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Geometry and Algebra of Prime Fano 3-folds of Genus 12

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Abstract. The connection between these Fano 3-folds and plane quartic curves is explained.

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1. Introduction

According to Mukai (1992, 1995) every prime Fano three-fold of genus 12 has geometric realizations in three different ways:

(a) Let V be a seven-dimensional vector space and let $\eta\colon \Lambda^2 V \to N$ be a net of alternating forms on V, i.e. N is three-dimensional. Denote by

$$\mathbb{G}(3, V, \eta) = \{E \in \mathbb{G}(3, V) | \Lambda^2 E \subset \ker(\eta\colon \Lambda^2 V \to N)\}$$

the Grassmannian of isotropic 3-spaces of V.

(b) Let $f \in S_4(U)$ be the equation of a plane quartic $F \subset \mathbb{P}(U) = \mathbb{P}^2$. A polar hexagon Γ of F is the union of six lines $\Gamma = \{l_1 \cdot \ldots \cdot l_6 = 0\}$ such that $f = l_1^4 + \ldots + l_6^4$. We can also identify Γ with a sextuple of points in $\check{\mathbb{P}}^2 = \mathbb{P}(U^*)$, i.e. a point of $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)$. Then the variety of sums of powers presenting f is

$$VSP(F, 6) = \overline{\{\Gamma \in \mathrm{Hilb}_6(\check{\mathbb{P}}^2) | \Gamma \text{ is polar to } F\}},$$

the closure of the set of polar hexagon of F.

(c) Let W be a four-dimensional vector space, and let $q\colon U^* \hookrightarrow S_2 W^*$ be a net of quadrics in $\check{\mathbb{P}}^3 = \mathbb{P}(W^*)$. Consider the Hilbert scheme $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3) = H_1 \cup H_2$ and the component H_1 containing the twisted cubic curves $C \subset \mathbb{P}(W) = \mathbb{P}^3$. Let $H \subset \mathbb{G}(3, S_2 W)$ be the image of H_1 under the map

$$\mathrm{Hilb}_{3t+1}(\mathbb{P}^3) \to \mathbb{G}(3, H^0(\mathbb{P}^3, \mathcal{O}(2))) = \mathbb{G}(3, S_2 W),$$

which sends $C \mapsto H^0(\mathbb{P}^3, I_C(2))$. Denote by

$$H(q) = H \cap \mathbb{G}(3, \ker(S_2 W \to U)) \subset \mathbb{G}(3, S_2 W)$$

the variety of twisted cubics, whose quadratic equations are annihilated by $U^* \subset S_2 W^*$.

THEOREM 1.1 (Mukai, 1992; Mukai, 1995). *Let X be a prime Fano 3-fold of genus 12 over an algebraically closed field of characteristic 0. Then there exist (a) a net of alternating forms $\eta: \Lambda^2 V \to N$, (b) a plane quartic F, (c) a net of quadrics $q: U^* \hookrightarrow S_2(W)^*$ such that*

$$X \cong \mathbb{G}(3, V, \eta) \cong VSP(F, 6) \cong H(q).$$

Conversely a general net of alternating forms, a general quartic or a general net of quadrics gives a smooth prime Fano 3-fold of genus 12.

COROLLARY 1.2. *The moduli spaces $\mathcal{M}_{Fano}$ of prime Fano 3-folds of genus 12, $\mathcal{M}_3$ of curves of genus 3, $\mathcal{M}_q$ of nets of quadrics, and $\mathcal{M}_{3,\vartheta^{ev}}$ of curves of genus 3 together with a non-vanishing theta characteristic are birational to each other.*

$\mathcal{M}_{3,\vartheta^{ev}}$ occurs, since a net of quadrics in $\mathbb{P}^3$ is determined by its discriminant, a plane quartic, together with the associated non-vanishing theta characteristic. The connection between (a) and (b) is sketched in Mukai (1992). The surprising fact that $\mathcal{M}_3$ and $\mathcal{M}_{3,\vartheta^{ev}}$ are birational, is actually an old result due to Scorza (1899) recently reconsider by Dolgachev and Kanev (1993). The purpose of this paper is to give a detailed description, how the three realizations are related to each other. In particular it is proved that Mukai's and Scorza's constructions give the same birational transformation. For non-algebraically closed ground fields our investigation gives that the models of type (a) and (b) exists over the field of definition of the V_{22}, while the space curve model (c) is in general only defined after a field extension.

2. A Polarity and Sums of Powers

Let k be a field of characteristic zero. Consider $S = k[x_0, \dots, x_r]$ and $T = k[\partial_0, \dots, \partial_r]$. T acts on S by differentiation:

$$\partial^\alpha(x^\beta) = \alpha! \frac{\beta}{\alpha} x^{\beta-\alpha}$$

if $\beta \geqslant \alpha$ and 0 otherwise. Here α and β are multi-indices, $\binom{\beta}{\alpha} = \prod \beta_i/\alpha_i$ and so on. In particular we have a perfect pairing, *apolarity*, between forms of degree n and homogeneous differential operators of order n.

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Note that the polar of a form $f \in S$ in a point $a \in \mathbb{P}^r$ is given by $P_a(f)$ for $a = (a_0, \dots, a_r)$ and $P_a = \sum a_i \partial_i \in T$.

One can interchange the role of S and T by defining

$$x^\beta(\partial^\alpha) = \beta! \binom{\alpha}{\beta} \partial^{\alpha-\beta}.$$

With this notation we have for forms of degree n

$$P_a^n(f) = f(P_a^n) = n! f(a).$$

Moreover

$$f(P_a^m) = 0 \Longleftrightarrow f(a) = 0 \tag{2.1}$$

if $m \geqslant n$.

A polarity allows us to define Artinian Gorenstein graded quotient rings of T via forms: For f a homogeneous form of degree n define

$$f^\perp = \{D \in T | D(f) = 0\}$$

and

$$A^f = T/f^\perp.$$

The socle of A^f is in degree n. Indeed $P_a(D(f)) = 0\ \forall P_a \in T_1 \Longleftrightarrow D(f) = 0$ or $D \in T_n$. In particular the socle of A^f is 1-dimensional, and A^f is indeed Gorenstein.

Conversely, for a graded Gorenstein ring $A = T/I$ with the socle in degree n multiplication in A induces a linear form $f : S_n(T_1) \to k$ which can be identified with a homogeneous polynomial $f \in S$ of degree n. This proves:

THEOREM 2.1 (Macaulay, 1916). *The map $F \mapsto A^F$ gives a bijection between hypersurfaces $F = \{f = 0\} \subset \mathbb{P}^r$ of degree n and Artinian graded Gorenstein quotient rings $A = T/I$ of T with socle in degree n.*

Note, that

$$(f^\perp : D) = D(f)^\perp \tag{2.2}$$

for any homogeneous $D \in T$.

DEFINITION 2.2 (Iarrobino, 1984; Sylvester, 1886). For forms f of even degree $2n$ the matrix $\mathrm{Cat}(f) = (D_i D_j(f))_{1 \leqslant i,j \leqslant \binom{n+r}{r}}$ with $D_1, \dots, D_{\binom{n+r}{r}} \in T_n$ a basis, is called catalectican matrix of f. f is called non-degenerate, if $Cat(f)$ has maximal rank.

The Hilbert function and syzygies of A^F depends on subtle properties of F (cf. Iarrobino, 1984; Iarrobino and Kanev, 1996). For example for plane quartics, we have (Clebsch, 1861):

THEOREM 2.3. *Let $F = \{f = 0\}$ be a plane quartic. The following integers are equal:*

(1) $\dim_k A_2^F$,
(2) rank Cat(f),
(3) *the minimal s such that over the algebraic closure $\overline{k}$ of k f lies in the closure of forms* $l_1^4 + \cdots + l_s^4$.

Proof. $\dim_k A_2^f = \operatorname{rank} \operatorname{Cat}(f)$ holds because multiplication in A^F gives perfect pairings $A_2^F \times A_2^F \to A_4^F \cong k$.

Suppose $f = l_1^4 + \cdots + l_s^4$. The corresponding lines $L_1, \ldots, L_s \in \check{\mathbb{P}}^2$, viewed as points in the dual space, impose at most s conditions on quadrics $\{D = 0\} \subset \check{\mathbb{P}}^2$. Hence $\dim_k f_2^{\perp} \geqslant 6 - s$ and $\dim_k A_2^F \leqslant s$. Since the Hilbert function of A^F varies semi-continuously with F, $\dim_k A_2^F \leqslant s$ holds for all forms in the closure of the set of forms of the sums of s powers.

