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Geometrization of Quantum Mechanics

T. W. B. Kibble
Blackett Laboratory, Imperial College, London SW7 2BZ, England

Abstract. Quantum mechanics is cast into a classical Hamiltonian form in terms of a symplectic structure, not on the Hilbert space of state-vectors but on the more physically relevant infinite-dimensional manifold of instantaneous pure states. This geometrical structure can accommodate generalizations of quantum mechanics, including the nonlinear relativistic models recently proposed. It is shown that any such generalization satisfying a few physically reasonable conditions would reduce to ordinary quantum mechanics for states that are "near" the vacuum. In particular the origin of complex structure is described.

1. Introduction

Geometrical ideas, especially symplectic structures, have come to play an increasingly important role in classical mechanics [1, 2]. The geometry of the classical phase space Γ also underlies the geometrical quantization programme of Souriau [3, 4] (see also Kostant [5]). Moreover it is known that quantum dynamics can be expressed in terms of a Hamiltonian structure on the Hilbert space $\mathscr{H}$ of state-vectors, where the imaginary part of the scalar product defines a symplectic structure [6].

However if one is seeking an axiomatic basis for quantum mechanics, it seems better to start from structures of direct physical significance, as in the operational approach of Haag and Kastler [7] and others [8, 9] or the work on the geometry of quantum logics [10, 11].

It is pointed out in Sect. 2 that this can be achieved by a slight modification of the Hamiltonian formalism. We have to consider not the Hilbert space $\mathscr{H}$ itself but the manifold Σ of "instantaneous pure states" which is (essentially but not quite) a projective Hilbert space. This formalism is closely akin to the work of Mielnik [12] on the geometry of the space of quantum states. It provides a convenient framework within which to discuss possible generalizations of quantum mechanics. In particular it can readily accommodate the relativistic model theories proposed in a recent paper [13], or at least a large class of them. It is worth noting that by formulating the theory on Σ rather than $\mathscr{H}$ we automatically ensure that it satisfies the scaling property shown in [13] to be necessary for a consistent measurement theory (see also [14]).

From this point of view, the essential difference between classical and quantum mechanics lies not in the set of states (save for the infinite dimensionality) nor in the dynamic evolution, but rather in the choice of the class of observables, which is far more restricted in quantum than in classical mechanics.

One motivation for the present work is the possibility that it might be of use in the unification of quantum mechanics with general relativity. The idea that this unification must be an essentially geometric one, so long championed by Einstein in his search for a unified theory, has recently been coming back into favour. It seems natural that as a prerequisite quantum mechanics itself should be cast in geometrical language. Moreover there are good reasons for seeking to generalize it, to free it from the restrictions of linearity just as general relativity has freed space-time from the limitations of flatness [12, 15]. The geometrical structure described in the present paper can easily be generalized to allow the space of quantum states to be an arbitrary infinite-dimensional symplectic manifold.

Obviously, any viable generalization of quantum mechanics must reduce to that theory for a wide range of phenomena. One of the main aims of this paper is to discuss how this might happen. I shall show, on the basis of some rather natural physical assumptions, that all the main features of conventional quantum mechanics would emerge naturally for states that are in a suitable sense near the vacuum, near enough to be represented by vectors in the tangent space.

In Sects. 2 and 3 we recall the main features of symplectic structures and Hamiltonian dynamics, and introduce the quantum phase space Σ. For conventional quantum mechanics, Σ is essentially a projective complex Hilbert space; more precisely, a dense subspace thereof. However the formalism can be applied to a much more general case, in which Σ is a real infinite-dimensional manifold with (weak) symplectic structure. The dynamics is discussed in Sect. 3 in terms of a Hamiltonian function E on Σ. For ordinary quantum mechanics E is the expectation value of the Hamiltonian operator, and the evolution equation reproduces Schrödinger's equation.

The concept of a symmetry is examined in Sect. 4 with particular emphasis on the space-time symmetries associated with the Poincaré group. Although an eventual aim is the unification of quantum theory with gravity, for the present a flat space-time is assumed. So too is the existence of a unique vacuum state v. The tangent space T_v to Σ at the vacuum plays an important role.

