Geometrically isolated/nonisolated solutions and their approximation

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RESUME

Une solution $x=x^o$ de l'équation $F(x)=0$ est dite "isolée" si la dérivée de Fréchet $F'(x^o)$ est non singulière. Elle est dite "géométriquement isolée" si il n'y a pas d'autre solution dans $\|x-x^o\| \leq \rho$ pour un certain $\rho > 0$. Les solutions isolées sont toujours géométriquement isolées. Dans ce travail on donne des conditions suffisantes pour qu'une solution non isolée soit néanmoins géométriquement isolée. On étudie ensuite des méthodes d'approximation de la forme générale

$$F_h(x_h) = 0$$

pour approcher les solutions non isolées qui sont géométriquement isolées. Les résultats obtenus sont, en un certain sens, négatifs, puisque l'on montre que sous des conditions de consistance assez fortes, le problème approché peut avoir un nombre pair (éventuellement zéro) ou un nombre pair de solutions au voisinage de x^o, ce nombre étant dépendant de la "multiplicité" de la solution x^o. Si la précision est en $O(h^p)$ et si la multiplicité est N, alors l'erreur d'approximation est en $O(h^{p/N})$. Les relations qui existent entre ces résultats et l'approximation des points de bifurcation et de retournement est brièvement analysée.

ABSTRACT

A solution $x=x^o$ of $F(x)=0$ is said to be "isolated if the Fréchet derivative $F'(x^o)$ is nonsingular. It is said to be "geometrically isolated" if no other solution is in : $\|x-x^o\| \leq \rho$ for some $\rho > 0$. Isolated solutions are always geometrically isolated. Sufficient conditions are obtained to insure that a nonisolated solution is also geometrically isolated. We then study the application of approximation methods, in the general form $F_h(x_h)=0$, to approximate nonisolated solutions which are geometrically isolated. Under strong consistency conditions the results are somewhat negative - the approximations may have an even number (including zero) or an odd number of roots near x^o, depending upon the "multiplicity" of x^o as a root. If the accuracy is $O(h^p)$ and the multiplicity is N, then the approximations have error $O(h^{p/N})$. The relation of these results to limit and bifurcation points is discussed briefly.

1. INTRODUCTION.

A solution $X = X^o$ of

$$(1.1) \qquad F(X) = 0,$$

in an appropriate Banach space setting, is said to be isolated if the Fréchet derivative $F'(X^o)$ is nonsingular. We shall say that a solution X^o is geometrically isolated if it is the only solution of (1.1) in some neighborhood of X^o. Of course an isolated solution is also geometrically isolated, as we show in §2, but the converse is not true. Indeed the terminology is somewhat unfortunate since a nonisolated solution, X^o, that is one for which $F'(X^o)$ is singular, may also be geometrically isolated. In §2 we derive sufficient conditions for this to occur. We do this for scalar equations and more generally for nonisolated solutions for which the null space of $F'(X^o)$ is one dimensional. We also consider the case of an n-dimensional null space in §2.

Computations of approximations to a solution of (1.1) can be studied in terms of a family of approximating problems, say in the form

$$(1.2) \qquad F_h(X_h) = 0 \ , \ \forall \ h \in (0,h_o] \ .$$

A general theory of such approximations for isolated solutions is developed in [5]. In particular sufficient conditions are given to insure that (1.2) has a unique solution X_h^o near each isolated solution X^o of (1.1) for each $h \in (0,h_o]$, and $X_h^o \to X^o$ as $h \to 0$. Further Newton's method is valid and converges quadratically to the X_h^o provided that $F(X)$ and $F_h(X_h)$ are "consistent" and that the Fréchet derivatives $F_h'(X_h)$ are Lipschitz continuous, with uniformly bounded inverses on $(0,h_o]$ at appropriate points related to X^o. This theory is not valid for nonisolated solutions and in §3 and 4 we seek to extend it to geometrically isolated nonisolated solutions.

In §3 we study real roots of scalar analytic equations. This simple but instructive case makes it clear that the situation for geometrically isolated nonisolated roots is rather complicated ; no solution or multiple solutions can exist for the approximating equations. The analysis is extended in §4 to the more general (Banach space) setting of nonisolated solutions with a one dimensional null space. Using the Lyapunov-Schmidt procedure and the theory of [5] for isolated solutions we reduce this case to that of the scalar case treated in §3.

Nonisolated solutions of equations whose linearizations, $F'(X^o)$, have one dimensional null spaces occur frequently in applications displaying limit point or bifurcation behaviour [6]. Indeed of all nonisolated solutions those with one dimensional null spaces are generic, so we have studied here the dominant case. In §5 we indicate how our theory is related to attempts to compute simple limit points or simple bifurcation points in nonlinear eigenvalue problems. The conclusion is unfortunately that direct attempts at such computations may easily fail or give inconclusive results. Indeed one way in which such points are located is not to approximate them directly but rather to approximate an entire arc of solutions on which they lie [1,6,7]. A more complete study of the approximation of such arcs is given in [8,12]. Alternative methods for approximating nonisolated solutions "directly", employ modifications of the problem such that the desired solution is contained as part of an isolated solution to another (inflated) problem [9,11].

In an Appendix we show how the simplest sufficient condition of §2, which insures geometric isolation of a nonisolated solution, also insures another geometric condition important for the application of Newton's method to the singular case [2,3,10].

2. GEOMETRICALLY ISOLATED SOLUTIONS.

A solution, X^o, of (1.1) is defined to be geometrically isolated if for some sufficiently small $r > 0$:

$$(2.1) \qquad F(X) \neq 0 \quad \forall\, X \in S_r(X^o) - \{X^o\},$$

where

$$S_r(X^o) \equiv \{X : \|X - X^o\| \leq r\}.$$

It is more or less well known, or at least generally accepted, that isolated solutions are geometrically isolated. Proofs of this are easily given when

$F'(X)$ is continuous on $S_r(X^o)$. However the condition of isolation is sufficient as we proceed to show.

Suppose X^o is an isolated but not geometrically isolated root of (1.1). Then X^o must be the limit point of some sequence of roots, say :

$$\lim_{\nu\to\infty} X_\nu = X^o \ ; \ F(X_\nu) = 0, \ \nu = 1,2,3,\ldots$$

From the definition of the Fréchet derivative at X^o we must have

$$\lim_{\nu\to\infty} \frac{\|F(X_\nu) - F(X^o) + F'(X^o)(X_\nu - X^o)\|}{\|X_\nu - X^o\|} = 0$$

This implies, since X^o and the X_ν satisfy (1.1), that

$$\lim_{\nu\to\infty} \|F'(X^o)\phi_\nu\| = 0,$$

where

$$\phi_\nu \equiv \frac{X_\nu - X^o}{\|X_\nu - X^o\|} \quad , \ \|\phi_\nu\| = 1 \ , \nu = 1,2,\ldots \ .$$

But this contradicts the nonsingularity of $F'(X^o)$. A proof using contractions is given in [5] but it imposes the additional hypothesis that $F'(X)$ is Lipschitz continuous in $S_r(X^o)$ for some $r > 0$.

