

Article

Geometrical Structure in a Perfect Fluid Spacetime with Conformal Ricci–Yamabe Soliton

Pengfei Zhang ¹, Yanlin Li ^{2,*} , Soumendu Roy ³ , Santu Dey ⁴  and Arindam Bhattacharyya ³ 

¹ College of Teacher Education, Harbin Normal University, Harbin 150025, China; zhangpengfei313@hrbnu.edu.cn

² School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China

³ Department of Mathematics, Jadavpur University, Kolkata 700032, India; myselfsoumendu.mathju@hotmail.com (S.R.); bhattachar1968@yahoo.co.in (A.B.)

⁴ Department of Mathematics, Bidhan Chandra College, Asansol 713304, India; santu.mathju@gmail.com

* Correspondence: liyl@hznu.edu.cn

Abstract: The present paper aims to deliberate the geometric composition of a perfect fluid spacetime with torse-forming vector field ξ in connection with conformal Ricci–Yamabe metric and conformal η -Ricci–Yamabe metric. We delineate the conditions for conformal Ricci–Yamabe soliton to be expanding, steady or shrinking. We also discuss conformal Ricci–Yamabe soliton on some special types of perfect fluid spacetime such as dust fluid, dark fluid and radiation era. Furthermore, we design conformal η -Ricci–Yamabe soliton to find its characteristics in a perfect fluid spacetime and lastly acquired Laplace equation from conformal η -Ricci–Yamabe soliton equation when the potential vector field ξ of the soliton is of gradient type. Overall, the main novelty of the paper is to study the geometrical phenomena and characteristics of our newly introduced conformal Ricci–Yamabe and conformal η -Ricci–Yamabe solitons to apply their existence in a perfect fluid spacetime.

Keywords: Ricci–Yamabe soliton; conformal Ricci–Yamabe soliton; conformal η -Ricci–Yamabe soliton; perfect fluid spacetime; torse-forming vector field; energy-momentum tensor; Einstein’s field equation



Citation: Zhang, P.; Li, Y.; Roy, S.; Dey, S.; Bhattacharyya, A.

Geometrical Structure in a Perfect Fluid Spacetime with Conformal Ricci–Yamabe Soliton. *Symmetry* **2022**, *14*, 594. <https://doi.org/10.3390/sym14030594>

Academic Editor: Egon Schulte

Received: 22 February 2022

Accepted: 15 March 2022

Published: 17 March 2022

Publisher’s Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Motivation and Introduction

In [1], R. S. Hamilton introduced the notions of Ricci flow, which is an evolution equation for metrics on a Riemannian manifold in 1982. The Ricci flow equation is the following:

$$\frac{\partial g}{\partial t} = -2S \quad (1)$$

on a compact Riemannian manifold M with Riemannian metric g .

The Ricci soliton, which is a self-similar solution to the Ricci flow, is given by [1–3]:

$$\mathcal{L}_V g + 2S + 2\Lambda g = 0, \quad (2)$$

where V is a vector field and Λ is a scalar, S is Ricci tensor, g is Riemannian metric, \mathcal{L}_V is the Lie derivative in the direction of V . We designate Ricci soliton as shrinking, steady and expanding accordingly as Λ is negative, zero and positive, respectively. The notion of conformal Ricci soliton [4] as:

$$\mathcal{L}_V g + 2S + \left[2\Lambda - \left(p + \frac{2}{n} \right) \right] g = 0, \quad (3)$$

where p is a scalar nondynamical field (time-dependent scalar field), n is the dimension of the manifold.

The notion of Conformal η -Ricci soliton was introduced by Mohd Danish Siddiqi [5] in 2018, which can be written as:

$$\mathcal{L}_\zeta g + 2S + \left[2\Lambda - \left(p + \frac{2}{n}\right)\right]g + 2\mu\eta \otimes \eta = 0, \tag{4}$$

where Λ, μ are constants.

A Yamabe soliton [6] corresponds to a self-similar solution of the Yamabe flow [3], and is defined on a Riemannian or pseudo-Riemannian manifold (M, g) as:

$$\frac{1}{2}\mathcal{L}_V g = (r - \Lambda)g, \tag{5}$$

where r is the scalar curvature and Λ is a constant [7].

Many authors ([8–13]) have studied Ricci soliton and Yamabe soliton on contact manifolds. Furthermore, some researchers have also studied conformal η -Ricci solitons, singular submanifolds, biharmonic submanifolds, warped product pointwise semislant submanifolds and so on [14–33]. In recent years, Kumara, H. A. studied and determined geometrical aspects of perfect fluid spacetime with torse-forming vector field and Ricci soliton in perfect fluid spacetime with torse-forming vector field ζ . They gave the conditions for the Ricci soliton to be expanding, steady or shrinking [34]. Singh, J. P. and Khatri, M. in [34], and Siddiqi, M. D. in [35] have considered conformal Ricci and Ricci–Yamabe solitons on general relativistic spacetime respectively. Motivated by these results, we will introduce and study more generalized versions of these solitons and discuss their existence on the perfect fluid spacetime. The differences in this paper to other studies is that, for example, [34] is about M-projective curvature tensor which has been studied in general relativistic spacetime, but we do not discuss any particular curvature tensor in this work. Moreover, [34,35] are about conformal Ricci and Ricci–Yamabe solitons on general relativistic spacetime, respectively, but we introduce and study more generalized versions of these solitons and discuss their existence on the perfect fluid spacetime. Moreover, there are some articles concerning the study of singularity theory, submanifolds and harmonic quasiconformal mappings and so on which are helpful to our present and future research. In our next work, we will consider taking the main results in this paper to connect the methods and techniques of singularity theory and submanifolds theory, etc., presented in [23–33,36–54] to explore new results and theorems related with more symmetric properties about this topic.

