LIE TRANSFORMATION GROUPS

by

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ABSTRACT

Suppose G is a Lie group and M is a manifold (G and M are not necessarily finite dimensional). Let $D(M)$ denote the group of diffeomorphisms on M and $V(M)$ denote the Lie algebra of vector fields on M. If X is a complete vector field then $\operatorname{Exp} tX$ will denote the one-parameter group of X. A local action ϕ of G on M gives rise to a Lie algebra homomorphism ϕ^+ from $L(G)$ into $V(M)$. In particular if G is a subgroup of $D(M)$ and $\phi : G \times M \longrightarrow M$ is the natural global action $(g,p) \longrightarrow g(p)$ then G is called a Lie transformation group of M. If M is a Hausdorff manifold and G is a Lie transformation group of M we show that ϕ^+ is an isomorphism of $L(G)$ onto $\phi^+(L(G))$ and $L = \phi^+(L(G))$ satisfies the following conditions :

(A) L consists of complete vector fields.

(B) L has a Banach Lie algebra structure satisfying the following two conditions :

- (B1) the evaluation map $ev : (X,p) \longrightarrow X(p)$ is a vector bundle morphism from the trivial bundle $L \times M$ into $T(M)$,
- (B2) there exists an open ball $B_r(0)$ of radius r at 0 such that $\operatorname{Exp} : L \longrightarrow D(M)$ is injective on $B_r(0)$.

Conversely, if L is a subalgebra of $V(M)$ (M Hausdorff) satisfying conditions (A) and (B) we show there exists a unique connected Lie transformation group with natural action $\phi : G \times M \longrightarrow M$ such that ϕ^+ is a Banach Lie algebra isomorphism of $L(G)$ onto L.

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Chapter 1

Preliminaries

All manifolds considered are real Banach manifolds of class C^K where $K = \infty$ or $K = \omega$. The word morphism will mean a C^K map between C^K manifolds. In this chapter, we collect the necessary facts on foliations of manifolds and on infinite dimensional Lie groups. Almost all of this material will come from Bourbaki [1, §9] or Bourbaki [2, Chapter 3].

§1 Foliations and Integrable Subbundles

Let M and S be manifolds and $p : M \longrightarrow S$ a submersion. We then have, for each $s \in S$, a manifold structure induced on the level set $p^{-1}(s)$ by M. Denote by M_p the manifold which is the disjoint union over S of $p^{-1}(s)$. Each $p^{-1}(s)$ is an open submanifold of M_p and topologically M_p is the topological sum of the topological spaces $p^{-1}(s)$.

Definition (1.1) Let M be a manifold. A foliation of M is a manifold Y having the same point set as M and satisfying the condition that for all $x \in M$, there exists an open submanifold U of M containing x, a manifold S, and a submersion $p : U \longrightarrow S$ such that the manifold U_p is an open submanifold of Y..

The inclusion map of Y into M is easily seen to be a bijective immersion.

We call the pair (M, Y) a foliated manifold. A set U is called a (connected) leaf if it is a (connected) open set in Y. The maximal connected leaves are therefore the connected components of Y.

Definition (1.2) If (M, Y) and (M', Y') are foliated manifolds, a morphism from (M, Y) into (M', Y') is a map which is a morphism of M into M' and at the same time a morphism of Y into Y'.

Using the inclusion map of Y into M, for each $x \in M$ we can identify the tangent space $T_x(Y)$ with a subspace of $T_x(M)$. With this identification we have the following propositions.

Proposition (1.3) *The spaces* $T_x(Y)$ *are the fibers of a subbundle* $T(M, Y)$ *of* $T(M)$. *Furthermore if* Y *is defined by a submersion* $p : M \longrightarrow S$, *then* $T(M, Y) = \ker T(p)$.

Proposition (1.4) *Let* (M, Y) *and* (M', Y') *be two foliated manifolds and* $f : M \longrightarrow M'$ *be a morphism. A necessary and sufficient condition that* f *is a morphism from* (M, Y) *into* (M', Y') *is that* $T(f)$ *takes* $T(M, Y)$ *into* $T(M', Y')$.

Let F be a subbundle of $T(M)$. We now examine the conditions on F which imply the existence of a manifold Y such that $T(M, Y) = F$. If this is the case then F is called an integrable subbundle of $T(M)$ and the foliation it defines is unique.

<u>Theorem of Frobenius</u> (1.5) <u>F is integrable if there exists a family $\{\xi_i\}_{i\in I}$ of sections of F such that</u>

(1) <u>for all</u> $x \in M$ <u>the set</u> $\{\xi_i(x) : i \in I\}$ <u>is a total subset of the fiber</u> F_x <u>of</u> F <u>above</u> x .

(2) <u>for all pairs</u> (i, j) <u>of elements of</u> I <u>and all</u> $x \in M$, $[\xi_i, \xi_j](x) \in F_x$.

§2 Total Differential Equations

We now construct a particular subbundle and examine what it means for it to be integrable. This will be the setting for discussing generalized differential equations.

Suppose M is the product of two manifolds A and B. Let $p_1 : M \longrightarrow A$ and $p_2 : M \longrightarrow B$ be the projections on the first and second factors. There are two subbundles, $p_1{*}T(A)$ and $p_2{*}T(B)$, of $T(M) = T(A) \times T(B)$ associated with p_1 and p_2 . The fiber $p_1{*}T(A)_{(a,b)}$ of $p_1{*}T(A)$ over (a,b) is $T_a(A) \times \{0_b\}$ where 0_b is the zero vector in $T_b(B)$. We identify this fiber with $T_a(A)$. Similarly the fiber $p_2{*}T(B)_{(a,b)}$ of $p_2{*}T(B)$ over (a,b) is $\{0_a\} \times T_b(B)$ which is identified with $T_b(B)$.

Let f be a vector bundle morphism from $p_1{*}T(A)$ into $p_2{*}T(B)$. Then for each $(a,b) \in M$, f is a continuous linear map $f_{(a,b)} : T_a(A) \longrightarrow T_b(B)$ (after identifying $T_a(A)$ with $p_1{*}T(A)_{(a,b)}$

and $T_b(B)$ with $p_2{*}T(B)_{(a,b)}$).

Proposition (2.1) The graphs of the $f_{(a,b)}$ are the fibers of a subbundle of $T(M)$ which we denote by F^f .

Definition (2.2) Let A' be an open set in A . A morphism $\phi : A' \longrightarrow B$ is called an integral of f if for all $a \in A'$ one has $T_a(\phi) = f_{(a,\phi(a))}$.

The following two propositions describe the local uniqueness of integrals.

Proposition (2.3) If ϕ_1 and ϕ_2 are two integrals of f taking the same value at a point $a \in A$, then they coincide in a neighbourhood of a.

Proposition (2.4) Let Z be a manifold, A' an open set in A, and $a \in A'$. Suppose ϕ_1 and ϕ_2 are morphisms of $Z \times A'$ into B such that ϕ_1 and ϕ_2 coincide on $Z \times \{a\}$ and for all $z \in Z$, the morphisms $a \longrightarrow \phi_1(z,a)$ and $a \longrightarrow \phi_2(z,a)$ are integrals of f . Then ϕ_1 and ϕ_2 coincide on a neighbourhood of $Z \times \{a\}$.

Suppose now that F^f is integrable and therefore defines a foliation Y of M with $T(M, Y) = F^f$. Let $\phi : A' \longrightarrow B$ be an integral for f and define $\psi : A' \longrightarrow M$ by $\psi(a) = (a, \phi(a))$. We have $T\psi(T(A')) \subset F^f$ since ϕ was an integral and Proposition (1.4) gives that ψ is also a morphism from A' into Y . Let $v_a \in T_aA'$. Now $T_a\psi(v_a) = (v_a, T_a\phi(v_a)) = (v_a, f_{(a,\phi(a))}(v_a))$ which implies $T_a\psi$ is an isomorphism (of Banach spaces) of T_aA' onto $F^f_{(a,\phi(a))}$. This means ψ is a local diffeomorphism into Y at a and as a was arbitrary we have proven the following result.

Proposition (2.5) If F^f is integrable and $\phi : A' \longrightarrow M$ is an integral for f then $\{(a, \phi(a)) : a \in A'\}$ is a leaf (open set) of the foliation defined by F^f.

We complete this section with the existence theorem for integrals.

Proposition (2.6) Suppose that F^f is integrable. Let $(z_o, a_o) \in Z \times A$ and ρ be a morphism from Z into B. Then there exists an open neighbourhood $Z' \times A'$ of (z_o, a_o) in $Z \times A$ and a morphism $\phi : Z' \times A' \longrightarrow B$ such that for every $z \in Z'$ the morphism $a \longrightarrow \phi(z,a)$ of A' into B is an integral for f and $\rho(z) = \phi(z,a_o)$.

We will mainly use this with $Z = B$ and ρ = identity.

§3 Lie Groups and Lie Algebras

A Lie group G is a group, which is also a Banach manifold (not necessarily finite dimensional) such that the operations of multiplication $G \times G \longrightarrow G$ and taking inverses $G \longrightarrow G$ are morphisms. G will be called finite (infinite) dimensional if its manifold structure is modelled on a finite (infinite) dimensional Banach space.

A Banach Lie Algebra L is a Lie algebra with a Banach space structure such that the bracket $[\ ,\ ] : L \times L \longrightarrow L$ is continuous. We call L finite (infinite) dimensional if the underlying vector space is finite (infinite) dimensional.

