *akataskeuos* here, but it is clear from the previous sentence and from other contexts that a postulate does require a construction, just not a complex one. See 181.4-15.

A little later, after stating that this distinction is one which Geminus (correctly) held,[77] Proclus goes on to inform of us of two different manners of distinguishing axiom and postulate, both of which he thinks are mistaken. First, some people restrict postulates to geometry, making axioms common to all sciences that deal with quantity and magnitude, but Proclus thinks that some axioms and postulates belong to arithmetic, some to geometry, and some are common to both. He does, however, insist that the manner in which axioms belong to one or the other of the two sciences is as an application of a more general principle to numbers or to magnitudes, respectively.[78] Secondly, he says that Aristotle thinks axioms are indemonstrable and that postulates are demonstrable, although they can be taken as beginning points (182.14-20). Although he does not immediately criticise Aristotle, he approves of Geminus' remark that some people have 'thought up demonstrations for indemonstrables' (183.15-16). This is directed at Apollonius, who is included in the Aristotelian 'party' just mentioned, even though he attempted to demonstrate axioms, not postulates. But Proclus' rejection of this second, Aristotelian position that postulates require demonstration is clear from his contention that the fifth postulate should be struck from the postulates altogether, because it stands in need of a proof, and so 'lacks the special character of a postulate' (193.1-3).[79]

As geometrical first principles, postulates and axioms are alike in being self-evident and indemonstrable. But they differ from each other in that axioms are simple to grasp while postulates are simple to construct. One can understand them by comparing them with what follows from them. In a theorem one moves from the enunciation to the conclusion by means of a demonstration, but an axiom is clear merely from its enunciation.[80] A problem, likewise, seeks to move between what is given and what is sought by means of its construction.

77) 182.3-6: 'the one is assumed because it is easy to construct, the other accepted because it is easy to know. This is the ground on which Geminus distinguishes postulate from axiom.'

78) 182.6-14; 184.11-29; 195.23-196.14. Therefore he excludes from the axioms the principle that two lines do not enclose an area, as applying only within geometry and as not being a geometrical application of something that is also true in arithmetic. See 196.21-197.5.

79) Proclus' relation to Geminus is complicated here. On the one hand, he has lain out three ways of distinguishing postulates and axioms, attributed the first to Geminus, and says more than once that he approves of Geminus' opinions (182.3-6; 183.14-15). On the other hand, he says clearly that Geminus' manner of distinguishing axiom from postulate bars not only Euclid's fifth, but also his fourth from the list of postulates (182.21-25; 183.22-4). I shall discuss this in detail below, and argue that while Proclus does accept Geminus' classification, his inclusion of a construction in his treatment of the fourth postulate is an attempt to make it 'count' as one.

80) E.g. 'Axiom I. Things which are equal to the same thing are equal to each other' (193.10-11).

The construction in a postulate, on the other hand, does not involve the kind of motion from what is given to what is sought characteristic of problems. We shall discuss this at length below.

### 3.2.2 The Epistemological Background of the Self-evident

Proclus' clearest explanation in this text of what he means by the self-evident and indemonstrable comes immediately after the passage quoted above (179.12-22):

> Principles must always be superior to their consequences in being simple, indemonstrable, and self-evident.[81] In general, says Speusippus, of the things which our discursive reason (*dianoia*) hunts for, some it projects (*proballei*) without making an elaborate excursus (*oudemian poikilên poiêsamenê diexodon*), preparing them for later inquiry, because it has a clearer apprehension of them than sight has of visible objects. But other things, because it is unable to grasp them immediately, discursive reason advances upon step by step (*kata metabasin*), and attempts to hunt for them through their consequents.

He gives an example, saying that our discursive reason easily grasps the construction of a straight line from one point to another, or the construction of a circle by the motion of one point bounding a line while the other remains stationary. A spiral, or an equilateral triangle, on the other hand, require more elaborate constructions.[82]

This is as explicit as Proclus gets in this commentary about what he means, philosophically, by the self-evident and the indemonstrable. We will examine below the actual axioms and postulates in order to fill in this picture. However, we first need to follow the track of his phrase 'of the things which our discursive reason hunts for, some it projects without making an elaborate excursus.'[83] His use of the technical terms *dianoia* and *proballei* here invoke his general theory of discursive reason, which the Euclid commentary in general is a rich source of information about, and which forms the epistemological context for

81) 179.12-14: δεῖ γὰρ δὴ πανταχοῦ τὰς ἀρχὰς τῶν μετὰ τὰς ἀρχὰς διαφέρειν τῇ ἁπλότητι, τῷ ἀναποδείκτῳ, τῷ αὐτοπίστῳ.

82) 179.22-180.8. Note that the two examples of what is more easily grasped are two of the first three postulates. See 185.1 *ff.*

83) 179.15-17: ὧν ἡ διάνοια τὴν θήραν ποιεῖται τὰ μὲν οὐδεμίαν ποικίλην ποιησαμένη διέξοδον προβάλλει.

his idea of the self-evident. So we will look at this very briefly before returning to the passage just quoted.

In the first prologue, Proclus says (45.6-46.3):

> Everything we call learning is remembering ... Although awakened by what appears to us, learning is projected (*proballomenê*) from within, from discursive reason itself reverting upon itself ... What remembers is the discursive part of the soul (*to dianoêtikon tês psuchês*). This part of the soul has its essence in these mathematical ideas, and it has a prior knowledge of them, even when it is not using them; it possesses them all in an essential, though latent, fashion (*ousiôdôs kai kruphiôs*) and brings each of them to light when it is set free of the hindrances that arise from sensation.