In the scalar case, say :

(2.2) $$f(\xi) = 0 \ , \ f(\cdot) : \mathbb{R} \to \mathbb{R}, \ f(\xi) \in C^\infty(\mathbb{R}) \ ;$$

it is easy to give sufficient conditions for a nonisolated root, $\xi = \xi^o$, to be geometrically isolated. Indeed we define

(2.3) $$a_\nu \equiv \frac{1}{\nu!} \frac{d^\nu f(\xi^o)}{d\xi^\nu} \ , \ \nu = 0,1,2,\ldots$$

and then we have

Lemma 2.4 : Let $f(\cdot) : \mathbb{R} \to \mathbb{R}$ and $f(\xi) \in C^{\infty}(\mathbb{R})$. Then $\xi = \xi^{o}$ is a geometrically isolated, nonisolated solution of (2.2) if for some integer $N \geq 2$:

(2.4) $$a_o = a_1 = \ldots = a_{N-1} = 0, \; a_N \neq 0 \; .$$

Proof : That ξ^{o} is a nonisolated root of (2.2) follows from $a_o = a_1 = 0$. The geometric isolation is obvious from the graph of $f(\xi)$ which simply has N^{th} order contact with the ξ-axis at $\xi = \xi^{o}$. A formal proof easily follows from the Taylor expansion and (2.4) which give :

(2.5) $$\begin{cases} f(\xi) = \sum_{\nu=0}^{N} a_{\nu}(\xi-\xi^{o}) + \mathcal{O}((\xi-\xi^{o})^{N+1}) \; , \\ \quad\;\; = (\xi-\xi^{o})^{N}[a_N + \mathcal{O}(\xi-\xi^{o})] \; . \end{cases}$$

Clearly the factor $[a_N + \mathcal{O}(\xi-\xi^{o})]$ cannot vanish in $|\xi-\xi^{o}| \leq r$ for r sufficiently small. ■

The conditions (2.4) are sufficient but not necessary for geometric isolation as is shown by

$$f(\xi) \equiv \begin{cases} 0 \; , & \xi = 0 \; , \\ e^{-1/\xi^2} , & \xi \neq 0 \; . \end{cases}$$

Note however that $f(\xi)$ is not analytic at $\xi=0$.

We turn now to the more general Banach space setting and consider :

(2.6) $$F(X) = 0, \; F(\cdot) : \mathbb{X} \to \mathbb{Y} \; , \; \mathbb{X} \subseteq \mathbb{Y}.$$

Here $\mathbb{X}$ and $\mathbb{Y}$ are appropriate Banach spaces and $F(X)$ is to be analytic in the present study. We assume that $X=X^{o}$ is a nonisolated solution of (2.6) and in particular that the Fréchet derivative $F'(X^{o})$ has a one dimensional null space, $\mathcal{N}$, and range, $\mathcal{R}$, satisfying :

(2.7) $$\begin{cases} \text{a)} \;\; \mathcal{N}(F'(X^{o})) = \text{span}\,\{\phi\}, \|\phi\| = 1 \; ; \\ \text{b)} \;\; \mathcal{R}(F'(X^{o})) \text{ is closed with codim} = 1 \; . \end{cases}$$

Thus $F'(X^o)$ is a Fredholm operator of index zero. Then the adjoint operator, say $F'(X^o)^* : \mathbb{Y}^* \to \mathbb{X}^*$, where $\mathbb{Y}^*$ and $\mathbb{X}^*$ are the dual spaces of $\mathbb{Y}$ and $\mathbb{X}$, has

$$(2.7) \quad \begin{cases} \text{c)}\ \mathcal{N}(F'(X^o)^*) = \text{span}\,\{\phi^*\} \ ; \\ \text{d)}\ \mathcal{R}(F'(X^o)) = \{y \in \mathbb{Y} : \phi^* y = 0\} \ . \end{cases}$$

We further assume that the zero eigenvalue of $F'(X^o)$ is simple (i.e. has algebraic multiplicity one) so that we may take

$$(2.7) \qquad \text{e)}\ \phi^*\phi = 1 .$$

The basic result for geometric isolation is given in

Theorem 2.8 : Let $F(\cdot) : \mathbb{X} \to \mathbb{Y}$, $\mathbb{X} \subseteq \mathbb{Y}$ have a nonisolated root $X = X^o$ of (2.6) satisfying (2.7). Let $F(X)$ be analytic on $S_r(X^o)$ for some $r > 0$ and set

$$(2.8) \qquad a_2 \equiv \frac{1}{2}\phi^* F''(X^o)\phi\phi .$$

Then X^o is geometrically isolated if $a_2 \neq 0$.

Proof : We first decompose $\mathbb{X}$ and $\mathbb{Y}$ into the direct sums :

$$(2.9) \qquad \text{a)}\ \mathbb{X} = \{\phi\} \oplus \mathbb{X}_1 , \quad \mathbb{Y} = \{\phi\} \oplus \mathbb{Y}_1$$

where

$$(2.9) \qquad \text{b)}\ \mathbb{X}_1 \equiv \{X \in \mathbb{X} : \phi^* X = 0\} , \quad \mathbb{Y}_1 \equiv \mathcal{R}(F'(X^o)) .$$

In addition we use the projections :

$$(2.9) \quad \begin{aligned} &\text{c)}\quad P \equiv \phi\phi^* : \mathbb{Y} \to \{\phi\} , \\ &\text{d)}\quad Q \equiv I - P : \mathbb{Y} \to \mathbb{Y}_1 . \end{aligned}$$

Now every point $X \in S_r(X^o) \subset \mathbb{X}$ has a unique representation in the form

$$(2.10) \qquad \text{a)}\ X = X^o + \xi\phi + v ,$$

where

(2.10) b) $\xi \in \mathbb{R}$, $v \in \mathbb{X}_1$, $\|\xi\phi+v\| \le r$.

If $F(X) = 0$ at a point in $S_r(X^o)$ we must have, using (2.9), (2.10) that :

$$\text{(2.11)} \quad \begin{aligned} &\text{a) } QF(X^o+\xi\phi+v) \equiv H(\xi,v) = 0 \ , \\ &\text{b) } PF(X^o+\xi\phi+v) = 0. \end{aligned}$$

We first examine all points satisfying (2.11a). Note that $H : \mathbb{R}\times \mathbb{X}_1 \to \mathbb{Y}_1$ has $H(0,0) = 0$. Furthermore the Frechet derivative with respect to v at $(\xi,v) = (0,0)$:

$$D_v H(0,0) = QF'(X^o) \equiv A$$

is an isomorphism on $\mathbb{X}_1 \to \mathbb{Y}_1$. Indeed this linear operator is just

$$A \equiv (F'(X^o)/\ \mathbb{X}_1) \ ,$$

the restriction of $F'(X^o)$ to $\mathbb{X}_1$ and it has a bounded inverse. Now the implicit function theorem can be applied to (2.11a) and it insures the existence of a unique one dimensional analytic manifold of solutions, say

(2.12) a) $(\xi,v) = (\xi,v(\xi))$, $\|\xi\phi+v(\xi)\| < r$,

where

(2.12) b) $v(0) = 0$, $v'(\xi) = 0$.

The last result follows from differentiating the identity

(2.12) c) $H(\xi,v(\xi)) = 0$

and evaluating at ξ=0 to get

$$QF'(X^o)[\phi+v((0)] = 0 \ .$$

But $F'(X^o)\phi = 0$ and $v'(0) \in \mathbb{X}_1$ since $v(\xi) \in \mathbb{X}_1$. So $QF'(X^o)v'(0) = 0$ implies $v'(0) = 0$. We have thus shown by the uniqueness part of the implicit function theorem, that all solutions of (2.11a) in $S_r(X^o)$ form the one dimensional manifold :

(2.13) a) $X^o(\xi) \equiv X^o+\xi\phi+v(\xi)$, $\|\xi\phi+v(\xi)\| \leq r$.

If any point in $S_r(X^o)$ is a solution of (2.6) it must lie on the manifold (2.13a). In addition such a solution must satisfy (2.11b). Thus we have shown that all solutions of (2.6) which lie in $S_r(X^o)$ have the form (2.13a) with a value of ξ that satisfies

(2.13) b) $f(\xi) \equiv \phi^* F(X^o(\xi)) = 0$.

Clearly $f(0) = 0$ so that $\xi = \xi^o = 0$ is one root of (2.13b). If $\xi^o = 0$ is a geometrically isolated root of (2.13b) then $X^o(0) = X^o$ is a geometrically isolated root of (2.6). Thus we have reduced the problem to the scalar case. Indeed we easily see that

(2.14) a) $a_o \equiv f(0) = 0,\ a_1 = f'(0) = 0$.

so that $\xi^o = 0$ is a nonisolated root. However we continue differentiating $f(\xi)$ using (2.13) to get that

(2.14) b) $a_2 \equiv \frac{1}{2} f''(0) = \frac{1}{2} \phi^* F''(X^o)\phi\phi$.