The aim of Sect. 5 is to show that for states close enough to the vacuum to be represented by vectors in the tangent space, the formalism necessarily reduces to ordinary linear quantum mechanics. In particular, I shall show how, although Σ is a real manifold, the complex structure of quantum mechanics would naturally appear on the tangent space.

The conclusions are summarized in Sect. 6 and a number of unresolved questions described.

2. The Quantum Phase Space

Axiomatic treatments of quantum mechanics often begin with the set of all mixed states or ensembles [7, 11, 16, 17]. Pure states are regarded merely as extremal elements of this convex set. However we shall deal exclusively with the set $\mathscr{S}$ of pure states, and assume at the outset that all others can be expressed as (presumably countable) mixtures of them.

Moreover instead of the Heisenberg-picture approach we shall use the Schrödinger picture. We assume that, with respect to a given choice of time axis each pure state can be represented by a path, or "history" in a space Σ of "instantaneous pure states". For convenience, we reserve the term *state* for an element of Σ; elements of $\mathscr{S}$ will be called *histories*. Of course, the dynamics establishes a one-to-one correspondence between elements of Σ at a specified time and elements of $\mathscr{S}$.

Although the formalism will be more general, it will be useful to begin by discussing the structure of Σ in the special case of standard quantum mechanics. Indeed this is the only space we shall actually need in this paper because the generalized models considered use the same set of instantaneous states and differ from the standard theory only in the dynamics. Later, however, more general theories will be considered.

In classical mechanics, Σ is of course the phase space Γ, normally a finite-dimensional manifold, but in quantum mechanics it is infinite-dimensional. Essentially it is a projective Hilbert space, the set of rays in a complex Hilbert space $\mathscr{H}$ of state vectors. However there are technical problems associated with the fact that the Hamiltonian operator H is unbounded. We have to choose between two alternative formalisms, each with its peculiar advantages and disadvantages. One is to work with the rays of $\mathscr{H}$ itself and accept that the Hamiltonian vector field which specifies the dynamics is defined only on a dense subspace [6]. The other, which we shall use here, is to work from the start with a dense subspace $\mathscr{K}$ of $\mathscr{H}$ equipped with a finer topology that makes H continuous.

The chief drawback of this second alternative is that Σ possesses only a weak symplectic structure (see below). If we were trying to prove existence theorems for solutions of the time-evolution equation, this might be a major defect, but for our present purposes it is not important.

Let us assume then that there is a dense linear subspace $\mathscr{K}$ of $\mathscr{H}$ which forms a common invariant domain for all the operators we wish to consider, including in particular the Hamiltonian and other symmetry generators. For example, in a nonrelativistic many-body theory we may take $\mathscr{K}$ to be the subspace of states whose wave functions belong to some test-function space, say the Schwartz space $\mathscr{S}$ (see for instance [18]). In a field theory we might take it to be the subspace generated by applying some algebra of observables to the vacuum state. Physically, it is only states in this subspace that we can actually prepare, so $\mathscr{K}$ is of more direct physical relevance than $\mathscr{H}$.

The space $\mathscr{K}$ is assumed to be equipped with a topology finer than that of $\mathscr{H}$, defined for example by a countable family of norms [18], which makes H and the other symmetry generators continuous on $\mathscr{K}$. It is possible to do this in such a way that $\mathscr{K}$ becomes a Fréchet-Schwartz space.

Now let $\mathscr{K}^0$ be the set of all nonzero vectors in $\mathscr{K}$, and let $\mathbb{C}^0$ be the multiplicative group of all nonzero complex numbers. Then we choose as our standard quantum phase space the projective space $\Sigma = \mathscr{K}^0/\mathbb{C}^0$, which we shall regard as a real manifold.

