

## Research Article

Yanlin Li, Santu Dey, Sampa Pahan, and Akram Ali\*

# Geometry of conformal $\eta$ -Ricci solitons and conformal $\eta$ -Ricci almost solitons on paracontact geometry

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**Abstract:** We prove that if an  $\eta$ -Einstein para-Kenmotsu manifold admits a conformal  $\eta$ -Ricci soliton then it is Einstein. Next, we proved that a para-Kenmotsu metric as a conformal  $\eta$ -Ricci soliton is Einstein if its potential vector field  $V$  is infinitesimal paracontact transformation or collinear with the Reeb vector field. Furthermore, we prove that if a para-Kenmotsu manifold admits a gradient conformal  $\eta$ -Ricci almost soliton and the Reeb vector field leaves the scalar curvature invariant then it is Einstein. We also construct an example of para-Kenmotsu manifold that admits conformal  $\eta$ -Ricci soliton and satisfy our results. We also have studied conformal  $\eta$ -Ricci soliton in three-dimensional para-cosymplectic manifolds.

**Keywords:** conformal  $\eta$ -Ricci soliton, Kenmotsu manifold, Einstein manifold, infinitesimal contact transformation, para-cosymplectic manifold

**MSC 2020:** 53C15, 53C25, 53D15

## 1 Introduction

In recent years, geometric flows, in particular, the Ricci flow, have been an interesting research topic in differential geometry. The concept of Ricci flow was first introduced by Hamilton and developed to answer Thurston's geometric conjecture. A Ricci soliton can be considered as a fixed point of Hamilton's Ricci flow (see details in [1]) and a natural generalization of the Einstein metric (i.e., the Ricci tensor  $[\text{Ric}]$  is a constant multiple of the pseudo-Riemannian metric  $g$ ), defined on a pseudo-Riemannian manifold  $(M, g)$  by

$$\frac{1}{2}\mathcal{L}_V g + \text{Ric} = \lambda g,$$

where  $\mathcal{L}_V$  denotes the Lie derivative in the direction of  $V \in \chi(M)$ ,  $\text{Ric}$  is the Ricci tensor of  $g$ , and  $\lambda$  is a constant. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as  $\lambda$  is negative, zero, and positive, respectively. Otherwise, it will be called indefinite. A Ricci soliton is trivial if  $V$  is either zero or Killing on  $M$ . First, Pigola et al. [2] assumed the soliton constant  $\lambda$  to be a smooth function on  $M$  and named as Ricci almost soliton. After that, Barros et al. studied Ricci almost soliton detailed in [3,4]. Recently, Cho and Kimura [5] generalized the notion of Ricci soliton to  $\eta$ -Ricci soliton, and Calin and Crasmareanu [6] studied this in Hopf hypersurfaces of complex space forms.

\* **Corresponding author: Akram Ali**, Department of Mathematics, College of Science, King Khalid University, 61421 Abha, Saudi Arabia, e-mail: akramali133@gmail.com

**Yanlin Li:** School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China, e-mail: liyl@hznu.edu.cn

**Santu Dey:** Department of Mathematics, Bidhan Chandra College, Asansol 713304, India, e-mail: santu.mathju@gmail.com

**Sampa Pahan:** Department of Mathematics, Mrinalini Datta Mahavidyalaya, Kolkata 700 051, India, e-mail: sampapahan25@gmail.com

A Riemannian or pseudo-Riemannian metric  $g$ , defined on a smooth manifold  $M^n$  of dimension  $n$ , is said to be an  $\eta$ -Ricci soliton if there exists a vector field  $V$  and constants  $\lambda, \mu$  such that

$$\frac{1}{2}\mathcal{L}_V g + \text{Ric} + \lambda g + \mu \eta \otimes \eta = 0. \quad (1.1)$$

If  $\lambda, \mu : M \rightarrow \mathbb{R}$  are smooth functions, then  $(M, g)$  is called  $\eta$ -Ricci almost soliton. If the potential vector field  $V$  is a gradient of a smooth function  $f$  on  $M$ , then the manifold is called a gradient  $\eta$ -Ricci almost soliton. In this case, equation (1.1) can be exhibited as

$$\text{Hess } f + \text{Ric} + \lambda g + \mu \eta \otimes \eta = 0, \quad (1.2)$$

where  $\text{Hess } f$  denotes the Hessian of  $f$ . The function  $f$  is known as the potential function.

In 2005, Fischer [7] has introduced conformal Ricci flow which is a mere generalization of the classical Ricci flow equation that modifies the unit volume constraint to a scalar curvature constraint. The conformal Ricci flow equation was given by

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg, \quad r(g) = -1,$$

where  $r(g)$  is the scalar curvature of the manifold,  $p$  is the scalar nondynamical field, and  $n$  is the dimension of the manifold. Corresponding to the conformal Ricci flow equation in 2015, Basu and Bhattacharyya [8] introduced the notion of conformal Ricci soliton equation as a generalization of Ricci soliton equation given by

$$\frac{1}{2}\mathcal{L}_V g + \text{Ric} + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]g = 0.$$

Recently, Siddiqi [9] established the definition of conformal  $\eta$ -Ricci soliton which generalizes the conformal Ricci soliton and  $\eta$ -Ricci soliton. The definition of conformal  $\eta$ -Ricci soliton is given by

$$\frac{1}{2}\mathcal{L}_V g + \text{Ric} + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]g + \mu \eta \otimes \eta = 0. \quad (1.3)$$

If the potential vector field  $V$  is a gradient of a smooth function  $f$  on  $M$ , then the manifold is called a gradient conformal  $\eta$ -Ricci almost soliton. In this case, equation (1.3) can be exhibited as

$$\text{Hess } f + \text{Ric} + \left[\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right]g + \mu \eta \otimes \eta = 0, \quad (1.4)$$

where  $\text{Hess } f$  denotes the Hessian of  $f$ . The function  $f$  is known as the potential function.

In the literature, many authors studied Ricci soliton and  $\eta$ -Ricci soliton in the framework of contact metric manifolds. For instance, Sharma [10] considered a  $K$ -contact and  $(\kappa, \mu)$ -contact metric as Ricci soliton; Cho and Sharma considered a contact metric as Ricci soliton [11];  $\eta$ -Einstein almost Kenmotsu metric as Ricci soliton by Wang and Liu [12]. Furthermore, Ghosh considered a noncompact almost contact metric, in particular, a Kenmotsu metric as Ricci soliton (see [13,14]). The interest in Ricci solitons and  $\eta$ -Ricci solitons has risen among theoretical physicists in relation with string theory and connection to general relativity and therefore these have been extensively studied in pseudo-Riemannian settings (see [15,16]). So, several authors studied Ricci soliton and  $\eta$ -Ricci soliton on paracontact metric manifolds, for instance, Patra et al. [17,16] considered a paracontact metric as a Ricci soliton and Naik et al. [18] considered a para-Sasakian metric as  $\eta$ -Ricci soliton. In [19], Welyczko introduced notion of para-Kenmotsu manifold, which is the analogous of Kenmotsu manifold [20] in paracontact geometry and studied in detail by Zamkovoy [21]. Furthermore, Balaga studied some aspects of  $\eta$ -Ricci solitons on para-Kenmotsu and Lorentzian para-Sasakian manifolds (see [22–24]).