In [55], Crasmareanu, M. and Güler, S. presented a new geometric flow which is a scalar combination of Ricci and Yamabe flow under the name Ricci–Yamabe map in 2019. This new geometric flow is known as Ricci–Yamabe flow of the type (α, β) . Also in [55], the authors characterized that the (α, β) -Ricci–Yamabe flow is said to be:

- Ricci flow [1] if $\alpha = 1, \beta = 0$;
- Yamabe flow [3] if $\alpha = 0, \beta = 1$;
- Einstein flow [13] if $\alpha = 1, \beta = -1$.

A soliton to the Ricci–Yamabe flow is called a Ricci–Yamabe soliton as long as it moves by only one parameter group of diffeomorphism and scaling. The metric of the Riemannian manifold $(M^n, g), n > 2$ is said to admit (α, β) -Ricci–Yamabe soliton or simply Ricci–Yamabe soliton (RYS) $(g, V, \Lambda, \alpha, \beta)$ if it satisfies the equation:

$$\mathcal{L}_V g + 2\alpha S = [2\Lambda - \beta r]g, \tag{6}$$

where Λ, α, β are real scalars.

In the above equation, if the vector field V is the gradient of a smooth function f (denoted by Df , D denotes the gradient operator) then Equation (6) is called gradient Ricci–Yamabe soliton (GRYS) and it is defined as:

$$Hessf + \alpha S = \left[\Lambda - \frac{1}{2}\beta r \right] g, \tag{7}$$

where $Hessf$ is the Hessian of the smooth function f .

Now, using (6) and (3), we introduce the notion of conformal Ricci–Yamabe soliton as [7]:

Definition 1. A Riemannian manifold (M^n, g) , $n > 2$ is said to admit conformal Ricci–Yamabe soliton if

$$\mathcal{L}_V g + 2\alpha S + \left[2\Lambda - \beta r - \left(p + \frac{2}{n} \right) \right] g = 0, \tag{8}$$

We call the conformal Ricci–Yamabe soliton expanding, steady or shrinking depending on Λ being positive, zero or negative, respectively. If the vector field V is of gradient type, that is to say $V = grad(f)$, for f is a smooth function on M , then we call Equation (8) a conformal gradient Ricci–Yamabe soliton. Also using (6) and (4), we extend the concept of conformal η -Ricci–Yamabe soliton by the following:

Definition 2. A Riemannian manifold (M^n, g) , $n > 2$ is said to admit conformal η -Ricci–Yamabe soliton if

$$\mathcal{L}_\xi g + 2\alpha S + \left[2\Lambda - \beta r - \left(p + \frac{2}{n} \right) \right] g + 2\mu\eta \otimes \eta = 0, \tag{9}$$

If the vector field ξ is of gradient type, that is to say $\xi = grad(f)$, for f is a smooth function on M , then we call Equation (9) the conformal gradient η -Ricci–Yamabe soliton.

A perfect fluid is a fluid which could be completely characterized by its rest-frame mass density and isotropic pressure. A perfect fluid has no shear stress, viscosity or heat conduction and it is distinguished by an energy-momentum tensor T of the form [56]:

$$T(X, Y) = \rho g(X, Y) + (\sigma + \rho)\eta(X)\eta(Y), \tag{10}$$

where ρ, σ are the isotropic pressure and energy-density, respectively, and $\eta(X) = g(X, \xi)$ is 1-form, which is equivalent to the unit vector ξ and $g(\xi, \xi) = -1$. The field equation governing the perfect fluid motion is Einstein’s gravitational equation [56]:

$$S(X, Y) + \left[\lambda - \frac{r}{2} \right] g(X, Y) = \kappa T(X, Y), \tag{11}$$

where λ is the cosmological constant and κ is the gravitational constant, which can be considered as $8\pi G$, where G is the universal gravitational constant.

Using (10), the above equation takes the form:

$$S(X, Y) = \left[-\lambda + \frac{r}{2} + \kappa\rho \right] g(X, Y) + \kappa(\sigma + \rho)\eta(X)\eta(Y). \tag{12}$$

Let (M^4, g) be a relativistic perfect fluid spacetime which satisfies (12). Then, by contracting (12) and considering $g(\xi, \xi) = -1$, we obtain

$$r = 4\lambda + \kappa(\sigma - 3\rho). \tag{13}$$

Using the value of r from the above equation, (12) becomes

$$S(X, Y) = \left[\lambda + \frac{\kappa(\sigma - \rho)}{2} \right] g(X, Y) + \kappa(\sigma + \rho)\eta(X)\eta(Y). \tag{14}$$

Hence the Ricci operator Q can be written as:

$$QX = \left[\lambda + \frac{\kappa(\sigma - \rho)}{2} \right] X + \kappa(\sigma + \rho)\eta(X)\xi, \tag{15}$$

where $g(QX, Y) = S(X, Y)$.

Example 1. A radiation fluid is a perfect fluid with $\sigma = 3\rho$ and so the energy momentum tensor T becomes,

$$T(X, Y) = \rho[g(X, Y) + 4\eta(X)\eta(Y)], \tag{16}$$

From (13), we can say that a radiation fluid has constant scalar curvature r equal to 4λ . Now we take a special case when ξ is a torse-forming vector field [57,58] of the form:

$$\nabla_X \xi = X + \eta(X)\xi. \tag{17}$$

Moreover, if the vector field ξ on a perfect fluid spacetime is torse-forming, then the following relations hold [58]:

$$\nabla_\xi \xi = 0, \tag{18}$$

$$(\nabla_X \eta)(Y) = g(X, Y) + \eta(X)\eta(Y) \tag{19}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{20}$$

$$\eta(R(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z), \tag{21}$$

for all vector fields X, Y, Z . Using (17), we have,

$$\begin{aligned} (\mathcal{L}_\xi g)(X, Y) &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \\ &= 2[g(X, Y) + \eta(X)\eta(Y)], \end{aligned} \tag{22}$$

for all vector fields X, Y .