For the general theory, all we shall assume is that Σ is a real infinite-dimensional paracompact manifold modelled on some Fréchet-Schwartz space $\mathscr{V}$. What this means [19] is that Σ is a Hausdorff topological space that can be covered by a countable family of open sets U_α on each of which a homeomorphism ϕ_α is defined to an open set in $\mathscr{V}$, and that whenever $U_\alpha \cap U_\beta \neq \emptyset$, the map $\phi_\alpha \phi_\beta^{-1}$ restricted to $\phi_\beta(U_\alpha \cap U_\beta)$ is twice continuously differentiable. In a general infinite-dimensional non-Banach space, the definition of differentiability is quite problematic [20]. However in the particular case of Fréchet-Schwartz spaces, there is a perfectly serviceable definition [21] which allows the construction of C^k manifolds (see also [22, 23]).

It might not be unreasonable to require that Σ be a C^∞ manifold, but we shall not need that assumption here.

In addition to its manifold structure, Σ is required to have a weak symplectic structure. This means [6] that there is defined on Σ a closed two-form ω which is weakly non-degenerate in the sense that if $\omega_u(X, Y)=0$ for all tangent vectors $Y \in T_u\Sigma$, then $X=0$. This form can be used to "lower" indices [1]. It defines a map $X \mapsto X^\flat$ from the tangent bundle $T\Sigma$ to the cotangent bundle $T^*\Sigma$: if X is any tangent vector (or vector field) the corresponding cotangent vector (or differential one-form) is

$$X^\flat = i_X \omega \,, \tag{1}$$

where i_X is the interior product (evaluation function) defined by

$$i_X\omega(Y) = \omega(X, Y) \,.$$

(Note that there is a difference of a factor 2 between conventions used here and in references [1, 2, 6]. I follow the conventions of Choquet-Bruhat et al. [24]).

The map $X \mapsto X^\flat$ is injective. However, ω is only weakly nondegenerate in the sense that $X \mapsto X^\flat$ is not in general bijective. There may be one-forms that are not images of any vector field.

In the special case of ordinary quantum mechanics, ω is defined in terms of the inner product. (As well its finer topology $\mathscr{K}$ retains the inner product $\langle \cdot, \cdot \rangle$ of $\mathscr{H}$.)

To define ω we need to introduce a local coordinate system in Σ. Let π be the canonical projection of $\mathscr{K}^0$ onto Σ. Thus if $\mathbf{u}$ is any nonzero vector in $\mathscr{K}$, $\pi\mathbf{u} = u$ denotes the corresponding state. Let $\mathbf{v}$ be any normalized vector in $\mathscr{K}$. Then in a neighbourhood of $\pi\mathbf{v} \in \Sigma$ we can represent states uniquely by vectors $\mathbf{u}$ in the hyperplane

$$\mathscr{K}_\mathbf{v} = \{\mathbf{u} \in \mathscr{K} : \langle \mathbf{v}, \mathbf{u} \rangle = 1\} \,.$$

Thus we have a homeomorphism ϕ from this neighbourhood to an open set in $\mathscr{K}_\mathbf{v}$. We then define ω at $\pi\mathbf{v}$ by

$$\omega_{\pi\mathbf{v}}(X, Y) = 2\,\mathrm{Im}\,\langle \phi_* X, \phi_* Y \rangle \tag{2}$$

for any $X, Y \in T_{\pi\mathbf{v}}\Sigma$, or more generally at any point in this coordinate patch by

$$\omega_u(X, Y) = \frac{2\,\mathrm{Im}\,\langle \phi_* X, P_{\phi u} \phi_* Y \rangle}{\langle \phi u, \phi u \rangle} \,, \tag{3}$$

where $X, Y \in T_u\Sigma$ and $P_\mathbf{u}$ is the orthogonal projector onto the subspace normal to $\mathbf{u}$.

This two-form ω is obviously skew-symmetric and weakly nondegenerate. But while ω is readily seen to be closed, it is not exact. (It is easy to construct a closed surface over which its integral is nonzero, although locally $\omega = -d\theta$ with

$$\theta_u = \frac{\mathrm{Im}\,\langle d\phi u, \phi u\rangle}{\langle \phi u, \phi u\rangle}\,.)$$

This is another interesting difference between quantum and classical mechanics. Classically, the phase space has the structure of a cotangent bundle and $\omega = -d\theta$ where θ is the canonical one-form $\theta = \Sigma p dq$. No such form exists globally on the quantum phase space Σ.