Recently, Patra [16] considered Ricci soliton on para-Kenmotsu manifold and proved that a para-Kenmotsu metric as a Ricci soliton is Einstein if it is  $\eta$ -Einstein or the potential vector field  $V$  is infinitesimal paracontact transformation. Many authors studied Conformal Ricci solitons and their generalizations in the

framework of almost contact and paracontact geometries, e.g., Kenmotsu manifold by [8,25–27], Sasakian manifold by [28,29], generalized Sasakian space forms by [30],  $f$ -Kenmotsu manifold by [31,32], para-Kähler manifold by [33], and  $(\kappa, \mu)$ -Paracontact metric manifolds by Siddiqi [34]. Also, recently Venkatesha et al. [35] considered the metric of  $\eta$ -Einstein para-Kenmotsu manifold as  $*$ -Ricci soliton and proved that the manifold is Einstein. Erken [36] considered Yamabe solitons on three-dimensional para-cosymplectic manifold and proved some vital results like the manifold are either  $\eta$ -Einstein or Ricci flat. Motivated by these results we consider a para-Kenmotsu metric as conformal  $\eta$ -Ricci solitons and conformal  $\eta$ -Ricci almost solitons. Li and Ali et al. also did several relevant investigations [25,37–43]. Based on the aforementioned facts and discussions in the research of para-contact geometry, a natural **question** arises.

*Are there paracontact metric almost manifolds, whose metrics are conformal  $\eta$ -Ricci soliton?*

We will give the answer of the above question very affirmatively in the following sections. This paper is organized as follows. After collecting some of the basic definitions and formulas on para-Kenmotsu manifold in Section 2, we prove in Section 3 that a para-Kenmotsu metric as a conformal  $\eta$ -Ricci soliton is Einstein if it is  $\eta$ -Einstein or the potential vector field  $V$  is infinitesimal paracontact transformation or  $V$  is collinear with the Reeb vector field  $\xi$ . In Section 4, we consider conformal  $\eta$ -Ricci almost solitons on para-Kenmotsu manifold and find some  $\eta$ -Einstein and Einstein manifolds using conformal  $\eta$ -Ricci almost solitons. We draw an example of para-Kenmotsu manifold that admits conformal  $\eta$ -Ricci soliton. In the last section, we consider three-dimensional para-coisymplectic manifold as a conformal  $\eta$ -Ricci soliton and deduce some relations on the scalar curvature of the manifold.

## 2 Preliminaries

In this section, we give a brief review of several fundamental notions and formulas which we will need later on. We refer to [15,21,44,45] for more details as well as some examples. A  $(2n + 1)$ -dimensional smooth manifold  $M^{2n+1}$  has an almost paracontact structure  $(\varphi, \xi, \eta)$  if it admits a  $(1, 1)$ -tensor field  $\varphi$ , a vector field  $\xi$ , and a 1-form  $\eta$  satisfying the following conditions:

$$(i) \quad \varphi^2 = I - \eta \circ \xi, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1, \quad (2.1)$$

(ii) There exists a distribution  $\mathcal{D} : p \in M \rightarrow \mathcal{D}_p \subset T_p M : \mathcal{D}_p = \text{Ker}(\eta) = \{x \in T_p M : \eta(x) = 0\}$ , called paracontact distribution generated by  $\eta$ . If an almost paracontact manifold  $M^{2n+1}$  with a structure  $(\varphi, \xi, \eta)$  admits a pseudo-Riemannian metric  $g$  such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

for all  $X, Y \in \chi(M)$ , then we say that  $M$  has an almost paracontact metric structure and  $g$  is called compatible metric. The fundamental 2-form  $\Phi$  of an almost paracontact metric structure  $(\varphi, \xi, \eta, g)$  defined by  $\Phi(X, Y) = g(X, \varphi Y)$  for all vector fields  $X, Y$  on  $M$ . If  $\Phi = d\eta$ , then the manifold  $M^{2n+1}(\varphi, \xi, \eta, g)$  is called a paracontact metric manifold. In this case,  $\eta$  is a contact form, i.e.,  $\eta \wedge (d\eta)^n \neq 0$ ,  $\xi$  is its Reeb vector field, and  $M$  is a contact manifold (see [46]). An almost paracontact metric manifold is said to be para-Kenmotsu manifold (see [21]) if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad (2.2)$$

for any  $X, Y \in \chi(M)$ . On para-Kenmotsu manifold the following formulas hold [21]:

$$\nabla_X \xi = X - \eta(X)\xi, \quad (2.3)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.4)$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.5)$$

$$Q\xi = -2n\xi, \quad (2.6)$$

for all  $X, Y \in \chi(M)$ , where  $\nabla$ ,  $R$ , and  $Q$  denote, respectively, the Riemannian connection, the curvature tensor, and the Ricci operator of  $g$  associated with the Ricci tensor given by  $\text{Ric}(X, Y) = g(QX, Y)$  for all  $X, Y \in \chi(M)$ . Now, we prove the following lemma on para-Kenmotsu manifold. In [47], Sarkar et al. have proved the following lemma in different techniques. Here we have used some short techniques and different methods to prove the lemma.

**Lemma 2.1.** *On para-Kenmotsu manifold  $M^{2n+1}(\varphi, \xi, \eta, g)$  the following formulas hold for any  $X, Y \in \chi(M)$ ,*

$$(\nabla_X Q)\xi = -QX - 2nX, \quad (2.7)$$

$$(\nabla_\xi Q)X = -2QX - 4nX. \quad (2.8)$$

**Proof.** Using (2.3) and (2.6), we acquire

$$\begin{aligned} (\nabla_X Q)\xi &= \nabla_X Q\xi - Q\nabla_X \xi \\ &= \nabla_X(-2n\xi) - Q(X - \eta(X)\xi) \\ &= -2n(X - \eta(X)\xi) - QX + \eta(X)Q\xi \\ &= -2nX + \eta(X)2n\xi - QX - 2n\xi\eta(X) \\ &= -QX - 2nX, \end{aligned} \quad (2.9)$$

which gives proof of the first part of the lemma. Now differentiating (2.4) along  $Z$ , we obtain

$$(\nabla_Z R)(X, Y)\xi = -R(X, Y)Z + g(X, Z)Y - g(Y, Z)X. \quad (2.10)$$

Now, let  $\{e_i\}_{i=1}^{2n+1}$  be an orthonormal basis on  $M$ . Take  $X = Z = e_i$  in (2.10) and summing up over  $i$ , we achieve

$$\sum_{i=1}^{2n+1} g((\nabla_{e_i} R)(e_i, Y)\xi, W) = S(Y, W) + 2ng(Y, W). \quad (2.11)$$