Perfect fluid is frequently considered to be a sharp tool in general relativity to model the idealized distribution of matter; for example, the interior of a star or an isotropic universe. In general relativity and symmetries of space time, one often employs a perfect fluid energy momentum tensor (10) to represent the source of the gravitational field. A perfect fluid has two thermodynamic degrees of freedom. The outline of the article is as follows: In Section 2, we discuss some properties of perfect fluid that will be used in the paper. In Section 3, we provide some applications of conformal Ricci–Yamabe soliton structure in perfect fluid spacetime with torse-forming vector field. In this section we have contrived the conformal Yamabe soliton in perfect fluid spacetime with torse-forming vector field to accessorize the nature of this soliton on the mentioned spacetime. We have also considered the potential vector field V of the soliton as a conformal Killing vector field to characterize the vector field. Sections 4–6, are devoted to finding the nature of the conformal Ricci–Yamabe soliton in a dust fluid spacetime, dark fluid spacetime and radiation era, respectively. Finally, in the last section, we give the conclusion of the paper.

2. Conformal Ricci–Yamabe Soliton Structure in Perfect Fluid Spacetime with Torse-Forming Vector Field

In this section, we study conformal Ricci–Yamabe soliton structure in a perfect fluid spacetime whose timelike velocity vector field ξ is torse-forming.

Taking V as a torse-forming vector field ξ in the soliton Equation (8) and putting $n = 4$, we obtain,

$$(\mathcal{L}_\xi g)(X, Y) + 2\alpha S(X, Y) + \left[2\lambda - \beta r - \left(p + \frac{1}{2} \right) \right] g(X, Y) = 0. \tag{23}$$

Using (22), the above equation becomes

$$2[g(X, Y) + \eta(X)\eta(Y)] + 2\alpha S(X, Y) + \left[2\Lambda - \beta r - \left(p + \frac{1}{2}\right)\right]g(X, Y) = 0. \quad (24)$$

In view of (14), we obtain

$$\left[\Lambda - \frac{\beta r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right) + \alpha\lambda + \frac{\alpha\kappa(\sigma - \rho)}{2} + 1\right] + [\alpha\kappa(\sigma + \rho) + 1]\eta(X)\eta(Y) = 0. \quad (25)$$

Taking $X = Y = \xi$ in the above equation, we acquire

$$\Lambda = \frac{\alpha\kappa(\sigma + 3\rho)}{2} + \frac{\beta r}{2} - \alpha\lambda + \frac{1}{2}\left(p + \frac{1}{2}\right). \quad (26)$$

Using (13), we have

$$\Lambda = \frac{\kappa}{2}[(\alpha + \beta)\sigma + 3(\alpha - \beta)\rho] + (2\beta - \alpha)\lambda + \frac{1}{2}\left(p + \frac{1}{2}\right). \quad (27)$$

Therefore, we can state the following:

Theorem 1. *If a perfect fluid spacetime with torse-forming vector field ξ admits a conformal Ricci–Yamabe soliton $(g, \xi, \Lambda, \alpha, \beta)$, then the soliton is expanding, steady or shrinking according as, $\frac{\kappa}{2}[(\alpha + \beta)\sigma + 3(\alpha - \beta)\rho] + (2\beta - \alpha)\lambda + \frac{1}{2}\left(p + \frac{1}{2}\right) \gtrless 0$.*

Remark 1. *In (27), if we take $\left(p + \frac{1}{2}\right) = 0$, then $\Lambda = \frac{\kappa}{2}[(\alpha + \beta)\sigma + 3(\alpha - \beta)\rho] + (2\beta - \alpha)\lambda$ and in this case the conformal Ricci–Yamabe soliton becomes Ricci–Yamabe soliton and we obtain that the soliton is expanding, steady or shrinking according as, $\frac{\kappa}{2}[(\alpha + \beta)\sigma + 3(\alpha - \beta)\rho] + (2\beta - \alpha)\lambda \gtrless 0$.*

A spacetime symmetry of physical interest is the conformal Killing vector, as it preserves the metric up to a conformal factor. We call a vector field V a conformal Killing vector field if, and only if, the following relation holds:

$$(\mathcal{L}_V g)(X, Y) = 2\Phi g(X, Y), \quad (28)$$

here, Φ is some function of the coordinates (conformal scalar).

Furthermore, if Φ is not constant the conformal Killing vector field V is called proper. Moreover, when Φ is constant, we call V a homothetic vector field and when the constant Φ becomes non-zero, V is called a proper homothetic vector field. If $\Phi = 0$ in the above equation, we call V a Killing vector field, if $\Phi = 0$ in the above equation. Let us assume that in Equation (8), the potential vector field V is a conformal Killing vector field. Then, using (28) and (8), we obtain

$$\alpha S(X, Y) = -\left[\Lambda + \Phi - \frac{\beta r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right)\right]g(X, Y), \quad (29)$$

which leads to the fact that the spacetime is Einstein, provided $\alpha \neq 0$.

Conversely, assuming that the perfect fluid spacetime with torse-forming vector field ξ is Einstein space time, i.e., $S(X, Y) = \theta g(X, Y)$.

Then, Equation (8) becomes

$$(\mathcal{L}_V g)(X, Y) = -\left[2\Lambda + 2\alpha\theta - \beta r - \left(p + \frac{1}{2}\right)\right]g(X, Y), \quad (30)$$

which can be written as,

$$(\mathcal{L}_V g)(X, Y) = 2\Psi g(X, Y), \tag{31}$$

where $\Psi = -\left[\Lambda + \alpha\theta - \frac{\beta r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right)\right]$.

Thus from (31), V becomes a conformal Killing vector field.