An application of Lemma 2.4 with N=2 concludes our proof.
It is clear from the above proof how we can obtain additional conditions to insure geometric isolation when $a_2 = 0$. We state these results in

Corollary 2.15 : Let the hypothesis of Theorem 2.8 hold with $a_2 = 0$ and set :

(2.15) a) $a_3 \equiv \frac{1}{3!} \phi^* F'''(X^o)\phi\phi\phi + \frac{1}{2}\phi^* F''(X^o)\phi\psi_2$,

where ψ_2 is the unique solution of

(2.15) b) $F'(X^o)\psi_2 = -F''(X^o)\phi\phi$, $\psi_2 \in \mathbb{X}_1$.

Then X^o is geometrically isolated if $a_3 \neq 0$.

Proof : We simply continue as in the proof of Theorem 2.8 to find that

$$a_3 = \frac{1}{3!}\, f'''(0).$$

If $a_2 = 0$ then $F''(X^o)\phi\phi \in \mathcal{R}(F'(X^o))$ and $\psi_2 \equiv v''(0)$ is uniquely defined by (2.15b).

We need not state additional corollaries but if $a_2=a_3=0$ we get that

$$\text{(2.16) a)}\quad \begin{cases} a_4 \equiv \frac{1}{4!}\, f^{iv}(X^o) \\ \quad = \frac{1}{4!}\, F^{iv}(X^o)\phi\phi\phi\phi + \frac{1}{4}\, F'''(X^o)\phi\phi\psi_2 + F''(X^o)[\frac{1}{8}\psi_2\psi_2 + \frac{1}{6}\phi\psi_3] \;. \end{cases}$$

Here $\psi_3 \equiv v'''(0)$ is the unique solution of

$$\text{(2.16) b)}\qquad F'(X^o)\psi_3 = -[F'''(X^o)\phi\phi\phi + 3F''(X^o)\phi\psi_2]\;,\quad \psi_3 \in \mathbf{X}_1 \;.$$

Clearly $a_4 \neq 0$ implies geometric isolation of X^o when $a_2=a_3=0$. And so it goes - but a general formula for the a_ν does not seem worthwhile ; in practical problems one seldom goes higher than a_3. Analyticity is not required in any of the above results but we use it in § 4.

It is perhaps somewhat remarkable that the condition $a_2 \neq 0$ has a completely different geometric implication. Specifically it implies that the manifold on which $F'(X)$ is singular is transversal at $X = X^o$ to the direction of $\mathcal{N}(F'(X^o))$. This is discussed in Appendix I and related to Newton's method for nonisolated solutions.

The methods employed in the proof of Theorem 2.8 can easily be extended to obtain sufficient conditions for geometric isolation when the null space is higher dimensional. To do this we assume that $X = X^o$ is a nonisolated solution of (2.6) for which $F'(X^o)$ has : an m-dimensional null space

$$\text{(2.17) a)}\qquad \mathcal{N}(F'(X^o)) = \text{span}\,\{\phi_1,\ldots,\phi_m\},\ \|\phi_j\| = 1\;;$$

a closed range with codim $\mathcal{R}(F'(X^o)) = m$ so that

(2.17) b) $\mathcal{N}(F'(X^o)^*) = \text{span}\,\{\phi_1^*,\ldots,\phi_m^*\}$;

the eigenvalue zero of $F'(X^o)$ with geometric multiplicity m so that

(2.17) c) $\phi_j^*\phi_k = \delta_{jk}$, $j,k=1,2,\ldots,m$.

Now we state

Theorem 2.18 : Let $X=X^o$ be a nonisolated solution of (2.6) satisfying (2.17). Set :

(2.18) a) $a_{ijk} = a_{ikj} \equiv \phi_i^* F''(X^o)\phi_k\phi_j$, $i,j,k=1,2,\ldots,m$.

For any $\zeta \equiv (\zeta_1,\ldots,\zeta_m) \in \mathbb{R}^m$ define the m×m matrix $A(\zeta)$ by

(2.18) b) $A_{ij}(\zeta) \equiv \sum_{k=1}^{m} a_{ijk}\zeta_k$, $i,j=1,2,\ldots,m$.

Then X^o is geometrically isolated if

(2.18) c) $A(\zeta)\zeta^T \neq 0 \quad \forall \|\zeta\| = 1$.

Proof : The proof is a rather clear generalization of that of Theorem 2.8 so we merely sketch the details. Points near X^o are represented a

$$X = X^o + \sum_1^m \xi_j\phi_j + v$$

where

$$v \in \mathbb{X}_m \equiv \{v \in \mathbb{X} : \phi_i^* v = 0,\ 1 \le i \le m\} .$$

The projections

$$P_m \equiv \sum_1^m \phi_j\phi_j^* \quad , \quad Q_m \equiv I - P_m$$

are used to decompose $F(X) = 0$ into :

$$Q_m F(X) = 0 \ , \ P_m F(X) = 0.$$

By the implicit function theorem we find all solutions of

$$H(\xi,v) \equiv Q_m F(X^o + \sum_1^m \xi_j \phi_j + v) = 0$$

near X^o. This gives some m-dimensional manifold

$$X(\xi) = X^o + \sum_1^m \xi_j \phi_j + v(\xi),$$

containing $X(0) = X^o$. The problem is thus reduced to showing that $\xi=0$ is a geometrically isolated root of

$$f(\xi) = P_m F(X(\xi)) = 0.$$

However the jacobian matrix

$$\left.\frac{Df(\xi)}{D\xi}\right|_{\xi=0} = (\phi_i^* F'(X^o) \frac{\partial v(0)}{\partial \zeta_j}) = 0 ,$$

so that $\xi=0$ is not an isolated root of $f(\xi) = 0$.

We represent the points $\xi \in S_\rho(0) \subset \mathbb{R}^m$ using polar coordinates, say :

$$\xi = \varepsilon\zeta, \|\zeta\| = 1, \; \zeta \in \mathbb{R}^m \; ; \; 0 \le \varepsilon \le \rho.$$

Then we define

$$g(\varepsilon,\zeta) \equiv f(\varepsilon\zeta)$$

and consider the Taylor expansion :

$$g(\varepsilon,\zeta) = g(0,\zeta) + \varepsilon g_\varepsilon(0,\zeta) + \frac{\varepsilon^2}{2} g_{\varepsilon\varepsilon}(0,\zeta) + \mathcal{O}(\varepsilon^3).$$

We find that

$$g(\varepsilon,\zeta) = \frac{\varepsilon^2}{2} A(\zeta)\zeta^T + \mathcal{O}(\varepsilon^3).$$

Thus if (2.18c) holds it follows that $g(\varepsilon,\zeta)$ has a geometrically isolated zero at $\varepsilon=0$. ∎

Note that condition (2.18c) is simply $a_2 \neq 0$ in (2.8) when m=1. It is not difficult to extend the above results when $A(\zeta)\zeta^T = 0$ has geometrically isolated roots, $\pm\zeta^o$, on $||\zeta|| = 1$. We need then only examine the vectors

$$\underset{\sim}{a}_\nu(\pm\zeta^o) \equiv \left.\frac{d^\nu g(\varepsilon,\pm\zeta^o)}{d\varepsilon^\nu}\right|_{\varepsilon=0} ,$$

to find some $\underset{\sim}{a}_\nu(\pm\zeta^o) \neq 0$ for $\nu \geq 3$. However the resulting conditions get rather complicated and we have not bothered to include them here.