From second Bianchi identity, we infer

$$\sum_{i=1}^{2n+1} g((\nabla_{e_i} R)(W, \xi)Y, e_i) = g((\nabla_W Q)\xi, Y) - g((\nabla_\xi Q)W, Y). \quad (2.12)$$

Putting equation (2.12) in (2.11), we obtain

$$\begin{aligned} g((\nabla_W Q)\xi, Y) - g((\nabla_\xi Q)W, Y) &= S(Y, W) + 2ng(Y, W), \\ g(-QW - 2nW, Y) - g((\nabla_\xi Q)W, Y) &= S(Y, W) + 2ng(Y, W) \text{ (using (2.7))}, \\ -S(Y, W) - S(W, Y) - 2ng(Y, W) - 2ng(Y, W) &= g((\nabla_\xi Q)W, Y), \\ (\nabla_\xi Q)W &= -2QW - 4nW, \end{aligned} \quad (2.13)$$

which gives our complete proof.  $\square$

### 3 On conformal $\eta$ -Ricci soliton

In this section, we study the conformal  $\eta$ -Ricci soliton on para-Kenmotsu manifold and find some important conditions so that a para-Kenmotsu metric as a conformal  $\eta$ -Ricci soliton is Einstein. First, we recall a definition: a contact metric manifold  $M^{2n+1}$  is said to be  $\eta$ -Einstein, if the Ricci tensor  $\text{Ric}$  can be written as

$$\text{Ric}(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y), \quad (3.1)$$

where  $\alpha, \beta$  are smooth functions on  $M$ . For an  $\eta$ -Einstein  $K$ -contact manifold (see Yano and Kon [48]) and para-Sasakian manifold [45] of dimension  $> 3$ , it is well known that the functions  $\alpha, \beta$  are constants, but for a  $\eta$ -Einstein para-Kenmotsu manifold this is not true. So, we continue  $\alpha, \beta$  as functions. In [13], Ghosh studied three-dimensional Kenmotsu metric as a Ricci soliton and for higher dimension in [14]. Recently, Patra [16]

considered Ricci soliton on para-Kenmotsu manifold and proved that an  $\eta$ -Einstein para-Kenmotsu metric as a Ricci soliton is Einstein and therefore here we consider  $\eta$ -Einstein para-Kenmotsu metric as a conformal  $\eta$ -Ricci soliton. Before obtaining our main results first we derive the following lemma.

**Lemma 3.1.** *Let  $M^{2n+1}(\varphi, \xi, \eta, g)$  be a para-Kenmotsu manifold. If  $g$  represents a conformal  $\eta$ -Ricci soliton with potential vector field  $V$ , then we have for any  $X \in \chi(M)$ ,*

$$(\mathcal{E}_V R)(X, \xi)\xi = 0. \quad (3.2)$$

**Proof.** Taking the covariant derivative of (1.3) along an arbitrary vector field  $Z$  on  $M$  and using (2.3) we obtain

$$(\nabla_Z \mathcal{E}_V g)(X, Y) = -2(\nabla_Z \text{Ric})(X, Y) - 2\mu\{g(X, Z)\eta(Y) + g(Y, Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)\}, \quad (3.3)$$

for any  $X, Y \in \chi(M)$ . Next, recalling the following commutation formula (see Yano [49], p. 23)

$$(\mathcal{E}_V \nabla_Z g - \nabla_Z \mathcal{E}_V g - \nabla_{[V, Z]} g)(X, Y) = -g((\mathcal{E}_V \nabla)(Z, X), Y) - g((\mathcal{E}_V \nabla)(Z, Y), X), \quad (3.4)$$

for all  $X, Y, Z \in \chi(M)$ . In view of the parallel Riemannian metric  $g$ , it follows from (2.6) that

$$(\nabla_Z \mathcal{E}_V g)(X, Y) = g((\mathcal{E}_V \nabla)(Z, X), Y) + g((\mathcal{E}_V \nabla)(Z, Y), X), \quad (3.5)$$

for all  $X, Y, Z \in \chi(M)$ . Plugging it into (3.3) we achieve

$$\begin{aligned} g((\mathcal{E}_V \nabla)(Z, X), Y) + g((\mathcal{E}_V \nabla)(Z, Y), X) &= -2(\nabla_Z \text{Ric})(X, Y) - 2\mu\{g(X, Z)\eta(Y) + g(Y, Z)\eta(X) \\ &\quad - 2\eta(X)\eta(Y)\eta(Z)\}, \end{aligned} \quad (3.6)$$

for any  $X, Y, Z \in \chi(M)$ . Interchanging cyclicly the roles of  $X, Y, Z$  in (3.6) we can compute

$$\begin{aligned} g((\mathcal{E}_V \nabla)(X, Y), Z) &= (\nabla_Z \text{Ric})(X, Y) - (\nabla_X \text{Ric})(Y, Z) - (\nabla_Y \text{Ric})(Z, X) \\ &\quad - 2\mu\{g(X, Y)\eta(Z) - \eta(X)\eta(Y)\eta(Z)\}, \end{aligned} \quad (3.7)$$

for all  $Y, Z \in \chi(M)$ . Now, substituting  $\xi$  for  $Y$  in (3.7) and using (2.8), (2.7) yields

$$(\mathcal{E}_V \nabla)(X, \xi) = 2QX + 4nX, \quad (3.8)$$

for all  $X \in \chi(M)$ . Next, using (2.3), (3.8) in the covariant derivative of (3.8) along  $Y$  provides

$$(\nabla_Y \mathcal{E}_V \nabla)(X, \xi) = (\mathcal{E}_V \nabla)(X, Y) - 2(\nabla_Y Q)X + 2\eta(Y)(QX + 2nX)$$

for any  $X \in \chi(M)$ . Using this in the following commutation formula (see Yano [49], p. 23)

$$(\mathcal{E}_V R)(X, Y)Z = (\nabla_X \mathcal{E}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{E}_V \nabla)(X, Z), \quad (3.9)$$

we can derive

$$(\mathcal{E}_V R)(X, Y)\xi = 2\{(\nabla_Y Q)X - (\nabla_X Q)Y\} + 2\{\eta(X)QY - \eta(Y)QX\} + 4n\{\eta(X)Y - \eta(Y)X\} \quad (3.10)$$

for all vector fields  $X, Y \in \chi(M)$ . Substituting  $Y$  by  $\xi$  in (3.10) and using (2.6), (2.8), and (2.7) we have the required result.  $\square$

**Theorem 3.1.** *Let  $M^{2n+1}(\varphi, \xi, \eta, g)$ ,  $n > 1$ , be a  $\eta$ -Einstein para-Kenmotsu manifold. If  $g$  represents a conformal  $\eta$ -Ricci soliton with potential vector field  $V$ , then  $g$  is Einstein with constant scalar curvature  $r = -2n(2n + 1)$ .*