Hence we can state the following:

Theorem 2. *Let a perfect fluid spacetime with torse-forming vector field ξ admit a conformal Ricci–Yamabe soliton $(g, V, \Lambda, \alpha, \beta)$. The potential vector field V is a conformal Killing vector field if and only if the spacetime is Einstein, provided $\alpha \neq 0$.*

Now, in view of (29) and (14), we obtain,

$$\left[\Lambda + \Phi + \alpha\lambda + \frac{\alpha\kappa(\sigma - \rho)}{2} - \frac{\beta r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right)\right]g(X, Y) + [\alpha\kappa(\sigma + \rho)]\eta(X)\eta(Y) = 0. \tag{32}$$

Taking $Y = \xi$ in the above equation and considering $\eta(\xi) = -1$, we have

$$\left[\Lambda + \Phi + \alpha\lambda - \frac{\alpha\kappa(\sigma + 3\rho)}{2} - \frac{\beta r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right)\right]\eta(X) = 0. \tag{33}$$

Since $\eta(X) \neq 0$, then we obtain

$$\Lambda + \Phi + \alpha\lambda - \frac{\alpha\kappa(\sigma + 3\rho)}{2} - \frac{\beta r}{2} - \frac{1}{2}\left(p + \frac{1}{2}\right) = 0. \tag{34}$$

Substituting the value of r from (13), the above equation reduces to

$$\Phi = \frac{\kappa}{2}\left[(\alpha + \beta)\sigma + 3(\alpha - \beta)\rho\right] + (2\beta - \alpha)\lambda - \Lambda + \frac{1}{2}\left(p + \frac{1}{2}\right). \tag{35}$$

Hence we can state the following:

Theorem 3. *Let a perfect fluid spacetime with torse-forming vector field ξ admit a conformal Ricci–Yamabe soliton $(g, V, \Lambda, \alpha, \beta)$. The potential vector field V is a conformal Killing vector field, then V is*

- (i) *proper conformal Killing vector field if α, β, p are not constant.*
- (ii) *homothetic vector field if α, β, p are constant.*

Take advantage of the property of Lie derivative we can write

$$(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X) \tag{36}$$

for any vector fields Y, X .

Thus, by using (14) and (36), (8), we have

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + \left[2\Lambda - \beta r - \left(p + \frac{1}{2}\right) + 2\alpha\left\{\lambda + \frac{\kappa(\sigma - \rho)}{2}\right\}\right]g(X, Y) + 2\alpha\kappa(\sigma + \rho)\eta(X)\eta(Y) = 0. \tag{37}$$

Suppose ω is a 1-form, that is metrically equivalent to V and is given by $\omega(X) = g(X, V)$ for an arbitrary vector field X . Furthermore, the exterior derivative $d\omega$ of ω can be given by:

$$2(d\omega)(X, Y) = g(\nabla_X V, Y) - g(\nabla_Y V, X). \tag{38}$$

As $d\omega$ is skew-symmetric, so if we define a tensor field F of type (1,1) by

$$(d\omega)(X, Y) = g(X, FY), \tag{39}$$

then F is skew self-adjoint that is $g(X, FY) = -g(FX, Y)$.

Therefore, the (39) can be given by:

$$(d\omega)(X, Y) = -g(FX, Y) \tag{40}$$

Using (40), (38) becomes,

$$g(\nabla_X V, Y) - g(\nabla_Y V, X) = -2g(FX, Y). \tag{41}$$

We add (41) and (37) together and factor out Y to yield

$$\nabla_X V = -FX - \left[\Lambda - \frac{\beta r}{2} - \frac{1}{2} \left(p + \frac{1}{2} \right) + \alpha \left\{ \lambda + \frac{\kappa(\sigma - \rho)}{2} \right\} \right] X - \alpha\kappa(\sigma + \rho)\eta(X)\xi. \tag{42}$$

Substituting the above equation in $R(X, Y)V = \nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X, Y]}V$, we have

$$\begin{aligned} R(X, Y)V &= (\nabla_Y F)X - (\nabla_X F)Y + \alpha\kappa(\sigma + \rho)[Y\eta(X) - X\eta(Y)] \\ &+ \beta \frac{Y}{2}(Xr) - \beta \frac{X}{2}(Yr). \end{aligned} \tag{43}$$

Since $d\omega$ is closed, we acquire

$$g(X, (\nabla_Z F)Y) + g(Y, (\nabla_X F)Z) + g(Z, (\nabla_Y F)X) = 0. \tag{44}$$

Making inner product of (43) with respect to Z , then, we obtain

$$\begin{aligned} g(R(X, Y)V, Z) &= g((\nabla_Y F)X, Z) - g((\nabla_X F)Y, Z) \\ &+ \alpha\kappa(\sigma + \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ &+ \beta \frac{Y}{2}(Xr) - \beta \frac{X}{2}(Yr). \end{aligned} \tag{45}$$

Since F is skew self-adjoint, then $\nabla_X F$ is skew self-adjoint. Then using (44), (45) takes the form

$$\begin{aligned} g(R(X, Y)V, Z) &= \alpha\kappa(\sigma + \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] - g(X, (\nabla_Z F)Y) \\ &+ \beta \frac{g(X, Dr)}{2}g(Y, Z) - \beta \frac{g(Y, Dr)}{2}g(X, Z). \end{aligned} \tag{46}$$

We take $X = Z = e_i$ in the above equations, and here, e_i s are a local orthonormal frame and summing over $i = 1, 2, 3, 4$, then, we find

$$S(Y, V) = -3\alpha\kappa(\sigma + \rho)\eta(Y) - (divF)Y - \frac{3\beta}{2}g(Y, Dr). \tag{47}$$

here $divF$ is the divergence of the tensor field F .