3. APPROXIMATING PROBLEMS : SCALAR CASE.

We examine first the simple but important case of approximating the root, ξ^o, of some scalar analytic equation (2.2) by the roots of a family of approximating analytic functions. Thus in place of (2.2) we consider an approximating family of problems, say :

$$(3.1)\qquad \begin{cases} f_h(\xi) = 0 \ , \ f_h(\cdot) = \mathbb{R} \to \mathbb{R} \ , \\ f_h(\xi) \text{ analytic on } D_{r(h)}(\xi^o) \equiv \{\xi \in C \ : \ |\xi-\xi^o| \leq r(h)\} \ . \end{cases}$$

The parameter $h \in (0,h_o]$ lables the approximation and h_o is to be sufficiently small. Further the radius $r(h)$ is a monotone increasing function of h with $r(h_o)$ sufficiently small. Further details on h_o and $r(h)$ are given in (3.3). Approximations of the form (3.1) to scalar equations of the form (2.2) are common if $f(\xi)$ involves transcendental functions and digital computer evaluation is contemplated. Obviously smaller h corresponds to more accurate approximations obtained, say, from truncation of infinite series, discretizations of differential equations, numerical quadrature, etc. As a consequence it is reasonable to assume that the consistency between $f(\xi)$ and the $f_h(\xi)$ is such that $\forall\ h \in (0,h_o]$ and for some $p > 0$:

$$(3.2)\qquad \begin{aligned} &\text{a)}\ |f_h(\xi)-f(\xi)| \leq Mh^{p_o} \ , \ p_o \geq p, \quad \forall\xi \in D_{r(h)}(\xi^o) \ ; \\ &\text{b)}\ \frac{1}{\nu!}|f_h^{(\nu)}(\xi^o)-f^{(\nu)}(\xi^o)| \leq Mh^{p_\nu}, \ p_\nu \geq p \ , \ \nu=1,2,\dots,m \ ; \\ &\text{c)}\ \frac{1}{m!}\ |f_h^{(m)}(\xi)-f_h^{(m)}(\xi^o)| \leq K_m|\xi-\xi^o| \ , \ \forall\xi \in D_{r(h)}(\xi^o). \end{aligned}$$

We also require that $f^{(N)}(\xi)$ satisfy a Lipschitz condition similar to (3.2c), say :

$$(3.2)\quad d)\qquad \frac{1}{N!}\,|f^{(N)}(\xi)-f^{(N)}(\xi^o)| \le K_N|\xi-\xi^o|\ ,\quad \forall\xi\in D_{r(h)}(\xi^o).$$

The roots of (3.1) as approximations to the roots of (2.2) are as described in

<u>Theorem 3.3</u> : <u>Let</u> $f(\xi)$ <u>be analytic on</u> $D_{r(h_o)}(\xi^o)$ <u>and satisfy</u> (2.2)-(2.4) (<u>i.e. have a real</u> N-<u>fold zero at</u> $\xi=\xi^o$). <u>Let</u> $f(\xi)$ <u>and the family</u> $\{f_h(\xi)\}$ <u>satisfy</u> (3.1)-(3.2) <u>with</u> m=N. <u>Define</u> r_o, h_o <u>and</u> $r(h)$ <u>by</u> :

$$(3.3)\qquad r_o \equiv \left(\frac{4M}{|a_N|}\right)^{1/N},\quad h_o \equiv \left(\frac{1}{r^o}\min[r_o, \frac{1}{2}, \frac{|a_N|}{2K_N}]\right)^{N/p},\quad r(h) \equiv r_o h^{p/N}.$$

<u>Then for each</u> $h\in(o,h_o]$ <u>the approximate equation</u> $f_h(\xi)=0$ <u>has exactly</u> N <u>roots in the open disk</u> : $|\xi-\xi^o|<r(h)$. <u>When</u> N=2n (<u>or</u> N=2n+1) <u>there are</u> 2ν (or $2\nu+1$) <u>real roots in</u> $(\xi^o-r(h),\xi^o+r(h))$ <u>for some</u> $\nu=0,1,\dots,n$.

<u>Proof</u> : By Taylor's expansion about ξ^o we can write

$$(3.4)\quad a)\qquad f_h(\xi) = g_N(\xi,h) + q_N(\xi,h)\ ;$$

where we use (2.4) to get :

$$(3.4)\quad b)\qquad \begin{aligned} g_N(\xi,h) &\equiv \left(a_N + \frac{1}{N!}[f_h^{(N)}(\zeta(\xi))-f_h^{(N)}(\xi^o)]\right)(\xi-\xi^o)^N,\\ q_N(\xi,h) &\equiv \sum_{\nu=0}^{N}\frac{1}{\nu!}[f_h^{(\nu)}(\xi^o)-f^{(\nu)}(\xi^o)]\ (\xi-\xi^o)\ . \end{aligned}$$

Although we have written the coefficient of the remainder term as $f_h^{(N)}(\zeta(\xi))$ both $g_N(\xi,h)$ and $q_N(\xi,h)$ are analytic in $D_{r(h_o)}(\xi^o)$. We have used this form, which does not show the analyticity, since we use (3.2b) to obtain the lower bound :

$$(3.4)\quad c)\qquad \begin{aligned} |g_N(\xi,h)| &\ge (|a_N|-K_N|\xi-\xi^o|)|\xi-\xi^o|^N\ ,\\ &\ge \frac{|a_N|}{2}\,|\xi-\xi^o|^N\ ,\quad \forall\xi\in D_{r(h)}(\xi^o). \end{aligned}$$

Here we have used the last term in the definition of h_o given in (3.3) and (3.4c) is valid $\forall\ h\in(0,h_o]$.

It clearly follows from this result that $g_N(\xi,h) = 0$ has only the N-fold root $\xi=\xi^o$ in $D_{r(h)}(\xi^o)$.
On the circle $|\xi-\xi^o| = r(h)$, bounding the disk $D_{r(h)}(\xi^o)$, we use (3.2a,b) to get :

$$|q_N(\xi,h)| \le \sum_{\nu=0}^{N} Mh^{p\nu} r^{\nu}(h),$$

$$\le Mh^p \frac{1-r^{N+1}(h)}{1-r(h)} ,$$

(3.4) d) $$\le 2Mh^p , \quad \forall h \in (0,h^o] .$$

Here we have noted that $r(h) \le 1/2$ by (3.3). From (3.4c,d) it follows that $|g_N(\xi,h)| > |q_N(\xi,h)|$ on $|\xi-\xi^o| = r(h)$. Then Roche's theorem implies that $f_h(\xi) = 0$ has exactly N roots in $|\xi-\xi^o| < r(h)$.

To determine how many of these roots are real we consider $f(\xi)$ and $f_h(\xi)$ on $\mathbb{R} \to \mathbb{R}$. We claim that

(3.5) a) $$f(\xi^o+r(h))\cdot f(\xi^o-r(h)) \begin{cases} > 0 \text{ for } N=2n, \text{ even} ; \\ < 0 \text{ for } N=2n+1, \text{ odd.} \end{cases}$$

This follows since, exactly as in the derivation of (3.4c), but using (3.2c) :

$$|f(\xi^o \pm r(h))| > r^N(h)\,\big|\,|a_N| - K_N r(h)\big| ,$$

(3.5) b) $$> \frac{a_N}{2} r^N(h).$$

We also claim that

(3.5) c) $$f_h(\xi^o+r(h))\cdot f_h(\xi^o-r(h)) \begin{cases} >0 \text{ for } N=2n, \\ <0 \text{ for } N=2n+1. \end{cases}$$

This follows from (3.5a) and

(3.5) d) $$|f_h(\xi^o \pm r(h)) - f(\xi^o \pm r(h))| < Mh^p.$$

Here we have used (3.2a) and recalled from (3.3) that $|a_N| r^N(h)/2 > Mh^p$ for all $h \in (0,h_o]$. Thus for N even (or odd) $f_h(\xi)$ must have an even (or odd) number of zeros in $[\xi^o-r(h),\xi^o+r(h)]$. However $f_h(\xi)$ has at most N zeros in this interval.

If in the hypothesis of Theorem 3.3 the approximations (3.1) only satisfy consistency in (3.2) for some $m < N$ we can say nothing as to how many roots $f_h(\xi) = 0$ has in $D_{r(h)}(\xi^o)$ for any $h > 0$.