**Proof.** First, tracing (3.1) gives  $r = (2n + 1)\alpha + \beta$  and putting  $X = Y = \xi$  in (3.1) and using (2.6) we obtain  $\alpha + \beta = -2n$ . Therefore, by computation, (3.1) transforms into

$$\text{Ric}(X, Y) = \left(1 + \frac{r}{2n}\right)g(X, Y) - \left\{(2n + 1) + \frac{r}{2n}\right\}\eta(X)\eta(Y), \quad (3.11)$$

for all  $X, Y$  on  $M$ . This gives

$$(\nabla_Y Q)X = \frac{(Yr)}{2n}\{X - \eta(X)\xi\} + \left\{(2n+1) + \frac{r}{2n}\right\}\{g(X, Y)\xi + \eta(X)(Y - 2\eta(Y)\xi)\}, \quad (3.12)$$

for all  $X, Y \in \chi(M)$ . By virtue of this, equation (3.10) provides

$$(\mathcal{E}_V R)(X, Y)\xi = \frac{1}{n}\{(Xr)(Y - \eta(Y)\xi) - (Yr)(X - \eta(X)\xi)\}, \quad (3.13)$$

for all  $X, Y \in \chi(M)$ . Setting  $Y = \xi$  in (3.13) and using Lemma 3.1 we obtain  $(\xi r)\varphi^2 X = 0$  for any  $X \in \chi(M)$ . Using this in the trace of (2.8) we obtain  $r = -2n(2n+1)$ . It follows from (3.11) that  $M$  is Einstein and hence the proof.  $\square$

Now, taking the Lie derivative of  $g(\xi, \xi) = 1$  along the potential vector field  $V$  and applying (1.3) one can obtain

$$\eta(\mathcal{E}_V \xi) = \lambda + \mu - 2n - \frac{1}{2}\left(p + \frac{2}{2n+1}\right). \quad (3.14)$$

Furthermore, from (2.3) we obtain  $R(X, \xi)\xi = -X + \eta(X)\xi$  and the Lie derivative of this along  $V$  yields

$$(\mathcal{E}_V R)(X, \xi)\xi + R(X, \mathcal{E}_V \xi)\xi + R(X, \xi)\mathcal{E}_V \xi = \{(\mathcal{E}_V \eta)X\}\xi + \eta(X)\mathcal{E}_V \xi \quad (3.15)$$

for any  $X \in \chi(M)$ . If  $g$  represents a conformal  $\eta$ -Ricci soliton with potential vector field  $V$ , then Lemma (3.1) holds, i.e.,  $(\mathcal{E}_V R)(X, \xi)\xi = 0$ . Plugging it into (3.15) and using (2.4) provides

$$(\mathcal{E}_V g)(X, \xi) + 2\eta(\mathcal{E}_V \xi)X = 0, \quad (3.16)$$

for any  $X \in \chi(M)$ . Again, applying (1.3) and (3.14) in (3.16) yields  $\left(2n - \lambda + \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - \mu\right)\varphi^2 X = 0$  for any  $X \in \chi(M)$ . Next using (2.1) and then tracing yields  $2n\left(2n - \lambda + \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - \mu\right) = 0$ . This implies that

$$\lambda + \mu = 2n + \frac{1}{2}\left(p + \frac{2}{2n+1}\right). \quad (3.17)$$

Next we consider a para-Kenmotsu metric as a conformal  $\eta$ -Ricci soliton with nonzero potential vector field  $V$  is collinear with  $\xi$  and prove the following result.

**Theorem 3.2.** *Let  $M^{2n+1}(\varphi, \xi, \eta, g)$  be a para-Kenmotsu manifold. If  $g$  represents a conformal  $\eta$ -Ricci soliton with nonzero potential vector field  $V$  is collinear with  $\xi$ , then  $g$  is Einstein with constant scalar curvature  $r = -2n(2n+1)$ .*

**Proof.** Since the potential vector field  $V$  is collinear with  $\xi$ , i.e.,  $V = v\xi$  for some smooth function  $v$  on  $M$ . Using (2.3) in the covariant derivative of  $V = v\xi$  along  $X$  yields

$$\nabla_X V = (Xv)\xi + v\{X - \eta(X)\xi\},$$

for any  $X \in \chi(M)$ . By virtue of this, the soliton equation (1.3) reduces to

$$2\text{Ric}(X, Y) + (Xv)\eta(Y) + (Yv)\eta(X) + 2\left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + v\right)g(X, Y) - 2(v - \mu)\eta(X)\eta(Y) = 0, \quad (3.18)$$

for all  $X, Y \in \chi(M)$ . Setting  $X = Y = \xi$  in (3.18) and using (2.6), (3.17) we obtain  $\xi v = 0$ . It follows from (3.18) that  $Xv = 0$ . Putting it into (3.18) provides

$$\text{Ric}(X, Y) = -\left(v + \lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right)g(X, Y) + (v - \mu)\eta(X)\eta(Y), \quad (3.19)$$

for all  $X, Y \in \chi(M)$ . This shows that  $M$  is  $\eta$ -Einstein and therefore from Theorem 3.1 we conclude that  $M$  is Einstein. Thus, from (3.18) we have  $v = \mu$  and therefore  $v + \lambda = 2n + \frac{1}{2}\left(p + \frac{2}{2n+1}\right)$  (follows from (3.17)). Hence, we have from (3.19) that  $\text{Ric} = -2ng$  and therefore  $r = -2n(2n+1)$ , as required. Hence the proof.  $\square$

**Remark 3.1.** In Theorem 3.2, we see that  $X\nu = 0$  for any  $X \in \chi(M)$  and therefore the smooth function  $\nu$  reduces to a constant and it equals to  $\mu$  and hence  $V = \mu\xi$ .

In particular, we can also say that if a para-Kenmotsu manifold admits a conformal  $\eta$ -Ricci soliton with the nonzero potential vector field  $V$  is  $\xi$ , then it is Einstein with constant scalar curvature  $r = -2n(2n + 1)$ .

On paracontact metric manifold  $M$ , a vector field  $X$  is said to be infinitesimal paracontact transformation if it preserves the paracontact form  $\eta$ , i.e., there exists a smooth function  $\rho$  on  $M$  that satisfies

$$\mathcal{L}_X\eta(Y) = \rho\eta(Y), \quad (3.20)$$

for any  $Y \in \chi(M)$  and if  $\rho = 0$  then  $X$  is said to be strict. Here we consider that a para-Kenmotsu metric as a conformal  $\eta$ -Ricci soliton with potential vector field  $V$  is infinitesimal paracontact transformation and prove the following.

**Theorem 3.3.** Let  $M^{2n+1}(\varphi, \xi, \eta, g)$ ,  $n > 1$ , be a para-Kenmotsu manifold. If  $g$  represents a conformal  $\eta$ -Ricci soliton with the potential vector field  $V$  is infinitesimal paracontact transformation, then  $V$  is strict and  $g$  is Einstein with constant scalar curvature  $r = -2n(2n + 1)$ .