Equating (14) and (47), we obtain

$$(divF)Y = -\kappa(\sigma + \rho)[3\alpha + \eta(V)]\eta(Y) - \left[\lambda + \frac{\kappa(\sigma - \rho)}{2} \right] \omega(Y) - \frac{3\beta}{2}g(Y, Dr). \tag{48}$$

We give the covariant derivative of the squared g-norm of V taking (42) as below:

$$\begin{aligned} \nabla_X |V|^2 &= 2g(\nabla_X V, V) \\ &= -2g(FX, V) - \left[2\Lambda - \beta r - \left(p + \frac{1}{2}\right)\right. \\ &\quad \left.+ 2\alpha \left\{\lambda + \frac{\kappa(\sigma - \rho)}{2}\right\}\right]g(X, V) - 2\alpha\kappa(\sigma + \rho)\eta(X)\eta(V). \end{aligned} \tag{49}$$

From (14), (8) becomes

$$\begin{aligned} (\mathcal{L}_V g)(X, Y) &= -\left[2\Lambda - \beta r - \left(p + \frac{1}{2}\right) + 2\alpha \left\{\lambda + \frac{\kappa(\sigma - \rho)}{2}\right\}\right]g(X, Y) \\ &\quad - 2\alpha\kappa(\sigma + \rho)\eta(X)\eta(Y). \end{aligned} \tag{50}$$

Using the above equation, (49) takes the form

$$\nabla_X |V|^2 + 2g(FX, V) - (\mathcal{L}_V g)(X, V) = 0. \tag{51}$$

Therefore, we can state the following:

Theorem 4. *If a perfect fluid spacetime with torse-forming vector field ξ admits a conformal Ricci–Yamabe soliton $(g, V, \Lambda, \alpha, \beta)$, then the vector V and its metric dual 1-form ω satisfies the relation*

$$(\operatorname{div}F)Y = -\kappa(\sigma + \rho)[3\alpha + \eta(V)]\eta(Y) - \left[\lambda + \frac{\kappa(\sigma - \rho)}{2}\right]\omega(Y) - \frac{3\beta}{2}g(Y, Dr)$$

and

$$\nabla_X |V|^2 + 2g(FX, V) - (\mathcal{L}_V g)(X, V) = 0.$$

3. Dust Fluid Spacetime with Conformal Ricci–Yamabe Soliton

In pressureless fluid spacetime or a dust, the energy-momentum tensor is the following [52]:

$$\mathcal{T}(X, Y) = \sigma\eta(X)\eta(Y), \tag{52}$$

where σ is the energy density of the dust-like matter and η is same as defined in (10).

Using (11) and (52), we have,

$$S(X, Y) = \left[-\lambda + \frac{r}{2}\right]g(X, Y) + \kappa\sigma\eta(X)\eta(Y). \tag{53}$$

Let (M^4, g) be a dust fluid spacetime which satisfies (53). Then by contracting (53) and considering $g(\xi, \xi) = -1$, we obtain,

$$r = 4\lambda + \kappa\sigma. \tag{54}$$

Taking contraction in (8) and using (54), we have,

$$\Lambda = -\frac{\operatorname{div}V}{4} + (2\beta - \alpha)\left[\lambda + \frac{\kappa\sigma}{4}\right] + \frac{1}{2}\left(p + \frac{1}{2}\right), \tag{55}$$

where $\operatorname{div}V$ is the divergence of the vector field V .

Then we have,

Theorem 5. *If a dust fluid spacetime admits a conformal Ricci–Yamabe soliton $(g, V, \Lambda, \alpha, \beta)$, then the soliton is expanding, steady, shrinking according as,*

$$-\frac{\operatorname{div}V}{4} + (2\beta - \alpha)\left[\lambda + \frac{\kappa\sigma}{4}\right] + \frac{1}{2}\left(p + \frac{1}{2}\right) \begin{matrix} \geq \\ = \\ \leq \end{matrix} 0.$$

4. Dark Fluid Spacetime with Conformal Ricci–Yamabe Soliton

In a dark fluid spacetime $\rho = -\sigma$, then the energy–momentum tensor (10) gives,

$$\mathcal{T}(X, Y) = \rho g(X, Y). \tag{56}$$

Using (11) and (56), we obtain,

$$S(X, Y) = \left[\kappa\rho - \lambda + \frac{r}{2} \right] g(X, Y). \tag{57}$$

Let (M^4, g) be a dark fluid spacetime which satisfies (57). Then by contracting (57) and considering $g(\zeta, \zeta) = -1$, we obtain,

$$r = 4(\lambda - \kappa\rho). \tag{58}$$

Taking contraction in (8) and using (58), we have,

$$\Lambda = -\frac{\text{div}V}{4} + (2\beta - \alpha)(\lambda - \kappa\rho) + \frac{1}{2}\left(p + \frac{1}{2}\right). \tag{59}$$

Therefore we can state the following:

Theorem 6. *If a dark fluid spacetime admits a conformal Ricci–Yamabe soliton $(g, V, \Lambda, \alpha, \beta)$, then the soliton is expanding, steady, shrinking according as,*

$$-\frac{\text{div}V}{4} + (2\beta - \alpha)(\lambda - \kappa\rho) + \frac{1}{2}\left(p + \frac{1}{2}\right) \begin{matrix} \geq 0 \\ = 0 \\ \leq 0 \end{matrix}.$$

5. Radiation Era in Perfect Fluid Spacetime with Conformal Ricci–Yamabe Soliton

In perfect fluid spacetime, radiation era is characterized by $\sigma = 3\rho$, so in that case the energy–momentum tensor (10) takes the form [35]:

$$\mathcal{T}(X, Y) = \rho[g(X, Y) + 4\eta(X)\eta(Y)]. \tag{60}$$

Using (11) and (60), we obtain

$$S(X, Y) = \left[\kappa\rho - \lambda + \frac{r}{2} \right] g(X, Y) + 4\kappa\rho\eta(X)\eta(Y). \tag{61}$$

Let (M^4, g) be a radiation fluid spacetime which satisfies (61). Then by contracting (61) and considering $g(\zeta, \zeta) = -1$, we obtain,

$$r = 4\lambda. \tag{62}$$

Taking contraction in (8) and using (62), we have,

$$\Lambda = -\frac{\text{div}V}{4} + \lambda(2\beta - \alpha) + \frac{1}{2}\left(p + \frac{1}{2}\right). \tag{63}$$

Then we have,

Theorem 7. *If a radiation fluid spacetime admits a conformal Ricci–Yamabe soliton $(g, V, \Lambda, \alpha, \beta)$, then the soliton is expanding, steady, shrinking according as,*

$$-\frac{\text{div}V}{4} + \lambda(2\beta - \alpha) + \frac{1}{2}\left(p + \frac{1}{2}\right) \begin{matrix} \geq 0 \\ = 0 \\ \leq 0 \end{matrix}.$$

6. Conformal η -Ricci-Yamabe Soliton Structure in Perfect Fluid Spacetime

Let (M^4, g) be a general relativistic perfect fluid spacetime and $(g, \xi, \Lambda, \mu, \alpha, \beta)$ be a conformal η -Ricci-Yamabe soliton in M .