3. Dynamics on the Quantum Phase Space

Let us now consider the specification of dynamics on Σ, assumed to be a real C^2 Fréchet-Schwartz manifold equipped with a weak symplectic structure.

The dynamics is described by a Hamiltonian flow, that is to say a one-parameter group of diffeomorphisms $\tau_t : \Sigma \to \Sigma$ (with $\tau_0 = 1$ and $\tau_{t+s} = \tau_t \circ \tau_s$) which preserve the symplectic structure, i.e.

$$\tau_t^* \omega = \omega\,. \tag{4}$$

The τ_t depend continuously and differentiably [21] on t and can thus be written

$$\tau_t = \exp tK\,,$$

where K is a C^1 vector field on Σ, which of course also leaves invariant the symplectic structure:

$$L_K \omega = 0\,, \tag{5}$$

where L_K is the corresponding Lie derivative. (Had we adopted the alternative approach mentioned in Sect. 2 we could not have required τ_t to be differentiable in t and would have had to allow K to be defined only on a dense subset of Σ, see [6].)

A history, i.e. an element of $\mathscr{S}$, is an integral curve of this flow in Σ, a curve c everywhere tangent to K:

$$\frac{dc(t)}{dt} = K_{c(t)}\,, \tag{6}$$

or equivalently

$$c(s+t) = \tau_t c(s)\,. \tag{7}$$

By virtue of the identity $L_K = i_K d + d i_K$ and the fact that ω is closed, it follows from (5) and the definition (1) that

$$d(K^\flat) = 0\,, \tag{8}$$

i.e. $K^\flat$ is closed.

In standard quantum mechanics, the space Σ defined in Sect. 2 is a simply connected paracompact manifold, and so its first de Rham cohomology group is trivial. (De Rham's theorem in general is hard to prove but here we need only the comparatively straightforward result that a closed form in such a space is exact.) This is an important feature, because it means that from (8) we can conclude that $K^\flat$ is exact, i.e. that a Hamiltonian function exists. We shall assume that the same is true in any generalized quantum theory we consider, though it might indeed be intriguing to consider a theory with a non-simply-connected quantum phase space, a toroidal space say.

With this assumption it follows from (8) that there exists a Hamiltonian function E, unique up to a constant, such that

$$K^\flat = dE \,. \tag{9}$$

It follows that any history is a solution of Hamilton's equations

$$\left(\frac{dc(t)}{dt}\right)^\flat = dE_{c(t)} \,. \tag{10}$$

Of course the function E uniquely determines the vector field K and therefore the one-parameter group τ_t. What is much less obvious is whether, given E, any such flow exists, or in other words to prove an existence theorem for solutions of the differential Eq. (10). We shall not attempt to treat this problem here (see [6]).

In ordinary quantum mechanics, the function E is identified with the expectation value of the Hamiltonian operator,

$$E(u) = \langle H \rangle_u \equiv \frac{\langle \phi u, H\phi u\rangle}{\langle \phi u, \phi u\rangle} \,. \tag{11}$$

It is easy to verify that with the definition (3) for ω this reproduces Schrödinger's equation. We seek a curve ψ in $\mathscr{H}$ such that $\pi\circ\psi = c$. Equation (10) holds as an equation for one-forms on Σ i.e. for arbitrary variations of c, but it can be pulled back by π to yield an equation for one-forms on $\mathscr{H}^0$, i.e. for arbitrary variations of ψ. The equation in fact involves only transverse variations, and thus yields

$$P_{\psi(t)}\left\{i\frac{d\psi(t)}{dt} - H\psi(t)\right\} = 0 \,,$$

where P is the orthogonal projector defined in Sect. 2. Equivalently

$$i\frac{d\psi(t)}{dt} = H\psi(t) + \alpha(t)\psi(t) \,, \tag{12}$$

where $\alpha(t)$ is an undetermined complex function. Its presence signifies our freedom to multiply ψ by an arbitrary time-varying factor without changing c. Normally of course we choose $\operatorname{Im}\alpha = 0$ to preserve normalization and fix $\operatorname{Re}\alpha$ (equivalent to a variable zero-point of energy) by some other convention.