Conversely, suppose $\dim_k A_2^F = s$. $A^F = T/I$ is Gorenstein of codimension 3. Hence the structure theorem of Buchsbaum and Eisenbud (1977) applies: A^F has syzygies

$$0 \longleftarrow A^F \longleftarrow T \longleftarrow F_1 \overset{\phi}{\longleftarrow} F_2 \longleftarrow T(-7) \longleftarrow 0,$$

with $F_1 = \bigoplus_{i=1}^{2r+1} T(-a_i)$, $F_2^* \cong F_1(7)$, ϕ skew-symmetric, and I generated by the $2r \times 2r$ pfaffians of ϕ. Conversely, any sufficiently general skew-symmetric homomorphism $\phi \in \operatorname{Hom}_T(F_2, F_1)$ defines via its pfaffians a graded Artinian Gorenstein ring $A = T/I$ with the same Hilbert function as A^f. Therefore it suffices to establish the sum presentation $f = l_1^4 + \cdots + l_s^4$ for an f corresponding to an $A = A^F$ with sufficiently general syzygy matrix ϕ for each possible numerical type of syzygies.

There are only a few number of numerical cases: We argue in each of the cases separately but similarly: Let n be the number of cubic generators of $I = f^{\perp}$ and $m = \lfloor n/2 \rfloor$. Consider the $n \times n$ submatrix $\tilde{\phi}$ of ϕ corresponding to the linear coefficients of the quartic syzygies.

Suppose there is a $m \times m$ (skew) symmetric submatrix of zeroes in $\tilde{\phi}$. The corresponding quartic syzygies and all the syzygies which involve only equations of degree $\leqslant 2$ give $r = m + p$ syzygies of degrees $(b_{r+1}, \ldots, b_{2r+1})$ as indicated above between equations of degrees $(a_1, \ldots, a_{r+1})$. Here p is the number of syzygies which involve only equations of degree $\leqslant 2$, and $m + p = r$. Thus we obtain a block decomposition

$$\phi = \begin{pmatrix} \begin{matrix} 0 & \cdots & * \\ \vdots & \ddots & \vdots \\ -* & \cdots & 0 \end{matrix} & \psi \\ -\psi^t & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}$$

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Table 1. Numerical types of syzygies

Hilbert function	$(a_1,\dots,a_{2r+1})$	m	p	$(b_{r+1},\dots,b_{2r+1})$
(1, 3, 6, 3, 1)	(3, 3, 3, 3, 3, 3, 3)	3	0	(4, 4, 4)
(1, 3, 5, 3, 1)	(2, 3, 3, 3, 3)	2	0	(4, 4)
(1, 3, 4, 3, 1)	(2, 2, 3)	0	1	(4)
(1, 3, 4, 3, 1)	(2, 2, 3, 3, 4)	1	1	(3, 4)
(1, 3, 3, 3, 1)	(2, 2, 2, 4, 4)	0	2	(3, 3)
(1, 2, 3, 2, 1)	(1, 3, 3)	1	0	(4)
(1, 2, 2, 2, 1)	(1, 2, 4)	0	1	(3)
(1, 1, 1, 1, 1)	(1, 1, 5)	0	1	(2)

with a $(r+1)\times r$ matrix ψ. The $r\times r$ minors of ψ are among the pfaffians of ϕ; they are precisely the generators involved in our r syzygies. By Hilbert and Burch (Eisenbud, 1995, Thm 20.15) these minors generate the homogeneous ideal J_Γ of a set $\Gamma\subset\check{\mathbb{P}}^2$ of distinct points with syzygies

$$0\longleftarrow J_\Gamma\longleftarrow\bigoplus_{i=1}^{r+1}T(-a_i)\overset{\psi}{\longleftarrow}\bigoplus_{j=r+1}^{2r+1}T(-b_j)\longleftarrow 0,$$

if ψ is sufficiently general. J_Γ is 2-regular, since $b_j\leqslant 4$ for $j\geqslant r+1$. Hence the Hilbert function of $R=R_\Gamma=T/J_\Gamma$ takes the values

$$h_R(t)=\dim_k(R_\Gamma)_t=\deg\Gamma$$

for $t\geqslant 2$. On the other hand $\dim_k(R_\Gamma)_2=\dim_k A_2^f=s$ as $(J_\Gamma)_{\leqslant 2}=I_{\leqslant 2}$ by construction. Thus, for sufficiently general ψ, the scheme Γ consists of s points $L_1=\{l_1=0\},\dots,L_s=\{l_s=0\}$. To prove that there exists a sum presentation $f=\lambda_1 l_1^4+\cdots+\lambda_s l_s^4$ we consider $T\to R\to A$ and the induced inclusions

$$\mathrm{Hom}(A_4,k)\subset\mathrm{Hom}(R_4,k)\subset\mathrm{Hom}(T_4,k).$$

The linear forms $\{D\mapsto D(l_i^4)\}$ are contained in $\mathrm{Hom}(R_4,k)$. Moreover, since Γ imposes s independent conditions on quartics, these linear forms span the image. In particular $\{D\mapsto D(f)\}\in\mathrm{Hom}(A_4,k)$ is contained in this space, i.e. $D(f)=D(\lambda_1 l_1^4+\cdots+\lambda_s l_s^4)$ for all $D\in T_4$ for suitable $\lambda_1,\dots,\lambda_s\in k$. Hence $f=\lambda_1 l_1^4+\cdots+\lambda_s l_s^4$ as desired. Taking roots of the λ_i's we can put them into the equations l_i.

It remains to prove the existence of a $m\times m$ block of zeroes in $\tilde\phi$, possibly after row and column operations. Let $V_f:=\mathrm{Tor}_2^T(A^f,k)_4$. Then $\tilde\phi$ corresponds to a net of alternating forms $\Lambda^2V_f\to T_1$ and we are looking for a subspace $E\in\mathbb{G}(m,n)=\mathbb{G}(m,V_f)$ such that $\Lambda^2E\subset\ker(\Lambda^2V_f\to T_1)$. If $m=1$ there is nothing to prove. If $m\geqslant 2$ then E exists, because for $j=\binom{m}{2}$ and $c_j=c_j(\Lambda^2\mathcal{E}^*)$ the jth Chern class, where $\mathcal{E}$ denotes the universal subbundle on $\mathbb{G}(m,n)$, we have $c_j^3\neq 0$. □

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Notice that we expect a three-dimensional family in the case $s = 6$, a one-dimensional family of sum presentations for f with $s = 5$, or $s = 4$ and $r = 3$, or $s = 3$ and $h_A(1) = 2$, and a unique presentation otherwise.

Remarks 2.4. (1) The fact that, despite the dimension count, a general plane quartic is not the sum of 5 powers, goes back to Clebsch (1861).

(2) It is not true, that every f with $\dim_k A_2^f = s$ is a sum of s powers. For s with an unique presentation examples are rather obvious. But even in case $s = 5$ this occurs: e.g.

$$f = \left(1 - \frac{1}{t^2}\right)x_1^4 + x_1^3\left(x_0 - \frac{4}{t}x_2\right) + \frac{1}{t^2}(x_1 + tx_2)^4 + x_2^3(x_0 - 4tx_1) + (1 - t^2)x_2^4$$

is not the sum of five powers. The reason is that the quadric in $I = f^\perp$ is a double line. Hence, distinct points in a Γ would give a linear form in I, a contradiction. For f the one-dimensional family of sum decompositions degenerates to the family parametrized by t of decompositions into five summands as above.

DEFINITION 2.5. For $F = \{f = 0\} \subset \mathbb{P}^n$ a hypersurface we call a scheme $X \subset \check{\mathbb{P}}^n$ apolar to F if $I_X \subset F^\perp$. The family of zero-dimensional apolar subschemes of degree s of F is denoted by $VPS(F, s)$.

Note that with this definition $VSP(F, s) \subset VPS(F, s)$ is an open subscheme and equality holds if $VPS(F, s)$ is irreducible and $VSP(F, s)$ non-empty.