Then writing explicitly the Lie derivative $(\mathcal{L}_\xi g)$ as $(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)$ and from (9) and (14), we acquire,

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2\alpha \left[\left\{ \lambda + \frac{\kappa(\sigma - \rho)}{2} \right\} g(X, Y) + \kappa(\sigma + \rho)\eta(X)\eta(Y) \right] + \left[2\Lambda - \beta r - \left(p + \frac{1}{2} \right) \right] g(X, Y) + 2\mu\eta(X)\eta(Y) = 0 \tag{64}$$

for any vector fields X, Y .

Then the above equation can be written as,

$$\left[\Lambda - \frac{\beta r}{2} - \frac{1}{2} \left(p + \frac{1}{2} \right) + \alpha\lambda + \frac{\alpha\kappa(\sigma - \rho)}{2} \right] g(X, Y) + \left[\mu + \alpha\kappa(\sigma + \rho) \right] \eta(X)\eta(Y) + \frac{1}{2} \left[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \right] = 0. \tag{65}$$

Consider $\{e_i\}_{1 \leq i \leq 4}$ an orthonormal frame field and $\xi = \sum_{i=1}^4 \xi^i e_i$. We have from [58], $\sum_{i=1}^4 \epsilon_{ii}(\xi^i)^2 = -1$ and $\eta(e_i) = \epsilon_{ii}\xi^i$.

Multiplying (65) by ϵ_{ii} and summing over i for $X = Y = e_i$, we obtain,

$$4\Lambda - \mu = 4(2\beta - \alpha)\lambda + \kappa(2\beta - \alpha)(\sigma - 3\rho) + 2 \left(p + \frac{1}{2} \right) - \text{div}(\xi), \tag{66}$$

where $\text{div}(\xi)$ is the divergence of the vector field ξ .

Putting $X = Y = \xi$ in (65), we obtain,

$$\Lambda - \mu = (2\beta - \alpha)\lambda + \frac{\kappa}{2} \left[(2\beta + \alpha)\sigma - 3(2\beta - \alpha)\rho \right] + \frac{1}{2} \left(p + \frac{1}{2} \right). \tag{67}$$

Then calculating Λ, μ from (66) and (67), we achieve,

$$\Lambda = (2\beta - \alpha)\lambda + \frac{\kappa}{2} \left[\left(\frac{2\beta - 3\alpha}{3} \right) \sigma - (2\beta - \alpha)\rho \right] + \frac{1}{2} \left(p + \frac{1}{2} \right) - \frac{\text{div}(\xi)}{3} \tag{68}$$

and

$$\mu = -\kappa \left[\left(\frac{2\beta + 3\alpha}{3} \right) \sigma - (2\beta - \alpha)\rho \right] - \frac{\text{div}(\xi)}{3}. \tag{69}$$

Then we can state the following:

Theorem 8. Let (M^4, g) be a 4-dimensional pseudo-Riemannian manifold and η be the g -dual 1-form of the gradient vector field $\xi := \text{grad}(f)$, with $g(\xi, \xi) = -1$, where f is a smooth function. If $(g, \xi, \Lambda, \mu, \alpha, \beta)$ is a conformal η -Ricci-Yamabe soliton on M , then the Laplacian equation satisfied by f becomes:

$$\Delta(f) = -3 \left[\mu + \kappa \left\{ \left(\frac{2\beta + 3\alpha}{3} \right) \sigma - (2\beta - \alpha)\rho \right\} \right]. \tag{70}$$

Example 2. A conformal η -Ricci-Yamabe soliton $(g, \xi, \Lambda, \mu, \alpha, \beta)$ in a radiation fluid is given by:

$$\Lambda = (2\beta - \alpha)\lambda - \kappa\alpha\rho + \frac{1}{2} \left(p + \frac{1}{2} \right) - \frac{\text{div}(\xi)}{3},$$

and

$$\mu = -4\kappa\alpha\rho - \frac{\text{div}(\xi)}{3}.$$

7. Conclusions

The main study of the paper is to obtain the geometrical phenomena and characteristics of our newly introduced conformal Ricci–Yamabe and conformal η -Ricci–Yamabe solitons to apply their existence in a perfect fluid spacetime. We first give the geometric composition of a perfect fluid spacetime with torse-forming vector field ξ in connection with conformal Ricci–Yamabe metric and conformal η -Ricci–Yamabe metric. Moreover, the conditions required for the conformal Ricci–Yamabe soliton to be expanding, steady or shrinking have been given. We have contrived the conformal Yamabe soliton in perfect fluid spacetime with torse-forming vector field to accessorize the nature of this soliton on the mentioned spacetime. We have also considered the potential vector field V of the soliton as conformal Killing vector field to characterized the vector field. Furthermore, we find the nature of the conformal Ricci–Yamabe soliton in a dust fluid spacetime, dark fluid spacetime and radiation era, respectively.

