Commun. Math. Phys. 155, 249–260 (1993)

Communications in Mathematical Physics

Geometry of Batalin–Vilkovisky Quantization*

Albert Schwarz
Department of Mathematics, University of California, Davis, CA 95616, USA
ASSCHWARZ@UCDAVIS.EDU

Received May 20, 1992; in revised form November 24, 1992

Abstract. The geometry of P-manifolds (odd symplectic manifolds) and SP-manifolds (P-manifolds provided with a volume element) is studied. A complete classification of these manifolds is given. This classification is used to prove some results about Batalin–Vilkovisky procedure of quantization, in particular to obtain a very general result about gauge independence of this procedure.

0. Introduction

A very general and powerful approach to quantization of gauge theories was proposed by Batalin and Vilkovisky [1, 2]. The present paper is devoted to the study of geometry of this quantization procedure. The main mathematical objects under consideration are P-manifolds and SP-manifolds (supermanifolds provided with an odd symplectic structure and, in the case of SP-manifolds, with a volume element). The Batalin–Vilkovisky procedure leads to consideration of integrals of the form $\int_L Hd\lambda$, where L is a Lagrangian submanifold of an SP-manifold M and H satisfies the equation $\Delta H = 0$, where Δ is an odd analog of the Laplacian. The choice of L can be interpreted as a choice of gauge condition; Batalin and Vilkovisky proved that in some sense their procedure is gauge independent. Namely they proved that $\int_{L_0} Hd\lambda_0 = \int_{L_1} Hd\lambda_1$ if Lagrangian submanifolds L_0 and L_1 are connected by a continuous family L_t of Lagrangian submanifolds. We will prove that the same conclusion can be made in the much more general case when the bodies $m(L_0)$ and $m(L_1)$ of submanifolds L_0 and L_1 are homologous in the body $m(M)$ of M. This theorem leads to a conjecture that one can modify the quantization procedure in such a way as to avoid the use of the notion of the Lagrangian submanifold. In the next paper we will show that this is really so at least in the semiclassical approximation. Namely if H is written in the form $\exp \hbar^{-1}S$, where $S = S_0 + \hbar S_1 + \dots$ we will find the asymptotics of $\int_L Hd\lambda$ as an integral over some set of critical points of S_0 with the integrand expressed in terms

* Research supported in part by NSF grant No. DMS-9201366

of Reidemeister torsion. This leads to a simplification of quantization procedure and to the possibility to get rigorous results also in the infinite-dimensional case, using the results of [4]. (We are talking about the semiclassical approximation.)

The present paper contains also a complete classification of P-manifolds and SP-manifolds. The classification is interesting by itself, but in this paper it plays also a role of an important tool in the proof of other results.

1. Main Definitions and Theorems

Let us consider a domain U in a superspace $R^{n|n}$ with coordinates $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$. An odd Poisson bracket (antibracket) of functions F and G on U can be defined by the formula

$$\{F, G\} = \frac{\partial_r F}{\partial x^a} \cdot \frac{\partial_l G}{\partial \xi_a} - \frac{\partial_r F}{\partial \xi_a} \cdot \frac{\partial_l G}{\partial x^a}, \tag{1}$$

where ∂_r and ∂_l denote the right derivative and the left derivative correspondingly. (We suppose usually that $x^1, \ldots, x^n$ are even and $\xi_1, \ldots, \xi_n$ are odd. However one can weaken this assumption by requiring only that x^a and ξ_a have opposite parity.) The transformations of U preserving the bracket (1) will be called P-transformations (or odd symplectic transformations). Volume preserving P-transformations (i.e. P-transformations having unimodular Jacobian) will be called SP-transformations. (Of course we have in mind the supervolume. Unimodularity of the Jacobian matrix means that the Berezinian of this matrix is equal to 1.) P-manifold (or odd symplectic manifold) is by definition a manifold pasted together from $(n|n)$-dimensional superdomains by means of P-transformations. Replacing in this definition P-transformations by SP-transformations we get the definition of SP-manifold.[1] In a general local coordinate system $(z^1, \ldots, z^{2n})$ one can write the Poisson bracket (1) in the form

$$\{F, G\} = \frac{\partial_r F}{\partial z^i} \omega^{ij}(z) \frac{\partial_l G}{\partial z^j}, \tag{2}$$

where $\omega^{ij}(z)$ is an invertible matrix. Its inverse matrix $\omega_{ij}(z)$ determines a differential form

$$\omega = dz^i \omega_{ij} dz^j . \tag{3}$$

It is easy to check that this form is closed ($d\omega = 0$). As in standard symplectic geometry one can construct a vector field K_H corresponding to a function H on a P-manifold M by the formula

$$K_H^i(z) = \omega^{ij}(z) \frac{\partial_l H}{\partial z^j} . \tag{4}$$

---

[1] Using the language of G-structures we can say that P-manifold is a supermanifold provided with locally flat P-structure where P is a group consisting of linear transformations of $R^{n|n}$ preserving the bilinear form $x^i \xi_i$. To get the definition of SP-manifold we have to replace here the group P by its subgroup $SP = P \cap SL(n|n)$. We will not use the language of G-structures; speaking about P-structure or SP-structure we will have in mind the structure of P-manifold or SP-manifold