The technical term 'projection' (*proballein*) here describes the drawing forth of geometrical ideas. Proclus' general theory of discursive reason (*dianoia*) is that the soul's participation in *Nous*, the divine mind, is through its own essence (*ousia*), which is a 'fullness of logoi' (*plêrôma tôn logôn*). We possess these *logoi* initially in an unconscious, hidden or latent fashion. In knowing, we project these *logoi* which are our essence, drawing out their concentrated content and producing the various sciences. So for Proclus all discursive knowledge is self-knowledge,[84] and all mathematical principles, including axioms and postulates, were always possessed by the soul. They are *logoi* which are part of Soul's essence, and the activity of geometry is discursive reason's projection of these *logoi* into the intelligible matter of imagination.[85]

Although these *logoi* are the soul's own content, it possesses them in different manners, depending on whether or not it has an active relation to them. In the passage above, Proclus states that the soul possesses mathematical ideas 'in an essential, though latent, fashion'. This language of the secret or latent (*kruphiôs*) possession of *logoi*, in the context of a discussion of knowledge as recollection, is very similar to fairly well known passages in his commentary on *Alcibiades I*, where Proclus uses the metaphors of our breathing and of our heartbeat to explain how our essence can be always cognitively active without our noticing it:

84) See MacIsaac 2010.

85) For the relation between *dianoia* and *phantasia* see MacIsaac 2001 and Nikulin 2010. Note that Nikulin suggests incorrectly (at 147) that I say that *phantasia* is part of *dianoia*.

> Because they possess the *logoi* of things, as a sort of heartbeat, they have notions (*ennoias*) of those things, but because they are conquered by the draught of oblivion they are unable make their own notions articulate and send them forth towards knowledge. Thus they carry them around as if suffocating, and scarcely drawing breath.[86]

A little later he says: 'we possess the *logoi* in our essence and knowledge of these *logoi* as a sort of breathing, but we do not possess them as projected and actualised.'[87]

What does it mean to possess these *logoi* as a cognitive heartbeat or breathing? I have argued elsewhere that this unconscious possession of *logoi* informs even our experience of the sensible world, according to Proclus.[88] Without being a mathematician, I can employ my unconscious understanding of 'one', 'many', 'part', 'whole', 'active', or 'passive', etc. to know things such as whether or not I have more or fewer chocolates than my friend. With regard to geometrical ideas, I can distinguish a point from a line, circle or triangle, even if I have never studied geometry. Proclus occasionally uses the term 'notion' or 'common notion' (*koinê ennoia*) to refer to the untutored possession of geometrical knowledge: the idea of the line, surface and volume coming from our experience of roads, fields and wells; the idea of the shortest distance between two points being in accord with what seems to be a common saying: 'those who go in a straight line travel only the distance they need to cover, as men say, whereas those who do not go in a straight line travel farther than is necessary.'[89]

The mere possession of these notions, however, is not the elaborated science of geometry. Geometry as a science requires the conscious projection of its ideas into imagination: 'For the understanding contains the ideas but, being

86) *In Alc. I* 189.6-11 Westerink. See also *in Alc. I* 280.24-281.8 Westerink.

87) *In Alc. I* 192.2-5 Westerink. Note that the context of this doctrine of 'breathing thought' indicates that the unconscious possession of *logoi*, while providing a basic sort of intelligibility for the soul, can also lead it to think it knows when it in fact does not. This is the source of error and what Proclus calls double ignorance. See MacIsaac 2011. For an earlier treatment of 'breathing thought', see Steel 1997.

88) See MacIsaac *forthcoming*.

89) 100.6-19; 109.17-20. See also 114.2; 266.9; 266.16; and especially 119.4 for the idea that we possess the ideas of the straight line, circle, plane and sphere without being taught (αὐτόθεν τὰς ἐννοίας ἔχομεν). Proclus uses the phrase *koinas epinoias* as a synonym for *koinas ennoias* at 188.10, although his use of *epionoia* in the text does not usually have this sense. He also refers to Euclid's use of the term 'common notion' to refer more or less to geometrical axioms, and sometimes uses the term that way himself: 76.15; 182.12; 194.9; 240.12; 254.25; 255.2; 264.11.

unable to see them when they are wrapped up, unfolds and exposes them and presents them to the imagination sitting in the vestibule' (54.27-55.4). Nor, according to Proclus, is this unconscious possession of *logoi* even how we possess only the principles of geometry. If I know that it is a shorter route to go directly to my friend's house than to take a detour to the shops, I am not thereby an incipient geometer. Instead, in speaking about geometrical principles Proclus seems to be pointing to a sort of minimal projection, based on, but lying beyond, the sort of immediate and unreflective grasp of mathematical ideas that everyone has, but preceding the fullness of geometrical science. It is in this minimal projection that his conception of the self-evident lies.

We shall now look again at the passage where Proclus talks about the self-evidence of axioms and postulates (179.12-22):

> Principles must always be superior to their consequences in being simple, indemonstrable, and self-evident. In general, says Speusippus, of the things which our discursive reason hunts for, some it projects (*proballei*) without making an elaborate excursus (*oudemian poikilên poiêsamenê diexodon*), preparing them for later inquiry, because it has a clearer apprehension of them than sight has of visible objects. But other things, because it is unable to grasp them immediately, discursive reason advances upon step by step (*kata metabasin*), and attempts to hunt for them through their consequents.

As I mentioned above, he then gives the examples of a straight line or circle as things that are easily constructed, and of a spiral or equilateral triangle as requiring a more elaborate construction. Notice that he does not say that the circle or straight line is possessed by the geometer *without* a projection, just that the projection required in these cases is not an 'elaborate excursus' (*poikilên . . . diexodon*). Proclus' theory of projection has the sense of expansion, unrolling and drawing out hidden content, but it also has the sense of conscious attention to the self. In a passage I quoted above, Proclus uses his technical term 'reversion' (*epistrophê*) to refer to the turn inwards by which the soul investigates its own content, as a precursor to drawing its ideas out of itself.[90] Proclus' description of awakening to one's own *logoi* implies that there is a moment or period of time in which a soul first becomes conscious of its own content, prior to its self-elaboration through discursive thinking. If this is

90) 45.10-12: 'Although awakened by what appears to us, learning is projected from within, from discursive reason itself reverting upon itself' (ἀπ' αὐτῆς τῆς διανοίας εἰς ἑαυτὴν ἐπιστρεφομένης).

the case, then Proclus' description of an immediate recognition of the truth of geometrical principles, a sort of minimal projection without an elaborate excursus, seems to be a description of this beginning-point.