THEOREM 2.6. *Let $F = \{f = 0\} \subset \mathbb{P}(U)$ be a non-degenerate plane quartic. Then $VPS(F, 6) \cong \mathbb{G}(3, V_f, \eta_f)$, where $V_f = (f^\perp)_3^*$, $N_f = U^*$ and $\eta\colon \Lambda^2 V_f \to N_f$ the skew-symmetric syzygy matrix of A^f. Conversely, for a net $\eta\colon \Lambda^2 V \to N$ of skew-forms on a seven-dimensional vector space, whose pfaffians define a ideal I of codimension 3 in $S(N)$, the dual socle quartic $F = F(V, \eta) \subset \mathbb{P}(N^*)$ is a non-degenerate quartic, and $VPS(F(V, \eta), 6) \cong \mathbb{G}(3, V, \eta)$.*

Proof. By 2.1, the structure theorem of Buchsbaum and Eisenbud (1977), and 2.3 $F \mapsto (\Lambda^2 V_f \to \eta_f)$ and $(\eta\colon \Lambda^2 V \to N) \mapsto F(V, \eta)$ give bijections between

$$\{F \mid \det(\mathrm{Cat}(f)) \neq 0\} \longleftrightarrow \{\eta\colon \Lambda^2 V \to N \mid \mathrm{codim}\, I = 3\}.$$

Moreover points $p \in \mathbb{G}(3, V, \eta)$ correspond to block decompositions

$$\phi = \begin{pmatrix} 0 & \phi_{12} & \cdots & \phi_{15} & \phi_{16} & \phi_{17} \\ -\phi_{12} & 0 & \cdots & \phi_{25} & \phi_{26} & \phi_{27} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\phi_{15} & -\phi_{25} & \cdots & 0 & 0 & 0 \\ -\phi_{16} & -\phi_{26} & \cdots & 0 & 0 & 0 \\ -\phi_{17} & -\phi_{27} & \cdots & 0 & 0 & 0 \end{pmatrix}$$

of the syzygy matrix ϕ. We claim that for the 4×3 submatrix

$$\psi = \begin{pmatrix} \phi_{15} & \phi_{26} & \phi_{17} \\ \vdots & \vdots & \vdots \\ \phi_{45} & \phi_{46} & \phi_{47} \end{pmatrix}$$

of every such block decomposition the ideal of minors $I(\psi)$ has codimension 2, hence defines of a subscheme of length 6 in $\check{\mathbb{P}}^2$.

Assume that, $I(\psi)$ has not depth 2. Then by Hilbert–Burch, the corresponding minors have a common factor. Since the minors are minimal generators of I_{pf}, the factor has to be a linear form $t \in T_1$. So ψ is a matrix of syzygies among 4 quadrics without a common factor. The quadrics generate an ideal J of codimension $\geqslant 2$. Let $B = T/J$. B has Hilbert function $(1, 3, 2, 1, \ldots)$. If $\dim B = 0$ then 3 general quadrics in J form a regular sequence, whose quotient has Hilbert function $(1, 3, 3, 1, 0)$ and the fourth quadric cuts down to a ring with Hilbert function $(1, 3, 2, 0)$. This is not the case. So $\dim B = 1$ and B has Hilbert function $(1, 3, 2, 1, 1, \ldots)$. Such quotients B exist: B is defined by four quadrics in the homogeneous ideal of a point $p \in \check{\mathbb{P}}^2$. However such a ψ does not occur as part of a skew-symmetric matrix ϕ, whose pfaffians have codimension 3. The syzygies of B start

$$0 \longleftarrow B \longleftarrow T \longleftarrow 4T(-2) \longleftarrow 3T(-3) \oplus T(-4) \oplus \cdots \longleftarrow \cdots.$$

Since $tJ \subset I_{pf}$ the syzygy $4T(-2) \leftarrow T(-4)$ gives a relation among the pfaffians. But this relation is not in the space generated by the columns of ϕ, since the sequence

$$3T(-3) \xleftarrow{-\psi^t} 4T(-4) \longleftarrow T(-6) \longleftarrow 0$$

is exact. This contradicts the exactness of the pfaffian complex.

Thus we have a well-defined morphism $\alpha\colon \mathbb{G}(3, V, \eta) \to \mathrm{Hilb}_6(\check{\mathbb{P}}^2)$. To prove that α is an isomorphism onto its image, consider the open part $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)^o$ of the Hilbert scheme of length 6 subschemes, which impose independent conditions on quadrics, and the embedding $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)^o \hookrightarrow \mathbb{G}(4, T_3)$. The diagram

$$\begin{array}{ccc} \mathrm{Hilb}_6(\check{\mathbb{P}}^2)^o & \hookrightarrow & \mathbb{G}(4, T_3) \\ {\scriptstyle\alpha}\uparrow & & \uparrow \\ \mathbb{G}(3, V, \eta) & \hookrightarrow & \mathbb{G}(4, V^*) \end{array}$$

commutes, where $V^* = (f^{\perp})_3 \subset T_3$.

Finally, note that the image of α contains all polar hexagons of F. Indeed, if $f = l_1^4 + \ldots + l_6^4$ for distinct lines $\Gamma = \{L_1, \ldots, L_6\} \subset \check{\mathbb{P}}^2$, then Γ imposes independent conditions on quadrics by Thm 2.3. Hence syzygies of Γ are of type:

$$0 \longleftarrow R_\Gamma \longleftarrow T \longleftarrow 4T(-3) \xleftarrow{\psi} 3T(-4) \longleftarrow 0. \tag{2.3}$$

By (2.1) the ideal $J_\Gamma \subset f^\perp$. Hence we have a sequence

$$0 \longleftarrow A^f \longleftarrow R_\Gamma \longleftarrow I_{A/R} \longleftarrow 0. \tag{2.4}$$

Since A^f and $R = R_\Gamma$ have Hilbert functions $(1, 3, 6, 3, 1)$ and $(1, 3, 6, 6, 6, \dots)$ respectively, $I_{A/R}$ has 3 cubic generators with 4 linear relations:

$$0 \longleftarrow I_{A/R} \longleftarrow 3T(-3) \longleftarrow 4T(-4).$$

The minors of the presentation matrix are contained in the annihilator, which is J_Γ. Hence, this matrix is ψ^t again, and $I_{A/R} \cong \omega_R(-4)$. A mapping cone between the complex (2.3) and its dual over the sequence (2.4), gives syzygies of A^f with the desired block structure.

We do not claim at this point, that every non-degenerate f has a non-degenerate polar hexagon. However, if there is one, then the points in the image of α corresponding to them, form an open subset. □

COROLLARY 2.7. *For a general plane quartic F the variety of polar hexagons $VSP(F, 6)$ is a smooth Fano 3-fold of genus 12.*

Proof. Since for the tautological subbundle $\mathcal{E}$ on $\mathbb{G}(3, V)$ the sheaf $\Lambda^2\mathcal{E}^*$ is globally generated by $\Lambda^2 V^*$, a general net N of skew-forms defines a smooth subscheme $\mathbb{G}(3, V, \eta)$ of codimension 9. Since $\omega_{\mathbb{G}(3,V)} \cong \mathcal{O}_{\mathbb{G}(3,V)}(-7)$ and $\Lambda^9(3\Lambda^2\mathcal{E}^*) \cong \mathcal{O}_{\mathbb{G}(3,V)}(-6)$ one has

$$\omega_{\mathbb{G}(3,V,\eta)} \cong \mathcal{O}_{\mathbb{G}(3,V,\eta)}(-1).$$

By degree reasoning $\mathbb{G}(3, V, \eta)$ is irreducible and hence it is a Fano 3-fold. □

3. The Scorza Map

In this section we recall some results of Scorza from Dolgachev and Kanev (1993).

A plane cubic C is called anharmonic, if C lies in the PGL(3)-orbit closure of $\{x_0^3 + x_1^3 + x_2^3 = 0\}$. The reason is that the cross-ratio of the Fermat cubic is anharmonic. Let $\mathbf{A} \subset \mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(3))^*) \cong \mathbb{P}^9$ denote the variety of anharmonic cubics. The PGL-orbit closure of a general cubics is a hypersurface of degree 12. Due to the additional automorphism of the Fermat cubic, $\mathbf{A}$ is hypersurface of degree 4. In terms of coordinates $(a, \dots, j)$ of $\mathbb{P}^9$,

$$ax_0^3 + bx_1^3 + cx_2^3 + 3dx_0^2x_1 + 3ex_0^2x_2 + 3fx_1^2x_0 + 3gx_1^2x_2 + 3hx_2^2x_0 +$$
$$+ 3ix_2^2x_1 + 6jx_0x_1x_2,$$

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$\mathbf{A}$ is defined by the Aronhold invariant

$$\begin{aligned}\mathbf{I}_4 = {} & abcj - (bcde + cafg + abhi) - j(agi + bhe + cdf) + \\ & + (afi^2 + ahg^2 + bdh^2 + bie^2 + cgd^2 + cef^2) - \\ & - j^4 + 2j^2(fh + id + eg) - 3j(dgh + efi) - \\ & - (f^2h^2 + i^2d^2 + e^2g^2) + (ideg + egfh + fhid).\end{aligned}$$