By definition K_H is a Hamiltonian vector field with Hamiltonian H. If the function H is odd then K_H is even and vice versa. The bracket (2) determines the structure of a Lie superalgebra on the linear (super)space F of (super)functions on M. It is easy to check that the map $H \to K_H$ is a homomorphism of F into the Lie superalgebra diff M of vector fields on M. A submanifold $L \subset M$ is called isotropic if ω vanishes on L (i.e. $t^\alpha \omega_{ab}(x)\tilde{t}^b = 0$ for every pair $t, \tilde{t}$ of tangent vectors to L at the point $x \in L$). A Lagrangian manifold L is by definition an isotropic manifold of dimension $(k|n-k)$, $0 \leqq k \leqq n$.

One can give an invariant definition of P-manifold. Namely such a manifold can be defined as an $(n|n)$-dimensional supermanifold provided with a non-degenerate closed odd 2-form ω. This definition is equivalent to a previous one because one can prove an analog of Darboux theorem: a non-degenerate closed odd 2-form ω locally can be written as $dx^a d\xi_a$ by an appropriate choice of coordinates $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$ – Darboux coordinates. Moreover if L is a Lagrangian submanifold of M one can choose Darboux coordinates in the neighborhood of a point $a \in L$ in such a way that in this neighborhood L is singled out by the equations $x^{k+1} = \ldots = x^n = 0$, $\xi_1 = \ldots = \xi_k = 0$. If we don't require that x^i are even, ξ_i are odd, then we always can define a Lagrangian submanifold locally by the equations $\xi_1 = \ldots = \xi_n = 0$.

The volume element in arbitrary coordinates $(z^1, \ldots, z^{2n})$ can be specified by means of the density function $\rho(z)$. In such a way SP-structure on M is determined by non-degenerate closed odd 2-form ω and by density ρ. (It is necessary to emphasize that ρ is not arbitrary; one has to require that in the neighborhood of every point in M one can make $\rho \equiv 1$ by means of approrpiate choice of Darboux coordinates.) As usual the volume element in M determines the divergence of vector field K^a by the formula

$$\operatorname{div} K = \rho^{-1} \frac{\partial_r(\rho K^a)}{\partial z^a} = \frac{\partial_r K^a}{\partial z^a} + \frac{\partial_r \ln \rho}{\partial z^a} K^a . \tag{5}$$

Therefore one can define an operator Δ on the space F of functions on SP-manifold M by the formula

$$\Delta H = \frac{1}{2} \operatorname{div} K_H . \tag{6}$$

One can check that $\Delta^2 = 0$ using the existence of local coordinates with $\omega = dx^a d\xi_a$, $\rho = 1$. In these coordinates

$$\Delta = \frac{\partial_r}{\partial x^a} \frac{\partial_l}{\partial \xi_a} \tag{7}$$

and the relation $\Delta^2 = 0$ is evident. In a general coordinate system the relation $\Delta^2 = 0$ leads to conditions on ρ. One can prove that these conditions are sufficient to assert that the non-degenerate closed odd 2-form ω and the density function $\rho(z)$ determine an SP-structure; see Theorem 5 below. In the formula (7) we suppose that the variables x^a, ξ_a have opposite parity; if $x^1, \ldots, x^n$ are even and $\xi_1, \ldots, \xi_n$ are odd as we assume usually the right derivative with respect to x^a in (7) is of course the standard derivative. If L is a Lagrangian submanifold of SP-manifold M one can define a volume element in L (up to a sign). Namely, if in Darboux coordinates $x^1, \ldots, x^n, \xi_1, \ldots, \xi_n$ the manifold L is singled out by the equations

$x^{k+1} = \ldots = x^n = 0$, $\xi_1 = \ldots = \xi_k = 0$ then the volume element in L can be defined as $dx^1 \ldots dx^k d\xi_{k+1} \ldots d\xi_n$. It is easy to check that this volume element is well defined up to a sign. (This fact follows immediately from another description of the volume element in L given in the proof of Lemma 4.) One has to impose some global conditions to define the volume element globally. Namely, we will prove that it suffices to require that M be orientable, i.e. that $m(M)$ be orientable. We denote by $m(M)$ the body of M, i.e. the bosonic part of M.