A large class of generalized models can be described by this same formalism, using the same set Σ and the same symplectic structure but with a more general choice for the Hamiltonian function E. For example, let us consider as in [13] a model of a scalar field ϕ. Let H be the Hamiltonian operator for the usual linear quantum field theory with ϕ^4 interaction. Let us take

$$E(u) = \langle H \rangle_u + \tfrac{1}{8}\lambda \int d^3\mathbf{x} \langle :\phi^2(\mathbf{x}): \rangle_u^2 \tag{13}$$

in place of (11), where $:\phi^2:$ is defined by subtracting an (infinite) constant, as in [13]. Then instead of (12) we obtain the Schrödinger equation

$$i\frac{d\psi(t)}{dt} = H\psi(t) + \tfrac{1}{4}\lambda \int d^3\mathbf{x} \langle :\phi^2(\mathbf{x}): \rangle_{\psi(t)} :\phi^2(\mathbf{x}): \psi(t) + \alpha(t)\psi(t) \,. \tag{14}$$

(We write $\langle \cdots \rangle_\psi$ for $\langle \cdots \rangle_{\pi\psi}$.) Except for the fact that the arbitrary time-dependent factor is made explicit, (14) is precisely the evolution equation of one of the models proposed in [13]. (There is also a difference of a factor of 2 in λ.) This model can be thought of as one in which the particle mass becomes state-dependent and position-dependent,

$$m^2 \mapsto m^2 + \tfrac{1}{2}\lambda \langle :\phi^2(\mathbf{x}): \rangle \,.$$

It is clear that many other generalized models can be constructed in the same way. For example, if we add to (13) the extra term

$$+\tfrac{1}{48}\mu \int d^3\mathbf{x} \langle :\phi^2(\mathbf{x}): \rangle_u \langle :\phi^4(\mathbf{x}): \rangle_u \,.$$

we obtain a model in which both m^2 and the ϕ^4 coupling constant g acquire a state-dependence,

$$m^2 \mapsto m^2 + \tfrac{1}{2}\lambda \langle :\phi^2(\mathbf{x}): \rangle + \tfrac{1}{24}\mu \langle :\phi^4(\mathbf{x}): \rangle \,,$$

$$g \mapsto g + \tfrac{1}{2}\mu \langle :\phi^2(\mathbf{x}): \rangle \,.$$

Not all the models of [13] can be expressed in this way because some of them do not possess a conserved energy functional that can serve as our Hamiltonian function E. However it seems natural to require the existence of such a function, so this is not a severe restriction.

As we noted in the introduction, by formulating the theory on Σ rather than $\mathscr{H}$ we automatically ensure the invariance under scaling transformations $\psi \mapsto \lambda\psi$ which was shown in [13] to be a necessary prerequisite for a consistent measurement theory (see also [14]).

4. Symmetries

As a preliminary to the discussion of how conventional quantum mechanics might emerge as a linear approximation to a more general theory, it will be useful to examine the representation of symmetries.

A one-parameter group of (time-independent) symmetries is a group of diffeomorphisms ϕ_s of Σ onto itself which leave invariant both the symplectic structure and the Hamiltonian function,

$$\phi_s^*\omega = \omega \,, \qquad \phi_s^* E = E \,. \tag{15}$$

It follows that ϕ_s commutes with the time evolution τ_t. The generator of the group is a C^1 vector field X on Σ which also leaves invariant the symplectic structure and commutes with the generator of time evolution:

$$[X, K] = 0 \,. \tag{16}$$

Just as for K itself, it follows that $X^\flat$ is closed, hence exact, so that there exists a generating function G_X, unique up to an additive constant, such that

$$X^\flat = dG_X \,. \tag{17}$$

If X and Y are symmetry generators, the effect of one on the generating function of the other is

$$L_X G_Y = G_{[X,Y]} \,, \tag{18}$$

in the sense that the left hand side is a possible generating function for $[X, Y]$ i.e. this equation holds up to an arbitrary constant.

We can easily generalize the discussion to include time-dependent symmetries, generated by vector fields that do not satisfy (16). Equation (18) is still valid if one or other vector field is K.