In short, Proclus' theory of the self-evident possession of geometrical principles is that these are the *logoi* that we always already possess, as our cognitive heartbeat or breathing, *looked at consciously* by the geometer. Because they are the first things that we become conscious of, and are intelligible as they are before being subjected to discursive argument, Proclus describes them as simple and self-evident. Proclus seems to think that the soul's knowledge of their truth by a simple awareness of them is analogous to the functioning of our sense organs (181.4-15):

> Both of these, postulate and axiom, must be simple and easy to grasp. But a postulate prescribes that we construct or provide some simple or easily grasped object for the exhibition of a character, while an axiom asserts some inherent attribute that is known at once to one's auditors—such as that fire is hot, or some other quite evident truth about which we say that they who are in doubt lack sense organs or must be prodded to use them. So a postulate has the same general character as an axiom but differs from it in the manner described. For each of them is an undemonstrated starting-point (*archê anapodeiktos*), one in one way, the other in another, as we have explained.

The self-evident is grasped by the mind once it attends to itself, according to this passage, as vision is aware of red by seeing or touch is aware of heat by feeling.

### 3.2.3 The Self-evidence of Particular Principles

If it is relatively clear from this that Proclus' theory of self-evidence is part of his wider theory of discursive projection, it is more difficult to see just how he distinguishes geometrical truths that are simple enough to be self-evident from those that are not. We can, however, come to some understanding of what he means if we look at his treatment of Euclid's Postulates I to V.

Proclus accepts I to III as fulfilling two necessary conditions for being a postulate: (1) self-evidence and (2) the need for construction. His discussion of these three postulates shows what is meant by one of the necessary conditions for self-evidence within postulates, namely ease of construction.[91] We will see below that Postulates IV and V add another necessary condition for

91) For this condition, see 182.3-4.

self-evidence, that a postulate not ask for anything beyond what is immediately constructed. Proclus gives us an idea of what ties these two conditions for self-evidence together when he uses the phrase *kata metabasin*, indicating a transition from one sort of thing to another, to characterise what cannot be easily constructed.[92] *Metabasis* is important because it is Proclus' normal word to describe the discursivity of *dianoia,*[93] so something will be easy to construct if it does not involve the kind of movement characteristic of the soul's discursivity. Although he is speaking in this passage about ease of construction, I shall argue below that lack of *metabasis* also characterises the requirement for self-evidence drawn from Postulates IV and V.

Postulates I to III require the construction only of the straight line and the circle.[94] With regard to their ease of construction, if we look at his classification of lines into simple and mixed we can see that the lack of *metabasis* in question has to do with the character, rather than the number of motions involved in a construction. Among lines, the circular and the straight line are simple and correspond to the principles of Limit and Unlimited respectively, while all spirals fall under the Mixed, whether lying in a plane or around a solid, as are the lines produced by conic sections.[95] Simple lines, whether Limited or Unlimited, are easily constructed because they are produced by one or more similar motions. Mixed lines, on the other hand, are constructed by two or more dissimilar motions and so are not so easy to construct.[96] So the circle is produced by the motion of one point around another, and the straight line by the motion from one stationary point to another.[97] A plane spiral, on the other hand, is produced by two dissimilar motions: 'a straight line one end of which is fixed and the other revolving about it, while a point is moving along it from the stationary end' (180.9-12). Here we have the circular rotation of the point at one end of the line and the rectilinear motion of the point along the line.

92) 179.19-21: 'But other things, because it is unable to grasp them immediately, discursive reason advances upon step by step' (τὰ δὲ ἐκ τοῦ εὐθέως αἵρειν ἀδυνατοῦσα κατὰ μετάβασιν ἐπ' ἐκεῖνα διαβαίνουσα).

93) See, for example *in Tim*. i. 246.5-9 Diehl.

94) 185.1-5: 'Post. I-III. Let it be postulated to draw a straight line from any point to any point, to produce a finite straight line continuously in a straight line, and to describe a circle with any center and distance.' Proclus approves these postulates in the next lines (185.6-8).

95) Proclus' classification has to do with which of the three primal metaphysical principles govern these lines, hence the capitalisation of each.

96) 105.26-106.3; 118.24-120.6; 179.22-180.8; 185.6-25.

97) For the priority of the circle over the straight line see 106.20-107.10, where he states that the circle is produced by two points, not by a line rotating around a point.

Although what is produced is a non-homoeomeric line, i.e. a line whose parts are not similar to each other, the complexity of the spiral does not lie primarily in this, but in its production by dissimilar motions. The cylindrical helix, i.e. a spiral line drawn on the surface of a cylinder, is homoeomeric, yet it is complex because it is produced by two dissimilar motions: 'a point moving uniformly along a straight line that is moving around the surface of a cylinder' (105.2-4). The other example he gives of something too complex to be easily done, the construction of an equilateral triangle, also requires dissimilar motions: the description both of circles and of straight lines (180.15-19). On the other hand, the product of two similar motions is a simple line, as when we imagine 'a square undergoing two motions of equal velocity, one lengthwise and the other sidewise; a diagonal motion in a straight line will result. But this does not make the line a mixed one' (106.3-8). So among lines a construction that moves from one type of motion to another is too complex to satisfy the necessary condition of self-evidence of being 'easily done'; a construction that has only one, or more than one motion of the same type, does satisfy this condition. On these grounds the first three postulates satisfy this necessary condition for self-evidence.

From the first three postulates, we saw what is meant by one of the necessary conditions of self-evidence within postulates. Ease of construction is understood as not involving a *metabasis* from one kind of motion to another. From Proclus' treatment of the fourth and fifth postulates, the former of which he seems to accept and the latter of which he rejects, we shall be able to understand a second necessary condition for self-evidence: that the postulate not ask for the soul to move beyond what is immediately constructed.