The Batalin–Vilkovisky approach to quantization is based on the following theorem: if L_0 and L_1 are closed oriented Lagrangian submanifolds connected with a smooth family of closed oriented Lagrangian submanifolds L_t and an even function H on M satisfies the condition $\Delta H = 0$ then $\int_{L_0} H\, d\lambda_0 = \int_{L_1} H\, d\lambda_1$. For completeness we will sketch a proof of this statement. As usual it is sufficient to consider an infinitesimal deformation of the Lagrangian manifold L; moreover one can assume that L is deformed only in a domain where (after appropriate change of coordinates) it is singled out by equations $\xi_1 = \ldots = \xi_n = 0$ and where $\rho = 1$. Then the deformed manifold can be specified by means of an odd function $\Psi(x^1, \ldots, x^n)$ that vanishes outside of this domain. Namely, the deformed manifold can be defined by the equations $\xi_j = \dfrac{\partial_l \Psi}{\partial x^j}$. The variation of the integral $\int_L H\, d\lambda$ by this deformation can be written as

$$\int \frac{\partial_r H}{\partial x^j} \frac{\partial_l \Psi}{\partial \xi_j} dx^1 \ldots dx^n .$$

Integrating by parts and using $\Delta H = 0$ we obtain that this variation is equal to zero.

In the formulation of Batalin–Vilkovisky theorem we assume that the volume elements $d\lambda_0$ and $d\lambda_1$ in L_0 are chosen in an appropriate way; namely we require the existence of volume elements $d\lambda_t$ in L_t depending continuously on t and connecting $d\lambda_0$ and $d\lambda_1$. A similar assumption must be made about the orientation of L_0 and L_1. Our aim is to prove a generalization of this theorem. Namely, we will prove the following:

Theorem 1. *Let L_0 and L_1 be closed oriented Lagrangian submanifolds of an orientable SP-manifold M. If the cycles $m(L_0)$ and $m(L_1)$ are homologous in $m(M)$ over R* (i.e. $m(L_0)$ and $m(L_1)$ determine the same element of $H_k(m(M), R)$) *then*

$$\int_{L_0} H\, d\lambda_0 = \int_{L_1} H\, d\lambda_1 \tag{8}$$

for every function H satisfying $\Delta H = 0$.

We will prove also

Theorem 2. *If $H = \Delta K$ then for every closed Lagrangian manifold L*

$$\int_L H\, d\lambda = 0 . \tag{9}$$

The proof of these theorems will be based on an explicit description of P-manifolds and their Lagrangian submanifolds. I don't know any direct proof of these theorems.

2. Classification of *P*-Manifolds and Description of Lagrangian Submanifolds

We begin with the remark that every transformation $\tilde{x} = f(x)$ of n-dimensional domain with coordinates $x^1, \ldots, x^n$ can be extended to a P-transformation $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n) \to (\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{\xi}_1, \ldots, \tilde{\xi}_n)$ by means of the formula

$$\tilde{\xi}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \xi_j \, . \tag{10}$$

This means that a cotangent bundle T^*N to an n-dimensional manifold N has a natural structure of P-manifold (the formula (10) coincides with the transformation law of covectors). We will prove

Theorem 3. *Every $(n|n)$-dimensional P-manifold M is equivalent to a P-manifold of the form T^*N. Namely, one can take $N = m(M)$.*

Let us begin with a remark that for every m-dimensional vector bundle α over an n-dimensional manifold N one can construct an $(m|n)$-dimensional supermanifold considering the fibres as odd linear spaces. More precisely, if a vector bundle over N has transition functions $\tilde{x}^i = f^i(x^1, \ldots, x^n)$, $\tilde{\eta}^l = \alpha^l_k(x^1, \ldots, x^n)\eta^k$, where x^i are coordinates in the base, η^k are coordinates in the fibre, one can construct a supermanifold pasted together by means of the same formulas where η^k are considered as odd coordinates. It is well known that every real $m|n$-dimensional supermanifold can be obtained by means of this construction [5]; therefore we can assume that P-manifold M is a bundle α over $N = m(M)$. (The bundle α has an invariant description as the so-called conormal bundle [5]; we will not use this description.) Sometimes we will use the notation N_α for the supermanifold corresponding to the bundle α over N. Let us restrict the form ω specifying the P-structure in $M = N_\alpha$ to $N \subset M$ (i.e. we take $\eta = 0$). The expression

$$\omega|_{\eta=0} = \omega_{ij}(x) dx^i d\eta^j \tag{11}$$

determines a non-degenerate pairing between fibres of α and tangent spaces to N. The existence of this pairing permits us to identify α with cotangent bundle and M with T^*N. However it is possible a priori that the P-structure on T^*N arising from this identification and the standard P-structure on T^*N are different. To show the equivalence of these P-structures we note that corresponding forms ω and ω_0 can be connected by a smooth family $\omega_t = (1-t)\omega_0 + t\omega$ of closed non-degenerate odd forms. (To check that the forms ω_t are non-degenerate we use the fact that ω and ω_0 coincide on N imbedded in standard way into T^*N. Non-degeneracy of ω_t on T^*N follows from non-degeneracy on $N \subset T^*N$.) To finish the proof we utilize the following:

Lemma 1. *If ω is a non-degenerate closed odd 2-form and σ is a closed odd 2-form then one can find a vector field V in such a way that $\sigma = L_V\omega$, where $L_V\omega$ is the Lie derivative of ω with respect to ω (the change of ω by the infinitesimal transformation V).*

To prove this lemma we note that $L_V\omega$ can be represented as

$$L_V\omega = (d\omega)_V - d\omega_V \, , \tag{12}$$

where for every k-form σ we denote by σ_V the $(k-1)$-form obtained from σ by contraction with the vector field V. For example, if $\omega = dz^i\omega_{ij}dz^j$ then

$\omega_V = V^i\omega_{ij}dz^j$. If ω is closed then $L_V\omega = -d\omega_V$. Every closed odd 2-form σ is exact: $\sigma = d\lambda$, where $\lambda = \lambda_j dz^j$. It remains to say that V^i can be found from the equation

$$\lambda_j = -V^i\omega_{ij}\,. \tag{13}$$

This equation always has a solution because ω_{ij} is non-degenerate. Moreover this solution is unique.