We shall be interested in particular in space-time symmetries. We shall assume that there is a realization by symmetries

$$(a, \Lambda) \mapsto \tau_{a,\Lambda}$$

of the connected component of the Poincaré group, or rather[1], if we wish to accommodate fermions, of its two-fold universal covering group P. There is then a corresponding realization by vector fields of its Lie algebra $\mathscr{P}$. Let us denote by K_μ and $R_{\mu\nu}$ the generators of translations and rotations, obeying the usual commutation rules, for example

$$[K_\lambda, K_\mu] = 0 \,,$$
$$[K_\lambda, R_{\mu\nu}] = \eta_{\lambda\mu} K_\nu - \eta_{\lambda\nu} K_\mu \,,$$

where $\eta_{\lambda\mu}$ is the Minkowski-space metric tensor. Of course K_0 is precisely the time-translation vector field K introduced earlier.

The associated generating functions are the energy-momentum and angular momentum functions,

$$P_\mu = G_{K_\mu} \,, \qquad J_{\mu\nu} = G_{R_{\mu\nu}} \,.$$

P_0 is the Hamiltonian function E. By virtue of (18) these functions transform in the correct way under translations and rotations. For example,

$$\begin{aligned} L_{K_\mu} P_\lambda &= 0 \\ L_{R_{\mu\nu}} P_\lambda &= \eta_{\lambda\mu} P_\nu - \eta_{\lambda\nu} P_\mu \,. \end{aligned} \tag{19}$$

Next, let us assume the existence of a unique vacuum state $v \in \Sigma$ invariant under all operations of the Poincaré group

$$\tau_{a,\Lambda} v = v \,. \tag{20}$$

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1 This point requires further justification. An argument for it was presented in an appendix to the preprint version of this paper (ICTP/77–78/22), but cannot be included here for reasons of space

Equivalently

$$K_\mu(v)=0\,, \qquad R_{\mu\nu}(v)=0\,. \tag{21}$$

The uniqueness is not essential. We could easily accommodate a discretely degenerate vacuum state. A continuous degeneracy, however, would be unacceptable.

Because v is unique, it must be invariant under any symmetry. For any one-parameter group ϕ_s of symmetries, $\phi_s v=v$, or if X is the corresponding generator

$$X_v=0\,. \tag{22}$$

The generating function $G=G_X$ satisfies

$$dG_v=0\,. \tag{23}$$

It is convenient to fix the arbitrary constants in generating functions by the condition

$$G(v)=0\,. \tag{24}$$

In particular

$$P_\mu(v)=0\,, \qquad J_{\mu\nu}(v)=0\,. \tag{25}$$

Note that these conditions are consistent with the validity of Eq. (19) without extra constant terms.

An important role will be played by the tangent space $T_v=T_v\Sigma$ to Σ at the vacuum state. Because of (22) any symmetry generator X induces on T_v a linear transformation X'_v which of course preserves the symplectic structure:

$$X_v'^*\omega=0\,.$$

Because of (23) and (24), there is a quadratic approximation to G on the tangent space. We may define the second derivative G''_v as a symmetric bilinear function of vectors in T_v. If c is any curve through v, with

$$c(0)=v, \qquad \frac{dc}{ds}(0)=Y\,,$$

then

$$\lim_{s\to 0} s^{-2}G(c(s))=\tfrac{1}{2}G''_v(Y,Y)\,.$$

From (17) there follows a relation between X'_v and G''_v, namely

$$G''_v=(X'_v)^\flat \tag{26}$$

in the sense that for all $Y,Z\in T_v$,

$$G''_v(Y,Z)=\omega_v(X'_vY,Z)\,.$$

In particular we may define the second derivative E''_v of the energy function $E=P_0$. We shall need one further important assumption, relating to the positivity of energy. (Remember that the zero point is fixed at the energy of the vacuum.) We shall assume specifically that the vacuum is a nondegenerate local minimum of the energy function, i.e. that E_v'' is a positive definite bilinear function:

$$E_v''(Y, Y) > 0 \quad \text{for all} \quad Y \neq 0 . \tag{27}$$

Presumably the vacuum should also be a global minimum of E, but we shall not need that condition here.