LEMMA 3.1. *Let $g \in k[x_0, x_1, x_2]$ be a plane cubic. The following are equivalent:*

(1) *$A^g \cong T/(D_1, D_2, D_3)$ is a complete intersection of three quadrics,*
(2) *g is not an anharmonic cubic.*

Proof. If g is not a cone, then A^g has Hilbert function $(1, 3, 3, 1)$, and there are precisely three quadrics in $I = g^{\perp}$. By Buchsbaum and Eisenbud (1977), either A^g is a complete intersection of three quadrics, or A^f has syzygies

$$0 \longleftarrow A^g \longleftarrow T \longleftarrow \bigoplus_{i=1}^{5} T(-a_i) \overset{\psi}{\longleftarrow} \bigoplus_{j=1}^{5} T(-b_j) \longleftarrow T(-6) \longleftarrow 0,$$

with $(a_1, \ldots, a_5) = (2, 2, 2, 3, 3)$ and $b_i = 6 - a_i$. In the second case the three quadrics of $I = f^{\perp}$ intersect in 3 points $\{L_1, L_2, L_3\} \in \check{\mathbb{P}}^2$ (possibly infinitesimal near), and as in Section 2 we obtain $g = \lambda_1 l_1^3 + \lambda_2 l_2^3 + \lambda_3 l_3^3$. Conversely, if g is a smooth anharmonic cubic all three quadrics of I vanish in $\{L_1, L_2, L_3\} \in \check{\mathbb{P}}^2$, and A^g is not a complete intersection.

Since all cones are anharmonic cubics, this proves the lemma. □

Let $F = \{f = 0\} \subset \mathbb{P}^2$ be a non-degenerate plane quartic. Consider

$$S_F = \{a \in \mathbb{P}^2 | P_a(F) \in \mathbf{A}\}.$$

Then either $S_F = \mathbb{P}^2$ or S_F is a plane quartic. The first case does not occur for non-degenerate quartics (Dolgachev and Kanev, 1993, 6.6.3). We call S_F the covariant quartic of F. Consider

$$T_F = \{(a, b) \in \mathbb{P}^2 \times \mathbb{P}^2 | \text{rank } P_{a,b}(F) \leqslant 1\}.$$

LEMMA 3.2. (Dolgachev and Kanev, 1993, 6.8.1). *Let $F \subset \mathbb{P}^2$ be a general quartic. Then S_F is a smooth quartic and T_F is a smooth symmetric correspondence of type* $(3, 3)$ *on $S_F \times S_F$ without united points.*

Proof. For a complete proof see Dolgachev and Kanev (1993). The reason, why T_F is such a correspondence on $S_F \times S_F$ is the following:

Suppose $(a, b) \in T_F$, say $P_{a,b}(f) = h^2$. Set $t_0 = P_b \in T_1$ and $(t_1, t_2) = (h^{\perp})_1 \subset T_1$. Then $t_0 t_1, t_0 t_2 \in (P_a(f)^{\perp})_2$ and $(P_a(f)^{\perp})$ is not a complete intersection of quadrics. By Lemma 3.1 $P_a(F)$ is an anharmonic cubic. So $a \in S_F$.

For general F and general $a \in S_F$, $P_a(F)$ is a smooth Fermat cubic, and then the points $b \in \mathbb{P}^2$ such that rank $P_{a,b}(F) \leqslant 1$ are the 3 vertices of the Hessian triangle of $P_a(F)$. So T_F is symmetric of type (3,3). Since

$$2\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2}{\partial x_i \partial x_j} P_{a,a}(f)$$

by (2.1), T_F has no united points, if

$$\operatorname{rank}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right) \geqslant 2 \quad \text{for all } a \in S_F.$$

This is the case for general F, because then the Hessian

$$He(F) = \left\{\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = 0\right\}$$

is smooth. □

Let S be a smooth plane quartic and ϑ an even theta characteristic on S. Consider the ϑ-correspondence

$$T_\vartheta = \{(a, b) \in S \times S \mid h^0(S, \vartheta(a - b)) \geqslant 1\}.$$

ϑ is not effective. $\deg \vartheta(a) = 3$, and

$$h^0(S, \vartheta(a - b)) = h^1(S, \vartheta(a - b)) = h^0(S, \vartheta(b - a)).$$

So T_ϑ is a symmetric correspondence of type $(3, 3)$ without united points.

THEOREM 3.3 (Dolgachev and Kanev, 1993, 7.6). *Let k be a algebraically closed field. If F is a quartic such that S_F is a smooth quartic, then there exists a unique theta characteristic $\vartheta = \vartheta_F$ on S_F such that $T_F = T_\vartheta$.*

Proof. For a general F and a general $(a, b) \in T_F$ consider the polar Hessian triangles $T_F(a) = b + b_1 + b_2$ and $T_F(b) = a + a_1 + a_2$, i.e. the vertices of the Hessian of $P_a(F)$ and $P_b(F)$ respectively. All 6 points are different. If $P_{a,b}(f) = h^2$ then both b_1, b_2 and a_1, a_2 span the line $\{h = 0\}$. So $S_F \cap \{h = 0\} = \{a_1, a_2, b_1, b_2\}$ and

$$T_F(a) - a + T_F(b) - b = b_1 + b_2 + a_1 + a_2 \tag{3.1}$$

is a canonical divisor on S_F. Moreover

$$T_F(a) - a \equiv T_F(b) - b \tag{3.2}$$

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for any 2 points $a, b \in S_F$. To establish this consider the map

$$S_F \to \mathrm{Pic}^2(S_F), a \mapsto \mathcal{O}(T_F(a) - a).$$

Since even for degenerate polar Hessians the three (possibly infinitesimally near) points $T_F(a) = b_1 + b_2 + b_3$ are not collinear, $h^0(S_F, \mathcal{O}(T_F(a))) = 1$ for all $a \in S_F$. So $h^0(S_F, \mathcal{O}(T_F(a) - a)) = 0$, as T_F has no united points. It follows, that the image of S_F does not intersect the Θ-divisor of $\mathrm{Pic}^2(S_F)$. Since Θ is ample, S_F maps to a point. By (3.1), (3.2) $\vartheta = \mathcal{O}(T_F(a) - a))$ is an non-vanishing theta characteristic. □

Remark 3.4. Although the $[\vartheta] \in \mathrm{Pic}\, C$ is a point defined over the ground field k, the line bundle ϑ may not be defined over the ground field. For an example see 6.7 below.

THEOREM 3.5 (Scorza, (Dolgachev and Kanev, 1993, 7.8, 7.11)). *Let k be algebraically closed. The rational map induced by $s\colon F \mapsto (S_F, \vartheta_F)$ from the moduli spaces of curves of genus 3 $s\colon \mathcal{M}_3 \to \mathcal{M}_{3,\vartheta^{ev}}$ to the moduli of curves of genus 3 together with a even theta characteristic is birational.*

Remarks 3.6. (1) The projection $\mathcal{M}_{3,\vartheta^{ev}} \to \mathcal{M}_3$ is a finite cover of degree 36:1, since a curve of genus 3 has precisely 36 even theta characteristics.

(2) For a pair (S, ϑ) of a plane quartic together with a non-vanishing theta characteristic, the quartic $F = s^{-1}(S, \vartheta)$ is called Scorza quartic of (S, ϑ). Dolgachev and Kanev give two description of s^{-1}. In Section 5 we will give another one.

(3) More general, for a canonical curve $S \subset \mathbb{P}^{g-1}$ of genus g, and a non-vanishing theta characteristic ϑ with some plausible, but yet unproven hypothesis Scorza constructs a quartic hypersurface $F \subset \mathbb{P}^{g-1}$. See Scorza (1899b) and Dolgachev and Kanev (1993).

4. Rank 2 Vector Bundles on $\mathbb{P}^3$ with $c_1 = 0$ and $c_2 = 3$

Let W be a four-dimensional vector space, $\check{\mathbb{P}}^3 = \mathbb{P}(W^*)$ and $q\colon U^* \hookrightarrow S_2 W^*$ a net of quadrics in $\check{\mathbb{P}}^3$, whose general element is smooth. Let $W \to U \otimes W^*$ the associated symmetric matrix with entries in U, and $b_q\colon W \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \to W^* \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ the associated map of sheaves on $\mathbb{P}^2 = \mathbb{P}(U)$ twisted.