The proof of the lemma repeats the standard proof of the fact that an even symplectic structure on closed manifold does not change if the 2-form defining it changes, but corresponding cohomology class remains intact.

If ω_t is a smooth family of non-degenerate closed odd 2-forms on $M = T^*N$ coinciding on $N \subset M$ it follows immediately from the lemma that all these forms determine equivalent P-structures. The lemma shows that an infinitesimal variation of form ω gives an equivalent P-structure. The study of a smooth deformation of ω can be reduced to the study of infinitesimal variation. To construct the transformations proving the equivalence we have to solve the differential equation

$$\dot{z}(t) = V(t)z(t)\,, \tag{14}$$

where the vector field $V(t)$ satisfies

$$\dot{\omega}_t = L_{V(t)}\omega_t\,. \tag{15}$$

It follows from the proof of the lemma that one can find $V(t)$ in such a way that it will be differentiable with respect to t. This assumption guarantees the existence of solution to (14). (In the case of even symplectic structure it is necessary to assume compactness of symplectic manifold to guarantee the existence of solution to the analog of (14). In the case at hand we don't need this assumption because V^i generates a zero vector field on the body N of $M = T^*N$.)

In what follows we restrict ourselves by the case when the P-manifold M is realized as T^*N with standard P-structure; as we proved this can be made without loss of generality. Let us define standard Lagrangian submanifolds of T^*N in the following way. Let us suppose that K is a k-dimensional submanifold of N. Then we can construct an $(n-k)$-dimensional bundle λ over K consisting of covectors orthogonal to K. Supermanifold L_K corresponding to this bundle is naturally imbedded into T^*N and can be considered as $(k|n-k)$-dimensional Lagrangian submanifold of T^*N.

Theorem 4. *For every Lagrangian submanifold of T^*N one can find a smooth deformation of this submanifold into a standard Lagrangian submanifold* (i.e. *into a submanifold of the form L_K*).

To prove this theorem we consider at first the group G_M of all transformations of arbitrary supermanifold M. Without loss of generality we assume that $M = N_\sigma$, where σ is a vector bundle over a manifold N. Let us denote by G_σ the group of automorphisms of the bundle σ. In local coordinates these automorphisms are given by formulas $\tilde{x}^i = F^i(x^1, \ldots, x^n)$, $\tilde{\eta}^j = a_i^j(x^1, \ldots, x^n)\eta^i$, where a_i^j is a non-degenerate matrix. The same formulas determine transformations of a supermanifold N_σ; therefore we have a natural imbedding i of G_σ into G_M. There exists also a natural map π of G_M onto G_σ. In local coordinates a transformation of M can be written as

$$\tilde{x}^i = f^i(x^1, \ldots, x^n) + \sum_{k=1} \sum_{j_1, \ldots, j_{2k}} f^i_{j_1, \ldots, j_{2k}}(x^1, \ldots, x^n)\eta^{j_1} \ldots \eta^{j_{2k}} . \tag{16}$$

$$\tilde{\eta}^j = a^j_i(x^1, \ldots, x^n)\eta^i + \sum_{k=1} \sum_{i_1, \ldots, i_{2k+1}} a^j_{i_1, \ldots, i_{2k+1}}(x^1, \ldots, x^n)\eta^{i_1} \ldots \eta^{i_{2k+1}} . \tag{17}$$

Leaving only the first term in (16), (17) we get an automorphism of σ. (In more invariant words one can say that the transformation of a supermanifold generates naturally an automorphism of corresponding conormal bundle). It is easy check that the maps i of G_σ into G_M and π of G_M onto G_σ generate a homotopy equivalence between G_M and G_σ. The main fact leading to this conclusion is that (16), (17) determine a transformation of M by any choice $f^i_{j_1, \ldots, j_{2k}}$, $a^j_{i_1, \ldots, i_{2k+1}}$ for $k \geqq 1$ if $f^i(x^1, \ldots, x^n)$ and $a^j_i(x^1, \ldots, x^n)$ determine an automorphism of σ. Therefore we can simply multiply all these functions by τ, $0 \leqq \tau \leqq 1$, to obtain a family Q_τ, $0 \leqq \tau \leqq 1$, of transformations of M obeying $Q_1 = i\alpha$, $Q_0 = i\pi$, $\pi Q_\tau = \pi$.