5. Complex Structure

As noted earlier, one of the most important questions to ask about any proposed generalization of quantum mechanics is how the ordinary linear theory can emerge as an approximation. The suggestion made here is that states that are, in a sense to be defined, near the vacuum can be represented by vectors in the tangent space T_v, and that on T_v one has all the usual structure of linear quantum mechanics, expressed of course in a particular local coordinate system like the one used in defining ω.

To validate this suggestion, we have to do two things – to show how the structures of linear quantum mechanics emerge on T_v, and to specify what is meant by "nearness" to the vacuum. As far as the second question is concerned, no general answer is possible; it must depend on the specific model. In the generalized field theories discussed earlier [13] the answer is reasonably clear. A state u is near the vacuum if the expectation value $\langle :\phi^2(\mathbf{x}): \rangle_u$ is everywhere small compared to a scale constant m^2. For such states the nonlinear time-evolution equation can be well approximated by its linear counterpart. Note however that there is no guarantee that a state initially near the vacuum will remain so during its subsequent history. In general, it will not.

In order for states near the vacuum to be represented by vectors in the tangent space, one also needs of course a well defined map from some neighbourhood of v in Σ into T_v. Here again, it is only in the context of a specific model that one can expect to specify it. In our generalized models, it poses no problem because the nonlinear and linear theories share the same space Σ and the same T_v. In the linear theory there is of course a natural map from Σ to T_v defined as in the definition of ω in Sect. 2.

Let us now turn to the other outstanding problem, that of recovering the structures of linear quantum theory on T_v. Much of the structure of course already exists. In particular we have a linear representation of the Poincaré group. What is lacking however is the complex structure of ordinary quantum mechanics, and the existence of a hermitean inner product.

A very important role in recovering this structure is played by the positive definite symmetric bilinear function E_v''. It is convenient for the moment to think of it as defining a real inner product on T_v. (However, it is important to realise that this is *not* directly related to the hermitean inner product we shall eventually construct.) Thus T_v becomes a pre-Hilbert space.

Now any one-parameter group of time-independent symmetries ϕ_s leaves invariant the energy function E, and so is represented on T_v by orthogonal transformations ϕ_{s*}. In particular this is true of the time-translation operators τ_{t*}.

Consequently, the generator K'_v is antisymmetric. We shall use it to introduce a complex structure on T_v, in terms of a linear operator J satisfying $J^2 = -1$.

It is convenient (though not essential) to consider the complexified space $\mathbf{T}_v = T_v \oplus i T_v$ and to define first a corresponding operator $\mathbf{J}$ on it. We may extend K'_v in an obvious way to a complex-linear operator $\mathbf{K}'_v$ on $\mathbf{T}_v$. Then $i\mathbf{K}'_v$ is essentially self-adjoint, and possesses a purely real spectrum. It may be regarded as an energy operator. We seek to define an operator $\mathbf{J}$ which takes the value $+i$ on the subspace where $i\mathbf{K}'_v$ is positive and $-i$ on that where it is negative, i.e.

$$\mathbf{J} = i\,\mathrm{sign}\,(i\mathbf{K}'_v)\,. \tag{28}$$

There is a slight technical problem here. This construction is certainly possible if the spectrum of $i\mathbf{K}'_v$ does not include the point 0. If it does include it, $\mathbf{J}$ could still be defined on the Hilbert space obtained by completion of $\mathbf{T}_v$ but there seems to be no reason to suppose that it would necessarily leave $\mathbf{T}_v$ invariant. Physically, the condition that 0 is excluded from the spectrum corresponds to the requirement that there be a nonzero minimum mass in the theory. For the moment we shall assume that either 0 is excluded or, if it is not, that $\mathbf{J}$ can still be defined.