Let $S = S_q = \{\det(b_q) = 0\} \subset \mathbb{P}^2$ denote the discriminant of the net, and $\vartheta = \vartheta_q = \mathrm{coker}\,(b_q)$. Since b_q is symmetric,

$$\vartheta \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^2}}(\vartheta, \omega_{\mathbb{P}^2}) \cong \mathcal{H}om_{\mathcal{O}_S}(\vartheta, \omega_S). \tag{4.1}$$

If S is smooth, then ϑ is an invertible $\mathcal{O}_S$-module, and hence ϑ is a non-vanishing theta characteristic on S.

Conversely, given a plane quartic S and a torsion free rank 1 $\mathcal{O}_S$-module ϑ which satisfies (4.1), we denote by $W^* = H^0(S, \vartheta(1))$ and $q = q(S, \vartheta)\colon U^* \hookrightarrow S_2W^*$ the corresponding net of quadrics in $\mathbb{P}(W^*) = \check{\mathbb{P}}^3$.

If S is smooth, then $\phi_{\vartheta(1)}\colon S \hookrightarrow \check{\mathbb{P}}^3$ is an embedding, and the image $\tilde{S} = \phi_{\vartheta(1)}(S)$ is the variety of vertices of the singular quadrics in the net. Equation of $\tilde{S} \subset \check{\mathbb{P}}^3 = \mathbb{P}(W^*)$ are given by the 3×3 minors of $\tilde{b}_q\colon W \otimes \mathcal{O}_{\check{\mathbb{P}}^3}(-1) \longrightarrow U \otimes \mathcal{O}_{\check{\mathbb{P}}^3}$.

If S is not smooth, one can take these equations to define $\tilde{S}$. From another point of view, $\tilde{b}_q$ is the jacobian matrix of q.

Now we consider the apolarity pairing between $\mathbb{P}^3 = \mathbb{P}(W)$ and $\check{\mathbb{P}}^3 = \mathbb{P}(W^*)$. Let $R = SW$ and $T = SW^*$ denote the homogeneous coordinate rings respectively. Consider

$$q^{\perp} = \{D \in R \mid D(Q) = 0 \quad \forall Q \in q(U^*) \subset S_2W^*\},$$

and $A^q = R/q^{\perp}$. A^q is an Artinian ring with Hilbert function $(1, 4, 3, 0, \ldots)$. More precisely $A_1^q = W$, $A_2^q = U$ and multiplication given by $q\colon S_2W \to U$.

LEMMA 4.1. *Let q be a general net of quadrics. Then A^q has syzygies:*

$$0 \leftarrow A^q \leftarrow R \leftarrow 7R(-2) \xleftarrow{(\phi_1,\phi_2)} 8R(-3) \oplus 3R(-4) \leftarrow 8R(-5) \leftarrow 3R(-6) \leftarrow 0.$$

Proof. The number of syzygy $\dim \operatorname{Tor}_i^R(A^q, k)_j$ in the above sequence take the minimal possible values for an Artinian ring A with Hilbert function $(1, 4, 3, 0, \ldots)$. Thus by semi-continuity it suffices to establish the existence of one example q with such syzygies. The Kleinian net

$$q_{Klein} = (\tfrac{1}{2}z_1^2 - z_0z_2, \tfrac{1}{2}z_2^2 - z_0z_3, \tfrac{1}{2}z_3^2 - z_0z_1)$$

where $T = k[z_0, z_1, z_2, z_3]$, has this property. □

Consider the map ϕ_1 in the complex above, and its kernel sheafivied and twisted by $\otimes\mathcal{O}_{\mathbb{P}^3}(5)$:

$$\mathcal{E} = \mathcal{E}_q = \ker(7\mathcal{O}_{\mathbb{P}^3}(3) \xleftarrow{\phi_1} 8\mathcal{O}_{\mathbb{P}^3}(2))$$

PROPOSITION 4.2. *Let q be a general net of quadrics. Then $\mathcal{E}_q$ is a stable rank 2 vector bundle with Chern classes $c_1 = 0$, $c_2 = 3$ and syzygies*

$$0 \longleftarrow \mathcal{E}_q \longleftarrow 8\mathcal{O}_{\mathbb{P}^3}(-2) \longleftarrow 7\mathcal{O}_{\mathbb{P}^3}(-3) \longleftarrow \mathcal{O}_{\mathbb{P}^3}(-5) \longleftarrow 0. \tag{4.2}$$

Its H^2-cohomology module is

$$A^q(5) = \bigoplus_n H^2(\mathbb{P}^3, \mathcal{E}_q(n)).$$

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Proof. Since A^q Artinian the first syzygy module sheafivied is a rank 6 vector bundle,

$$0 \longleftarrow \mathcal{O}_{\mathbb{P}^3}(5) \longleftarrow 7\mathcal{O}_{\mathbb{P}^3}(3) \longleftarrow \mathcal{F} \longleftarrow 0. \tag{4.3}$$

$\mathcal{E}_q$ is the kernel of by ϕ_1,

$$(0 \longleftarrow)\mathcal{F} \longleftarrow 8\mathcal{O}_{\mathbb{P}^3}(2) \longleftarrow \mathcal{E}_q \longleftarrow 0. \tag{4.4}$$

One expects that $\mathcal{F} \longleftarrow 8\mathcal{O}_{\mathbb{P}^3}(2)$ is surjective outside a set of codimension $8-6+1=3$, hence that $\mathcal{E}_q$ is a vector bundle of rank 2 outside a finite set of points. The expected number of these points is 0 by Porteous formula. Thus either ϕ_1 has rank $\leqslant 5$ along at least a curve, or $\mathcal{E}_q$ is a rank 2 vector bundle with Chern polynomial

$$c_t(\mathcal{E}_q) = \frac{(1+5t)(1+2t)^7}{(1+3t)^8} \equiv 1 + 3t^2 \quad \mod \quad t^4.$$

For a general q the second alternative takes place, as one can check by considering an example, e.g. the Kleinian net.

Since $\mathcal{E}_q$ has rank 2 and $c_1 = 0$, wedge product $\mathcal{E}_q \otimes \mathcal{E}_q \to \Lambda^2\mathcal{E}_q \cong \mathcal{O}_{\mathbb{P}^3}$ gives $\mathcal{E}_q^* \cong \mathcal{E}_q$. Thus the dual of the sequences (4.3) and (4.4) give the exact sequence (4.2). Since this complex is short enough to stay exact on global sections for arbitrary twists, this is the minimal resolution. The last statement follows from the cohomology sequence of (4.3) and (4.4). $\mathcal{E}$ is stable, because $H^0(\mathbb{P}^3, \mathcal{E}) = 0$, cf. Okonek *et al.*, (1980), Lemma II 1.2.5. □

COROLLARY 4.3. *If q is general, then there are natural isomorphism*

$$U \cong \mathrm{Tor}_4^R(A^q, k)_6 \cong (\mathrm{Tor}_2^R(A^q, k)_4)^*$$

and a skew-symmetric self-duality

$$\mathrm{Tor}_3^R(A^q, k)_5 \cong (\mathrm{Tor}_3^R(A^q, k)_5)^*.$$

The maps $3R(-4) \leftarrow 8R(-5)$ *and* $8R(-5) \leftarrow 3R(-6)$ *in the resolution of 4.1 are dual to each other under these isomorphisms.*

Proof. The matrices yield minimal presentations

$$0 \leftarrow \bigoplus_n H^1(\mathbb{P}^3, \mathcal{E}(n)) \leftarrow 3R(1) \leftarrow 8R$$

and

$$0 \leftarrow \mathrm{Ext}_R^4(A^q, R) \leftarrow 3R(6) \leftarrow 8R(5)$$

respectively. By Serre duality and $A^q(5) = \bigoplus_n H^2(\mathbb{P}^3, \mathcal{E}(n))$,

$$\mathrm{Ext}_R^4(A^q(5), R) = \bigoplus_n H^1(\mathbb{P}^3, \mathcal{E}^*(n)).$$

Since $\mathcal{E} \cong \mathcal{E}^*$, the modules are isomorphic, hence the desired isomorphisms follow by comparison of the presentations. The self-duality on $\mathrm{Tor}_3^R(A^q, k)_5$ is skew, since the isomorphism $\mathcal{E} \cong \mathcal{E}^*$ has this property. Finally, we note

$$U \cong A_2^q \cong H^2(\mathbb{P}^3, \mathcal{E}(-3)) \cong H^1(\mathbb{P}^3, \mathcal{E}^*(-1))^*$$
$$\cong H^1(\mathbb{P}^3, \mathcal{E}(-1))^* \cong \mathrm{Tor}_2^R(A^q, k)_4^* \cong \mathrm{Tor}_4^R(A^q, k)_6.$$