**Proof.** First, recalling the well-known formula (see page no. 23 of [49]):

$$\mathcal{L}_V\nabla_X Y - \nabla_X \mathcal{L}_V Y - \nabla_{[V,X]} Y = (\mathcal{L}_V\nabla)(X, Y), \quad (3.21)$$

for all  $X, Y \in \chi(M)$ .

Using (3.15), (3.20), Lemma 3.1 in the Lie derivative of  $\eta(X) = g(X, \xi)$  along  $V$ , we acquire  $\mathcal{L}_V\xi = \rho\xi$ . Thus, equations (3.14) and (3.17) entail that  $\rho = 0$  and therefore  $\mathcal{L}_V\xi = 0$  and  $V$  is strict. Furthermore, equation (3.20) gives  $\mathcal{L}_V\eta = 0$ . So, we obtain that  $(\mathcal{L}_V\nabla)(X, \xi) = 0$  for any  $X \in \chi(M)$ . Thus, from (3.8) we conclude the rest part of this theorem. Hence the proof.  $\square$

## 4 On conformal $\eta$ -Ricci almost soliton

In this section, we consider conformal  $\eta$ -Ricci almost soliton on para-Kenmotsu manifold. It follows from (1.4) that conformal  $\eta$ -Ricci almost soliton is the generalization of conformal Ricci almost soliton because it involves two smooth functions  $\lambda$  and  $\mu$ . First, we consider gradient conformal  $\eta$ -Ricci almost soliton on para-Kenmotsu manifold in order to extend the result of gradient Ricci almost soliton on para-Kenmotsu manifold [16] taking into account of equations (1.3) and (1.4) hold for smooth functions  $\lambda, \mu$ . First we prove the following.

**Theorem 4.1.** Let  $M^{2n+1}(\varphi, \xi, \eta, g)$  be a para-Kenmotsu manifold. If  $M$  admits a gradient conformal  $\eta$ -Ricci almost soliton and the Reeb vector field  $\xi$  leaves the scalar curvature  $r$  invariant, then it is Einstein with constant scalar curvature  $-2n(2n + 1)$ .

**Proof.** Equation (1.4) can be exhibited as

$$\nabla_X Df + QX + \left\{ \lambda - \frac{1}{2} \left( p + \frac{2}{2n+1} \right) \right\} X + \mu\eta(X)\xi = 0, \quad (4.1)$$

for any  $X \in \chi(M)$ . Using this in  $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ , we can easily obtain the curvature tensor expression in the following form:

$$R(X, Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y + (Y\lambda)X - (X\lambda)Y + (Y\mu)\eta(X)\xi - (X\mu)\eta(Y)\xi + \mu\{\eta(Y)X - \eta(X)Y\} \quad (4.2)$$

for all  $X, Y \in \chi(M)$ . Taking contraction of (4.2) over  $X$  with respect to an orthonormal basis  $\{e_i : i = 1, 2, \dots, 2n + 1\}$ , we compute



$$\operatorname{Ric}(Y, Df) = - \sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)Y, e_i) + (Yr) + 2n(Y\lambda) + (Y\mu) + \{(-\xi\mu) + 2n\mu\}\eta(Y)$$

for any  $Y \in \chi(M)$ . Now, contracting Bianchi's second identity we have  $\sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)Y, e_i) = \frac{1}{2}(Yr)$ . Plugging it into the previous equation gives

$$\operatorname{Ric}(Y, Df) = \frac{1}{2}(Yr) + 2n(Y\lambda) + (Y\mu) + \{(-\xi\mu) + 2n\mu\}\eta(Y) \quad (4.3)$$

for any  $Y \in \chi(M)$ . Now, substituting  $\xi$  for  $Y$  in (4.2) and using Lemma 2.1 provides

$$R(X, \xi)Df = -QX - 2nX + (\xi\lambda)X - (X\lambda)\xi - (X\mu)\xi + (\xi\mu)\eta(X)\xi + \mu\varphi^2X \quad (4.4)$$

for any  $X \in \chi(M)$ . Next, taking inner product of (4.4) with  $\xi$  and using (2.6), we obtain  $g(R(X, \xi)Df, \xi) = \xi\left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + \mu\right)\eta(X) - X\left(\lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right) + \mu\right)$  for any  $X \in \chi(M)$ . By virtue of (2.5), the preceding equation reduces to

$$X(f + \lambda + \mu) = \xi(f + \lambda + \mu)\eta(X) \quad (4.5)$$

for any  $X \in \chi(M)$ . Furthermore, using (2.5) in (4.4), we infer

$$X(f + \lambda + \mu)\xi = -QX + \{\xi(f + \lambda) + \mu - 2n\}X + \{-\mu + (\xi\mu)\}\eta(X)\xi, \quad (4.6)$$

for any  $X \in \chi(M)$ . By virtue of (4.5), equation (4.6) reduces to

$$QX = \{-\mu + \xi(\lambda + f) - 2n\}X - \{\mu + \xi(\lambda + f)\}\eta(X)\xi, \quad (4.7)$$

for any  $X \in \chi(M)$ . Thus,  $M$  is  $\eta$ -Einstein. Substituting  $Y$  by  $Df$  in (2.5) and contracting we obtain  $\operatorname{Ric}(\xi, Df) = -2n(\xi f)$ . Plugging it into (4.3) yields  $(\xi r) + 4n\xi(f + \lambda) = -4n\mu$ . Using it in the trace of (2.8) we have  $\xi(\lambda + f) = -\mu + \left\{2n + 1 + \frac{r}{2n}\right\}$ . By virtue of this, equation (4.7) transforms into

$$QX = \left(1 + \frac{r}{2n}\right)X - \left\{(2n + 1) + \frac{r}{2n}\right\}\eta(X)\xi, \quad (4.8)$$

for any  $X \in \chi(M)$ . By hypothesis:  $\xi r = 0$  and therefore, the trace of (2.8) gives  $r = -2n(2n + 1)$ . It follows from (4.8) that  $QX = -2nX$ , as required. So, we complete the proof.  $\square$

Note that, Theorem 4.1 is a more general version, where  $\lambda, \mu$  are smooth functions on  $M$ , and therefore it also holds for gradient conformal  $\eta$ -Ricci soliton, where  $\lambda, \mu$  are constants.

**Corollary 4.1.** *Let  $M^{2n+1}(\varphi, \xi, \eta, g)$  be a para-Kenmotsu manifold. If  $M$  admits  $g$  represents a gradient conformal  $\eta$ -Ricci soliton and the Reeb vector field  $\xi$  leaves the scalar curvature  $r$  invariant, then  $M$  is Einstein with constant scalar curvature  $-2n(2n + 1)$ .*

Next, considering a para-Kenmotsu metric as a conformal  $\eta$ -Ricci almost soliton with the potential vector field  $V$  is pointwise collinear with the Reeb vector field  $\xi$ , we extend Theorem 4.1 from gradient conformal  $\eta$ -Ricci almost soliton to conformal  $\eta$ -Ricci almost soliton and prove the following.