It is not immediately clear from his discussion whether or not Proclus accepts the fourth postulate: 'Post. IV. And that all right angles are equal to one another' (188.1-2). He begins by discussing whether or not this would be accepted as a postulate by Geminus or Aristotle. What he says adds up to the following. If we say that the equality of right angles is self-evident, it will not be accepted by either Geminus or Aristotle, but for different reasons—Geminus will reject it because it fails condition (2) for being a postulate, positing an intrinsic property of right angles rather than asking for any construction; and Aristotle will reject it because he does not agree with condition (1), thinking instead that postulates are not self-evident and so need to be demonstrated. On the other hand, if we say that it can be demonstrated (and satisfy Aristotle), Geminus will now reject it also because it fails condition (1), which he accepts (188.3-11).

Proclus then begins his own discussion by saying that 'the equality of right angles is manifest from our common notions (*koinas … epinoias*)' (188.11-12 ).

This statement, combined with his tendency to agree with Geminus, makes the reader expect a rejection of this postulate on the grounds that it does not require a construction. He follows this, however, with a proof seemingly given under duress: 'But if we must provide a graphic proof (*apodeixin ... grammikên*) of this postulate ...'[98] Following this is a discussion of the converse of the postulate, and a conclusion which both calls it a postulate and seems to approve of it (189.11-191.4; 191.5-15).

I think what Proclus is doing here is massaging the postulate a bit to show how it satisfies both the first and second conditions. On the one hand, he asserts that it satisfies the first condition because it is evident from our common notions, and his statements immediately following imply that Geminus is correct in holding that it asserts an intrinsic quality of right angles. However, in following this with a 'graphic proof', he shows that, while it may be self-evident without a diagram, it becomes even clearer once you try to construct two right angles that are unequal. So while he implies that the construction is not needed, he also implies that the equality of right angles invites it. This being the case, the equality of right angles just squeaks in as a postulate. In other words, since Proclus states clearly that postulates must ask for a construction, and since the fourth postulate clearly does no such thing, he supplies one anyway in order to make the fourth postulate 'count'.

The relevance of this for our discussion of self-evidence lies in comparing the sort of construction he gives for the fourth postulate with his comments on the fifth postulate, which he rejects.[99] Proclus states that the fifth postulate is not self-evident. The main reason why people think it is self-evident is that they see that the two straight lines in question are convergent, and assume as self-evident the principle that convergent lines must meet (192.1-29). This is in fact not only not self-evident, it is false, as certain sorts of lines converge indefinitely but do not meet, such as a straight line and a hyperbola (Cf. 176.18-177.25). Because the intersection of the straight lines goes beyond what is immediately given it cannot be something that is immediately grasped, and so Proclus states that this is a theorem, as it must be demonstrated that convergent straight lines meet.

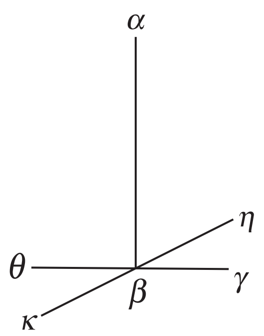
The construction that he gives for the fourth postulate, on the other hand, does not consist in much more than drawing one pair of adjacent right angles

98) 188.20-22: εἰ δὲ δεῖ καὶ ἀπόδειξιν αὐτοῦ παραθέσθαι γραμμικήν. The proof runs from 188.20-189.10.

99) 191.16-20: 'Post. V. And that, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which are the angles less than the two right angles.'

on top of another pair of adjacent right angles: Draw two pairs of adjacent right angles, sharing the vertical line αβ. If one of the right angles (αβη) is smaller than the corresponding right angle on the other pair (αβγ), then its adjacent angle (αβϰ) is larger than its corresponding angle (αβθ). By the definition of a right angle, adjacent right angles are equal to each other (131.3-8). Therefore angle αβθ is equal to angle αβγ, and angle αβη is equal to αβϰ. But then αβη is both smaller and larger than αβγ. This is impossible, therefore right angle αβη cannot be smaller than right angle αβγ. The converse obviously holds, as αβγ was assumed to be larger than αβη. And so right angles must be equal to each other.

When written out like this, it seems like an elaborate proof, and does not seem to satisfy the criterion of self-evidence. However, if you actually construct it, you can see why Proclus grants to it the requisite simplicity:[100]




In essence, all you need to see in order to grasp the truth of the postulate is that what lies on the right side of the vertical line αβ should be a mirror image of what lies on the left side. But this is clearly not the case if we suppose the original two angles are not equal. Therefore the postulate must be true.[101]

If we compare Postulates IV and V we can see that V is no more difficult to construct than IV—it simply asks for three straight lines in a particular arrangement. But it seeks to indicate something that is not given in the construction, namely the intersection of the lines. Postulate IV, on the other hand,

100) The diagram is reproduced from Friedlein 1873, 188.

101) In defence of Proclus' provision of this 'proof' of the fourth postulate, it must be said that the construction really does allow one to grasp why the postulate is true, even if Geminus is correct in stating that this postulate neither asks for nor requires a construction.

does not demand anything beyond what is constructed in order to show the property in question.

We said above (Section 3.2.1 *ad fin.*) that the construction in a postulate does not involve the kind of movement between what is given and what is sought that we find in a problem. Taken together, Proclus' treatment of the five postulates seems to indicate two necessary conditions for something to be self-evident, by indicating two types of *metabasis* that would make something a problem rather than a postulate. First, a postulate cannot involve an internal movement from one kind of object to another kind of object, as for example from a straight to a curved line; and secondly, a postulate cannot ask for any movement beyond what is given in the initial construction. On both of these conditions Euclid's first problem, to take one example, could not be a postulate.[102] First, although what is given and what is sought both involve only straight lines, it is necessary to move from one to the other by means of the construction of circles. Secondly, the equilateral triangle that is sought is not given in the initial construction, which is only a straight line. On the other hand, all four postulates that Proclus accepts do satisfy these two conditions. None of their constructions require dissimilar motions, and none requires the soul to move beyond the initial construction. We have garnered these conditions from his discussion of geometrical postulates, but it is not hard to see how analogues of these conditions could be given for geometrical axioms, or for arithmetical principles.