4. APPROXIMATING PROBLEMS : GENERAL CASE.

We now turn to the more general case in which (2.6) is approximated by a family of problems, say :

$$(4.1) \qquad F_h(X_h) = 0, \; F_h(\cdot) : \mathbb{X}_h \to \mathbb{Y}_h, \; \mathbb{X}_h \subseteq \mathbb{Y}_h ,$$

for all $h \in (0,h_o]$. Here the Banach spaces $\mathbb{X}_h$ and $\mathbb{Y}_h$, usually finite dimensional, are related to $\mathbb{X}$ and $\mathbb{Y}$ by means of appropriate bounded linear mappings, π_h^X and π_h^Y, such that :

$$(4.2) \quad \begin{aligned} &\text{a)} \quad \pi_h^X : \mathbb{X} \to \mathbb{X}_h , \quad \pi_h^Y = \mathbb{Y} \to \mathbb{Y}_h ; \\ &\text{b)} \quad \|\pi_h^X X\| = [1+O(h^p)] \, \|X\| , \; \|\pi_h^Y y\| = [1+O(h^p)] \, \|y\| . \end{aligned}$$

Norms of any quantity are assumed to be those of the appropriate space to which the quantity belongs. We also find it convenient to use the notation

$$(4.2) \quad \text{c)} \quad \pi_h^X X \equiv [X]_h , \; \pi_h^Y y \equiv [y]_h .$$

We also require bounded linear mappings, say $\pi_h^{Y^*}$, between the dual spaces $\mathbb{Y}^*$ and $\mathbb{Y}_h^*$ of $\mathbb{Y}$ and $\mathbb{Y}_h$, respectively. These are defined such that for all

$$\phi^* \in \mathbb{Y}^* , \; y \in \mathbb{Y} ,$$

and corresponding images

$$\pi_h^{Y^*} \phi^* \equiv [\phi^*]_h \in \mathbb{Y}_h^* , \; [y]_h \in \mathbb{Y}_h ,$$

there is a $p > 0$ for which :

$$(4.2) \quad \text{d)} \quad |\phi^* y - [\phi^*]_h [y]_h| = O(h^p), \; \forall h \in (0,h_o] .$$

The implied uniform consistency over $\mathbb{Y}^*$ and $\mathbb{Y}$ is easily replaced by that over spheres centered about special elements to be introduced later. However we do not seek here the weakest conditions but rather to explain the basic ideas. The interested reader should have no trouble in relaxing many of our constraints. Further we do not attempt to keep track of all the constants which effect the size of h_o and $r(h)$ as was done in § 3. This is already made clear in the estimates (4.2b) as well as in (4.2d).

We will consider a nonisolated root of (2.6), $X^o \in \mathbb{X}$, for which (2.7) holds. The approximations (4.1) are required to satisfy $\forall h \in (0,h_o]$:

$$\text{a)} \quad F_h(X_h) \text{ analytic on } S_{r(h)}([X^o]_h) \;;$$

$$(4.3) \quad \text{b)} \quad \|F_h([X]_h)-[F(X)]_h\| = O(h^p), \quad \forall\, X \in S_{r(h)}(X^o) \;;$$

$$\text{c)} \quad \|F_h'(X_h)-F_h'(y_h)\| \le K\,\|X_h-y_h\|, \quad \forall X_h, y_h \in S_{r(h)}([X^o]_h).$$

In addition to the simple consistency between F and F_h expressed in (4.3b) we require much more complicated consistency conditions between the Fréchet derivatives F' and F_h' (and their duals). These conditions are expressed in terms of corresponding eigenvalues and eigenfunctions. There are at least two distinct ways in which this can be done. We adopt the one which maintains the closer analogy to the continuous theory in § 2. Specifically we require that $F_h'(X_h)$ and $F_h'(X_h)^*$ have an eigenvalue, $\alpha(X_h)$, and corresponding eigenfunctions, $\phi_h(X_h)$ and $\phi_h^*(X_h)$, such that $\forall h \in (0,h_o]$, $X_h \in S_{r(h)}([X^o]_h)$:

$$(4.4) \quad \text{a)} \quad \begin{aligned} &F_h'(X_h)\phi_h(X_h) = \alpha(X_h)\phi_h(X_h), \quad \|\phi_h(X_h)\| = 1 : \\ &F_h'(X_h)^*\phi_h^*(X_h) = \alpha(X_h)\phi_h^*(X_h), \quad \phi_h^*(X_h)\phi_h(X_h) = 1. \end{aligned}$$

Further these quantities satisfy for some $p > 0$:

$$(4.4) \quad \text{b)} \quad |\alpha(X_h)| = O(h^p) \quad \left\{ \begin{aligned} &\|\phi_h([X^o]_h)-[\phi]_h\| = O(h^p), \\ &\|\phi_h^*([X^o]_h)-[\phi^*]_h\| = O(h^p), \end{aligned} \right.$$

where ϕ and ϕ^* are as in (2.7 a,c).

Using $\phi_h^*([X^o]_h)$ we introduce, $\forall h \in (0,h_o]$, the subspaces :

(4.5) a) $\mathbb{X}_{h,1} \equiv \{X_h \in \mathbb{X}_h : \phi_h^*([X^o]_h)X_h = 0\}$, $\mathbb{Y}_{h,1} \equiv \{y_h \in \mathbb{Y}_h : \phi_h^*([X^o]_h)y_h = 0\}$.

Then we have the direct sum decompositions :

(4.5) b) $\mathbb{X}_h = \{\phi_h([X^o]_h)\} \oplus \mathbb{X}_{h,1}$, $\mathbb{Y}_h = \{\phi_h([X^o]_h)\} \oplus \mathbb{Y}_{h,1}$,

and the restriction of $F_h'(X_h)$ to $\mathbb{X}_h$, is defined as

(4.5) c) $A_h(X_h) \equiv (F_h'(X_h) / \mathbb{X}_{h,1})$.

It is required that these restrictions have uniformly bounded inverses at $X_h = [X^o]_h$, say

(4.5) d) $\|A_h^{-1}([X^o]_h)\| \leq C$, $\forall h \in (0,h_o]$.

A final consistency condition involving the second Fréchet derivative is given in terms of

(4.6) a) $a_{2,h} \equiv \frac{1}{2} \phi_h^*([X^o]_h)F_h''([X^o]_h)\phi_h([X^o]_h)\phi([X^o]_h)$.

It is required that, with a_2 from (2.8),

(4.6) b) $|a_2 - a_{2,h}| = O(h^p)$.

Now our basic approximation result for the general case can be stated as

Theorem 4.7 : Let X^o be a nonisolated solution of (2.6) for which (2.7) holds. Let $a_2 \neq 0$ in (2.8) so that X^o is a geometrically isolated solution. Let the family of approximations $\{F_h(\cdot)\}$ in (4.1)-(4.2) satisfy (4.3)-(4.6) for some $h_o > 0$ and $r(h) = r_o h^{p/2}$ with h_o and r_o sufficiently small. Then $\forall h \in (0,h_o]$ the approximation

(4.7) a) $F_h(X_h) = 0$

has either two or no solutions in $S_{r(h)}([X^o]_h)$. In the former case each of the two roots has the form :

(4.7) b) $X_h^o(\xi) = [X^o]_h + \xi\phi_h([X^o]_h) + v_h(\xi)$

<u>where</u>

(4.7) c) $v_h(\xi) \in \mathbb{X}_{h,1}$, $\|\xi\phi([X^o]_h) + v_h(\xi)\| \leq r(h)$.

<u>Furthermore there is a manifold of the form</u> (4.7b,c) <u>with</u> $v_h(\xi)$ <u>analytic such that with the manifold in</u> (2.13a) :

(4.7) d) $\|[X^o(\xi)]_h - X_h^o(\xi)\| = O(h^p)$.