**Theorem 4.2.** *Let  $M^{2n+1}(\varphi, \xi, \eta, g)$  be a para-Kenmotsu manifold. If  $M$  admits a conformal  $\eta$ -Ricci almost soliton with nonzero potential vector field  $V$  collinear with  $\xi$ , then  $g$  is  $\eta$ -Einstein. Moreover, if the Reeb vector field  $\xi$  leaves the scalar curvature  $r$  invariant, then  $g$  is Einstein with constant scalar curvature  $-2n(2n + 1)$ .*

**Proof.** By hypothesis:  $V = \sigma\xi$  for some smooth function  $\sigma$  on  $M$ . It follows that

$$(\mathcal{L}_V g)(X, Y) = (X\sigma)\eta(Y) + (Y\sigma)\eta(X) + 2\sigma\{g(X, Y) - \eta(X)\eta(Y)\},$$

for all  $X, Y \in \chi(M)$ . By virtue of this, the soliton equation (1.3) transforms into

$$2\operatorname{Ric}(X, Y) + (X\sigma)\eta(Y) + (Y\sigma)\eta(X) + 2\left(\sigma + \lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right)g(X, Y) = 2(\sigma - \mu)\eta(X)\eta(Y), \quad (4.9)$$



for all  $X, Y \in \chi(M)$ . Now, putting  $X = Y = \xi$  in (4.9) and using (2.6) yields  $\xi\sigma = 2n - \lambda + \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - \mu$ . Thus, equation (4.9) gives  $X\sigma = \left(2n - \lambda + \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - \sigma\right)\eta(X)$ . Using this in (4.9) entails that

$$\text{Ric}(X, Y) = -\left(\sigma + \lambda - \frac{1}{2}\left(p + \frac{2}{2n+1}\right)\right)g(X, Y) - (2n - \lambda + \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - \sigma)\eta(X)\eta(Y), \quad (4.10)$$

for all  $X, Y \in \chi(M)$ . Hence,  $M$  is  $\eta$ -Einstein. Moreover, if the Reeb vector field  $\xi$  leaves the scalar curvature  $r$  invariant, i.e.,  $\xi r = 0$ . Now, tracing (2.7) yields  $(\xi r) = -2\{r + 2n(2n+1)\}$  and therefore,  $r = -2n(2n+1)$ . Using this in the trace of (4.10) gives  $\lambda - \sigma = 2n + \frac{1}{2}\left(p + \frac{2}{2n+1}\right)$ . Thus, from (4.10) we have  $QX = -2nX$  and therefore  $M$  is Einstein. This completes the proof.  $\square$

If we consider  $V = \sigma\xi$  for some constant  $\sigma$  instead of a function, then (4.9) holds good and therefore inserting  $X = Y = \xi$  in (4.9) and using (2.6) gives  $\xi\sigma = 2n - \lambda + \frac{1}{2}\left(p + \frac{2}{2n+1}\right) - \mu$ . Using this in (4.9) yields  $\lambda - \sigma = 2n + \frac{1}{2}\left(p + \frac{2}{2n+1}\right)$ , where we have used  $\sigma$  is a constant. Thus, from (4.10) we can conclude the following corollary.

**Corollary 4.2.** *Let  $M^{2n+1}(\varphi, \xi, \eta, g)$  be a para-Kenmotsu manifold. If  $M$  admits a nontrivial conformal  $\eta$ -Ricci almost soliton with  $V = \sigma\xi$  for some constant  $\sigma$ , then it is Einstein with constant scalar curvature  $r = -2n(2n+1)$ .*

Now, we present an example of para-Kenmotsu manifold that admits a gradient conformal  $\eta$ -Ricci soliton.

**Example 4.1.** Let  $(x, y, z)$  be the standard coordinates in  $\mathbb{R}^3$  and  $M^3 = \{(x, y, z) \in \mathbb{R}^3\}$  be a three-dimensional manifold. Now, consider a orthonormal basis  $\{e_1, e_2, e_3\}$  of vector fields on  $M^3$ , where  $e_1 = \frac{\partial}{\partial x}$ ,  $e_2 = \frac{\partial}{\partial y}$ ,  $e_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ . Define  $(1, 1)$  tensor field  $\varphi$  as follows:

$$\varphi(e_2) = e_1, \quad \varphi(e_1) = e_2, \quad \varphi(e_3) = 0.$$

The pseudo-Riemannian metric is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and  $\eta(X) = g(X, e_3)$  for any  $X \in \chi(M)$ . Then  $\eta(e_3) = 1$ ,  $\varphi^2 = X - \eta(X)e_3$ , and  $g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y)$  for all  $X, Y \in \chi(M)$ . Thus,  $(\varphi, \xi, \eta, g)$  is an almost paracontact structure. The nonzero components of the Levi-Civita connection  $\nabla$  (using Koszul's formula) are

$$\nabla_{e_1}e_3 = e_1 \quad \nabla_{e_1}e_1 = -\nabla_{e_2}e_2 = -e_3 \quad \nabla_{e_2}e_3 = e_2. \quad (4.11)$$

By virtue of this we can verify (2.2) and therefore  $M^3(\varphi, \xi, \eta, g)$  is a para-Kenmotsu manifold. Using the well-known expression of curvature tensor  $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ , we now compute the following nonzero components

$$\begin{aligned} R(e_1, e_2)e_1 &= e_2, & R(e_1, e_2)e_2 &= e_1, & R(e_1, e_3)e_1 &= e_3, \\ R(e_1, e_3)e_3 &= -e_1, & R(e_2, e_3)e_2 &= -e_3, & R(e_2, e_3)e_3 &= -e_2. \end{aligned}$$

Using these, we compute the components of the Ricci tensor

$$\text{Ric}(e_i, e_i) = -2 \quad \text{for } i = 1, 3 \quad \text{Ric}(e_2, e_2) = 2.$$

Therefore, the Ricci tensor is given by  $\text{Ric} = -2g$  and the scalar curvature  $r = -6$ . Hence,  $(M^3, g)$  is Einstein with constant scalar curvature  $r = -2n(2n+1)$  for  $n = 1$ . Let us consider a potential vector field  $V = (x-1)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$  on  $M^3$ . Then using (4.11) we obtain

$$\frac{1}{2}(\mathcal{L}_V g)(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}, \quad (4.12)$$

for all  $X, Y \in \chi(M^3)$ . If we choose the potential function  $f(x, y, z) = \frac{(x-1)^2}{2} + \frac{(y-1)^2}{2} + z$ , then from  $\text{Ric} = -2g$  and (4.12) we can conclude that the metric  $g$  is a gradient conformal  $\eta$ -Ricci soliton with constants  $\lambda = \frac{p}{2} + \frac{4}{3}$  and  $\mu = 1$ .