Almost all of the standard finite dimensional Lie group theory carries over to infinite dimensions. If G is a Lie group then there is a Banach Lie algebra L(G) corresponding to G and an exponential map from L(G) into G, which is a local diffeomorphism at 0 . (We break with the usual convention of having L(G) equal to the set of left invariant vector fields on G and instead it will be the set of right invariant vector fields. Defining L(G) to be the right invariant vector fields will make the definition of an infinitesimal action in Chapter 2 easier. This is a slight change since if we identify L(G) with $T_e(G)$, the tangent space at the identity, then the only difference between the right invariant Lie algebra structure and left invariant Lie algebra structure is that the bracket differs by a sign.)

The major difference between the finite and infinite dimensional theories is that there exist infinite dimensional Banach Lie algebras L for which there does not exist any Lie group G such that L = L(G). If a Lie group G does exist such that L = L(G) then the Banach Lie algebra L is called enlargeable. For an example of a non-enlargeable Banach Lie algebra see Est and Korthagen [4]. Although L may not be enlargeable, a Banach Lie algebra closely related to L is always enlargeable. This Lie algebra is the path space of L which we now examine.

Let $\mathcal{BL}$ denote the category of Banach Lie algebras with continuous homomorphisms as morphisms. Then we have the path functor $\Lambda : \mathcal{BL} \longrightarrow \mathcal{BL}$ which takes L to $\Lambda L = \{f \mid f : [0,1] \longrightarrow L$ continuous with $f(0) = 0\}$ with the following Lie algebra structure. If $f, g \in \Lambda L$ then the norm of f is $\max_{t\in[0,1]} ||f(t)||$ and the bracket is defined pointwise, $[f, g](t) \equiv [f(t), g(t)]$. If $\phi : L \longrightarrow L'$ is a morphism of Banach Lie algebras then $\Lambda\phi : \Lambda L \longrightarrow \Lambda L'$ is given by $\Lambda\phi(f) = \phi \circ f$.

<u>Theorem</u> (3.1) <u>Let</u> L <u>be a Lie algebra and</u> ΛL <u>be as above</u>. <u>Then</u>

(1) <u>the endpoint evaluation map</u> $f \longrightarrow f(1)$ <u>from</u> ΛL <u>into</u> L <u>is continuous</u>.

(2) ΛL <u>is enlargeable</u>.

Proof : The proof of (1) is obvious from the definition of ΛL. The reader is referred to Swierczkowski [8] for a proof of (2).

For later reference, we now list some facts on subgroups and subalgebras of Lie groups and Banach Lie algebras. The proofs are in Bourbaki [2, Chapter 3].

<u>Definition</u> (3.2) A subset H of G is a <u>Lie subgroup</u> of G if it is a subgroup and a submanifold of G.

<u>Proposition</u> (3.3) <u>Let</u> H <u>be a subgroup of a Lie group</u> G. <u>A necessary and sufficient condition for</u> H <u>to be a Lie subgroup is that there exists a point</u> $h \in H$ <u>and an open neighbourhood</u> U <u>of</u> h <u>in</u> G <u>such that</u>

$H \cap U$ is a submanifold of G .

Let L be a Banach Lie algebra. A Banach Lie subalgebra of L is a closed vector subspace of L which is closed under the bracket operation, i.e. a subalgebra. If H is a Lie subgroup of a Lie group G then using the inclusion we identify $L(H)$ with a Banach Lie subalgebra of $L(G)$ which splits in $L(G)$. (A closed subspace F of a Banach space E is said to split if there exists a closed subspace F_1 such that $F + F_1 = E$ and $F \cap F_1 = 0$). If in addition H is normal then $L(H)$ is an ideal in $L(G)$, i.e. $[L(G), L(H)] \subset L(H)$.

Proposition (3.4) Let G be a Lie group and H be a normal Lie subgroup of G . Then there exists a structure of a Lie group on G/H such that the projection map is a submersion and $L(G/H) \cong L(G)/L(H)$.

Proof : Bourbaki [2, prop. 11, p.105 and p.141] .

Chapter 2

Local and Infinitesimal Group Actions

We determine the correspondence between local group actions and infinitesimal group actions in this chapter. Our treatment of this subject follows that of Palais [7].

Before proceeding we establish some notation conventions. G will denote a connected Lie group and $L(G)$ will be its Banach Lie algebra of right invariants vector fields. Right multiplication by an element $g \in G$ will be denoted by $R(g)$. The identity element in G will be denoted by e . M will denote a manifold and $V(M)$ will be the Lie algebra of vector fields on M .

§4 Local Group Actions

Definition (4.1) A local (left) action of G on M is a morphism ϕ from an open set D containing $\{e\} \times M$ in $G \times M$ into M satisfying the following conditions :

(1) $\phi(e, p) = p$ for all $p \in M$.

(2) If (h, p), $(g, \phi(h, p))$ and (gh, p) all belong to D then $\phi(gh, p) = \phi(g, \phi(h, p))$.

If $D = G \times M$ then ϕ is called a global action of G on M .

Let $D^p = \{ g : (g, p) \varepsilon D \}$. The morphism $g \longrightarrow \phi(g, p)$ of D^p into M will be denoted by ϕ^p .

The definition of local action we have given is from Palais [7]. Bourbaki [2, p.118] gives what appears to be a different definition of local action as follows.

Definition (4.1(a)) (Bourbaki) A local (left) action of G on M is a morphism ψ defined on an open set Ω of $G \times M$ containing $\{e\} \times M$, with values in M, possessing the following properties

(1) $\psi(e, p) = p$ for all $p \varepsilon M$;

(2) there exists a neighbourhood Ω_1 of $\{e\} \times \{e\} \times M$ in $G \times G \times M$ such that, for $(g, g', p) \varepsilon \Omega_1$, the elements (g', p), (gg', p), $(g, \psi(g', p))$ are in Ω and $\psi(g, \psi(g', p)) = \psi(gg', p)$.

This is slightly different from the version in Bourbaki since we aren't considering actions of "grouplets".

Proposition (4.2) Definition (4.1) and Definition (4.1(a)) are equivalent.

Proof : Def.(4.1) implies Def.(4.1(a))

Let $\psi = \phi$ and $\Omega = D$. We have to find an open set Ω_1 in $G \times G \times M$ satisfying condition (2) in Def.(4.1(a)). Define δ from $G \times D$ into $G \times M$ by $\delta(g, h, p) = (g, \phi(h, p))$, then $\delta^{-1}(D)$ is open and contains $\{e\} \times \{e\} \times M$. Define γ from $G \times D$ into $G \times M$ by $\gamma(g, h, p) = (gh, p)$ then $\gamma^{-1}(D)$ is open and contains $\{e\} \times \{e\} \times M$.

Let $\Omega_1 = \delta^{-1}(D) \cap \gamma^{-1}(D)$, then Ω_1 is an open neighbourhood of $\{e\} \times \{e\} \times M$ and if $(g, h, p) \in \Omega_1$ we have $(h, p) \in D$; $(g, \phi(h, p)) \in D$ since $(g, h, p) \in \delta^{-1}(D)$; and $(gh, p) \in D$ since $(g, h, p) \in \gamma^{-1}(D)$. Then Def.(4.1) (2) gives

$$\psi(g, \psi(h, p)) = \phi(g, \phi(h, p)) = \phi(gh, p) = \psi(gh, p)$$

and condition (2) of Def.(4.1(a)) is satisfied.

Def.(4.1(a)) implies Def.(4.1)

Let $\phi = \psi$. We will find D such that condition (2) of Def.(4.1) is satisfied. Let Ω_1 be as in Def.(4.1(a)). Since Ω_1 is an open neighbourhood of $\{e\} \times \{e\} \times M$ we can find neighbourhoods V_p and U_p of e in G and W_p of p in M such that $V_p \times U_p \times W_p \subset \Omega_1$. For each $p \in M$, let $G_p = \exp(B_r(0))$ where $B_r(0)$ is the ball of radius r centered at 0 in $L(G)$ and r is so small that $G_p^2 \subset V_p \cap U_p$. Then G_p is connected, $G_p = G_p^{-1}$, $G_p \subset U_p$, $G_p \subset V_p$, and $G_p \times G_p \times W_p$ is an open neighbourhood of (e, e, p) contained in Ω_1. Also $\{G_p\}_{p \in M}$ are ordered by inclusion so if we have G_x and G_y then either $G_x \subset G_y$ or $G_y \subset G_x$. Define $D = \bigcup_{p \in M} G_p \times W_p$ and suppose (h, p), $(g, \phi(h, p))$ and $(gh, p) \in D$. Since D is "symmetric" (each G_p was symmetric) we have $(h, p) \in D$ implies $(h^{-1}, p) \in D$. Now by the definition of D; (gh, p) and (h^{-1}, p) belonging to D means there exists $x \in M$ such that $(h^{-1}, p) \in G_x \times W_x$ and there exists $y \in M$ such that $(gh, p) \in G_y \times W_y$. By the remark above either $G_x \subset G_y$ or $G_y \subset G_x$ so (without loss of generality) assuming the latter we have $(gh, p) \in G_x \times W_x$ also. Now $G_x^2 \subset V_x \cap U_x$ implies $((gh)(h^{-1}), p) \in G_x^2 \times W_x \subset V_x \times W_x$, i.e. $(g, p) \in V_x \times W_x$. We also have $(h, p) \in G_x \times W_x \subset U_x \times W_x$ which means $(g, h, p) \in V_x \times U_x \times W_x \subset \Omega_1$ and condition (2) of Def.(4.1(a)) gives

$$\phi(g, \phi(h, p)) = \psi(g, \psi(h, p)) = \psi(gh, p) = \phi(gh, p).$$

Examples of local actions

Example (4.3) : Let M be a paracompact manifold and ξ be a vector field on M. Then the flow (see Bourbaki [1, §9]) of ξ is a local left action of $\mathbb{R}$ on M.