Author Contributions: Conceptualization, Y.L., S.R., S.D. and A.B.; methodology, Y.L., S.R., S.D. and A.B.; investigation, Y.L., S.R., S.D. and A.B.; writing—original draft preparation, P.Z., Y.L., S.R., S.D. and A.B.; writing—review and editing, P.Z., Y.L., S.R., S.D. and A.B.; All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China (Grant No. 12101168) and Zhejiang Provincial Natural Science Foundation of China (Grant No. LQ22A010014).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We gratefully acknowledge the constructive comments from the editor and the anonymous referees.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hamilton, R.S. Three Manifold with positive Ricci curvature. *J. Differ. Geom.* **1982**, *17*, 255–306. [\[CrossRef\]](#)
2. Topping, P. *Lecture on the Ricci Flow*; Cambridge University Press: Cambridge, UK, 2006.
3. Hamilton, R.S. The Ricci flow on surfaces. *Commun. Contemp. Math.* **1988**, *71*, 237–261.
4. Basu, N.; Bhattacharyya, A. Conformal Ricci soliton in Kenmotsu manifold. *Glob. J. Adv. Res. Class. Mod. Geom.* **2015**, *4*, 15–21.
5. Siddiqi, M.D. Conformal η -Ricci solitons in δ -Lorentzian Trans Sasakian manifolds. *Int. J. Maps Math.* **2018**, *1*, 15–34.
6. Barbosa, E.; Ribeiro E. On conformal solutions of the Yamabe flow. *Arch. Math.* **2013**, *101*, 79–89. [\[CrossRef\]](#)
7. Roy, S.; Bhattacharyya, A. A Kenmotsu Metric as \ast -conformal Yamabe Soliton with Torse Forming Potential Vector Field. *Acta Math. Sci.* **2021**, *37*, 1896–1908. [\[CrossRef\]](#)
8. Cao, H.D.; Sun, X.; Zhang, Y. On the structure of gradient Yamabe solitons. *arXiv* **2011**, arXiv:1108.6316v2.
9. Dey, S.; Roy, S. \ast - η -Ricci Soliton within the framework of Sasakian manifold. *J. Dyn. Syst. Geom. Theor.* **2020**, *18*, 163–181. [\[CrossRef\]](#)
10. Ghosh, A. Yamabe soliton and Quasi Yamabe soliton on Kenmotsu manifold. *Math. Slovaca* **2020**, *70*, 151–160. [\[CrossRef\]](#)
11. Singh, A.; Kishor, S. Some types of η -Ricci Solitons on Lorentzian para-Sasakian manifolds. *Facta Univ. (NIŠ)* **2018**, *33*, 217–230. [\[CrossRef\]](#)
12. Cho, J.T.; Kimura, M. Ricci solitons and real hypersurfaces in a complex space form. *Tohoku Math. J.* **2009**, *61*, 205–212. [\[CrossRef\]](#)
13. Catino, G.; Mazzieri, L. Gradient Einstein solitons. *Nonlinear Anal.* **2016**, *132*, 66–94. [\[CrossRef\]](#)
14. Abedi, E.; Ziabari, R.B. Slant submanifolds of a conformal Sasakian manifold. *Acta Univ. Apulensis* **2014**, *40*, 35–49.
15. Abedi, E.; Ziabari, R.B.; Tripathi, M.M. Ricci and scalar curvatures of submanifolds of a conformal Sasakian space form. *Arch. Math.* **2016**, *52*, 113–130. [\[CrossRef\]](#)
16. Alegre, P.; Carriazo, A.; Kim, Y.H.; Yoon, D.W. B.Y. Chen’s inequality for submanifolds of generalized space forms. *Indian J. Pure Appl. Math.* **2007**, *38*, 185–201.
17. Akram, A.; Alkhalidi, A.H. Chen inequalities for warped product pointwise bi-slant submanifolds of complex space forms and its applications. *Symmetry* **2019**, *11*, 200.
18. Akram, A.; Mofarreh, F.; Laurian-Ioan, P.; Alluhaibi, N. Geometry of k -Yamabe Solitons on Euclidean Spaces and Its Applications to Concurrent Vector Fields. *Symmetry* **2021**, *13*, 222.