If we remember that Proclus' theory of discursive reason characterises it essentially as *metabatikos*, i.e. as a thinking that moves from one object of thought to another, we can see the significance of Proclus' requirement that in a postulate there be no movement from one kind of object to another kind of object within a construction, or from an initially constructed object to an object yet to be constructed. It indicates, as we would expect, that he thinks of the self-evident as what discursive reason grasps before it begins its characteristic movement from one sort of object of thought to another, and serves as that from which this movement begins. I think that is what he means by the 'lack of an elaborate excursus' and what I called above a 'minimal projection'. Such a projection does not produce more than one sort of object, and it does not yet relate this object to anything further. Instead, it simply rests in the awareness of its initial production. So if the two necessary conditions which we have examined rule out both possible types of *metabasis*, either internal to

102) 200.19-21: 'Prop. I. Prob. I. On a given finite straight line to construct an equilateral triangle.'

the object or beyond the object, then satisfying them both can be taken as a sufficient condition for self-evidence.

I think this is why Proclus groups together self-evidence with simplicity and indemonstrability in his description of geometrical first principles (179.12-14). Their simplicity consists in not presenting different sorts of things to our discursive reason, and so in not requiring a movement of thought within the object. Their indemonstrability consists in not requiring our discursive reason to move beyond them in order to grasp their truth. These two taken together make them self-evident.

### 3.2.4 The Self-evident and Definition

We should examine briefly whether or not Proclus thinks that geometrical definitions are self-evident, and in the next section draw our conclusions about the self-evident and the hypothetical character of geometry. On the one hand, Proclus' general statements that mathematical first principles are self-evident should apply to definitions as well, not just to axioms and postulates. Further, as we saw in our discussion of his theory of the unconscious possession of its own *logoi*, Proclus thinks that we possess the ideas of the straight line, circle, plane, and sphere without being taught.[103] But on the other hand, a general idea of a line or circle is not a definition, and Proclus never actually describes definitions as either self-evident or indemonstrable, in the way that he does with the other two sorts of principle. Despite this, there must be a certain amount of self-evidence at work in geometrical definitions. As we said above, although definitions do not serve as the hypotheses of geometrical arguments, the objects that are defined are used within axioms, postulates, theorems and problems, so it would make sense for the definitions at least of the objects used within axioms and postulates to be self-evident. We have argued above that a definition is primarily an activity of thinking by which the geometer perceives clearly the boundaries of his object. So obviously the constructions of a straight line or a circle in the postulates cannot be self-evident if it is not equally evident to the geometer what a straight line or a circle are.

It is probable that all definitions would satisfy the second requirement for self-evidence which we saw above, that they not require a movement of thought beyond themselves. But it is not so clear which ones would satisfy the first requirement, that they not require an internal movement of thought from one sort of thing to another. It is probable that Proclus thinks that an object which itself is only composed of one sort of element would satisfy both requirements, and so would have a self-evident definition, while an object composed of two

103) 119.4: αὐτόθεν τὰς ἐννοίας ἔχομεν.

or more different sorts of elements would not. In favour of this interpretation is the fact that, according to Proclus, the objects which Euclid defines exhibit an increasing complexity, and only the first few are simple enough that their constructions are postulates rather than problems.

Further, in his discussion of the definition of the semicircle, Proclus seems to apply this criterion, and classifies figures as monadic, dyadic, triadic, etc. The circle is monadic, because it is composed of one sort of line, while the semicircle, as composed of both a straight and a circumference, is dyadic (159.12-25). This passage, if read quickly, gives the false impression that Proclus considers the definitions of monadic figures to require no internal movement of thought from one sort of thing to another, and so that the definitions of monadic figures and all other such monadic objects are self-evident. Dyadic, triadic, etc. figures, on the other hand, would require such a movement, and so are not.

Proclus' classification is not initially clear. In his discussion of the semicircle he seems to be indicating a progression from one to more than one *sort* of line. But in actuality, the progression from monadic to dyadic and triadic only indicates the number of lines enclosing a figure. And as he himself points out, two similar lines can make a figure, if they are circumferences, as is the case with the lunule and the area between two concentric circles (160.1-5), and it is likely that these are dyadic. This is because he indicates that the next topic of discussion is triadic figures, and what actually follows is his discussion of Definitions XX to XXIII: Rectilinear Figure, Triangle, Quadrilateral, and Multilateral. Because a triangle, as the first rectilinear figure, is only composed of one sort of line, 'triadic' must refer not to the number of sorts of lines, but just to the number of lines that enclose a figure. If we go back to his characterization of the semicircle, moreover, it becomes clear that he has said not that the semicircle is dyadic *because* it is composed of dissimilar elements (*ex anomoiôn huphestêke*), but that it is dyadic *and* that it is composed of dissimilar elements.

The idea of dissimilar elements points forward to another classification scheme (162.27-164.8), which he presents in his discussion of Definitions XX to XXIII, and which gives us what we are looking for, the ability to distinguish between figures which are composed of one or more than one kind of element. The present scheme recalls his earlier classification of angles, figures, planes and solids, into simple and mixed, and subdivides the simple into the species of Limit and Unlimited.[104] Here he classifies plane figures into those bounded by (a) simple lines and (b) mixed lines. Of those bounded by simple lines, some are bounded by (i) similar lines, and others by (ii) dissimilar lines (*hupo tôn*

104) See my discussion below, and 103.21-104.25; 118.24-120.6.