□

COROLLARY 4.4. *$\tilde{S} = \tilde{S}_q \subset \check{\mathbb{P}}^3$ is the variety of unstable planes of $\mathcal{E}_q$. $\tilde{S}$ determines $\mathcal{E}_q$ up to isomorphism. The moduli space $\mathcal{M}_{\mathbb{P}^3}(2; 0, 3)$ of rank 2 vector bundles on $\mathbb{P}^3$ with $c_1 = 0$ and $c_2 = 3$ has a component birational to $\mathbb{G}(3, S_2 W^*)$.*

Proof. A plane $H = \{h = 0\} \subset \mathbb{P}^3$ is unstable for $\mathcal{E}$, if and only if $H^0(H, \mathcal{E}\mid_H) \neq 0$, equivalently, if multiplication with h is not injective on

$$H^1(\mathbb{P}^3, \mathcal{E}(-1)) \xrightarrow{h} H^1(\mathbb{P}^3, \mathcal{E}).$$

$H^1(\mathbb{P}^3, \mathcal{E}(-1)) \cong U^*$ and $H^1(\mathbb{P}^3, \mathcal{E}) \cong W^*$. A quadric $q_1 \in U^* \subset S_2 W^*$ is annihilated by $h \in W$, iff $Q_1 = \{q_1 = 0\} \subset \check{\mathbb{P}}^3$ is a cone with vertex $H \in \check{\mathbb{P}}^3$. So the variety of unstable planes coincides with the variety $\tilde{S} \subset \check{\mathbb{P}}^3$ of vertices of the cones. $\tilde{S}$ determines q, which in turn determines A^q and $\mathcal{E}_q$. The vector bundles obtained from points $q \in \mathbb{G}(3, S_2 W^*)$ form an open part of the moduli scheme$\mathcal{M}_{\mathbb{P}^3}(2; 0, 3)$, since by semi-continuity and minimality, the cohomology modules $\bigoplus_n H^2(\mathbb{P}^3, \mathcal{E}(n))$ have the same numerical type of syzygies for an open part of $\mathcal{M}_{\mathbb{P}^3}(2; 0, 3)$. □

5. Twisted Cubics Annihilated by a Net of Quadrics

Let $q: U^* \hookrightarrow S_2 W^*$ be a net of quadrics as before. Let $H(q)$ denote the variety of twisted cubics $C \subset \mathbb{P}^3$, whose equations $H^0(\mathbb{P}^3, \mathcal{I}_C(2)) \subset S_2 W$ are annihilated by q. Let $V_q = (q^\perp)_2 \subset S_2 W$. Since a twisted cubic is defined by its quadrics and $h^0(\mathbb{P}^3, \mathcal{I}_C(2)) = 3$, $H(q)$ is a subset of $\mathbb{G}(3, V_q)$ in a natural way. We are looking for an net of alternating forms on V_q, which defines $H(q) \subset \mathbb{G}(3, V_q)$.

For a description of $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$ and the map

$$\mathrm{Hilb}_{3t+1}(\mathbb{P}^3) \to \mathbb{G}(3, S_2 W)$$

see (Ellingsrud *et al.*, 1987; Piene and Schlessinger, 1985).

Consider the syzygies of A^q. By definition of A^q we have

$$\mathrm{Tor}_1^R(A^q, k)_2 \cong V_q.$$

Define

$$N_q = \mathrm{Tor}_2^R(A^q, k)_4$$

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and consider

$$\eta_q\colon \Lambda^2 V_q \to N_q$$

given by multiplication in the algebra $\mathrm{Tor}_*^R(A^q, k)$, cf. (Eisenbud, 1995, Exercise A3.20).

$$\eta(p_1 \wedge p_2) = 0$$

for $p_1, p_2 \in V_q$, iff the Koszul syzygy

$$p_1 \otimes p_2 - p_2 \otimes p_1 \in \ker(R \leftarrow 7R(-2))$$

lies in Im $(7R(-2) \leftarrow 8R(-3))$.

THEOREM 5.1.

$$H(q) \cong \mathbb{G}(3, V_q, \eta_q)$$

for a general net q.

Proof. Let $C \subset \mathbb{P}^3$ be a rational normal curve, whose ideal $I_C = (p_1, p_2, p_3)$ is generated by three quadrics $p_1, p_2, p_3 \in V$. C has syzygies

$$0 \longleftarrow \mathcal{O}_C \longleftarrow \mathcal{O}_{\mathbb{P}^3} \longleftarrow 3\mathcal{O}_{\mathbb{P}^3}(-2) \longleftarrow 2\mathcal{O}_{\mathbb{P}^3}(-3) \longleftarrow 0. \tag{5.1}$$

Hence all syzygies among p_1, p_2, p_3 are generated by linear relations, and

$$E = (I_C)_2 = (p_1, p_2, p_3)_2 \in \mathbb{G}(3, V, \eta).$$

Conversely, suppose that $E = (p_1, p_2, p_3) \in \mathbb{G}(3, V, \eta)$. We will prove that p_1, p_2, p_3 generate the homogeneous ideal of a curve C of degree 3 and arithmetic genus 0. Choose $p_4, \ldots, p_7 \in V$, such that $p_1, \ldots, p_7$ form a basis. By definition of η there is a matrix $8R(-3) \overset{\psi}{\leftarrow} 3R(-4)$, such that

$$\phi_1 \cdot \psi = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \\ 0 & 0 & 0 \\ \vdots & & \vdots \end{pmatrix}$$

gives the matrix of Koszul relations, where ϕ_1 is the matrix of linear syzygies in Lemma 4.1. With

$$\tau = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

we have $\phi_1 \cdot \psi \cdot \tau = 0$. But for $\ker(7\mathcal{O}_{\mathbb{P}^3}(-2) \overset{\phi_1}{\longleftarrow} 8\mathcal{O}_{\mathbb{P}^3}(-3)) = \mathcal{E}(-5)$ we have $H^0(\mathbb{P}^3, \mathcal{E}(1)) = 0$ by Prop. 4.2. So $\psi \cdot \tau = 0$, i.e. p_1, p_2, p_3 are three quadrics with some

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linear relations

$$r_1 \cdot p_1 + r_2 \cdot p_2 + r_3 \cdot p_3 = 0. \tag{5.2}$$

The coefficients $r_1, r_2, r_3 \in R_1$ are linearly independent for a general linear combination of rows of ψ. Because otherwise any two elements of E would have a common linear factor, which implies, that all three elements p_1, p_2, p_3 have a common factor, and $\mathrm{Tor}_3^R(A^q, k)_4 \neq 0$. But this group is zero by Lemma 4.1. Thus

$$(p_1, p_2, p_3) = \Lambda^2 \tau_2$$

for a 2×3 matrix

$$\tau_2 = \begin{pmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \end{pmatrix}$$

of linear forms $r_i \in R_1$, cf. [Schreyer (1991), Lemma 4.3]. Since the minors p_1, p_2, p_3 have no common factor, the Hilbert–Burch complex of τ_2 is exact, and p_1, p_2, p_3 generate the ideal of a curve $C \subset \mathbb{P}^3$ of degree 3 and arithmetic genus 0. $C \in \mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$ lies in the component H_1, which contains the twisted cubics, (Piene and Schlessinger, 1985; Ellingsrud *et al.*, 1987).

Note, that boundary points corresponding to plane nodal cubics with an embedded point at the node, do not occur, since all C are arithmetically Cohen–Macaulay. □

PROPOSITION 5.2. *For a general net q: $U^* \hookrightarrow S_2 W^*$ of quadrics the pfaffians of the net of alternating forms η_q: $\Lambda^2 V_q \to N_q$ defines an Artinian Gorenstein ring with Hilbert function* (1, 3, 6, 3, 1) *with a smooth dual socle quartic $F_q = F(V_q, \eta_q)$.*

Proof. Since the desired property is an open condition on nets q of quadrics, it suffices to exhibit an example. For the Kleinian net q_{Klein} we obtain as dual socle quartic $F_{Klein} = \{x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0\}$. F_{Klein} is smooth for $char(k) \neq 7$. For $char(k) = 7$ one can take some other example. □

6. The Hilbert Schemes of Lines on *X*

Let $F = \{f = 0\} \subset \mathbb{P}^2 = \mathbb{P}(U)$ be a non-degenerate plane quartic. In this section we prove that the circle of constructions

$F \mapsto (S_F, \vartheta_F)$,	Scorza,
$(S, \vartheta) \mapsto q_{S,\vartheta}$,	net corresponding to ϑ,
$q \mapsto A^q$,	apolarity,
$A^q \mapsto (\eta_q: \Lambda^2 V_q \to N_q)$,	Tor multiplication,
$(\eta: \Lambda^2 V \to N) \mapsto A_{V,\eta}$,	pfaffians,
$A \mapsto F_A$,	dual socle quartic,

gives the identity transformation on an open set of quartics. Note that, since $N_q \cong U^*$ by Cor. 4.3, F_A is again a quartic in $\mathbb{P}(U)$.