19. Akram, A.; Piscoran, L.I. Geometric classification of warped products isometrically immersed into Sasakian space form. *Math. Nachr.* **2019**, *292*, 234–251. [[CrossRef](#)]
20. Akram, A.; Piscoran, L.I. Geometry of warped product immersions of Kenmotsu space forms and its applications to slant immersions. *J. Geom. Phys.* **2017**, *114*, 276–290.
21. Akram, A.; Uddin, S.; Othman, W.A.M. Geometry of warped product pointwise semi-slant submanifolds of Kahler manifolds. *Filomat* **2017**, *12*, 3771–3778.
22. Al-Solamy F.R.; Khan, V.A. Warped product semi-slant submanifolds of a Sasakian manifold. *Serdica Math. J.* **2008**, *34*, 597–606.
23. Yang, Z.C.; Li, Y.L.; Erdoğan, M.; Zhu, Y.S. Evolving evolutooids and pedaloids from viewpoints of envelope and singularity theory in Minkowski plane. *J. Geom. Phys.* **2022**, *104513*, 1–23.
24. Li, Y.L.; Liu, S.Y.; Wang, Z.G. Tangent developables and Darboux developables of framed curves. *Topol. Appl.* **2021**, *301*, 107526. [[CrossRef](#)]
25. Li, Y.L.; Wang, Z.G. Lightlike tangent developables in de Sitter 3-space. *J. Geom. Phys.* **2021**, *164*, 1–11. [[CrossRef](#)]
26. Li, Y.L.; Wang, Z.G.; Zhao, T.H. Geometric Algebra of Singular Ruled Surfaces. *Adv. Appl. Clifford Algebr.* **2021**, *31*, 1–19. [[CrossRef](#)]
27. Li, Y.L.; Zhu, Y.S.; Sun, Q.Y. Singularities and dualities of pedal curves in pseudo-hyperbolic and de Sitter space. *Int. J. Geom. Methods Mod. Phys.* **2021**, *18*, 1–31. [[CrossRef](#)]
28. Li, Y.L.; Ganguly, D.; Dey, S.; Bhattacharyya, A. Conformal η -Ricci solitons within the framework of indefinite Kenmotsu manifolds. *AIMS Math.* **2022**, *7*, 5408–5430. [[CrossRef](#)]
29. Li, Y.L.; Alkhalidi, A.H.; Ali, A.; Laurian-Ioan, P. On the Topology of Warped Product Pointwise Semi-Slant Submanifolds with Positive Curvature. *Mathematics* **2021**, *9*, 3156. [[CrossRef](#)]
30. Li, Y.L.; Lone, M.A.; Wani, U.A. Biharmonic submanifolds of Kähler product manifolds. *AIMS Math.* **2021**, *6*, 9309–9321. [[CrossRef](#)]
31. Li, Y.L.; Ali, A.; Ali, R. A general inequality for CR-warped products in generalized Sasakian space form and its applications. *Adv. Math. Phys.* **2021**, *2021*, 5777554. [[CrossRef](#)]
32. Li, Y.L.; Ali, A.; Mofarreh, F.; Abolarinwa, A.; Ali, R. Some eigenvalues estimate for the ϕ -Laplace operator on slant submanifolds of Sasakian space forms. *J. Funct. Space* **2021**, *2021*, 6195939.
33. Li, Y.L.; Ali, A.; Mofarreh, F.; Alluhaibi, N. Homology groups in warped product submanifolds in hyperbolic spaces. *J. Math.* **2021**, *2021*, 8554738. [[CrossRef](#)]
34. Kumara, H.A. Ricci soliton and geometrical structure in a perfect fluid spacetime with torse-forming vector field. *Afr. Mat.* **2019**, *30*, 725–736.
35. Siddiqi, M.D.; Siddiqi, S.A. Conformal Ricci soliton and Geometrical structure in a perfect fluid spacetime. *Int. J. Geom. Methods Mod. Phys.* **2020**, *17*, 2050083. [[CrossRef](#)]
36. Arsenović, M.; Manojlović, V.; Näkki, R. Boundary modulus of continuity and quasiconformal mappings. *Ann. Acad. Sci. Fenn. Math.* **2012**, *37*, 107–118. [[CrossRef](#)]
37. Cvetković, D.M.; Todorčević, V. Cospectrality graphs of Smith graphs. *Filomat* **2019**, *33*, 3269–3276. [[CrossRef](#)]
38. Klén, R.; Todorčević, V.; Vuorinen, M. Teichmüller’s problem in space. *J. Math. Anal. Appl.* **2017**, *455*, 1297–1316. [[CrossRef](#)]
39. Klén, R.; Manojlović, V.; Simić, S.; Vuorinen, M. Bernoulli inequality and hypergeometric functions. *Proc. Am. Math. Soc.* **2014**, *142*, 559–573. [[CrossRef](#)]
40. Koskela, P.; Manojlović, V. Quasi-nearly subharmonic functions and quasiconformal mappings. *Potential Anal.* **2012**, *37*, 187–196. [[CrossRef](#)]
41. Kojić, V.; Pavlović, M. Subharmonicity of $|f|^p$ for quasiregular harmonic functions, with applications. *J. Math. Anal. Appl.* **2008**, *342*, 742–746. [[CrossRef](#)]
42. Kojić, V. Quasi-nearly subharmonic functions and conformal mappings. *Filomat* **2007**, *21*, 243–249. [[CrossRef](#)]
43. Manojlović, V.; Vuorinen, M. On quasiconformal maps with identity boundary values. *Trans. Am. Math. Soc.* **2011**, *363*, 2367–2479. [[CrossRef](#)]
44. Manojlović, V. On bilipschicity of quasiconformal harmonic mappings. *Novi Sad J. Math.* **2015**, *45*, 105–109. [[CrossRef](#)]
45. Manojlović, V. Bilipschitz mappings between sectors in planes and quasi-conformality. *Funct. Anal. Approx. Comput.* **2009**, *1*, 1–6.
46. Manojlović, V. Bi-Lipschicity of quasiconformal harmonic mappings in the plane. *Filomat* **2009**, *23*, 85–89. [[CrossRef](#)]
47. Manojlović, V. On conformally invariant extremal problems. *Appl. Anal. Discret. Math.* **2009**, *3*, 97–119. [[CrossRef](#)]
48. Nezhad, A.D.; Radenović, S. Some aspects of $b_{(\alpha_n, \beta_n)}$ -hypermtric spaces over banach algebras. *Eur. J. Pure Appl. Math.* **2021**, *14*, 1148–1160. [[CrossRef](#)]
49. Nezhad, A.D.; Ahmadi, A. A novel approach to sheaves on diffeological spaces. *Topol. Appl.* **2019**, *263*, 141–153. [[CrossRef](#)]
50. Nezhad, A.D.; Davvaz, B. Universal hyperdynamical systems. *Bull. Korean Math. Soc.* **2010**, *47*, 513–526. [[CrossRef](#)]
51. Simić, S.; Todorčević, V. Jensen Functional, Quasi-Arithmetic Mean and Sharp Converse of Hölder’s Inequalities. *Mathematics* **2021**, *9*, 3104. [[CrossRef](#)]
52. Srivastava, S.K. *General Relativity and Cosmology*; Prentice-Hall of India Private Limited: New Delhi, India, 2008.
53. Todorčević, V. *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*; Springer International Publishing: Berlin/Heidelberg, Germany, 2019.
54. Todorčević, V. Subharmonic behavior and quasiconformal mappings. *Anal. Math. Phys.* **2019**, *9*, 1211–1225. [[CrossRef](#)]

55. Güler, S.; Crasmareanu, M. Ricci–Yamabe maps for Riemannian flows and their volume variation and volume entropy. *Turk. J. Math.* **2019**, *43*, 2361–2641. [[CrossRef](#)]
56. O’Neill, B.; *Semi-Riemannian Geometry with Applications to Relativity*; Academic Press: New York, NY, USA, 1983.
57. Yano, K. On the torse-forming directions in Riemannian spaces. *Proc. Imp. Acad. Tokyo* **1944**, *20*, 340–345. [[CrossRef](#)]
58. Blaga, A.M. Solitons and geometrical structures in a perfect fluid spacetime. *arXiv* **2017**, arXiv:1705.04094.