Example (4.4) : If E and F are Banach spaces then denote by $\mathrm{Hom}(E, F)$ the Banach space of continuous linear maps from E into F and by $GL(F)$ the Lie group of invertible elements in $\mathrm{Hom}(F, F)$. $GL(F)$ is open in $\mathrm{Hom}(F, F)$. (See Lang [5, p.5] for proofs). Let $M = \mathrm{Hom}(F, E)$, G be the additive Lie group $\mathrm{Hom}(E, F)$, and I_F be the identity in $GL(F)$. Define the morphism $\gamma : G \times M \longrightarrow \mathrm{Hom}(F, F)$ by $\gamma(g, p) = g \circ p + I_F$. Let $D = \gamma^{-1}(GL(F))$; then D is open and contains $\{0\} \times M$. Define the local action $\phi : D \longrightarrow M$ of G on M by $\phi(g, p) = p \circ (g \circ p + I_F)^{-1}$. ϕ is a local action for ;

(1) $\phi(0, p) = p \circ (0 + I_F)^{-1} = p$

(2) $\phi(g, \phi(h, p)) = \phi(h, p)\circ(g\circ\phi(h, p) + I_F)^{-1}$

$$= p\circ(h\circ p + I_F)^{-1}(g\circ p\circ(h\circ p + I_F)^{-1} + I_F)^{-1}$$

$$= p\circ((g\circ p\circ(h\circ p + I_F)^{-1} + I_F)(h\circ p + I_F))^{-1}$$

$$= p\circ(g\circ p + h\circ p + I_F)^{-1}$$

$$= p\circ((g + h)\circ p + I_F)^{-1}$$

$$= \phi(g + h, p) .$$

§5 Infinitesimal Actions

Let L be a Banach Lie algebra.

Definition (5.1) A (left) action of L on M is a Lie algebra homomorphism $\theta : L \longrightarrow V(M)$ satisfying the condition that the evaluation map $(x, p) \longrightarrow \theta(x)(p)$ is a vector bundle morphism from the trivial vector bundle $L \times M$ into $T(M)$.

Remarks : (1) If $L = L(G)$ for some Lie group G then θ is called an infinitesimal (left) action of G on M.

(2) If L is finite dimensional then the evaluation map is automatically a vector bundle morphism (Bourbaki [2, Remarque p.140]).

Example (5.2) : An infinitesimal group action

Suppose H is a real Hilbert space with scalar product $(\, , \,)$. Let $M = H$ and G be H with the additive group structure of H. Then $L(G) = H$ also. Define $\theta : L(G) \longrightarrow V(M)$ by $\theta(Y)(X) = 2(X,Y)X - (X,X)Y$. We show that θ is an infinitesimal action of G on M.

(1) The map $\varepsilon : (Y,X) \longrightarrow \theta(Y)(X)$ is a vector bundle morphism from $L(G) \times M$ into $T(M)$: ε is obviously a morphism. Let $\mathrm{Hom}(H, H)$ denote the continuous linear maps from H into H and let $\delta_X \in \mathrm{Hom}(H, H)$ be the map $Y \longrightarrow \theta(Y)(X)$. We need that the map $X \longrightarrow \delta_X$ of H into $\mathrm{Hom}(H, H)$ is continuous, but this is the case since $(\, , \,) : H \times H \longrightarrow H$ is continuous.

(2) θ is a Lie algebra homomorphism : θ is obviously linear. In order to prove that θ preserves brackets it suffices to show that $[\theta(Y), \theta(Z)] = 0$ for any Y and Z in $L(G)$ since $L(G) = H$ is abelian. By definition

$$[\theta(Y), \theta(Z)](X) = D\theta(Z)\big|_X(\theta(Y)(X)) - D\theta(Y)\big|_X(\theta(Z)(X)) .$$

A short calculation gives $D\theta(W)\big|_X(H) = 2\Big\{(X,Y)H + (H,Y)X - (X,H)\Big\}$ and substituting this into the above equation with $W = Z$ (and Y) and $H = \theta(Y)(X)$ (and $\theta(Z)(X)$) makes the equation identically zero. Therefore θ preserves brackets.

Suppose $\phi : D \longrightarrow M$ is a local action of G on M. Define $\phi^+ : L(G) \longrightarrow V(M)$ by $\phi^+(v)(p) = T(\phi)(v(e), 0_p)$ where 0_p is the zero vector in $T_p(M)$.

<u>Proposition</u> (5.3) ϕ^+ <u>is an infinitesimal action of</u> G <u>on</u> M.

Proof : <u>Evaluation map of</u> ϕ^+ <u>is a vector bundle morphism</u> :

We have the following sequence of maps

$$L(G) \times M \longrightarrow L(G) \times G \times M \xrightarrow{\beta\times\gamma} TD \xrightarrow{T(\phi)} TM$$

$$(v, p) \longrightarrow (v, e, p) \longrightarrow (v(e), 0_p) \longrightarrow T(\phi)(v(e), 0_p)$$

where β is the trivializing vector bundle isomorphism $(v, g) \longrightarrow v(g)$ of $L(G) \times G$ into $T(G)$ and γ is the zero section. The fact that the evaluation map is a vector bundle morphism then follows from the fact that β and $T(\phi)$ are.

ϕ^+ <u>is a Lie algebra homomorphism</u> :

ϕ^+ is obviously linear and therefore it remains to show that it preserves brackets. Let $p \in M$, suppose $(g, p) \in D$ and $\phi(g, p) = q$, then if $h \in D^p g^{-1} \cap D^q$ we have $(h, q) = (h, \phi(g, p))$, (hg, p) and $(g, p) \in D$ which implies $\phi(h, g) = \phi(h, \phi(g, p)) = \phi(hg, p)$ by Def.(4.1) (2). This means $\phi^q = \phi^p \circ R(g)$ on the open set $D^p g^{-1} \cap D^q$ containing e which implies $T(\phi^q) = T(\phi^p) \circ T(R(g))$ on $T(D^p g^{-1} \cap D^q)$ and that $T_e(G) \subset T(D^p g^{-1} \cap D^q)$. Then for $v \in L(G)$ we have

$$\phi^+(v)(\phi^p(g)) = \phi^+(v)(q) = T(\phi)(v(e), 0_q)$$

$$= T(\phi^q)(v(e)) = T(\phi^p)\circ T(R(g))(v(e)) = T(\phi^p)(v(g))$$

which implies that v and $\phi^+(v)$ are ϕ^p-related vector fields. Then $[v, v']$ is ϕ^p-related to $[\phi^+(v), \phi^+(v')]$ (Bourbaki [, 8.5.6 p.17]) . Then ϕ^+ is a Lie algebra homomorphism for

$$\phi^+([v, v'])(p) = \phi^+([v, v'])(\phi^p(e))$$

$$= T(\phi^p)([v, v'](e))$$

$$= [\phi^+(v), \phi^+(v')](\phi^p(e))$$

$$= [\phi^+(v), \phi^+(v')](p)$$

where p was an arbitrary point of M. This completes the proof.

ϕ^+ is called the <u>infinitesimal generator</u> of ϕ . If an infinitesimal action θ of G on M is equal to ϕ^+ for some local action ϕ then θ is called <u>generating</u>.

<u>Example</u> (5.4) Let ϕ be the local action considered in Example (4.4). Let $X \in L(G) = \mathrm{Hom}(E, F)$. Then

$$\phi^+(X)(p) = T(\phi)(X(0), 0_p)$$

$$= \left.\frac{d}{dt}\right|_{t=0} \phi(tX, p)$$

$$= \left.\frac{d}{dt}\right|_{t=0} p\circ(tX\circ p + I_F)^{-1} = -\, p\circ X\circ p \ .$$

§6 The Infinitesimal Graph

Let $\theta : L(G) \longrightarrow V(M)$ be an infinitesimal left action and let $p_G : G \times M \longrightarrow G$ and $p_M : G \times M \longrightarrow M$ be the canonical projections. Define f from $p_G{}^*T(G)$ into $p_M{}^*T(M)$ by

$$f_{(g,m)}(X(g)) = \theta(X)(m)$$

where $X(g)$ is the value of $X \varepsilon L(G)$ at g. (See §2 for definitions of $p_G{}^*T(G)$ and $p_M{}^*T(M)$). We have $f(X(g), 0_p) = (0_g, \theta(X)(p))$ and f is a vector bundle morphism since the evaluation map $(X, p) \longrightarrow \theta(X)(p)$ was assumed to be a vector bundle morphism of $L(G) \times M$ into $T(M)$. Then prop. (2.1) implies that the graphs of the $f_{(g,p)}$, $\{(X(g), \theta(X)(p)) : p \varepsilon M\}$, are the fibers of a subbundle F^f of $T(G) \times T(M)$. F^f is called the <u>infinitesimal graph</u> of θ.

<u>Proposition</u> (6.1) F^f <u>is an integrable subbundle of</u> $T(G) \times T(M)$.

Proof : Consider the family of sections $\{\xi_X\}_{X\varepsilon L(G)}$ of F^f where $\xi_X(g, p) = (X(g), \theta(X)(p))$. Then

(1) by definition of F^f the set $\{\xi_X(g, p)\}_{X\varepsilon L(G)}$ is total in the fiber $F^f_{(g,p)}$ above (g, p) in F^f, and

(2) if (X, Y) is any pair of elements of $L(G)$ and if $(g, p) \varepsilon G \times M$ then

$$[\xi_X, \xi_Y](g, p) = ([X, Y](g), [\theta(X), \theta(Y)](p))$$

$$= ([X, Y](g), \theta([X, Y](p))$$

since θ is a Lie algebra homomorphism. This shows $[\xi_X, \xi_Y](g, p) \in F^f_{(g,p)}$ and the Theorem of Frobenius (1.5) implies F^f is integrable.