If M is a P-manifold, we will denote by S_M the group of all P-transformations of M (transformations preserving the P-structure in M). In this case σ is a cotangent bundle and G_σ is imbedded in S_M. One can prove the following lemma which is interesting by itself.

Lemma 2. *The imbedding i of G_σ into S_M and the natural map π of S_M onto G_σ determine a homotopy equivalence between G_σ and S_M.*

To prove this statement we use the deformation Q_τ constructed above and the arguments used in the proof of Lemma 1. Namely, we will define the deformation $\tilde{Q}_\tau$ as $R_\tau Q_\tau$, where R_τ is a transformation of $M = T^*N$ satisfying $(R_\tau Q_\tau)^*\omega = \omega$. Such a transformation R_τ can be found by solving the equations (14), (15). To guarantee the continuity of R_τ with respect to τ we have to eliminate the freedom in the construction of R_τ. This can be made if we construct R_τ by means of (15) with V_t found as a solution of the equation $\lambda^t_j = - V^i_t \cdot \omega_{ij}$, where $\lambda^t = \lambda^t_j dz^j$ is specified by the formula $\lambda^t = (Q^*_t)^{-1}\lambda$, $d\lambda = \omega$.

Now we are able to prove Theorem 4. It is easy to check that for arbitrary Lagrangian submanifold L of P-manifold M one can find a map φ of T^*L into M preserving P-structure. Representing L as K_β, where $K = m(L)$, β is a vector bundle over K we can construct a map of L onto a standard Lagrangian submanifold of $M = T^*N$ and extend this map to a map $\tilde{\varphi}$ of T^*L into M preserving P-structure. Using Lemma 2 we can deform φ into $\tilde{\varphi}$ and therefore every Lagrangian submanifold into a standard Lagrangian submanifold.

Let us consider a manifold N provided with a volume element α (one can consider α as an n-form $\alpha(x^1, \ldots, x^n)dx^1 \wedge \cdots \wedge dx^n$, where $\alpha(x^1, \ldots, x^n)$ is a non-vanishing function on N). Differential forms on N can be considered as functions on a supermanifold TN, corresponding to the tangent bundle over N. If $x^1, \ldots, x^n$ are local coordinates in N then $x^1, \ldots, x^n, \eta^1 = dx^1, \ldots, \eta^n = dx^n$ can be considered as coordinates in TN. If ω is a differential form on N (a function on TN) one can define a function $\tilde{\omega} = F\omega$ on T^*N by the formula

$$\tilde{\omega}(x, \xi) = \int e^{\xi_i \eta^i} \omega(x, \eta)\alpha^{-1}(x) d^n\eta . \tag{18}$$

In other words $\tilde{\omega}$ is a Fourier transform of ω with respect to odd variables. (The functions on T^*N can be identified with antisymmetric multivector fields. In this interpretation Fourier transformation is simply the duality transformation, i.e. contraction of covector field $\omega_{i_1 \ldots i_k}$ with the universal antisymmetric tensor $\alpha^{-1}\varepsilon^{j_1 \ldots j_n}$.) It is easy to check that

$$\frac{\partial_l(F\omega)}{\partial \xi_i} = F(\eta^i \omega), \quad \frac{\partial(F\omega)}{\partial x^i} = F\left(\frac{\partial \omega}{\partial x^i}\right) - \alpha^{-1}\frac{\partial \alpha}{\partial x^i} F(\omega) . \tag{19}$$

Using these formulas we obtain that

$$F(d\omega) = \Delta F(\omega) , \tag{20}$$

where d denotes the exterior differential of ω $\left(\text{in the language of functions on } TN \text{ we have } d = \eta^i \frac{\partial}{\partial x^i}\right)$ and the operator Δ is constructed by means of SP-structure on P-manifold T^*N, specified by the volume element

$$\rho_0(x, \xi) d^n x d^n \xi = \alpha^2(x) d^n x d^n \xi . \tag{21}$$

(This connection between d and Δ was used in [3]).

Now we are able to prove Theorems 1 and 2 for the case when the manifold $M = T^*N$ is provided with standard P-structure and with the volume element (21). We will use the following statement that can be easily proved in this case.

Lemma 3. *If ω is a form on N and K is a closed oriented submanifold of N then*

$$\int_K \omega = \int_{L_K} F(\omega) d\lambda , \tag{22}$$

where L_K denotes the Lagrangian submanifold of T^*N corresponding to K.

The proof of Lemma 3 in the case when ω has a support in a domain where K in an appropriate coordinate system can be singled out by equations $x^{k+1} = 0, \ldots, x^n = 0$ is immediate. Without loss of generality one can assume that ω is a monomial with respect to $\eta^1, \ldots, \eta^n$. Only the monomial $\omega = \gamma(x)\eta^1, \ldots, \eta^k$ gives a non-zero contribution to the integrals in (22). For this monomial we have

$$F(\omega) = \gamma(x)\alpha^{-1}(x)\xi_{k+1} \ldots \xi_n .$$

The volume element $d\lambda$ on L_K induced by (21) can be written in the form

$$d\lambda = \alpha(x) dx^1 \ldots dx^k d\xi_{k+1} \ldots d\xi_n$$

(we omit the proof of this assertion because a more general fact will be proven later; see Lemma 4). Using the expressions for $F(\omega)$ and $d\lambda$ we obtain (22) in the case at hand. The general case can be reduced to this simplest case by means of standard technique (one has to use the partition of unity).