## 5 Three-dimensional para-cosymplectic metric as conformal $\eta$ -Ricci soliton

Dacko [50] introduced the notion of para-cosymplectic manifold. The fundamental 2-form  $\Phi$  is defined on an almost paracontact metric manifold  $(M, \phi, \xi, \eta, g)$  by  $\Phi(X, Y) = g(X, \phi Y)$  for any vector fields  $X$  and  $Y$  on  $M$ . Clearly, the skew-symmetricness of the 2-form  $\Phi$  inherits from  $\phi$ .

An almost paracontact metric manifold is said to be almost para-cosymplectic if the forms  $\eta$  and  $\Phi$  are closed, i.e.,  $d\eta = 0$  and  $d\Phi = 0$ , respectively. In addition, if the normality of almost para-cosymplectic manifold is fulfilled, then it is called para-cosymplectic manifold. Equivalently, we can say that an almost paracontact metric manifold is para-cosymplectic if the forms  $\eta$  and  $\phi$  are parallel with respect to the corresponding Levi-Civita connection  $\nabla$  of the metric  $g$ , i.e.,  $\nabla\eta = 0$  and  $\nabla\Phi = 0$ , respectively. We recall some useful relations which are satisfied by any para-cosymplectic manifold

$$R(X, Y)\xi = 0, \quad (5.1)$$

$$(\nabla_X \phi) = 0, \quad (5.2)$$

$$\nabla_X \xi = 0, \quad (5.3)$$

$$S(X, \xi) = 0, \quad (5.4)$$

$$Q\xi = 0, \quad (5.5)$$

where  $X$  is the arbitrary vector field and  $R, \nabla, S$ , and  $Q$  are the usual notations. For three-dimensional case we have additional Riemannian curvature property

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y - \frac{r}{2}\{g(Y, Z)X - g(X, Z)Y\} \quad (5.6)$$

for arbitrary vector fields  $X, Y, Z$ . Using this result we deduce that three-dimensional para-cosymplectic manifold satisfies

$$S(X, Y) = \frac{r}{2}\{g(X, Y) - \eta(X)\eta(Y)\}, \quad (5.7)$$

$$QX = \frac{r}{2}\{X - \eta(X)\xi\}, \quad (5.8)$$

for any  $X, Y \in \chi(M)$ .

A vector field  $V$  is said to be conformal Killing vector field or simply conformal vector field if there is a smooth function  $\rho$  such that

$$\mathcal{L}_V g = 2\rho g. \quad (5.9)$$

$\rho$  is called the conformal coefficient. If we consider the conformal coefficient  $\rho$  to be zero, then the conformal vector field reduces to Killing vector field. Now we first prove some lemmas whose results are used to deduce our main result.

**Lemma 5.1.** (See [49]) *If a  $n$ -dimensional Riemannian manifold admits a conformal vector field  $V$ , then we obtain*

$$(\mathcal{L}_V S)(X, Y) = -(n-2)g(\nabla_X D\rho, Y) + (\Delta\rho)g(X, Y), \quad (5.10)$$

$$\mathcal{L}_V r = 2(n-1)\Delta\rho - 2\rho r, \quad (5.11)$$

for any vector fields  $X$  and  $Y$ , where  $D$  and  $\Delta$  denote the gradient and Laplacian operator of  $g$ , respectively.

**Lemma 5.2.** *If the metric  $g$  of a three-dimensional para-cosymplectic manifold represents a conformal  $\eta$ -Ricci soliton, then the following properties hold*

$$\eta(\mathcal{L}_V \xi) = \lambda - \frac{p}{2} - \frac{1}{3} + \mu, \quad (5.12)$$

$$(\mathcal{L}_V \eta)\xi = -\lambda + \frac{p}{2} + \frac{1}{3} - \mu. \quad (5.13)$$

**Proof.** As the vector field  $\xi$  is a unit vector field we have  $g(\xi, \xi) = 1$ . Taking Lie derivative of the previous relation w.r.t. vector field  $V$  we have  $(\mathcal{L}_V g)(\xi, \xi) + 2\eta(\mathcal{L}_V \xi) = 0$ . Using (1.3), (2.1), and (5.7), we achieve

$$\eta(\mathcal{L}_V \xi) = \lambda - \frac{p}{2} - \frac{1}{3}.$$

Taking Lie derivative of (2.1) along the direction of  $V$  and using (5.12), we arrive at

$$(\mathcal{L}_V \eta)\xi = -\lambda + \frac{p}{2} + \frac{1}{3}. \quad \square$$

**Lemma 5.3.** *For a three-dimensional para-cosymplectic manifold, we have*

$$\xi(r) = 0. \quad (5.14)$$

**Proof.** For proof, we refer to [36].

A vector field  $V$  on an  $n$ -dimensional semi-Riemannian manifold  $(M, g)$  is said to be conformal vector field if

$$\mathcal{L}_V g = 2\rho g, \quad (5.15)$$

where  $\rho$  is called the conformal coefficient.  $\square$

**Lemma 5.4.** [49] *On an  $n$ -dimensional semi-Riemannian manifold  $(M, g)$  endowed with conformal vector field  $V$ , we obtain*

$$(\mathcal{L}_V S)(X, Y) = -(n-2)g(\nabla_X D\rho, Y) + (\Delta\rho)g(X, Y), \quad (5.16)$$

$$\mathcal{L}_V r = -2\rho r + 2(n-1)(\Delta\rho). \quad (5.17)$$

Now, we prove the following lemma.

**Lemma 5.5.** *For a three dimensional para-cosymplectic manifold which admits a conformal  $\eta$ -Ricci soliton and satisfies  $p = 2\lambda + 2\mu - \frac{2}{3}$  has always harmonic scalar curvature, i.e.,  $\Delta r = 0$ .*

**Proof.** Equations (1.3) and (5.15) enable us to obtain

$$2\rho g + 2\text{Ric} + \left[ 2\lambda - \left( p + \frac{2}{3} \right) \right] g + 2\mu\eta \otimes \eta = 0. \quad (5.18)$$

Now, we contract the previous equation to yield

$$2r + 6p + 6\lambda - \left( p + \frac{2}{3} \right) 3 + 2\mu = 0. \quad (5.19)$$

Using (5.19), we achieve

$$\rho = \frac{-\left(r + 3\lambda - \frac{3}{2}\left(p + \frac{2}{3}\right) + \mu\right)}{3}.$$

We plug  $n = 3$  into identity (5.16) to infer

$$(\mathcal{E}_V S)(X, Y) = -g(\nabla_X D\rho, Y) + (\Delta\rho)g(X, Y), \quad (5.20)$$

$$D\rho = -\frac{Dr}{3}.$$

Also,

$$\Delta\rho = -\frac{\Delta r}{3}.$$

So,

$$(\mathcal{E}_V S)(X, Y) = \frac{1}{3}g(\nabla_X Dr, Y) - \frac{\Delta r}{3}g(X, Y). \quad (5.21)$$