By the definition of integrability there is a foliation Y of $G \times M$ such that $T(G\times M, Y) = F^f$.

Proposition (6.2) For $g \in G$, let $\overline{R}(g)$ be the morphism of $G \times M$ into itself given by $\overline{R}(g)(h, p) = (hg, p)$, then $\overline{R}(g)$ is also a morphism of Y into Y where Y is the foliation defined by any infinitesimal action θ of G on M .

Proof :

$$T(\overline{R}(g))(T_{(h,p)}(G\times M, Y))$$
$$= T(\overline{R}(g))(\{(X(h), \theta(X)(p) : X \in L(G)\})$$
$$= \{(X(hg), \theta(X)(p) : X \in L(G)\}$$
$$= T_{(hg,p)}(G\times M, Y)$$

and prop.(1.4) implies that $\overline{R}(g)$ is a morphism of Y into Y .

Remark : Since $\overline{R}(g)$ is a diffeomorphism it takes a maximal connected leaf of Y diffeomorphically onto another maximal connected leaf of Y .

The next proposition explains the name "infinitesimal graph".

Proposition (6.3) If ϕ is any local left action with domain D and infinitesimal generator ϕ^+ then the morphism $\phi^p : g \longrightarrow \phi(g, p)$ of D^p

<u>into</u> M <u>is an integral (Def. (2.2)) of</u> f (<u>where</u> f <u>is defined as above with</u> $\theta = \phi^{+}$). <u>Also the graph of</u> ϕ^p <u>is a leaf containing</u> (e, p) <u>of the foliation</u> Y <u>and the morphism</u> $\pi_G : Y \longrightarrow G$ <u>given by</u> $\pi_G(g, p) = g$ <u>is a local diffeomorphism at each point of</u> Y .

Proof : Let $X \in L(G)$. Then

$$\begin{aligned} T_g(\phi^p)(X(g)) &= T_g(\phi^p)\circ T_e(R(g))(X(e)) \\ &= T_e(\phi^p\circ R(g))(X(e)) \\ &= T_e(\phi^{\phi(g,p)})(X(e)) \\ &= \phi^{+}(X)(\phi(g, p)) \ . \end{aligned}$$

Hence $T_g\phi^p = f_{(g,\phi^p(g))}$ and so ϕ^p is an integral of f . The fact that the graph of ϕ^p is a leaf containing (e, p) follows from prop.(2.5).

Let (g, p) be any point in Y . Then π_G is a local diffeomorphism at (e, p) because $N_p = \{(h, \phi^p(h)) : h \in D^p\}$ is an open neighbourhood of (e, p) in Y mapped diffeomorphically onto D^p by π_G . Now $\overline{R}(g)(N_p)$ is an open neighbourhood of (g, p) in Y by the remark after prop.(6.2) and π_G is a local diffeomorphism on $\overline{R}(g)(N_p)$ which completes the proof.

We now show that two local actions with the same infinitesimal generator coincide in a neighbourhood of $\{e\} \times M$. We need a lemma.

Lemma (6.4) If an infinitesimal action θ of G on M is generating then the foliation Y defined by the infinitesimal graph of θ is a Hausdorff manifold.

Proof : See Palais [7, Theorem VIII, p.44].

Note : Palais' definition of leaf differs slightly from ours.

Let ϕ and ψ be local actions of G on M with domains D_ϕ and D_ψ respectively. Let D_p be the connected component of e in $D_\phi^p \cap D_\psi^p$, then $D = \bigcup_{p\in M} D_p \times \{p\}$ is an open neighbourhood of $\{e\} \times M$ in $G \times M$ (Palais [7, Theorem 1, p.32]).

Uniqueness Theorem (6.5) If ϕ and ψ have the same infinitesimal generator θ then ϕ and ψ coincide on D.

Proof : By prop.(6.3) both ϕ^p and ψ^p are integrals of f (where f is defined as in prop.(6.3)). Let $A \subset D_p$ be the set of points on which ϕ^p and ψ^p agree. A is nonempty since $\phi^p(e) = \psi^p(e) = p$. Prop.(2.3) implies that A is open. Let Y as usual be the foliation defined by the infinitesimal graph of θ. A is closed in D_p since $A = \Phi^{-1}(\Delta)$ where Φ is the morphism from D_p into $Y \times Y$ given by $\Phi(g) = (\phi^p(g), \psi^p(g))$ and Δ is the diagonal in $Y \times Y$ which is a closed set since Y is Hausdorff (Lemma (6.4)). Then $A = D_p$ since D_p is connected.

§7 Existence Theorem

We now give necessary and sufficient conditions on M for an infinitesimal action of G on M to be generating.

Theorem (7.1) A necessary and sufficient condition that an infinitesimal action θ of G on M is generating is that the foliation defined by the infinitesimal graph of θ is a Hausdorff manifold.

Proof : This theorem is proven in Palais [7, pp.52-58] for finite dimensional M. The same proof works in infinite dimensions. A weaker theorem giving sufficient (but not necessary) conditions for θ to be generating is proven in Bourbaki [2, Cor.1, p.184].

Example (7.2) : Local action generated by an infinitesimal action

Consider the infinitesimal action defined in Example (5.2). Keeping the same notation, let $\exp : L(G) \longrightarrow G$ be the exponential map, then $\exp = \mathrm{id}$. If $X \in V(M)$, let $\delta_{X,t}$ denote the local one-parameter group defined by X. Now if ϕ is a local action of G on M such that $\phi^+ = \theta$ then $\phi(tY, p) = \phi(\exp tY, p) = \delta_{\phi^+(Y),t}(p)$ by the uniqueness theorem for differential equation and definition of ϕ^+. Therefore in order to find the local action ϕ corresponding to θ we must find the local one-parameter group corresponding to $\theta(Y)$. To shorten notation we will denote (p, p) by p^2 and $(p, p)(p, p)$ by p^4 for $p \in H$. Now we have

$$\delta_{\theta(Y),t}(p) = \frac{p - t(p, p)Y}{1 - 2t(p, Y) + t^2(p, p)(Y, Y)}$$

for

(1) $\delta_{\theta(Y),0}(p) = p$

(2) $\frac{d}{dt}\,\delta_{\theta(Y),t}(p) = \theta(Y)(\delta_{\theta(Y),t}(p))$.

Proof of (2) : We have

$$\frac{d}{dt}\,\delta_{\theta(Y),t}(p) = \left\{ \frac{-p^2}{1 - 2t(p, Y) + t^2p^2Y^2} \right\} Y$$

$$+ \frac{\{2(p, Y) - 2tp^2Y^2\}}{\{1 - 2t(p, Y) + t^2p^2Y^2\}^2}\{p - tp^2Y\}$$

and

$$\theta(Y)(\delta_{\theta(Y),t}(p)) = 2\{\delta_{\theta(Y),t}(p)\}(\delta_{\theta(Y),t}(p), Y)$$

$$- (\delta_{\theta(Y),t}(p), \delta_{\theta(Y),t}(p))Y$$

$$= 2\left\{ \frac{p - tp^2Y}{1 - 2t(p, Y) + t^2p^2Y^2} \right\} \left(\frac{p - tp^2Y}{1 - 2t(p, Y) + t^2p^2Y^2}, Y \right)$$

$$- \frac{1}{\{1 - 2t(p, Y) + t^2p^2Y^2\}^2}(p - tp^2Y, p - tp^2Y)Y$$

$$= 2\{ p - tp^2Y \}\left\{ \frac{\{(p, Y) - tp^2Y^2\}}{\{1 - 2t(p, Y) + t^2p^2Y^2\}^2} \right\}$$

$$+ \frac{\{- p^2\}}{\{1 - 2t(p, Y) + t^2p^2Y^2\}} Y$$

Comparing these two equations we see that (2) is true. Let

$$D = \left\{(Y, p) \in G \times M \mid 1 - 2(p, Y) + (p, p)(Y, Y) \neq 0\right\}.$$

D is open and contains $\{0\} \times M$. Finally define $\phi : D \longrightarrow M$ by

$$\phi(Y, p) = \frac{p - (p, p)Y}{1 - 2(p, Y) + (p, p)(Y, Y)} .$$

We complete this chapter with a discussion of a special type of infinitesimal action.

§8 Uniform Infinitesimal Actions

Let $\theta : L(G) \longrightarrow V(M)$ be an infinitesimal left group action and Σ_p be the maximal connected leaf through (e, p) of the foliation Y defined by θ. $\pi_G : Y \longrightarrow M$ is the morphism given by $\pi_G(g, p) = g$.

Definition (8.1) θ is called a uniform infinitesimal (left) action of G on M if there exists a connected neighbourhood V of e in G such that for each $p \in M$ the connected component containing (e, p) in $\Sigma_p \cap \pi_G^{-1}(V)$ is mapped one-to-one onto V by π_G. V is called a uniform neighbourhood for θ.