The statements of Theorems 1 and 2 follow immediately from (22) and (20) when the Lagrangian submanifolds are standard. The case of general Lagrangian submanifolds of T^*N can be reduced to this simplest case by means of Theorem 3. Therefore we can say that Theorems 1 and 2 are proved in the case when SP-structure in T^*N is determined by the density that does not depend on ξ. (The volume element corresponding to such a density can be represented up to a sign in the form (21).)

In the consideration above we did not pay sufficient attention to the choice of the sign of the volume element in Lagrangian submanifold. It suffices to analyze this question for standard Lagrangian submanifold L_K. Let us introduce the notation $\Lambda(E)$ for the one-dimensional linear space of translationally invariant real measures in the linear superspace E. In other words $\Lambda(E)$ consists of functions of bases in E having degree 1 (i.e. to specify an element $\alpha \in \Lambda(E)$ we have to assign to every basis $e \in E$ a real number $\alpha(e)$ in such a way that $\alpha(Ae) = \det A \cdot \alpha(e)$, where $\tilde{e} = Ae$ denote a basis obtained from e by means of a linear transformation: $\tilde{e}_i = A_i^j e_j$.) To specify the volume element in L_K one has to single out a non-zero element of $\Lambda(TL_K(z))$ – a non-zero measure in tangent space $TL_K(z)$ to L_K at every point $z \in L_K$. One can identify $\Lambda(TL_K(z))$ with $\Lambda(TK(m(z))) \otimes \Pi\Lambda(TN(m(z))/TK(m(z)))^* = \Lambda(TN(m(z))$. (Here Π denotes the parity reversion. We used that $\Lambda(\Pi E) = \Lambda(E)^*$, $\Lambda(E^*) = \Lambda(E)^*$, $\Lambda(E_1) = \Lambda(E_2) \otimes \Lambda(E_1/E_2)$ if $E_2 \subset E_1$). The spaces $\Lambda(TL_K(z))$ can be considered as fibres of a line bundle over L_K. If this bundle is trivial and locally the volume element is defined up to a sign then the volume element can be defined globally. Conversely if the volume element is defined globally it can be considered as a non-zero section of this bundle and this bundle is trivial. Using the identification $\Lambda TL_K(z) = \Lambda TL_K(m(z)) = \Lambda(TN(m(z)))$ we conclude that in the case when N is orientable (i.e. the bundle over N with the fibres $\Lambda(TN(x))$ is trivial) the volume element on every Lagrangian submanifold can be defined globally. In the general case Lagrangian submanifold L of SP-manifold M can be provided with global volume element if and only if its body $m(L)$ can be imbedded in an orientable submanifold of $m(M)$.

3. The Proof of Main Theorems

Now we have to give a proof of Theorems 1 and 2 for general SP-manifold. The proof is based on the following

Lemma 4. *Let us suppose that SP-structure in a SP-manifold M is specified by the density ρ. If the density $\tilde{\rho} = \rho e^{\sigma}$ in M also determines a SP-structure in M then*

$$\Delta_\rho \sigma + \frac{1}{4}\{\sigma, \sigma\} = 0 , \tag{23}$$

where Δ_ρ denotes the operator Δ corresponding to the SP-structure determined by the density ρ. The operator Δ corresponding to the density $\tilde{\rho}$ can be written in the form

$$\Delta_{\tilde{\rho}} = e^{-\sigma/2}\Delta_\rho(e^{\sigma/2}H) . \tag{24}$$

Volume elements $d\tilde{\lambda}$ and $d\lambda$ in the Lagrangian submanifold $L \subset M$ corresponding to SP-structures at hand are connected by the formula

$$d\tilde{\lambda} = e^{\sigma/2}d\lambda . \tag{25}$$

The statement of Lemma 4 is local and therefore we can simplify the proof using Darboux coordinates and assuming that $\rho = 1$. We can write in these coordinates

$$\Delta_{\tilde{\rho}}H = \Delta_\rho H + \frac{1}{2}\{\sigma, H\} = \frac{\partial}{\partial x^a}\frac{\partial_l}{\partial \xi^a}H + \frac{1}{2}\{\sigma, H\} . \tag{26}$$

Calculating $\Delta_{\tilde{\rho}}^2$ we get

$$\Delta_{\tilde{\rho}}^2 H = \left\{\Delta_\rho \sigma + \frac{1}{4}\{\sigma, \sigma\}, H\right\} = 0 . \tag{27}$$