<u>Proof</u> : By the decomposition (4.5b) every point $X_h \in S_{r(h)}([X^o]_h) \subset \mathbb{X}_h$ has a <u>unique</u> representation in the form

(4.8) a) $X_h = [X^o]_h + \xi\phi_h([X^o]_h) + v_h$.

where

(4.8) b) $v_h \in \mathbb{X}_{h,1}$, $\|\xi\phi_h([X^o]_h) + v_h\| \leq r(h)$.

Then in analogy with (2.9) we introduce the projections

$$\text{(4.9)}\quad \begin{aligned} &\text{a)}\quad P_h \equiv \phi_h([X^o]_h)\phi_h^*([X^o]_h) : \mathbb{Y}_h \to \{\phi_h([X^o]_h)\} , \\ &\text{b)}\quad Q_h \equiv I - P_h : \mathbb{Y}_h \to \mathbb{Y}_{h,1} . \end{aligned}$$

Now X_h in (4.8) is a solution of (4.7a) <u>iff</u> :

$$\text{(4.10)}\quad \begin{aligned} &\text{a)}\quad H_h(\xi, v_h) \equiv Q_h F_h([X^o]_h + \xi\phi_h([X^o]_h) + v_h) = 0, \\ &\text{b)}\quad P_h F_h([X^o]_h + \xi\phi_h([X^o]_h) + v_h) = 0. \end{aligned}$$

We first determine the solutions of (4.10a) in $\mathbb{R}\times\mathbb{X}_h$, near $(\xi, v_h) = (\xi, [v(\xi)]_h)$ using $v(\xi)$ from (2.12).
This is done by applying the theory of [5], uniformly in ξ, on $|\xi| \leq r(h)$, for all $h \in (0, h_o]$. Specifically we note from (4.3b) and (2.12b) that :

(4.11) a) $\|H_h(\xi,[v(\xi)]_h)\| = O(h^p)$.

Here we have used (4.4b) and the Lipschitz continuity of $F_h(X_h)$. From (4.10a) we get, recalling (4.5c),

$$D_{v_h}H_h(\xi,v_h) = X_hF_h'([X^o]_h+\xi\phi_h([X^o]_h)+v_h) ,$$

$$= A_h([X^o]_h+\xi\phi_h([X^o]_h)+v_h).$$

It follows from (4.3c) that

(4.11) b) $\|D_{v_h}H_h(\xi,v_h)-D_{v_h}H(\xi,w_h)\| \le K\,\|v_h-w_h\|,\ \forall\ \|v_h\|,\ \|w_h\| \le r(h)$;

and the Banach lemma with (4.5d) implies

(4.11) c) $\|(D_{v_h}H(\xi,[v(\xi)]_h))^{-1}\| <$ Const., $\forall h \in (0,h_o], |\xi| \le r(h)$.

Now Theorem 3.6 of [5] can be applied, using (4.11a,b,c), to insure that $\forall\ h \in (0,h_o]$ there exists a unique solution, $v_h(\xi)$, of

$$H_h(\xi,v_h(\xi)) = 0\ ,\ v_h(\xi) \in \mathbf{X}_{h,1}$$

and satisfying

(4.12) a) $\|v_h(\xi)-[v(\xi)]_h\| = O(h^p)$.

Further $v_h(\xi)$ is analytic in ξ for $|\xi| \le r(h)$. This follows from the uniqueness of $v_h(\xi)$ and an application of the implicit function theorem to $H_h(\xi,v_h) = 0$ at $(\xi,v_h) = (0,v_h(0))$.

We have thus shown that (4.7a) can have a solution in $S_{r(h)}([X^o]_h)$ only on the manifold

(4.12) b) $X_h^o(\xi) = [X^o]_h + \xi\phi_h([X^o]_h)+v_h(\xi)$

on which (4.10a) is satisfied. All solutions of (4.7a) on this manifold are determined by the solutions of the scalar equation, from (4.10b) :

(4.12) c) $f_h(\xi) \equiv \phi_h^*([X^o]_h)F_h(X_h^o(\xi)) = 0$.

We now use the scalar theory of § 3, with $f(\xi)$ as defined in (2.13), to determine the possible solutions of (4.12c). The hypothesis that $a_2 \neq 0$ in (2.8) or (2.14b) implies that $f(\xi)$ has a geometrically isolated zero at $\xi=\xi^o= 0$. So to apply Theorem 3.3 with N=2 we need only verify that (3.1)-(3.2) hold with m=2.

The analyticity of $f_h(\xi)$ about $\xi=0$ follows from that of $v_h(\xi)$. To verify (3.2a) we use (2.13),(2.14) and the abbreviations

$$\phi_h^* \equiv \phi_h^*([X^o]_h), F_h \equiv F_h(X_h^o(\xi)), \quad F \equiv F(X^o(\xi)),$$

to write :

$$f_h(\xi)-f(\xi) = (\phi_h^*-[\phi^*]_h)F_h+[\phi^*]_h(F_h-F_h([X^o(\xi)]_h))$$
$$+[\phi^*]_h(F_h([X^o(\xi)]_h)-[F]_h)+([\phi^*]_h[F]_h-\phi^*F).$$

The first, thrid and last terms of the above right hand side are estimated using (4.4b), (4.3b) and (4.2d), respectively, to get

$$\| f_h(\xi)-f(\xi) \| \leq O(h^p)+O(\| X_h^o(\xi)-[X^o(\xi)]_h \|).$$

However from (4.4b) again and (4.12a) :

$$\| X_h^o(\xi)-[X^o(\xi)]_h \| = \| \xi(\phi_h([X^o]_h)-[\phi]_h)+[v_h(\xi)]_h) \| ,$$
$$= O(h^p).$$

Thus (3.2a) holds. We also note that the above establishes (4.7d).

To estimate the quantities in (3.2b) we differentiate in (2.13b) and (4.12c) to get :

(4.13)

a) $$f'(\xi) = \phi^*F'(X^o(\xi))[\phi+v'(\xi)] ,$$

b) $$f_h'(\xi) = \phi_h^*([X^o]_h)F_h'(X_h^o(\xi))[\phi_h([X^o]_h)+v_h'(\xi)].$$

Since $X^o(0) = X^o$ it follows that $f'(0) = 0$. With (4.4a,b) we get

$$f_h'(0) = \phi_h^*([X^o]_h)F_h'([X^o]_h+v_h(0)[\phi_h([X^o]_h)+v_h'(0)] ,$$

$$= \alpha([X^o]_h)\ [1+O(h^p)+\phi_h^*([X^o]_h)v_h'(0)] ,$$

$$= O(h^p).$$

Here we have used $v_h(0) = O(h^p)$ and $v_h'(0) = O(h^p)$. The latter follows from differentiation of $H_h(\xi,v_h(\xi)) = 0$ to get

$$(4.14) \qquad Q_hF_h'([X^o]_h+v_h(0))[\phi_h^*([X^o]_h)+v_h'(0) = 0.$$

But since $Q_hF_h'([X^o]_h) = A_h([X^o]_h)$ has a bounded inverse the result easily follows. Thus (3.2b) holds for $\nu=1$. For $\nu=2$ we have

$$\text{a)} \qquad f''(\xi) = \phi^*F''(X^o(\xi))[\phi+v'(\xi)][\phi+v'(\xi)][\phi+v'(\xi)] +\phi^*F'(X^o(\xi))v''(\xi),$$

$$(4.15) \quad \text{b)} \qquad f_h''(\xi) = \phi_h^*([X^o]_h)F_h''(X_h^o(\xi))[\phi_h([X^o]_h)+v_h'(\xi)][\phi_h([X^o]_h+v_h'(\xi)]$$

$$+ \phi_h^*([X^o]_h)F_h'(X_h^o(\xi))v_h''(\xi).$$

As in the above derivation we easily obtain

$$f''(0) = \phi^*F''(X^o)\phi\phi = 2a_2$$

and

$$f_h''(0) = 2a_{2,h}+O(h^p)$$

So from (4.6) we find that (3.2b) holds for $\nu=2$ also.