Theorem (8.2) *Each maximal connected leaf Σ of Y is a covering space for G with covering map $\pi := \pi_G|_\Sigma$ if and only if θ is uniform.*

Proof : Suppose θ is uniform. Let V be a uniform neighbourhood. We have to show that for each $g \varepsilon G$ there exists an open neighbourhood W such that $\pi^{-1}(W)$ is a disjoint union of open sets in Σ, each of which is mapped diffeomorphically onto W by π. We first show that $\underline{\pi(\Sigma) = G}$: Let $(g, p) \varepsilon \Sigma$, then by prop. (6.2), $\overline{R}(g^{-1})(\Sigma) = \Sigma_p$ since $\overline{R}(g^{-1})(g, p) = (e, p)$ and $\pi(\Sigma) = \pi\circ\overline{R}(g)(\Sigma_p) = R(g)\circ\pi_G(\Sigma_p)$. So if $\pi_G(\Sigma_p) = G$ then $\pi(\Sigma) = G$ also. This will be proven by showing that for every positive integer n ; $V^n \subset \pi_G(\Sigma_p)$, then $\pi_G(\Sigma_p)$ will equal G since any neighbourhood of e in a connected group generates the group. Since V is a uniform neighbourhood for θ this is true for $n = 1$. Assume now that $V^{n-1} \subset \pi_G(\Sigma_p)$, we will show that $V^n \subset \pi_G(\Sigma_p)$ also. Let g be any point of V^{n-1} then by the induction hypothesis there exists $q \varepsilon M$ such that $(g, q) \varepsilon \Sigma_p$. By prop. (6.2), $\overline{R}(g^{-1})(\Sigma_p) = \Sigma_q$. Now $V \subset \pi_G(\Sigma_q)$ since V is uniform and so $V \subset \pi_G\circ\overline{R}(g^{-1})(\Sigma_p) = R(g^{-1})\circ\pi_G(\Sigma_p)$. This means $gV \subset \pi_G(\Sigma_p)$ for each $g \varepsilon V^{n-1}$, i.e. $V^n \subset \pi_G(\Sigma_p)$.

Now let g be any point of G. Let U be a symmetric connected neighbourhood of e in G such that $U^2 \subset V$, then $W = Ug$ is a neighbourhood of g. We will show that $\pi^{-1}(W)$ is a disjoint union of open sets in Σ, each of which is mapped diffeomorphically onto W by π. Since $\pi(\Sigma) = G$ we have $\pi^{-1}(W)$ is nonempty. Let C be any component in Σ of $\pi^{-1}(W) = \pi^{-1}(Ug)$. If (h, s) is any point of C then $h \varepsilon Ug$ which means $gh^{-1} \varepsilon U^{-1} = U$ and $Ugh^{-1} \subset UU \subset V$. This implies that Ugh^{-1} is a uniform neighbourhood for θ since V was. Prop. (6.2) gives $\overline{R}(h^{-1})(\Sigma) = \Sigma_s$ since $\overline{R}(h^{-1})(h, s) = (e, s)$ and $\overline{R}(h^{-1})(C)$ is the component of (e, s) in $\Sigma_s \cap \pi_G^{-1}(Ugh^{-1})$. But π_G maps the component of (e, s) in $\Sigma_s \cap \pi_G^{-1}(Ugh^{-1})$ diffeomorphically onto Ugh^{-1} since Ugh^{-1} is a uniform neighbourhood which means π maps C diffeomorphically onto $\overline{R}(h)(Ugh^{-1}) = Ug$ and therefore the pair (Σ, π) is a covering space for G.

Conversely suppose $\pi_G|_{\Sigma_q} : \Sigma_q \longrightarrow G$ is a covering map. Let V be any simply connected open neighbourhood of e. Then the component containing (e, q) in $\pi_G^{-1}(V) \cap \Sigma_q$ is a covering space for V and therefore must be mapped diffeomorphically onto V.

We will need the following theorem in Chapter 3.

Theorem (8.3) If G is simply connected and M is a Hausdorff manifold then a uniform infinitesimal left action $\theta : L(G) \longrightarrow V(M)$ generates a global action of G on M.

Proof : By the above theorem each leaf Σ is a covering space for G and since G is simply connected $\pi_G|_\Sigma : \Sigma \longrightarrow G$ is a diffeomorphism. For $p \in M$ denote this diffeomorphism of Σ_p onto G by π_G^p. As usual denote by f, the vector bundle morphism from $p_G{*}T(G)$ into $p_M{*}T(M)$ induced by θ, and by Y the foliation of $G \times M$ defined by the integrable subbundle F^f. Define $\phi^p : G \longrightarrow M$ to be $\phi^p(g) = \pi_M \circ (\pi_G^p)^{-1}(g)$ where $\pi_M : Y \longrightarrow M$ is $\pi_M(g, m) = m$. Finally define $\phi : G \times M \longrightarrow M$ to be $\phi(g, p) = \phi^p(g)$. We will show that ϕ is a global group action with infinitesimal generator

θ . Let $v \in L(G)$. Note that each ϕ^p is an integral for f since

$$T(\phi^p)(v(g)) = T(\pi_M)\circ T(\pi_G^p)^{-1}(v(g))$$

$$= T(\pi_M)(v(g), f_{(g,(\pi_G^p)^{-1}(g))}(v(g)))$$

$$= f_{(g,\phi^p(g))}(v(g)) \quad \text{as} \quad (\pi_G^p)^{-1}(g) = \phi^p(g) \ .$$

ϕ is a global action :

(1) $\phi(e, p) = \pi_M\circ(\pi_G^p)^{-1}(e) = \pi_M(e, p) = p$ since (e, p) is the unique point in Σ_p with first component equal to e .

(2) Show $\phi(g, \phi(h, p)) = \phi(gh, p)$ for all $g, h \in G$ and $p \in M$. Define $\psi_1(g) = \phi(g, \phi(h, p))$ and $\psi_2(g) = \phi(gh, p)$. By the definition of ϕ , the graph of ψ_1 is $\Sigma_{\phi(h,p)}$ and the graph of ψ_2 is $\overline{R}(h^{-1})(\Sigma_p) = \Sigma_{\phi(h,p)}$ since $\overline{R}(h^{-1})(h, \phi(h, p)) = (e, \phi(h, p))$. Then since $\pi_G^{\phi(h,p)}$ is one-to-one on $\Sigma_{\phi(h,p)}$ we have $\phi(g, \phi(h, p)) = \phi(gh, p)$.

We now show that $\phi : G \times M \longrightarrow M$ is a morphism. For $p \in M$, define

$$A^p = \left\{ g \in G : \begin{array}{l} \text{there exists some open neighbourhood } U \\ \text{of } g \text{ and some open neighbourhood } V \text{ of} \\ p \text{ such that } \phi \text{ is a morphism on } U \times V \end{array} \right\} .$$

(a) <u>A^p contains e and therefore $A^p \neq \emptyset$</u>.

Let $\rho : M \longrightarrow M$ be the identity. By prop.(2.6) there exists a connected open neighbourhood $U \times V$ of (e, p) in $G \times M$ and a morphism $\psi : U \times V \longrightarrow M$ such that for all $m \in V$ the morphism $\psi^m : g \longrightarrow \psi(g, m)$ is an integral for f with $\psi^m(e) = \rho(m) = m$. ϕ^m is also an integral for f on $U \times V$ with $\phi^m(e) = \psi^m(e)$. Since M is Hausdorff and U is connected it follows, just as in the proof of theorem (6.4) using the uniqueness of integrals, that $\phi^m = \psi^m$ on U ; i.e. $\phi = \psi$ on $U \times V$ and A^p contains e.

(b) <u>A^p is open in G</u> by definition.

(c) <u>A^p is closed in G</u>.

Let $g \in \overline{A^p}$, by (a) above there exists a connected neighbourhood $U \times V$ of $(e, \phi(g, p))$ such that ϕ is a morphism on $U \times V$. We denote ϕ by β on $U \times V$ to emphasize that it is a morphism. Furthermore we assume $U = U^{-1}$. Since $h \longrightarrow \phi(h, p)$ is an integral, and so in particular continuous, there exists a neighbourhood N of g such that $\phi(N, p) \subset V$. Let $h \in N \cap Ug \cap A^p$; h exists since $g \in \overline{A^p}$ and $N \cap Ug$ is a neighbourhood of g. By the definition of A^p there exists a connected neighbourhood $U_1 \times V_1$ of (h, p) on which ϕ is a morphism and since $\phi(h, p) \in V$ we can assume (shrinking if necessary) that $\phi(h, V_1) \subset V$. Define $\gamma : Uh \times V_1 \longrightarrow M$ by $\gamma(k, m) = \beta(kh^{-1}, \phi(h, m))$. γ is a morphism on $Uh \times V_1$ since it is a composition of morphisms; $\gamma = \beta \circ (R(h^{-1}) \times \phi^h)$. Now for $m \in V_1$, we have the morphism $\gamma^m : U_1 \longrightarrow M$ given by

$\gamma^m(k) = \gamma(k, m)$ with $\gamma(h, m) = \phi(h, m)$. We will show that γ^m is an integral for f and then (as in the proof of (a)) since ϕ^m is also an integral with the same value at h, ϕ^m will equal γ^m and ϕ will be a morphism on $Uh \times V_1$. Let $\beta^m : U \longrightarrow M$ be the morphism defined by $\beta^m(k) = \beta(k, m)$. Since $\beta = \phi$ on $U \times V$ we have $\beta^m = \phi^m$ on U. Let $X \in L(G)$, to show that γ is an integral we need that

$$T_g(\gamma^m)(X(g)) = f_{(g,\gamma^m(g))}(X(g)) = \theta(X)(\gamma^m(g)) .$$

We have

$$T_g(\gamma^m)(X(g)) = T_g(\beta^{\phi(h,m)} \circ R(h^{-1}))(X(g)) \qquad \text{(by def. of } \gamma)$$

$$= T_{gh^{-1}}(\beta^{\phi(h,m)})(X(gh^{-1})) \qquad \text{(since } X \in L(G))$$

$$= \theta(X)(\beta^{\phi(h,m)}(gh^{-1}))$$

(since $\beta^{\phi(h,m)} = \phi^{\phi(h,m)}$ is an integral of f)

$$= \theta(X)(\beta^{\phi(h,m)} \circ R(h^{-1})(g))$$

$$= \theta(X)(\gamma^m(g))$$

and so γ^m is an integral of f. Therefore ϕ is a morphism on $Uh \times V_1$ and $h \in Ug$ implies $g \in U^{-1}h = Uh$, i.e. $(g, p) \in Uh \times V_1$ and $g \in A^p$ showing that $A^p = \overline{A^p}$. Since G is connected (a), (b) and (c) imply that $A^p = G$. As p was arbitrary ϕ is a morphism on $G \times M$.