This equation shows that in the case when the density $\tilde{\rho}$ determines an SP-structure, $\Delta_\rho \sigma + \{\sigma, \sigma\}/4 = \text{const}$. This equation can be written also in the form

$$\Delta_\rho e^{\sigma/2} = \text{const} \cdot e^{\sigma/2} . \tag{28}$$

Applying Δ_ρ to (28) we obtain from $\Delta_\rho^2 = 0$ that the constant in this equation is equal to 0. In such a way

$$\Delta_\rho e^{\sigma/2} = 0 . \tag{29}$$

Using (29) one can check that (24) follows from (23). To prove (25) we will give another description of volume element in Lagrangian submanifold L. Let us fix a basis $(e_1, \ldots, e_n)$ in the tangent space $TL(z)$ to L at the point $z \in L$. Then one can find a basis $(e_1, \ldots, e_n, f^1, \ldots, f^n)$ in the tangent space $TM(z)$ to M satisfying $\omega(e_i, f^j) = \delta_i^j$ (P-structure in M determines an odd bilinear form ω on $TM(z)$). The volume element λ in L can be defined by the formula

$$\lambda(e_1, \ldots, e_n) = \mu(e_1, \ldots, e_n, f^1, \ldots, f^n)^{1/2} , \tag{30}$$

where μ denotes the volume element determined by SP-structure in M. Equation (25) follows immediately from (30).

Using Lemma 4 we can reduce the study of SP-structure with density function $\tilde{\rho} = \rho e^\sigma$ to the study of SP-structure with density function ρ. In particular we are able now to prove Theorems 1 and 2 for all SP-manifolds. As we mentioned already it is sufficient to consider manifolds of the form $M = T^*N$ with standard P-structure. If SP-structure in M is specified by the density

$$\rho(x, \xi) = \rho_0(x) + \sum_{k>1} \rho^{i_1 \ldots i_k}(x) \xi_{i_1} \ldots \xi_{i_k} \tag{31}$$

we consider another SP-structure in M determined by the density $\rho_0(x)$. It is clear that $\rho(x, \xi) = \rho_0(x) e^{\sigma(x, \xi)}$, where $\sigma(x, \xi) = 0$ for $\xi_1 = \cdots = \xi_n = 0$. It follows from (25) that for every Lagrangian submanifold $L \subset M$ we have

$$\int_L H \, d\lambda = \int_L H e^{\sigma(x, \xi)/2} \, d\lambda_0 . \tag{32}$$

Further if $\Delta H = 0$ we obtain from (24) that $\Delta_0(He^{\sigma/2}) = 0$ and if $H = \Delta K$ we get that $He^{\sigma/2} = \Delta_0(Ke^{\sigma/2})$. (We use the notations $d\lambda$ and $d\lambda_0$ for volume elements in L determined by the densities ρ and ρ_0; the notations Δ and Δ_0 have similar meaning.) Using these remarks we reduce the proof of Theorems 1 and 2 for the density ρ to the case of density ρ_0. This case was analyzed already.

4. Classification of *SP*-Manifolds

The consideraton above permits us to construct one-to-one correspondence between SP-structures in connected P-manifold M and cohomology classes $s \in H(m(M), R)$ satisfying $s^n \neq 0$. (Recall that by definition $H(N, R)$ is the direct sum of k-dimensional cohomology groups $H^k(N, R)$; we represent $s \in H(m(M), R)$ as $s^0 + s^1 + \cdots + s^n$, where $s^k \in H^k(m(M), R)$.) We suppose without loss of generality that M coincides with T^*N provided with standard P-structure. Let us fix a volume element α in N. If ω is a differential form $\omega = \sum_{k=0}^n \omega^k$ in N (i.e. a function $\omega(x, \eta)$ on TN) we define a function $\tilde{\omega}(x, \xi)$ on T^*N by means of Fourier transformation (18). Let us assume that the n-dimensional component ω^n of the form ω does not vanish (i.e. $\omega^n = \beta(x)dx^1 \wedge \ldots \wedge dx^n$, where $\beta(x) \neq 0$). We will define the density function $\rho_\omega(x, \xi)$ on T^*N by the formula

$$\rho_\omega(x, \xi) = \alpha^2(x)\tilde{\omega}(x, \xi)^2 . \tag{33}$$

It is easy to check that $\rho_\omega(x, \xi)$ does not depend on the choice of α:

$$\rho_\omega(x, \xi) = (\int e^{\xi_i \eta^i} \omega(x, \eta) d^n \eta)^2 . \tag{34}$$

Theorem 5. *The density function $\rho_\omega(x, \xi) = \alpha^2(x)\tilde{\omega}(x, \xi)^2$ determines an SP-structure in T^*N if and only if the form ω is closed and its n-dimensional component ω^n does not vanish. Every SP-structure in T^*N can be described by means of density function of such a kind. If ρ_ω and $\rho_{\omega'}$ are density functions corresponding to closed forms ω and ω' then corresponding SP-structures are equivalent only in the case when the form $\omega' - \omega$ is exact.*

We say here that two SP-structures on T^*N are equivalent if there exists a P-transformation connecting these SP-structures and homotopic to the identity mapping. (As we have seen the transformation of the supermanifold T^*N is homotopic to identity if and only if corresponding transformation of N is homotopic to identity.)