The Lipschitz continuity required for (3.2c) easily follows from the analyticity of $f_h(\xi)$. The hypothesis for Theorem 3.3 are thus established with N=2. ∎

It is clear to see how "higher order" cases can be included. For example if $a_2=0$ we have

<u>Corollary 4.16</u> : <u>Let the hypothesis of Theorem</u> 4.7 <u>hold modified only in that</u> $r(h) \equiv r_o hp/3$, $a_2=0$ <u>but</u> $a_3 \neq 0$ <u>in</u> (2.14). <u>Further</u>, <u>with</u> $\phi_h^* \equiv \phi_h^*([X^o]_h)$ <u>and</u> $\phi_h \equiv \phi_h([X^o]_h)$, <u>let</u> :

$$(4.16) \qquad a_{3,h} \equiv \frac{1}{3!}\, \phi_h^* F_h'''([X^o]_h)\phi_h\phi_h\phi_h + \frac{1}{2}\, \phi_h^* F_h''([X^o]_h)\phi_h v_h''(0)$$

satisfy

$$(4.16)\ \text{b)} \qquad |a_3 - a_{3,h}| = O(h^p) \qquad \forall\, h \in (0,h_o] \ .$$

<u>Then for each</u> $h \in (0,h_o]$ <u>the approximation</u> (4.7a) <u>has either one or three solutions in</u> $S_{r(h)}([X^o]_h)$. <u>They are all of the form</u> (4.7b,c) <u>and</u> (4.7d) <u>still holds</u>.

<u>Proof</u> : The proof is an obvious extension of the proof of Theorem 7.4. No new difficulties arise so we do not include the details. ∎

Clearly our results easily extend to the cases where $a_2=a_3=0$ and $a_4 \neq 0$; etc...

5. <u>LIMIT POINT AND BIFURCATION EXAMPLES</u>.

The implications of the theory in §4 are quite relevant in attempts to compute limit points and bifurcation points in nonlinear eigenvalue problems. We indicate the connection here and in a following paper [8] we develop a more complete theory of the approximation of such critical points on solution arcs. A related independent study has also been made in [12] where finite element approximations are considered.

We first recall, for nonlinear eigenvalue problems in the general form

$$(5.1) \qquad G(u,\lambda) = 0 \ , \ G : \mathbb{X} \times \mathbb{R} \to \mathbb{Y} \ , \ \mathbb{X} \subseteq \mathbb{Y} \ ,$$

that a critical point is a solution (u^o,λ^o) at which the Fréchet derivative

$$(5.2) \qquad G_u^o \equiv D_u G(u^o,\lambda^o)$$

is singular. In particular we consider here only cases in which

$$(5.3) \quad \begin{aligned} &\text{a)} \quad \mathcal{N}(G_u^o) = \text{span}\,\{\phi\}\,,\ \|\phi\| = 1\ ;\\ &\text{b)} \quad \mathcal{R}(G_u^o) \text{ is closed with codim} = 1\ ;\\ &\text{c)} \quad \mathcal{N}(G_u^{o*}) = \text{span}\,\{\phi^*\}\,,\ \phi^*\phi = 1. \end{aligned}$$

Such a critical point is called a "limit point" if in addition

(5.4) a) $\phi^* G^o_\lambda \neq 0.$

A branch or arc of solutions, say $[u(s),\lambda(s)]$, which contains the limit point, say $(u(0),\lambda(0)) = (u^o,\lambda^o)$, exists locally only for $\lambda > 0$ (or for $\lambda < 0$) if

(5.4) b) $a \equiv \phi^* G^o_{uu}\phi\phi \neq 0.$

This result was first established in [4] and has more recently been employed in [9] to aid in computing limit point solutions. A sketch of the local behavior of a solution arc of (5.1) through a limit point satisfying (5.4a,b) is given in Figure 1, as the solid curve $[u(s),\lambda(s)]$.

Now with the critical value of the parameter $\lambda = \lambda^o$ known and kept <u>fixed</u> we consider the problem of finding the value $x=u^o$ such that

(5.5) $$F(X) \equiv G(X,\lambda^o) = 0 .$$

From (5.2)-(5.3) and (5.4b) it is clear that $x=u^o$ is a geometrically isolated but nonisolated solution of (5.5). Thus if we seek to approximate u^o by solving some family of approximate problems, say

(5.6) $$F_h(X_h) \equiv G_h(X_h,\lambda^o) = 0 ,$$

we may, under appropriate assumptions on the approximations, apply the theory of §4. Our theory tells us that (5.6) has, for each h sufficiently small either no solution or two solutions within $O(h^{p/2})$ of $[u^o]_h$. This is clearly consistent with the arcs of solutions of the approximations

(5.7) $$G_h(X_h,\lambda) = 0$$

when λ is free to vary. Indeed under the more or less obvious consistency assumptions between $G_h(X_h,\lambda)$ and $G(X,\lambda)$ the solution arcs of (5.7) are as sketched in Figure 1 as the dotted curve $[u^I_h(s),\lambda^I(s)]$ or as the dot-dash curve $[u^{II}_h(s),\lambda^{II}(s)]$. The former yields no solution of (5.6) since the value λ^o is not attained by $\lambda^I(s)$ on this arc. The latter arc yields two solutions of (5.6) since two different values of $X^{II}_h(s)$ are attained when $\lambda^{II}(s) = \lambda^o$ at two distinct values of s.

The theory for the solution curves of (5.7) is presented in [8] and [12]. However none of these general results can distinguish between cases I and II. It is the relative magnitudes of the perturbing terms that determines these details. The same is true of course in our much simpler cases of §3 and §4. Thus the number of real roots of the quadratic

$$[a_2+0_a(h^P)]\xi^2 + 20_b(h^P)\xi + 0_c(h^P) = 0$$

with $a_2 \neq 0$ and h sufficiently small depends upon the sign of

$$a_2 0_c(h^P).$$

It is precisely such an analysis that must be made and since the required mangitudes can seldom be known, the exact behavior of the approximations can seldom be known.

For bifurcation phenomena the behavior is similar and slightly more complicated to analyse. We content ourselves here with the sketches in Figures 2 and 3 which show respectively, how the bifurcation point (u^o,λ^o) yields zero and two or one and three roots of the approximation (5.6). Again a more detailed study is contained in [8] and [12].

APPENDIX

Geometric Isolation and Transversality of the Singular Manifold.

We show here how the condition $a_2 \neq 0$ for (2.8) which insures that X^o is a geometrically isolated solution of (2.6) when (2.7) holds, also implies that another important geometric condition is satisfied. For ease of exposition and to validate the geometric interpretation of our results we here confine the analysis to the finite dimensional case, say $\mathbb{X} = \mathbb{Y} = \mathbb{E}^n$. However the extension to a Banach space setting is immediate.

Specifically we claim that $a_2 \neq 0$ insures that the n-1 dimensional manifold containing X^o, on which $F'(X)$ is singular, is transversal to the null space, span $\{\phi\}$, at $X=X^o$. That is the normal to the manifold at X^o and ϕ are not orthogonal to each other. This then insures the existence of a cone with vertex at X^o and axis in the direction of ϕ such that the Newton iterates for solving (2.6) stay in the "tip" of such a cone, if started there, and converge to X^o. These results on Newton's method in the singular case can be found in [2,10].