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For $X = \mathbb{G}(3, V, \eta)$ we denote by $\mathcal{H}_X$ the Hilbert scheme of lines in X with respect to the Plücker embedding $X \subset \mathbb{G}(3, V) \hookrightarrow \mathbb{P}(\Lambda^3 V^*)$.

THEOREM 6.1. *Let q be a general net of quadrics in $\check{\mathbb{P}}^3$. Let $\mathcal{E}_q$ be the corresponding vector bundle on $\mathbb{P}^3$, $X = H(q) = \mathbb{G}(3, V_q, \eta_q)$ and $F = F_q$ the dual socle quartic of A_{V_q,η_q}. The following curves are isomorphic:*

(a) *the discriminant S_q of q,*
(b) *the variety $\tilde{S}_q$ of unstable planes of $\mathcal{E}_q$,*
(c) *the Hilbert scheme $\mathcal{H}_X$ of lines on X,*
(d) *the covariant quartic S_F of F.*

Proof. $(a) \leftrightarrow (b)$: $S_q \cong \tilde{S}_q$ holds by Section 4. $(b) \leftrightarrow (c)$: Let $H = \{r = 0\} \subset \mathbb{P}^3$ be an unstable plane. Then

$$U^* = H^1(\mathbb{P}^3, \mathcal{E}_q(-1)) \xrightarrow{r} W^* = H^1(\mathbb{P}^3, \mathcal{E}_q)$$

is not injective, equivalently, $\mu_r\colon W \xrightarrow{r} U$ not surjective. So $\ker(\mu_r)$ is at least two dimensional, i.e. there are two elements $r_1, r_2 \in R_1$ such that $p_1 = r \cdot r_1$, $p_2 = r \cdot r_2 \in V_q$.

$(r_1, r_2) \cap V_q \subset S_2 W$ is at least $7 - 3 = 4$ dimensional. So there are further 2 quadrics

$$p_3 = a_1 r_1 + a_2 r_2, p_4 = b_1 r_1 + b_2 r_2 \in (r_1, r_2) \cap V_q.$$

Let $C_{(\alpha:\beta)} \subset \mathbb{P}^3$ be the curve (!) defined by

$$(p_1, p_2, \alpha p_3 + \beta p_4) = \Lambda^2 \begin{pmatrix} r_2 & -r_1 & 0 \\ \alpha a_1 + \beta b_1 & \alpha a_2 + \beta b_2 & -r \end{pmatrix}.$$

By Theorem 5.1

$$p_1 \wedge p_2 \wedge (\alpha p_3 + \beta p_4) \in \mathbb{G}(3, V_q, \eta_q) \subset \mathbb{P}(\Lambda^3 V_q^*), \quad (\alpha : \beta) \in \mathbb{P}^1, \tag{6.1}$$

gives a point in $\mathcal{H}_X$.

Conversely, every line in $H(q)$ is of type (6.1) for some $p_1, p_2, p_3, p_4 \in V_q$. p_1 and p_2 have a common factor r, since

$$\bigcup_{(\alpha:\beta)} C_{(\alpha:\beta)} \subset \{p_1 = p_2 = 0\} \subset \mathbb{P}^3,$$

and $H = \{r = 0\}$ is a unstable plane.

$(c) \to (d)$: $F = \{f = 0\}$ is non-degenerate by Corollary 5.2 and Theorem 2.3. Moreover

$$\mathbb{G}(3, V_q, \eta_q) = \mathbb{G}(3, V_f, \eta_f) \cong VSP(F, 6).$$

From this point of view a line

$$p_1 \wedge p_2 \wedge (\alpha p_3 + \beta p_4) \in \mathbb{G}(3, V_f, \eta_f)$$

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corresponds to a syzygy matrix

$$\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & a_{15} & a_{16} & a_{17} \\ 0 & 0 & 0 & 0 & a_{25} & a_{26} & a_{27} \\ 0 & 0 & 0 & a_{34} & a_{35} & a_{36} & a_{37} \\ 0 & 0 & -a_{34} & 0 & a_{45} & a_{46} & a_{47} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} & a_{57} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 & a_{67} \\ -a_{17} & -a_{27} & -a_{37} & -a_{47} & -a_{57} & -a_{67} & 0 \end{pmatrix}, \tag{6.2}$$

and the family of submatrices

$$\psi_{(\alpha:\beta)} = \begin{pmatrix} 0 & a_{15} & a_{16} & a_{17} \\ 0 & a_{25} & a_{26} & a_{27} \\ a_{34} & \alpha a_{35} + \beta a_{45} & \alpha a_{36} + \beta a_{46} & \alpha a_{37} + \beta a_{47} \end{pmatrix}$$

corresponds to one-parameter family of polar hexagons with three fixed lines $\in \check{\mathbb{P}}^2$ defined by

$$\begin{pmatrix} a_{15} & a_{16} & a_{17} \\ a_{25} & a_{26} & a_{27} \end{pmatrix}$$

and three moving lines through the common point $\langle a_{34}\rangle \in \mathbb{P}^2$. Hence the polar $P_{a_{34}}(F) \in \mathbf{A}$, i.e. $\langle a_{34}\rangle \in S_F \subset \mathbb{P}^2$.

$(c) \leftarrow (d)$: Conversely given $\langle a\rangle \in S_F$. Since F is non-degenerate $g = P_a(f)$ is not a cone. By Lemma 3.1. A^g is not a complete intersection. Hence there are three quadrics $b_1, b_2, b_3 \in (g^{\perp})_2$ with precisely two linear syzygies. Then $ab_1, ab_2, ab_3 \in (f^{\perp})_3$, and the two linear syzygies give two of the columns of ϕ with many zeroes. Thus this gives a decomposition of ϕ of shape (6.2). We only have to check that none of the six possibly non-zero entries of the two columns can lie on the diagonal. Suppose one or two entries lie on the diagonal. Then, since b_1, b_2, b_3 are the minors of the 3×2 matrix, either one quadrics is zero, or they have a common factor and a further syzygy. Both cases are impossible. Thus every point $\langle a\rangle \in S_F$ gives a uniquely determined decomposition of ϕ of shape (6.2). Hence a well-defined point in $\mathcal{H}_X$. Since $\langle a\rangle = \langle a_{34}\rangle$ in this correspondence this is the inverse of $(c) \rightarrow (d)$.

Notice, that under the isomorphisms of Sections 5 and 4.3

$$\eta_q(p_3 \wedge p_4) = a = a_{34} \in N \cong U^* \cong H^1(\mathbb{P}^3, \mathcal{E}(-1)).$$

Moreover (p_1, p_2, p_3, p_4) have the relations

$$\tilde{\phi}_1 = \begin{pmatrix} -r_2 & a_1 & b_1 & 0 \\ r_1 & a_2 & b_2 & 0 \\ 0 & -r & 0 & -p_4 \\ 0 & 0 & -r & +p_3 \end{pmatrix}.$$

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$\tilde{\phi}_1 \cdot \rho = 0$ for

$$\rho = \begin{pmatrix} a_1 b_2 - b_1 a_2 \\ p_4 \\ -p_3 \\ r \end{pmatrix}.$$

Thus, if we choose a basis for $8R(-3) \oplus 3R(-4)$ in the complex of Corollary 4.1 with these four relation corresponding basis elements, ρ gives one column of the matrix

$$8R(-3) \oplus 3R(-4) \longleftarrow 8R(-5).$$

Since

$$0 \longleftarrow \bigoplus_n H^1(\mathbb{P}^3, \mathcal{E}(n)) \longleftarrow (3R(-4) \longleftarrow 8R(-5)) \otimes R(5)$$

is a presentation, we obtain, that $\eta_q(p_3 \wedge p_4) = a \in H^1(\mathbb{P}^3, \mathcal{E}(-1))$ is annihilated by r. Since $h^0(H, \mathcal{E}|_H) = 1$ for $H = \{r = 0\}$ any unstable plane, there is only one quadric cone $Q \in U^*$ with vertex $H \in \check{\mathbb{P}}^3$ up to scalars. Thus under the isomorphisms of Corollary 4.3, section 5 and $(a) \leftrightarrow (d)$ the curves $S_q, S_F \in \mathbb{P}(U)$ are actually equal.□

Let $X = X_q = \mathbb{G}(3, V_q, N_q)$. Denote by

$$T_{\mathcal{H}} = \overline{\{(L_1, L_2) \in \mathcal{H}_X \times \mathcal{H}_X | L_1 \cap L_2 \neq \emptyset, L_1 \neq L_2\}}$$

the correspondence of intersecting lines in X.