*anomoeidôn*). Clearly a figure bounded by mixed lines (e.g. cissoid, arch), or one bounded by two or more dissimilar lines (e.g. semicircle) would exhibit the internal complexity that would rule out self-evidence. A figure bounded by two or more similar simple lines, however, would seem not to present more than one kind of element, and so their definitions might not require an internal movement of thought from one sort of thing to another. This would include the circle; all other figures composed only of circumferences like the lunule, the area between concentric circles; all figures formed by tangent circles; and all rectilinear figures.

Light may be shed on this by a similar scheme that Proclus gives classifying angles, figures, planes and solids, in each case dividing the simple ones into the species of Limit and Unlimited, and distinguishing them from the Mixed:[105]

| | **Limited** | **Unlimited** | **Mixed (examples of)** |
|---|---|---|---|
| **Lines** | Circular Line | Straight Line | Spirals; Sections |
| **Angles** | Circular Angle | Rectilinear Angle | Semicircular; Horned |
| **Plane Figures** | Circle | Rectilinear Figures | Semicircles and Arches; Cissoidal |
| **Surfaces** | Spherical | Plane | Cylindrical, Conical, Spiric |
| **Solids** | Sphere | [Rectilinear Solids] | Cones; Cylinders |

His enumeration in various places of the circle, straight line, plane and sphere as easily grasped indicates that he probably thinks this is the case for all simple, i.e. Limited and Unlimited geometrical objects. However, the fact that when he gives examples he juxtaposes a figure (circle) and a line (straight line), and a solid (sphere) and a surface (plane) indicates that things might be more

105) 103.21-104.25; 118.24-120.6. How this scheme applies to plane figures can be extrapolated from his later classification which we have just looked at (162.27-164.17). For this chart, note that a figure 'bounded by mixed lines' is either bounded by lines that are themselves Mixed, 'such as the area cut off by the curve of a cissoid', or is bounded by simple lines that are dissimilar to each other, as in an arch. See 161.18-164.17. Note also that in the table I have supplied the Unlimited analogue for solids because, although Proclus does not state explicitly what they would be, if he follows the pattern they will be as indicated. Note as well that it is not clear where Proclus would put figures enclosed by two circumferences, such as the lunule and concentric circles (160.2-5), or more than two circumferences, i.e. all figures formed by tangent circles. Presumably they would join the circle under Limit in Plane Figures.

complicated than he is letting on.[106] In any case, the scheme which we looked at in the previous paragraph implies that all geometrical objects falling under Limit and Unlimited in this classification are composed either of only one element, or more than one similar element, and so that their definitions would be self-evident, while all those falling under Mixed would not have self-evident definitions.

I present this analysis as a suggestion, because Proclus does not use the term self-evident with regard to definitions. It seems clear that the definitions of the figures which are used in Euclid's postulates—point, straight line, circle, right angle—and the characteristics used in his axioms—equality, addition, subtraction, part, whole, coincidence—must be self-evident. And it is clear that definitions of certain of the objects defined in Euclid's text cannot be self-evident, once a certain level of internal complexity is reached. But would Proclus consider self-evident the definition of a dodecagon as a twelve-sided figure, or a hecatagon as a one-hundred-sided figure? He might, and by his principles he should, because they do not present the sort of internal complexity that would rule this out. But if this is the case, we should note, it would mean that having a self-evident definition does not entail having a self-evident construction. For example, the definition of an equilateral triangle is likely self-evident, as it is a figure enclosed by three equal straight lines. But its construction is too complex to be a postulate, because it requires the description of both straight lines and circles.[107] In the end, whether or not Proclus would extend this degree of self-evidence to geometrical definitions, it is understandable that he makes both general statements—that geometrical first principles include definitions and that geometrical first principles are self-evident—because of the role that definitions play in making clear the components at least of axioms and theorems.

Why Proclus' silence on this matter? It might indicate that the category of self-evidence and indemonstrability does not really apply to definitions. Perhaps he thinks that a scientist's definitions are preparatory to his knowing, but not part of the activity of knowing itself. In his second Prologue, Proclus' statement that the principles of a science are self-evident invokes Aristotle's principle that a scientist knows his principles better than his conclusions, because the latter are the causes of the former.[108] So a geometer knows his conclusions, which are demonstrated, through his postulates and axioms,

106) 118.24-120.6; 179.22-180.8; 185.6-25.

107) Definition and discussion: 164.18-168.25. Problem and discussion: 200.19-21; 213.12-218.12.

108) 75.14-17; Cf. *An. Post.* I.2, 72a31-33.

principles which he knows more fully because they are indemonstrable and self-evident. In other words, it is possible that these terms only apply within the actual activity of knowing things through arguments, because the indemonstrable and the self-evident imply a comparison with the demonstrated. If this is the case, then it would be proper for Proclus to characterise the definitions of certain geometrical objects in a manner analogous to self-evidence, as for example when he says that we have a notion (*ennoia*) of simple lines and simple surfaces without being taught (118.24-119.6), but it would not be appropriate to describe this as self-evidence.

### 3.2.5 The Self-evident and the Hypothetical

As we have seen, when Proclus calls something 'hypothetical' he does not mean that the thing in question is unknown, nor moreover that it is known. Rather, he simply means that what is in question is used as the premise of an argument that falls short of *Nous*. Because of this, the hypothetical character of the principles of the special sciences is compatible with their self-evidence. So even though geometry is hypothetical, in comparison with dialectic, the geometer does not have to be a dialectician in order to understand his definitions, axioms, and postulates. He can have a clear grasp of his definitions, and take up his axioms and postulates as self-evident hypo-theses, from which he makes arguments which stay within the bounded field of geometrical reality (32.13-18):

> In the same way mathematics is second to the highest science and imperfect as compared with it but nevertheless is a science—not an unhypothetical science, but one which, being capable of knowing the specific *logoi* that are in the soul, gives an account of the causes of its conclusions and thus has an account for the matters known to it.