It is clear from its definition (28) that $\mathbf{J}$ commutes with the conjugation operation $X + iY \mapsto X - iY$ on $\mathbf{T}_v$, and so defines a corresponding operator J on T_v. In fact J could be defined directly on T_v by setting

$$J = -K'_v[-(K'_v)^2]^{-1/2}\,, \tag{29}$$

where the positive square root of the operator is implied. It follows incidentally that J commutes with all Poincaré-group generators. Moreover

$$\omega(JX, Y) = -\omega(X, JY)\,. \tag{30}$$

It may be helpful to describe one way of thinking about this operation in the case of ordinary quantum mechanics. A vector $X \in T_v$ may be represented by a density operator which in Dirac notation takes the form

$$X \mapsto |\mathbf{x}\rangle\langle\mathbf{v}| + |\mathbf{v}\rangle\langle\mathbf{x}|$$

where $|\mathbf{v}\rangle$ is a representative of the vacuum, and $\langle\mathbf{v}|\mathbf{x}\rangle = 0$. Then J is defined by

$$JX \mapsto i|\mathbf{x}\rangle\langle\mathbf{v}| - i|\mathbf{v}\rangle\langle\mathbf{x}|$$

Now that we have defined an operator J satisfying $J^2 = -1$ we may introduce a complex structure on T_v. We define a complex vector space T_v^c whose vectors are in one-to-one correspondence $X \leftrightarrow X^c$ with those of T_v. T_v^c is given a complex structure by defining

$$iX^c = (JX)^c\,. \tag{31}$$

Note that T_v^c has half as many complex dimensions as $\mathbf{T}_v$.

In terms of the representation above, we may regard the passage to T_v^c as projecting out the "positive-energy" part, i.e.

$$X^c \mapsto |\mathbf{x}\rangle\langle\mathbf{v}|\,.$$

On T_v^c we can define a hermitean inner product by setting

$$2\langle X^c, Y^c\rangle = \omega(X, JY) + i\omega(X, Y)\,. \tag{32}$$

It is easy to verify that it is indeed hermitean, linear in the second factor and antilinear in the first.

On the tangent space we have then all the usual structures of linear quantum theory, in particular the complex structure and hermitean inner product. Moreover the symmetry generators are antihermitean, and multiplying by i makes them essentially self-adjoint.

It should be remarked that the positivity of energy has played a vital role here. Without it K'_v would not have left invariant a positive symmetric form and so its complexification would not have had a pure imaginary spectrum.

6. Discussion

The principal results of this paper are two. First, I showed in Sect. 2 that quantum dynamics can be expressed in a simple and elegant Hamiltonian form in terms of a symplectic structure on the manifold Σ of instantaneous pure states. Thus classical and quantum mechanics, and generalizations thereof, can all be formulated in very similar ways.

Second, I have shown that if one makes some rather natural physical assumptions, such as existence of a unique vacuum and local positivity of the energy, then in any of the proposed generalizations of quantum mechanics, the conventional quantum theory will re-emerge as a linear approximation for states near the vacuum. In particular, the complex structure appears naturally.

Many problems remain. An important feature of the method adopted here is its use of the Schrödinger picture, but this has the obvious disadvantage of lacking manifest relativistic covariance. It would be useful to find a way of rewriting the formalism in an explicitly covariant manner.

There are several technical points that need to be clarified. The mathematical characterization of Fréchet-Schwartz manifolds is not at present very simple. It is not even clear that they constitute the correct class to choose. In particular the cohomology of such manifolds is not well known; and an existence theorem for histories is needed. Also the introduction of complex structure in Sect. 5 required a specific assumption about the spectrum of K'_v which needs to be eliminated or better understood.

The only generalizations of quantum mechanics we have considered have used the same quantum phase space Σ, but a different evolution law. However, one might also examine alternative choices for Σ. In particular, it would be intriguing to consider spaces with nontrivial first cohomology group.

Although the formalism was developed to permit generalizations of quantum mechanics, it also provides an interesting starting point for axiomatisation of the conventional theory. It would be useful to know what conditions imposed on the symplectic structure and Hamiltonian function would allow one to identify the theory with standard quantum mechanics.

One aim of this work was to rewrite quantum mechanics in a form better suited to unification with general relativity, but so far we have considered only flat space-time. The next step may be to examine what happens when both structures, space-time and the space of quantum states, are simultaneously freed from the restrictions of linearity. I hope to return to this question at a later date.

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