It remains to show that θ is the infinitesimal generator of ϕ.

Let $X \in L(G)$, then

$$T_e\phi^p(X(e)) = f_{(e,\phi^p(e))}(X(e)) \quad \text{(since } \phi^p \text{ is an integral for } f\text{)}$$

$$= \theta(X)(\phi^p(e)) = \theta(X)(p)$$

showing that θ is the infinitesimal generator of ϕ and completing the proof of the theorem.

The proof that ϕ is a morphism is essentially the same as the proof showing that the flow of a vector field is a morphism. (Cf. Lang [5, p.80]).

<u>Proposition</u> (8.4) <u>Let</u> $\phi : G \times M \longrightarrow M$ <u>be a global left action of</u> G <u>on a Hausdorff manifold</u> M. <u>Let</u> $X \in L(G)$ <u>and</u> $\{\delta_t\}$ <u>be the one-parameter group corresponding to</u> $\phi^+(X)$. <u>Then</u>

$$\delta_t(p) = \phi(\exp_G tX, p) \quad \text{for all } p \in M .$$

Proof : $$\phi(\exp_G 0 \cdot X, p) = \phi(e, p) = p$$

and

$$\frac{d}{dt}\phi(\exp_G tX, p) = \frac{d}{ds}\bigg|_{s=0} \phi(\exp_G(s+t)X, p)$$

$$= \frac{d}{ds}\bigg|_{s=0} \phi(\exp_G sX, \phi(\exp_G tX, p))$$

since ϕ is a global action

$$= \phi^+(X)(\phi(\exp_G tX, p)) \quad \text{by definition of } \phi^+ .$$

The result then follows from the uniqueness theorem for differential equations.

Chapter 3

Connected Lie Transformation Groups

Let $D(M)$ be the group of diffeomorphisms of the manifold M. A Lie group G is called a Lie transformation group of M if the underlying group of G is a subgroup of $D(M)$ and if the map $(g, p) \longrightarrow g(p)$ of $G \times M$ into M is a morphism. Of course one could give G the discrete topology and this would automatically be true. A nontrivial example is the group $I(M)$ of isometries of a finite dimensional Riemannian manifold, which is a Lie transformation group with respect to the compact open topology. Further examples of Lie transformation groups can be found in H. Chu and S. Kobayashi [3]. The main result of this chapter is to show that there is a one-to-one correspondence between connected Lie transformation groups of M and certain subalgebras of the Lie algebra of vector fields $V(M)$ where M is a Hausdorff manifold. In this chapter M will always denote a Hausdorff manifold.

§9 The Image of the Infinitesimal Generator of a Lie Transformation Group

Let G be a Lie transformation group of M. Then there is a global action ϕ of G on M with infinitesimal generator $\phi^+ : L(G) \longrightarrow V(M)$. We now examine the image of ϕ^+ in $V(M)$. Let $\exp : L(G) \longrightarrow G$ be the exponential map.

Proposition (9.1) The image $\phi^{+}(L(G))$ consists of complete vector fields and the one-parameter group corresponding to $\phi^{+}(X)$ is $\exp tX$.

Proof : $\exp tX$ is a one-parameter group and

$$\left.\frac{d}{dt}\right|_{t=0} \exp tX(p) = \phi^{+}(X)(p) .$$

The result then follows from the uniqueness theorem for differential equations.

Proposition (9.2) ϕ^{+} is injective.

Proof : If $\phi^{+}(X) = 0$ then

$$\left.\frac{d}{dt}\right|_{t=s} \exp tX(p) = \left.\frac{d}{dt}\right|_{t=0} \exp((s + t)(X)(p)$$

$$= \left.\frac{d}{dt}\right|_{t=0} \exp tX(\exp sX(p))$$

$$= \phi^{+}(X)(\exp sX(p))$$

$$= 0_{\exp sX(p)} \qquad \text{for all } p \in M .$$

This means $\exp tX(p) = p$ for all $t \in R$ and all $p \in M$, i.e. $\exp tX = id_M$ which implies that $X = 0$ since exp has a radius of injectivity at 0 in $L(G)$.

Proposition (9.3) $\phi^{+}(L(G))$ possesses a Banach Lie algebra structure such that the evaluation map $(Y, p) \longrightarrow Y(p)$ is a vector bundle morphism from

<u>the trivial vector bundle</u> $\phi^+(L(G)) \times M$ <u>into</u> $T(M)$ <u>and</u> $\phi^+ : L(G) \longrightarrow \phi^+(L(G))$ <u>is a Banach Lie algebra isomorphism.</u> <u>Furthermore, this Banach space structure is necessarily unique.</u>

Proof : By prop. (9.2) $\phi^+ : L(G) \longrightarrow \phi^+(L(G))$ is a Lie algebra isomorphism and hence induces a Banach Lie algebra structure on $\phi^+(L(G))$ making ϕ^+ a Banach Lie algebra isomorphism. Define $\beta : \phi^+(L(G)) \times M \longrightarrow L(G) \times M$ by $\beta(Y, p) = ((\phi^+)^{-1}(Y), p)$, then β is easily seen to be a vector bundle morphism. Now prop.(5.3) gives that the map $\alpha : (X, p) \longrightarrow \phi^+(X)(p)$ of $L(G) \times M \longrightarrow T(M)$ is a vector bundle morphism. The evaluation map $\phi^+(L(G)) \times M \longrightarrow T(M)$ is equal to $\alpha \circ \beta$ and therefore is a vector bundle morphism. The uniqueness of the Banach space structure comes from the following proposition.

<u>Proposition</u> (9.4) <u>Let</u> E <u>be a vector bundle over</u> M <u>and let</u> V <u>be a vector space of sections of</u> E. <u>If</u> V <u>admits two Banach space structures such that the evaluation map</u> $(X, p) \longrightarrow X(p)$ <u>of</u> $V \times M$ <u>into</u> E <u>is continuous with respect to both then the identity map from</u> V <u>into</u> V <u>is a homeomorphism,</u> <u>i.e.</u> <u>the two norms are equivalent</u>.

Proof : Let V_1 and V_2 denote V with respect to the two topologies and let $e_i : V_i \times M \longrightarrow E$ $(i = 1, 2)$ be the evaluation maps. By the closed graph theorem, in order to show that $id : V_1 \longrightarrow V_2$ is continuous, it is enough to show that the diagonal is closed in $V_1 \times V_2$. Let $\{(X_n^1, X_n^2)\}$ be a Cauchy sequence in the diagonal of $V_1 \times V_2$, i.e. $X_n^1 \in V_1$, $X_n^2 \in V_2$ and $X_n^1 = X_n^2$. Since $V_1 \times V_2$ is complete there exists a limit point (X, Y) of this sequence; but for all $p \in M$ we have

$$X(p) = \lim_{n\to\infty} e_1(X_n^1, p) = \lim_{n\to\infty} X_n^1(p)$$

$$= \lim_{n\to\infty} X_n^2(p)$$

$$= \lim_{n\to\infty} e_2(X_n^2, p) = Y(p) .$$

Therefore $X = Y$ and the diagonal is closed in $V_1 \times V_2$. Interchanging V_1 and V_2 above gives that $id : V_2 \longrightarrow V_1$ is continuous also and id is a homeomorphism.

If Y is a complete vector field then denote by $\mathrm{Exp}\, tY$, the one-parameter group generated by Y. Let $\phi : G \times M \longrightarrow M$ be the global action of a Lie transformation group G. Prop.(9.1) gives that $\exp_G(X) = \mathrm{Exp}(\phi^+(X))$ and this implies the following result.

<u>Proposition</u> (9.5) <u>Exp is injective on a neighbourhood of</u> 0 <u>in</u> $\phi^+(L(G))$ <u>in the topology induced on</u> $\phi^+(L(G))$ <u>by</u> ϕ^+.

§10 Banach Lie Algebras of Complete Vector Fields

We now consider when a Lie subalgebra L of $V(M)$ is the image of the infinitesimal generator of a connected Lie transformation group. In view of propositions (9.1), (9.3), and (9.5) we only consider Lie subalgebras L of $V(M)$ which satisfy the following conditions

(A) L consists of complete vector fields;

(B) L has a Banach Lie algebra structure, (necessarily unique by Prop.(9.4)) such that ;

(B1) the evaluation map ev : (X, p) ⟶ X(p) is a vector bundle morphism from the trivial bundle L × M into T(M).

(B2) there exists an open ball $B_r(0)$ of radius r at 0 such that Exp : L ⟶ D(M) is injective on $B_r(0)$.

Proposition (10.1) If L is finite dimensional and satisfies (A) then condition (B) is true also.

Proof : Since L is finite dimensiional it has a natural Banach space structure. A proof of (B1) is in Bourbaki [2, Remarque p.140] and a proof of (B2) is in Loos [6, p.182].