So Proclus' statement that geometry is a hypothetical science does not mean that the geometer merely assumes his principles without knowledge. When he takes up his principles he knows them through themselves, and can know his conclusions through his principles. Yet his entire science remains hypothetical in comparison with dialectic, because it neither begins from nor returns to *Nous*.

## 3.3 *Geometry as a Hypothetical Science and Dialectic*

If the geometer does not have to be a dialectician in order to know the principles of his science, what does Proclus mean by saying that geometry is hypothetical because it receives its principles from the one unhypothetical science, i.e. from dialectic?[109] It is clear that the geometer does not receive them as the conclusions of that higher science, because he holds them as self-evident. Instead, Proclus' language indicates that geometry makes use of principles which, while it grasps them sufficiently for its own purpose, belong more properly to the higher science. As a hypothetical science, geometry does not stray outside of the boundaries of geometrical being. But according to the principle of analogy the objects that geometry studies exist in various manners from the top of reality to the bottom of the cosmos, and dialectic has the entire series to which these objects belong as its purview.[110]

Throughout the text, Proclus turns from an explicitly mathematical treatment of his subject to things that are proper to dialectic. So, for example, in his discussion of the definition of right angles, acute angles and obtuse angles, he states that most geometers cannot give a reason for this threefold classification of angles. The Pythagoreans, on the other hand, know that the right angle derives from the principle of Limit while the other pair derives from Unlimited, and so are able to explain these as images of such things as the various forms of divine providence or the distinction between virtue and vice.[111] Early in his second prologue, Proclus gives a surprising description of the range of geometry (61.25-62.26):

> Let us now turn back for another look at the science of geometry as a whole (*holên geômetrian*), to see from what harbour it sets out (*hothen te hôrmêtai*)[112] and up to what point it proceeds, so as to get a view of

109) The unhypothetical is the science 'from which' (*aph' hês*) all the principles of other sciences come (31.22-32.1); mathematics 'takes' (*labousan*) its principles from it (32.5-7); and all other sciences 'receive' (*hupodechontai*) their principles from the unhypothetical (75.6-10). For the identification of this one unhypothetical science and dialectic, see MacIsaac 2010, 130-2.

110) See MacIsaac 2010.

111) 131.17-134.7. Cf. the classification of lines, angles, plane figures, surfaces and solids according to Limit, Unlimited and Mixed, given above.

112) *Hôrmêtai* is either from *hormaô* (to set in motion) or *hormêo* (to lie at anchor), and it is possible that Proclus is playing on both senses of the same form, and that the latter verb is derived from the former—the harbour as that from which one sets out. Cf. *in Tim*. i. 302.17-25 Diehl: 'For it is only when the soul has passed beyond the distraction of birth and the [process of] purification and beyond the illumination of scientific knowledge that its intellective activity and the intellect in us lights up, anchoring (*hormizôn*) the soul in the Father and establishing it immaculately in the demiurgic thoughts. It connects light with light, not in the manner of scientific knowledge, but in a manner that is more beautiful, more intellective and more unificatory. This is the Paternal harbour (*patrikos hormos*), the discovery of the Father, the immaculate unification with him' (trans. Runia and Share 2009).

> the ordered cosmos of its *logoi*. Let us note that it is coextensive with all beings, applies the reasonings of its *dianoia* to them all, and comprehends all their Forms (*eidê*) in itself. At its highest and most intellectual point (*noerôtaton*) it inspects from all sides the region of genuine being (*ta ontôs onta*), teaching us through images the special properties of the divine orders and the powers of the intellectual Forms (*noerôn eidôn*), for it contains even the *logoi* of these beings within its range of vision. Here it shows us what figures are appropriate to the gods, which ones belong to primary beings and which ones to the souls' manner of existing (*tais psuchais hupostasesi*). In the middle regions of knowledge it unfolds the *logoi* that are in *dianoia*; it unrolls and investigates their variety, exhibiting their modes of existence and their properties, their similarities and differences; and the forms of figures shaped from them in imagination it comprehends within fixed boundaries (*en perasin hôrismenois*) and refers back to the existence of the *logoi* in its essence (*ôusiôdê tôn logôn hupostasin*). At the third level of the progression of *dianoia* (*tas tritas tês dianêseôs diexodous*) it examines nature, that is, the Forms of the elements of perceptible bodies and the powers associated with them, and explains how they are contained in causal form in its own *logoi*. It contains images of all intelligible genera and paradigms of sensible ones; but the Forms of *dianoia* constitute its essence (*ousiôtai de kata ta eidê ta dianoêta*), and through these Middle Forms it rises up and it descends along the entirety of being and becoming (*eph' hola ta onta kai ta ginomena*).

Most of the things on this list are not objects of geometry as it is strictly understood. Geometry begins from the soul's 'harbour' in the noeric Forms, ranges through the entirety of the Soul's being, only a small part of which is geometrical in the strict sense, and proceeds into the philosophy of nature. He then continues to say that it produces the practical sciences of geodesy, mechanics and optics (63.7-8). What we have here in fact is a description not of geometry, despite the way in which he introduces this passage, but a description of how the science of dialectic understands the truths known by geometry and how these truths belong in a fuller sense within dialectic. We can see this at the end of the passage where he states explicitly that geometry knows all these things because it 'contains images of all intelligible genera and paradigms of sensible ones; but the Forms of *dianoia* constitute its essence'. So, for example, the circle described in the imagination is an object of the science of geometry, but it is also an image of the intelligibles and a paradigm of sensibles, because it is the same thing that exists analogically on all these levels.[113] So in that

113) See the series of the circle: 146.24-156.5.

sense, when the geometer knows his circle in imagination, he is also knowing the intelligibles and sensibles, and Proclus can say that geometry knows these things. But it is pretty clear that the only geometer who would realise that his geometrical arguments also tell him about the gods and about the cycles of nature is one who is also a dialectician, and who therefore knows about the analogical counterparts of his properly geometrical objects.[114]

Moreover, Proclus also states clearly that his own interest in Euclid's text is dialectical, at the end of his second prologue (84.8-23):

> As we begin our examination of details, we warn those who may encounter this book not to expect of us a discussion of matters that have been dealt with over and over by our predecessors, such as lemmas, cases and the like. We are surfeited with those topics and shall touch on them but sparingly. But whatever matters contain more substantial science and contribute to philosophy as a whole (*tên holên philosophian*), these we shall make it our chief concern to mention, emulating the Pythagoreans whose byword and proverb was 'a figure and a stepping stone, not a figure and three obols'. By this they meant that we must cultivate that science of geometry (*tên geômetrian ekeinên*) with which each theorem lays the basis for a step upward and draws the soul to the higher world, instead of letting it descend among sensibles to satisfy the common needs of mortals and, in aiming at these, neglect to turn away hence.