COROLLARY 6.2. *Let q be a general net of quadrics. Then $B_X = \bigcup_{L \in \mathcal{H}_X} L$ consists of all singular twisted cubics in $H(q)$, and $S_F \subset \mathbb{P}(U)$ is the set of the triple points of polar hexagons to $F = F_q$. The correspondences*

(a) T_{ϑ_q} *on* S_q,
(b) $T_{\mathcal{H}}$ *on* $\mathcal{H}_X$,
(c) T_F *on* S_F
are isomorphic.

Proof. From the proof of the theorem we see, that the curves $C_{(\alpha:\beta)} \in H(q)$ on a line

$$\{p_1 \wedge p_2 \wedge (\alpha p_3 + \beta p_4)\}_{(\alpha:\beta) \in \mathbb{P}^1} \in \mathcal{H} \tag{6.1}$$

are all singular, and that they have the component $\{r_1 = r_2 = 0\} \subset \mathbb{P}^3$ in common. Conversely, if a curve $C \in H(q)$ is singular, it is reducible and one of its components is a line $\{r_1 = r_2 = 0\}$ in the intersection of two reducible quadrics $p_1 = r_1 \cdot r, p_2 = r_2 \cdot r \in H^0(\mathbb{P}^3, I_C(2)) \subset V_q$. r defines a unstable plane of $\mathcal{E}_q$ and gives a line (6.1).

Now take the point of view from polar hexagons. If a point $\Gamma = \{L_1, \dots, L_6\}$ lies on a line in $VSP(F, 6)$,then three lines of the hexagon pass through a common point

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$a_{34} \in \mathbb{P}^2$. Conversely, if $\{L_1, L_2, L_3\} \subset \Gamma$ pass through a point $< a > \in \mathbb{P}^2$, then $P_a(F)$ is anharmonic, i.e. $\langle a \rangle \in S_F$ and $\tilde{f} = f - (\lambda_4 l_4^4 + \lambda_5 l_5^4 + \lambda_6 l_6^4)$ is a quartic with $h_{A^{\tilde{f}}}(1) = 2$ and $h_{A^{\tilde{f}}}(2) = 3$, since f is not a sum of five powers. So by Theorem 2.3 there is a pencil of 3-tuples of lines presenting $\tilde{f}$, and this gives the family $\Gamma_{(\alpha:\beta)}$ defined by (6.1). For three values of $(\alpha : \beta) \in \mathbb{P}^1$ one of the moving lines passes through an intersection point b of a pair of the fixed lines. This corresponds to an intersection of two different lines in $\mathcal{H}$, i.e. a point in $T_{\mathcal{H}}$, and also to the point $(a, b) \in T_F$.

Finally to prove $T_F \cong T_{\vartheta_q}$ note that we already know $S_F = S_q$. Thus T_F and T_{ϑ_q} correspond both to one of the 36 even theta characteristics on S_F. Since $\mathcal{M}_3$ and $\mathcal{M}_{3,\vartheta^{ev}}$ have the same dimension, this implies, that the circle of constructions induces a covering transformation of $\mathcal{M}_{3,\vartheta^{ev}} \to \mathcal{M}_3$ over an open set corresponding to general nets q. Since $\mathcal{M}_q$ hence also $\mathcal{M}_{3,\vartheta^{ev}}$ is irreducible, it suffices to verify $T_F = T_{\vartheta_q}$ in one example, where all steps are defined. For example one can check this for the Kleinian net q_{Klein}.

Note, that $F_{q_{Klein}} = S_{q_{Klein}}$ is the Klein curve. Thus, ϑ_{Klein} is the unique theta characteristic on the Klein curve invariant under the whole automorphism group G_{168}, cf. Burnside (1911), Section 232. □

COROLLARY 6.3. *Over an algebraically closed field the circle of constructions*

$$F \longmapsto (S_F, \vartheta_F), \qquad (S, \vartheta) \longmapsto q_{S,\vartheta}, \qquad q \longmapsto A^q, \qquad A^q \longmapsto (\eta_q : \Lambda^2 V_q \to N_q),$$
$$(\eta : \Lambda^2 V \to N) \longmapsto A_{V,\eta}, \qquad A \longmapsto F_A$$

gives the identity transformation on an open set of quartics.

COROLLARY 6.4. *Over an algebraically closed field the circle of constructions define birational transformations of the moduli spaces $\mathcal{M}_{Fano}$ of nets of alternating forms (equivalently of prime Fano 3-folds of genus 12 by Mukai's Theorem), $\mathcal{M}_3$ of curves of genus 3, $\mathcal{M}_{3,\vartheta^{ev}}$ of curves of genus 3 together with a non-vanishing theta characteristic, and $\mathcal{M}_q$ of nets of quadrics.*

COROLLARY 6.5. *Let K be an arbitrary field of characteristic 0 and X a smooth prime Fano 3-fold of genus 12. Then the Grassmannian model $\mathbb{G}(3, V, \eta)$ and the plane model $VSP(F, 6)$ are defined over K.*

Proof. By Mukai's (1992; 1995) Theorems all three models are defined over the algebraic closure of K. However the Hilbert scheme $\mathcal{H}_X$ of lines on X is defined over K, and so is the correspondence $T_{\mathcal{H}}$ of intersecting lines. Identifying $T_{\mathcal{H}} = T_F$ and $S_F = \mathcal{H}$ in its canonical embedding, we obtain a quadratic system of equations for the coefficients of the defining equation $F = \{f = 0\}$ with coefficients in K. So the by Mukai's result unique solution with a non-degenerate quartic is defined over K. The equivalence of the Grassmannian model and space model is defined over K. So also η is defined over K. □

The space model $H(q)$ is in general not defined over the ground field, due to the descent from T_ϑ to ϑ. For the real numbers we note:

Remark 6.6. (1) Let $X \cong VSP(F, 6)$ be a prime Fano 3-fold of genus 12 defined over k with smooth covariant quartic S_F. If the covariant quartic S_F contains a k-rational point, then the space model $H(q)$ is defined over k. Indeed if $a \in S_F$ is defined over k, then the fiber $T_F(a)$ and line bundle $\vartheta = \mathcal{O}(T_F(a) - a)$ are defined over k

(2) The quadrics

$$q_0 = w_0^2 + w_1^2 - w_2^2 - w_3^2, q_1 = w_0 w_2 + w_1 w_3,$$
$$q_2 = (w_1 + w_2 + w_3)^2 + (w_0 + w_1 - w_3)^2 - (w_0 + w_1 + w_3)^2 - (w_0 + w_1 - w_2)^2$$

span a net q, whose discriminant S_q contains no real point. Indeed the catalectican $\mathrm{Cat}(S_q)$ is positive definite. So the existence of a point is sufficient but not necessary for the existence of the space model, even for the ground field $\mathbb{R}$.

EXAMPLE 6.7. For the Mukai–Umemura quartic, cf. Mukai and Umemura (1983) there are two different plane models over $\mathbb{R}$,

$$F_{MU} = \{(x^2 + y^2 + z^2)^2 = 0\} \text{ and } F'_{MU} = \{(x^2 + y^2 - z^2)^2 = 0\}.$$

Their covariant quartics are equal to themselves.

For the space model $H(q_{MU})$ there is only one version. The net of quadrics q_{MU} is the ideal of the twisted cubic, which, if defined over $\mathbb{R}$, is isomorphic to $\mathbb{P}^1_{\mathbb{R}}$. Its discriminant is the indefinite F'_{MU}. Thus for $VSP(F_{MU}, 6)$ there is no $\mathbb{R}$-isomorphic space model.

Since the image of $\{q = (q_0, q_1, q_2)/\mathbb{R}\} \to \{(\eta : \Lambda^2 V \to \mathbb{R}^3)/\mathbb{R}\} \to \{F \subset \mathbb{P}^2/\mathbb{R}\}$ is closed (in a neighborhood of F_{MU}) and F_{MU} is not in the image, we obtain that for any quartic F over $\mathbb{R}$ nearby F_{MU}, the Fano 3-fold $VSP(F, 6)$ has no $\mathbb{R}$-isomorphic space model.

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