The rest of this section will be devoted to proving the following theorem.

Theorem (10.2) If M is a Hausdorff manifold and L is a Lie subalgebra of V(M) satisfying conditions (A) and (B) then there exists a unique connected Lie transformation group G with natural global action $\phi : G \times M \longrightarrow M$ such that ϕ^+ is a Banach Lie algebra isomorphism of L(G) onto L.

Remark : Palais [7] first proved this theorem in the case where L and M are finite dimensional. Using a different method, Loos [6] extended this result to the case where L is finite dimensional and M is a (not necessarily Hausforff) Banach mainfold. The proof given here is similar to Palais' .

In order to prove Theorem (10.2) we need the following theorem, which is of interest in itself.

Theorem (10.3). If M is a Hausdorff manifold and L is a Lie subalgebra of $V(M)$ satisfying condition (A) and admitting a Banach Lie algebra structure such that (B1) is true (but not necessarily (B2)) then there exists a simply connected Lie group $\tilde{G}$ with $L(\tilde{G}) = \Lambda L$ and a global action $\psi : \tilde{G} \times M \longrightarrow M$ such that $\psi^{+}(L(\tilde{G})) = L$, ψ^{+} is a continuous linear map into L, and for $C \in \Lambda L$; $\psi^{+}(C) = C(1)$.

Proof : By theorem (3.1) there exists a Lie group with Lie algebra ΛL. Let $\tilde{G}$ be the universal covering group (see Bourbaki [2, p.113]) of this group ; then $L(\tilde{G}) = \Lambda L$. We have an infinitesimal left action, which we call ψ^{+}, of $\tilde{G}$ on M given by the following sequence of vector bundle morphisms,

$$\Lambda L \times M \longrightarrow L \times M \xrightarrow{\text{ev}} T(M)$$

$$(C, p) \longrightarrow (C(1), p) \longrightarrow C(1)(p)$$

where ev is the evaluation map which is a vector bundle morphism by condition (B1). The map $\psi^{+} : C \longrightarrow C(1)$ is continuous by theorem (3.1). The existence of the global action ψ will follow from theorem (8.3) by showing that ψ^{+} is a uniform infinitesimal left action.

By condition (B1), $ev : L \times M \longrightarrow T(M)$ is a vector bundle morphism. Then the global version of the existence theorem for differential equations depending on a parameter (in this case the parameter space is L) implies that the map $(t, X, p) \longrightarrow Exp(tX)(p)$ from $\mathbb{R} \times L \times M$ into M is a morphism. The fact that this flow is defined on all of $\mathbb{R}$ follows from condition (A) .

Now let $B_\rho(0)$ be an open ball about 0 in ΛL on which $\exp_{\tilde{G}}$ is a diffeomorphism. We will show that ψ^+ is uniform on $V = \exp_{\tilde{G}}(B_\rho(0))$. Define, for each $p \in M$, the map $\delta^p : g \longrightarrow (g, Exp(\psi^+(\exp_{\tilde{G}}^{-1}(g)))(p))$ from V into $\tilde{G} \times M$. Let Σ_p be the maximal connected leaf containing (e, p) in the foliation Y of $\tilde{G} \times M$ defined by the infinitesimal graph of ψ^+. For $X \in L(G)$ and $p \in M$ define

$$\alpha_X^p : t \longrightarrow (\exp_{\tilde{G}}(tX), Exp(t\psi^+(X)(p))$$

from $\mathbb{R}$ into $\tilde{G} \times M$. Now

$$T(\alpha_X^p)\left(\left.\frac{d}{dt}\right|_s\right) = \left.\frac{d\alpha_X^p}{dt}\right|_s$$

$$= \left(\left.\frac{d}{dt}\right|_{t=0} \exp_{\tilde{G}}(tX)\exp_{\tilde{G}}(sX), \left.\frac{d}{dt}\right|_{t=s} Exp(t\psi^+(X))(p)\right)$$

$$= \left(X(\exp_{\tilde{G}}(sX), \psi^+(X)(Exp(s\psi^+(X))(p))\right)$$

belongs to $T(\tilde{G} \times M, Y)$. Then prop.(1.4) implies that the image of α_X^p is in Σ_p since $\alpha_X^p(0) = (e, p)$ and $\mathbb{R}$ is connected. In particular if

$g = \exp_{\tilde{G}}(X)$ for $X \in B_\rho(0)$ then

$$\alpha_X^p(1) = (\exp_{\tilde{G}}(X), \operatorname{Exp}(\psi^+(X))(p))$$

$$= (g, \operatorname{Exp}(\psi^+(\exp_{\tilde{G}}^{-1}(g))(p))$$

which shows that $\delta^p(V) \subset \Sigma_p$.

Denote by V^p the image of V under δ^p . The "projection" $\pi_{\tilde{G}} : Y \longrightarrow \tilde{G}$ obviously maps V^p one-to-one onto V . In fact we now show that V^p is the component containing (e, p) in $\pi_{\tilde{G}}^{-1}(V) \cap \Sigma_p$. We have $(e, p) = \delta^p(e) \in V^p$. Suppose (g, q) is any point in V^p and let U be an open set in Σ_p containing (g, q) on which $\pi_{\tilde{G}}$ is a diffeomoephism (prop.(6.3)). Let W be an open set in $V \cap \pi_{\tilde{G}}(U)$ containing g , then $\pi_{\tilde{G}}^{-1} : \pi_{\tilde{G}}(U) \longrightarrow U$ takes W onto an open set containing (g, q) and $\pi_{\tilde{G}}^{-1}\Big|_{\pi_{\tilde{G}}(U)}(W) \subset V^p$ since $\pi_{\tilde{G}}^{-1} = \delta^p$ on W . This proves that V^p is open. In order to show that V^p is the component containing (e, p) in $\pi_{\tilde{G}}^{-1}(V) \cap \Sigma_p$ it remains to prove that V^p is closed in $\pi_{\tilde{G}}^{-1}(V)$. Let (h, m) be any point in $\pi_{\tilde{G}}^{-1}(V) \cap \Sigma_p$ such that $(h, m) \notin V^p$. Now since $\pi_{\tilde{G}}$ is one-to-one on V^p there exists a unique point in V^p with first component h, say (h, n). Σ_p is Hausdorff since it is an open submanifold of Y and therefore we can find disjoint open neighbourhoods A and B of (h, m) and (h, n) respectively which $\pi_{\tilde{G}}$ maps diffeomorphically onto the same neighbourhood of h ; this is possible since $\pi_{\tilde{G}}$ is a local diffeomorphism. By restricting A and B further we can assume that $A \subset V^p$ and it then follows that $B \cap V^p = \emptyset$ since $\pi_{\tilde{G}}$ is one-to-one on

V^p . This completes the proof that ψ^+ is uniform and proves the theorem.

Proof of Theorem (10.2)

We keep the notations used above. L is assumed to be a subalgebra of V(M) satisfying conditions (A) and (B).

Consider the ideal $\ker \psi^+ = \{ C \in L(\tilde{G}) : C(1) = 0\}$ in $L(\tilde{G})$ which is the kernel of the map $\psi^+ : L(\tilde{G}) \longrightarrow V(M)$; it is closed in $L(\tilde{G})$ since $\ker \psi^+ = \bigcap_{p \in M} (\psi^+_p)^{-1}(0_p)$ where ψ^+_p is the continuous linear map $X \longrightarrow \psi^+(X)(p)$ from $L(\tilde{G})$ into $T_p(M)$ (0_p denotes the zero vector in $T_p(M)$). Therefore $\ker \psi^+$ is a Banach Lie subalgebra of $L(G)$. Let

$$L' = \left\{ C \in L(\tilde{G}) = \Lambda L : C(t) = tX \text{ for some } X \in L \right\},$$

then L' is a closed vector subspace in $L(\tilde{G})$ which complements $\ker \psi^+$, i.e. $\ker \psi^+$ splits in $L(\tilde{G})$ and we identify $L(\tilde{G})$ with $\ker \psi^+ \times L'$.

For $g \in \tilde{G}$, denote by ψ_g the diffeomorphism $p \longrightarrow \psi(g, p)$ of M into M. Let $\delta : \tilde{G} \longrightarrow D(M)$ be the group homomorphism $g \longrightarrow \psi_g$. Let $H = \ker \delta$, then we have a group isomorphism $\bar{\delta} : \tilde{G}/H \longrightarrow \delta(\tilde{G})$. We will show that H is a Lie subgroup of $\tilde{G}$. Condition (B2) gives the existence of a open neighbourhood N of 0 in L on which Exp is injective. Let $A \times B \subset \ker \psi^+ \times L' = L(\tilde{G})$ be an open neighbourhood of 0 on which $\exp_{\tilde{G}}$ is a diffeomorphism and such that $\psi^+(A \times B) \subset N$. Let $h \in H \cap \exp_{\tilde{G}} (A \times B)$, then $h = \exp_{\tilde{G}} (C)$ for some unique $C \in A \times B$. Now for all $p \in M$ we have

$$p = \psi(h, p) = \psi(\exp_{\tilde{G}}(C), p)$$

$$= \text{Exp}\,(\psi^{+}(C))(p) \qquad \text{(by prop.(8.4))}$$

$$= \text{Exp}\,(C(1))(p)\ .$$

Since $C(1) \varepsilon N$ and $\text{Exp}\,(C(1)) = \text{id}$ we have $C(1) = 0$, i.e. $C \varepsilon A \times \{0\}$. Also if $C \varepsilon A \times \{0\}$ then $\psi(\exp(C), p) = p$ for all p and $\exp(C) \varepsilon H$. The fact that H is a Lie subgroup then follows from prop.(3.3) since $\exp_{\tilde{G}}(A \times \{0\}) = H \cap \exp_{\tilde{G}}(A \times B)$. We also have $L(H) = \ker \psi^{+}$.