From the smoothness of $F(X)$ it follows that there exists an eigenvalue $\alpha_1(X)$ and eigenvectors $\phi(X), \phi^*(X)$ on $S_\rho(X^o) \subset \mathbb{E}^n$ such that

$$\text{(A.1)}\quad \begin{aligned} &\text{a)} \quad F'(X)\phi(X) = \alpha_1(X)\phi(X), \quad \|\phi(X)\| = 1 \ ; \\ &\text{b)} \quad \phi^*(X)F'(X) = \alpha_1(X)\phi^*(X) \ , \ \phi^*(X)\phi(X) = 1 \ ; \\ &\text{c)} \quad \alpha_1(X^o) = 0 \end{aligned}$$

We further assume that the gradient vector

$$\text{(A.2)}\qquad \underset{\sim}{\nabla}\alpha_1(X^o) \neq \underset{\sim}{0} \ ,$$

where $\underset{\sim}{\nabla} \equiv (\partial_1, \partial_2, \ldots, \partial_n)$. Then by the implicit function theorem applied to

$$\text{(A.3)}\qquad \alpha_1(X) = 0$$

at $X=X^o$ it follows that (A.3) holds on some smooth n-1 dimensional manifold $\mathcal{M}$ and $X^o \in \mathcal{M}$. Of course the vector $\underset{\sim}{\nabla}\alpha_1(X^o)$ is orthogonal to $\mathcal{M}$ at X^o.

Differentiating in (A.1a) multiplying by $\phi^*(X)$ and evaluating the result at $X = X^o$ yields

$$\nabla\alpha_1(X^o) = \phi^*(X^o)F''(X^o)\phi(X^o) .$$

It follows that

$$\text{(A.4)} \qquad a_2 = \nabla\alpha_1(X^o)\phi(X^o)$$

and so $a_2 \neq 0$ implies that some cone, $\mathbb{K}$, with vertex at X^o and axis parallel to $\phi(X^o)$ does not intersect the singular manifold $\mathcal{M}$ within $S_\rho(X^o)$ for some small $\rho > 0$. The semi-angle of the cone is any angle less that θ_o where $\cos\theta_o = a_2 | \, \|\nabla\alpha_1(X^o)\|$. Since $F'(X)$ is nonsingular in $\mathbb{K} \cap S_\rho(X^o)$ the Newton iterates are defined throughout this volume. Convergence proofs can then be given as in [2,10].

Essentially the same derivation as above is independently given in [3] to get (A.4). However the authors were not aware of the fact that $a_2 \neq 0$ implies geometric isolation and so this extra hypothesis was assumed in their work.

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CAPTIONS FOR FIGURES

Figure 1 - Limit point branch and two approximating families : I and II. For $\lambda=\lambda^o$ case I has no solution and case II has two solutions. Clearly X^o is a double root on the limit point branch.

Figure 2 - Bifurcation branches and two approximating families : I and II. For $\lambda=\lambda^o$ case I has no solution and case II has two solutions. Note that X^o is a double root since it lies on two intersecting branches.

Figure 3 - Bifurcation branches and two approximating families : I and II. For $\lambda=\lambda^o$ case I has one solution, labled Δ, and case II has three solutions, Θ. Note that X^o is a triple root since it lies on two intersecting branches and is a double root on one of these branches.

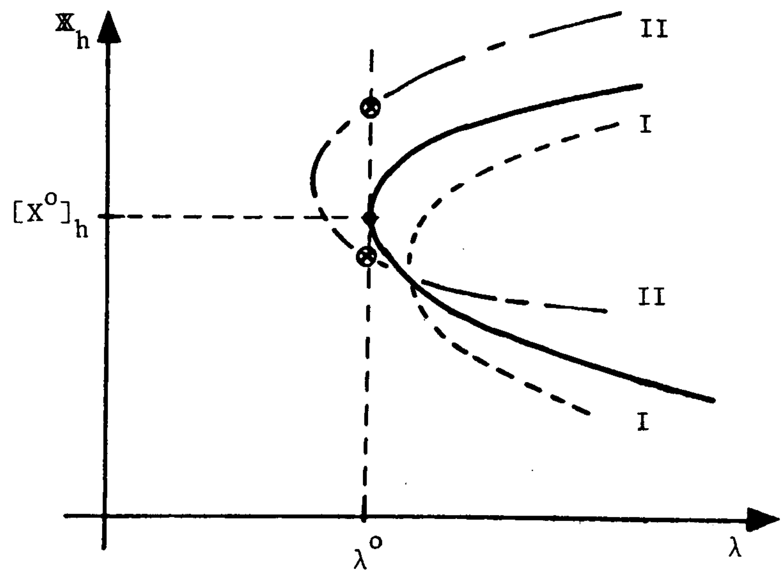

Figure 1

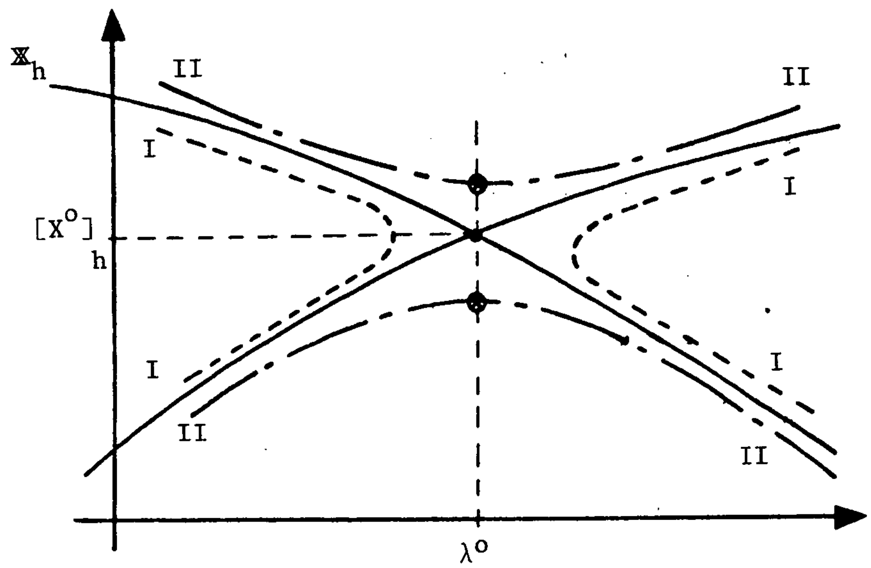

Figure 2

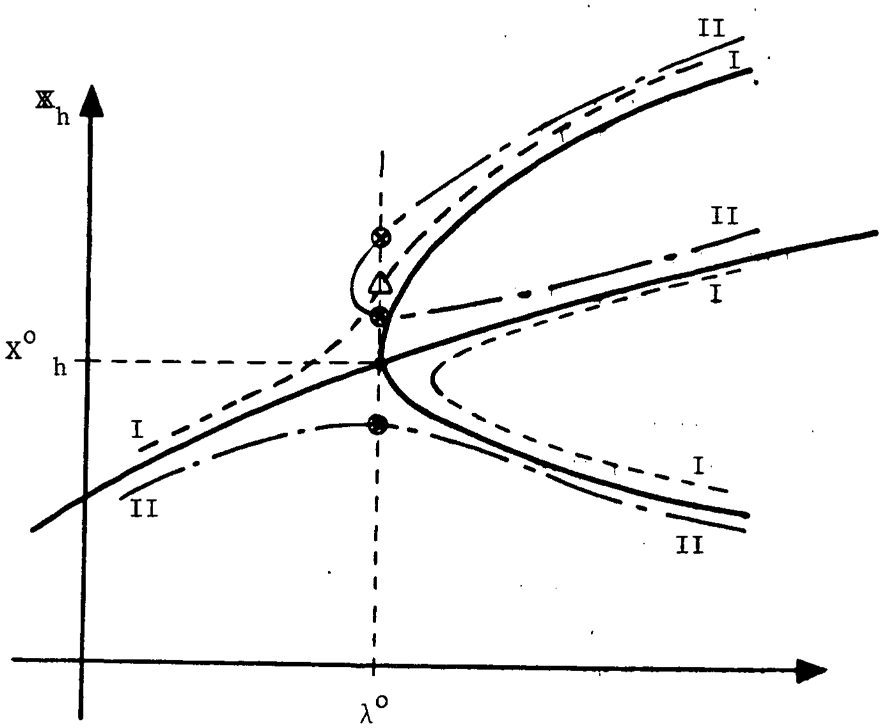

Figure 3