Let us begin the proof with the remark that the application of Lemma 4 to $\tilde{\rho} = \rho_\omega$, $\rho_0 = \alpha^2$, $\tilde{\omega}(x, \xi) = e^{\sigma/2}$ shows that the operator Δ corresponding to the density ρ_ω satisfies $\Delta^2 = 0$ if and only if the form ω is closed. Therefore if ρ_ω determines an SP-structure then ω is closed. To prove that in the case of closed ω the density ρ_ω determines an SP-structure we will construct a family ω_t of closed forms: $\omega_t = (1 - t)\omega^n + t\omega$, corresponding densities $\rho_t = \rho_{\omega_t}$ and operators Δ_t defined by the formula

$$\Delta_t H = e^{-\sigma_t/2} \Delta_0(e^{\sigma_t/} H) = \tilde{\omega}_t^{-1} \cdot \Delta_0(\tilde{\omega}_t H) . \tag{35}$$

Here Δ_0 is constructed by means of the density ρ_0 corresponding to the form $\omega_t|_{t=0} = \omega^n$. It is clear that the density ρ_0 determines an SP-structure. To prove that ρ_ω also determines an SP-structure it is sufficient to check that at least locally we can transform ρ_ω into ρ_0 by means of P-transformation. To find such a P-transformation we construct at first an infinitesimal P-transformation (Hamiltonian vector field) transforming ρ_t into ρ_{t+dt}. To verify the existence of such a field we note that the change of density ρ_t by the infinitesimal transformation generated by the vector field K_t can be written as

$$\frac{\partial_r}{\partial z^a}(\rho_t K_t^a) = \rho_t \operatorname{div} K_t . \tag{36}$$

If K_t is a Hamiltonian vector field with Hamiltonian H_t we obtain

$$\dot{\rho}_t = 2\rho_t \Delta_t H_t \tag{37}$$

or equivalently

$$\dot{\sigma}_t = 2\Delta_t H_t = 2e^{-\sigma_t/2} \Delta_0(e^{\sigma_t/2} H_t) . \tag{38}$$

From the other side we obtain from Lemma 4 that $\Delta_0 e^{\sigma_t/2} = 0$. Differentiating this equation with respect to t we get

$$\Delta_0(\dot{\sigma}_t e^{\sigma_t/2}) = 0 . \tag{39}$$

It follows from (39) that (38) considered as an equation for H_t can be solved at least locally. (One should make the Fourier transformation (18) and use the Poincaré lemma). As usual to find the transformation connecting ρ_ω and ρ_0 we have to integrate the equation $\dot{z} = K_t(z)$.

In such a way we proved that the density (33) determines an SP-structure if ω is closed. The same arguments can be used to check that the $\rho_{\omega'}$ and ρ_ω determine equivalent SP-structures if $\omega' - \omega$ is exact (it follows from the exactness of $\omega' - \omega$ that Eq. (38) for H_t can be solved globally). To finish the proof of Theorem 5 we have to check that in the case when $\omega' - \omega$ is not exact the densities $\rho_{\omega'}$ and ρ_ω cannot determine equivalent SP-structures. Let us suppose that there exists a P-transformation Q connecting $\rho_{\omega'}$ and ρ_ω and homotopic to identity. One can conclude from Lemma 2 that in this case we can find a smooth family Q_t of P-transformations connecting $Q = Q_1$ with the identity map Q_0. Let us denote by ρ_t the density obtained from ρ_ω by means of Q_t; corresponding form will be denoted by ω_t. The density ρ_{t+dt} can be obtained from the density ρ_t by means of infinitesimal P-transformation (Hamiltonian vector field $K_t = \dot{Q}_t Q_t^{-1}$) and we can apply (38). It follows from (38) that the form $\dot{\omega}_t$ is exact, therefore the form $\omega' - \omega = \int_0^1 \dot{\omega}_t dt$ is exact too.

Acknowledegement. I am indebeted to A. Givental, M. Kontsevich, A. Weinstein and E. Witten for useful discussions.

Note added in proof. As I was informed, O. Khudaverdian and A. Nersessian also considered manifolds provided with an odd symplectic structure and a volume element. They proved some results of the present paper. In particular, they checked that in the case when the operator Δ satisfies $\Delta^2 = 0$ one can construct Darboux coordinates where Δ takes the standard form (7) and proved some statements of Lemma 4.

References

1. Batalin, I., Vilkovisky, G.: Gauge algebra and quantization. Phys. Lett. **102B**, 27 (1981)
2. Batalin, I., Vilkovisky, G.: Quantization of gauge theories with linearly dependent generators. Phys. Rev. **D29**, 2567 (1983)
3. Witten, E.: A note on the antibracket formalism. Mod. Phys. Lett. **A5**, 487 (1990)
4. Schwarz, A.: The partition function of a degenerate functional. Commun. Math. Phys. **67**, 1 (1979)
5. Berezin, F.: Introduction to algebra and analysis with anticommuting variables. Moscow Univ., 1983 (English translation is published by Reidel)

Communicated by N.Yu. Reshetikhin