It follows from prop. (3.6) that there exists a connected Lie group structure on $\tilde{G}/H$ such that the projectiion $p : \tilde{G} \longrightarrow \tilde{G}/H$ is a submersion and $L(\tilde{G}/H) \cong L(\tilde{G})/L(H) \cong L$. Using the group isomorphism $\overline{\delta} : \tilde{G}/H \longrightarrow \delta(\tilde{G})$ we have a Lie group structure induced on $\delta(\tilde{G})$ such that δ is a submersion, $\ker L(\delta) = \ker \psi^{+}$, $L(\delta(\tilde{G})) \cong L$, and $\overline{\delta}$ is a diffeomorphism. With this Lie group structure $\delta(\tilde{G})$ will be denoted by G. Define $\phi : G \times M \longrightarrow M$ to be the natural action $\phi(g, p) = g(p)$. Let α be the submersion $(k, p) \longrightarrow (\delta(k), p)$ of $\tilde{G} \times M$ into $G \times M$. Then $\psi = \phi \circ \alpha$ and ϕ is a morphism since ψ is a morphism and α is a submersion.

We now show that ϕ^{+} is a Banach Lie algebra isomorphism. Let $C \varepsilon L(\tilde{G}) = \Lambda L$ and $p \varepsilon M$, then

$$C(1)(p) = \psi^{+}(C)(p)$$

$$= T_{(e,p)}\psi(C(e),\ 0_p)$$

$$= T_{(e,p)}(\phi\circ\alpha)(C(e),\ 0_p)$$

$$= T_{(id,p)}\phi(T_e\delta(C(e)),\ 0_p)$$

$$= \phi^+(L(\delta)(C))(p)\ ,$$

i.e. $\psi^+ = \phi^+\circ L(\delta)$.

We see that ϕ^+ maps $L(G)$ onto L since ψ^+ maps $L(\tilde{G})$ onto L (theorem (10.3)). Since $L(\delta)(\ker\psi^+) = 0$ we have that ϕ^+ is injective. The fact that ϕ^+ is continuous follows easily from the following;

(1) ψ^+ is continuous

(2) $L(\delta)$ is continuous, surjective and $\ker L(\delta) = \ker\psi^+$ splits in $L(\tilde{G})$ (since δ is a submersion).

It remains to prove the uniqueness of G. Let F be another Lie transformation group with the same properties as G and let $\beta : F\times M \longrightarrow M$ be the map $(f, p) \longrightarrow f(p)$. Now $\exp_F(tX) = \text{Exp}(t\beta^+(X))$ for $X \in L(F)$ by prop. (9.1) and therefore since F is connected it is generated by Exp (L). Similarly G is generated by Exp (L) which shows that the underlying groups of G and F are the same in $D(M)$. The following commutative diagram

$$\begin{array}{ccc} L(G) & \xrightarrow{(\beta^+)^{-1}\circ\phi^+} & L(F) \\ \downarrow{\scriptstyle \exp_G} & & \downarrow{\scriptstyle \exp_F} \\ G & \xrightarrow{\ \ Id\ \ } & F \end{array}$$

shows that Id from G into F is a morphism and completes the proof of the theorem.

§11 A Banach Lie algebra of complete vector fields which does not generate a connected Lie transformation group

If we have a subalgebra L of $V(M)$ which satisfies condition (A) and admits a Banach Lie algebra structure such that (B1) is true but not (B2) then prop.(9.4) implies that this L won't admit any Banach space structure such that (B2) is satisfied. Hence by prop.(9.5), L isn't the image of the infinitesimal generator of any connected Lie transformation group.

We now give an example of such an L which, although it doesn't generate a connected Lie transformation group, is still enlargeable. Let $M = \text{disjoint} \bigcup_{n=1}^{\infty} S_n^1$ where S_n^1 is the unit circle S^1. Define the vector field X_n by

$$X_n(p) = \begin{cases} 0_p & \text{if } p \in S_j^1 \text{ and } j \neq n \\ \frac{d\alpha}{dt}(p) & \text{if } p \in S_n^1 \end{cases}$$

where α is the curve on S_n^1 ; $t \longrightarrow e^{2\pi it}$.

Let L be the normed vector space consisting of all sums $\sum_{n=1}^{\infty} c_n X_n$, $c_n \in \mathbb{R}$, such that $\sum_{n=1}^{\infty} \frac{|c_n|}{n} < \infty$. If $C = \sum c_n X_n \in L$, define the norm of C to be $||C|| = \sum_{n=1}^{\infty} \frac{|c_n|}{n}$. We will show that L is a Banach Lie algebra satisfying (A) and (B1) but not (B2). L consists of complete vector fields since we have a disjoint union of compact manifolds. It is a vector space for

$$||\sum c_n X_n + \sum b_n X_n|| = \sum \frac{|c_n + b_n|}{n}$$

$$\leq \sum \frac{|c_n|}{n} + \sum \frac{|b_n|}{n} < \infty$$

if $\sum c_n X_n$ and $\sum b_n X_n$ belong to L. Similarly L is closed under scalar multiplication. L is closed under the bracket operation since

$$[\sum c_n X_n, \sum b_n X_n] = \sum_{i,j} c_i b_j [X_i, X_j] = 0 .$$

We now give the usual proof that a space of sequences is complete. Let $\{A^n\}$ be a Cauchy sequence of elements in L, i.e. given $\varepsilon > 0$ there exists N such that if $i, j > N$ we have

$$||A^i - A^j|| = \sum_k \frac{|A_k^i - A_k^j|}{k} < \varepsilon$$

where $A^n = \sum_k A_k^n X_k$. In particular this implies that for fixed k, $\{A_k^i\}$ is a Cauchy sequence. Let $A_k = \lim_{i\to\infty} A_k^i$ and $A = \sum A_k X_k$. We will show that $A \varepsilon L$ and $\lim_{n\to\infty} A^n = A$. From above we have

$$\sum_{k=1}^{s} \frac{|A_k^i - A_k^j|}{k} < \varepsilon$$

for all $s \geq 1$ and $i, j > N$, then

$$\lim_{i\to\infty} \sum_{k=1}^{s} \frac{|A_k^i - A_k^j|}{k} = \sum_{i=1}^{s} \frac{|A_k - A_k^j|}{k} < \infty \qquad \text{for all } s > 1.$$

Since this is true for all s we have

$$\sum_{k=1}^{\infty} \frac{|A_k - A_k^j|}{k} < \varepsilon .$$

But this implies that $A - A^j \in L$ which means $A = (A - A^j) + A^j \in L$. We also have for $j > N$, $||A - A^j|| < \varepsilon$; which shows that $\lim_{j\to\infty} A^j = A$. We have now shown that L is a Banach Lie algebra and it is trivially enlargeable since it is abelian.

It remains to show that the evaluation map $ev : L \times M \longrightarrow T(M)$ is a vector bundle morphism. Local coordinates on each of the S_n^1 's are given by the local inverse of the map $t \longrightarrow e^{2\pi it}$. We denote this map by $\log_n$. This induces local coordinates on $T(M)$ and we denote this map by $\log_n^T$. If $\pi : T(M) \longrightarrow M$ is the usual projection and if $Z = \sum z_n X_n$ then

$$\log_n^T(Z(p)) = (\log_n(\pi(Z(p)), z_n) = (\log_n(p), z_n) .$$

Denote by π_n the continuous linear map from L into R given by $\sum a_k X_k \longrightarrow a_n$. A local coordinates map at $(Y, p) \in L \times M$ is given by $(Y, p) \longrightarrow (Y, \log_n(p))$ if $p \in S_n'$. Let $Y = \sum y_k X_k$ and W be an open neighbourhood of $Y(p) = ev(Y, p)$ in $T(M)$. Now $\log_n^T(Y(p)) = (\log_n(p), y_n)$

and by the definition of the topology of $T(M)$ there exists an open set U containing $\log_n(p)$ and an interval $B_\rho(y_n)$ about y_n such that $(\log_n^T)^{-1}(U \times B_\rho(y_n)) \subset W$. Let $Z = \sum z_n X_n$, if $(Z, q) \in B_{\rho/n}(Y) \times (\log_n)^{-1}(U) \subset L \times M$ then $||Z - Y|| < \rho/n$ which implies $|z_n - y_n| < \rho$ which implies $\log_n^T(Z(q)) \in U \times B_\rho(y_n)$; which shows $ev(B_{\rho/n}(Y) \times U) \subset W$. Hence ev is continuous and these local coordinates ev is given by the map τ in the following diagram,

$$\begin{array}{ccc} (Y, p) & \xrightarrow{\ ev\ } & Y(p) \\ \Big\downarrow & & \Big\downarrow \\ (Y, \log_n(p)) & \xrightarrow{\ \tau\ } & (\log_n(p), \pi_n(p)) \end{array}$$

and τ is a morphism since π_n is. This proves that ev is a morphism of manifolds. It is a vector bundle morphism since the constant map $p \longrightarrow \pi_n$ is a morphism from U into the space of continuous linear maps from L into $\mathbb{R}$.

Condition (B2) doesn't hold because Exp doesn't have a radius of injectivity at 0 in L, for $\text{Exp } X_n$ = identity for all n and $\lim_{n\to\infty} X_n = 0$.

Although L doesn't generate a connected Lie transformation group theorem (10.3) ensures that it is the image of the infinitesimal generator of a global left action.

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