This is certainly his aim in the first part of his commentary. It is true that in his treatment of Euclid's propositions, and his relatively brief discussion of postulates and axioms, Proclus stays mostly within the boundaries of geometrical science.[115] But his two prologues and his extremely long discussion of Euclid's definitions, together occupying two-fifths of the book, are full of dialectical analyses. It is quite astonishing that Proclus goes on for almost a hundred

114) This is what Proclus is indicating by his statement that the Pythagoreans have a knowledge of the classification of angles superior to the common geometrician (131.17-134.7). In the passage at hand, Proclus' phrase 'the science of geometry as a whole' (*holên geômetrian*) is probably an indication of this 'broader' sense of geometrical knowledge.

115) These two sections occupy the final three fifths of the book. His few forays into dialectical analysis in them are discussions of the cosmic significance of the line drawn at right angles as compared to the perpendicular (290.14-291.19); of the right angle as a *horos* of acute and obtuse as an image of the primary causes which bound the influence of the indefinite dyad (294.2-294.14); and of how different triangles manifest the influence of Limit, Unlimited, and Mixed (314.12-315.4).

pages in Friedlein's text about Euclid's one-and-a-half pages of definitions, and that most of this material deals with the analogical counterparts of the objects that are defined. Proclus' aim in this first part of the book is often to show the full realities of which geometry itself only knows a small part. So while Proclus' commentary eventually settles down to being a work of geometry, it embeds its specifically geometrical investigations within the science of dialectic, specifying the boundaries of the science of geometry while not restricting itself to those boundaries. After all, while a geometer need not be a dialectician in order to know his principles, Proclus was certainly a dialectician. And one really does get the impression that Proclus thinks the significance of points, lines, and circles is made much more clear from his dialectical explanations than from within the carefully bounded field of geometry.

Therefore I think it is clear that for Proclus geometry receives its hypotheses from dialectic in the sense of using something that more properly belongs to the higher science. Although something like the circle is really the property of the more fundamental science of dialectic, which looks to its whole series, geometry takes a very small portion of that series and posits it as one of its beginning points. Dialectic knows that the circle exists also among the gods and in the cycles of nature, but the geometer's ignorance of this does not hinder his clear grasp of what his circle is, nor does it hinder him from describing circles in his imagination as the self-evident hypotheses of geometrical arguments.

Moreover, this points to a very important nuance that we must add to our conclusions about self-evidence. For Proclus, the same thing must be self-evident to varying degrees and in various ways. We would probably think that the self-evident is the most intelligible sort of thing that can be known. But Proclus has described a situation where the same things are self-evident in different ways in different contexts, and for people who are ignorant of much of the significance of what is evident to them. So as we saw above, 'that which has no parts' is self-evidently a monad for the arithmetician, a point for the geometer, one of the elements for the physician, and something else yet again for the physicist (93.11-94.4). The case seems similar to Proclus' idea of how the axioms of arithmetic and geometry are each applications of the same truth. In a similar way, he uses the axiom that there is nothing larger than infinity in the first proposition of the *Elements of Theology*, an axiom that is true in different ways in arithmetic and geometry. So for Proclus the self-evident must be something that is simple enough to be grasped from within each science in question, but the whole of ways in which analogues of this simple reality appear can be known only by dialectic.

## 4 Conclusion

So the solution to our original dilemma has to do with the manner in which the geometer projects the *logoi* that constitute his soul's essence. A geometer begins with an immediate grasp of things like a point or a circle, and uses these as beginning points for further demonstrations. Although he leaves these principles undemonstrated, in attending to them the geometer *knows* them, because they are self-evident, that is, their natures and characteristics are immediately comprehensible through the sort of minimal projection described above. And because he knows his principles he can know the conclusions which he draws from them. However, geometry is subordinate to dialectic because the geometer does not investigate the analogical counterparts of his principles in the series of which they are a part, i.e. the Forms which lie above and below them. In this sense the geometer is content to receive his principles from dialectic, to which they more properly belong, and merely uses them as beginning points, as hypotheses, for arguments. Moreover, because he starts from these determinate or defined points, his knowledge is bounded within (*horismenos*) or limited to that part of reality.[116] This analysis is phrased in terms of geometry, but it accurately describes Proclus' conception of arithmetic and all other sciences which fall short of dialectic.[117]

The first principle of dialectic, on the other hand, is unhypothetical. Dialectic begins from the non-discursivity of *Nous*, and so is not bound within one or another discursive science. Each particular science begins from a set of definitions, or bounded principles, but the inexhaustible fertility of *Nous* allows dialectic to surpass those boundaries.[118] Dialectic's grasp of the entire series to which particular principles belong would perhaps allow Proclus to say that it can give rise to new sciences or expand the principles of already existing ones. However, the real aim of dialectic is not a completed system of discursive science, but *noêsis*, a non-discursive grasp of the Forms in *Nous*. Its aim is transformative, and its culmination is the overcoming of its own discursivity.

116) In this Proclus agrees with Aristotle (*An. Post.* 1.10), despite his different account of principles.

117) Note that this rules out bodies of knowledge that do not know causes and so are not sciences. See 30.8-32.2; 75.19-26; 93.11-94.4.

118) See MacIsaac 2010, 137.

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