Again,

$$\mathcal{E}_V r = -2\rho r + 4\Delta\rho = \frac{2r}{3}\left(r + 3\lambda - \frac{3}{2}\left(p + \frac{2}{3}\right) + \mu\right) - \frac{4}{3}\Delta r. \quad (5.22)$$

Now, from the identity (5.7), we arrive at

$$\begin{aligned} (\mathcal{E}_V S)(X, Y) &= \frac{\mathcal{E}_V r}{2}g(X, Y) - \frac{\mathcal{E}_V r}{2}\eta(X)\eta(Y) + \frac{r}{2}[(\mathcal{E}_V g)(X, Y) - (\mathcal{E}_V \eta)(X)\eta(Y) - (\mathcal{E}_V \eta)(X)\eta(Y)] \\ &= \frac{1}{2}\left[\frac{2r}{3}\left(r + 3\lambda - \frac{3}{2}\left(p + \frac{2}{3}\right) + \mu\right) - \frac{4}{3}\Delta r\right]g(X, Y) \\ &\quad + \frac{r}{2}\left[-2S(XY) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right]g(X, Y) - 2\mu\eta(X)\eta(Y)\right] \\ &\quad - \frac{1}{2}\left[\frac{2r}{3}\left(r + 3\lambda - \frac{3}{2}\left(p + \frac{2}{3}\right) + \mu\right) - \frac{4}{3}\Delta r\right]\eta(X)\eta(Y) \\ &\quad - \frac{r}{2}[(\mathcal{E}_V \eta)(X)\eta(Y) - (\mathcal{E}_V \eta)(X)\eta(Y)]. \end{aligned} \quad (5.23)$$

Now, we put the value of (5.21) in (5.23) to yield

$$\begin{aligned} &\frac{1}{3}g(\nabla_X Dr, Y) - \frac{\Delta r}{3}g(X, Y) \\ &= \frac{\mathcal{E}_V r}{2}g(X, Y) - \frac{\mathcal{E}_V r}{2}\eta(X)\eta(Y) + \frac{r}{2}[(\mathcal{E}_V g)(X, Y) - (\mathcal{E}_V \eta)(X)\eta(Y) - (\mathcal{E}_V \eta)(X)\eta(Y)] \\ &= \frac{1}{2}\left[\frac{2r}{3}\left(r + 3\lambda - \frac{3}{2}\left(p + \frac{2}{3}\right) + \mu\right) - \frac{4}{3}\Delta r\right]g(X, Y) \\ &\quad + \frac{r}{2}\left[-2S(XY) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right]g(X, Y) - 2\mu\eta(X)\eta(Y)\right] \\ &\quad - \frac{1}{2}\left[\frac{2r}{3}\left(r + 3\lambda - \frac{3}{2}\left(p + \frac{2}{3}\right) + \mu\right) - \frac{4}{3}\Delta r\right]\eta(X)\eta(Y) - \frac{r}{2}[(\mathcal{E}_V \eta)(X)\eta(Y) - (\mathcal{E}_V \eta)(X)\eta(Y)]. \end{aligned} \quad (5.24)$$

Now, we plug  $X = Y = \xi$  into the identity (5.24) and using the condition  $p = 2\lambda + 2\mu - \frac{2}{3}$  to arrive  $\Delta r = 0$ .  $\square$

**Theorem 5.1.** *If the semi-Riemannian metric of a three-dimensional para-cosymplectic manifold is a conformal  $\eta$ -Ricci soliton satisfying  $p = 2\lambda + 2\mu - \frac{2}{3}$ , then it has a constant scalar curvature.*

- (i) *If  $r \neq 0$ , then the manifold is an Einstein manifold.*  
(ii) *If  $r = 0$ , then the manifold is Ricci-flat.*

**Proof.** We have

$$g(\nabla_X Dr, \xi) = 0. \quad (5.25)$$

From equations (5.21) and (5.23), we obtain

$$\frac{1}{3}g(\nabla_X Dr, Y) - \frac{\Delta r}{3}g(X, Y) = \frac{r}{2} \left[ -\left(2\lambda - \left(p + \frac{2}{3}\right)\right) - 2\mu \right] - \frac{r}{2}[(\mathcal{E}_V \eta)(X) + \eta(X)(\mathcal{E}_V \eta)(\xi)]. \quad (5.26)$$

Now, we utilize Lemma 5.3, Lemma 5.4, (5.25) and using the given condition to find

$$\frac{r}{2}[(\mathcal{E}_V \eta)(X)] = 0. \quad (5.27)$$

As  $r \neq 0$ , we arrive  $(\mathcal{E}_V \eta)(X) = 0$ . Again

$$\begin{aligned} \frac{1}{3}\nabla_X Dr &= \frac{1}{2} \left[ \frac{2r}{3} \left( r + 3\lambda - \frac{3}{2} \left( p + \frac{2}{3} \right) + \mu \right) \right] X + \frac{r}{2} \left[ -2QX - \left[ 2\lambda - \left( p + \frac{2}{3} \right) \right] X - 2\mu\eta(X)\xi \right] \\ &\quad - \frac{1}{2} \left[ \frac{2r}{3} \left( r + 3\lambda - \frac{3}{2} \left( p + \frac{2}{3} \right) + \mu \right) \right] \eta(X)\xi. \end{aligned} \quad (5.28)$$

Now using (5.8), we infer

$$\begin{aligned} \nabla_X Dr &= r \left[ r + 3\lambda - \frac{3}{2} \left( p + \frac{2}{3} \right) + \mu \right] [X - \eta(X)\xi] + \frac{3}{2} r \left[ -r(X - \eta(X)\xi) - \left( 2\lambda - \left( p + \frac{2}{3} \right) \right) X - 2\mu\eta(X)\xi \right] \\ &= \left[ \frac{r}{3}(3r + 5\lambda - \mu) + \frac{r}{2}(-3r + 6\mu) \right] [X - \eta(X)\xi]. \end{aligned} \quad (5.29)$$

Now, we take covariant derivative in the previous equation with respect to  $Y$  and achieve

$$\begin{aligned} \nabla_Y \nabla_X Dr &= \left[ \frac{Yr}{3}(3r + 5\lambda - \mu) + \frac{r}{3}(Yr) + \frac{Yr}{2}(-3r + 6\mu) + \frac{r}{2}(-3Yr) \right] [X - \eta(X)\xi] \\ &\quad + \left[ \frac{r}{3}(3r + 5\lambda - \mu) + \frac{r}{2}(-3r + 6\mu) \right] [\nabla_Y X - \eta(\nabla_Y X)\xi]. \end{aligned} \quad (5.30)$$

Similarly, we have

$$\begin{aligned} \nabla_X \nabla_Y Dr &= \left[ \frac{Xr}{3}(3r + 5\lambda - \mu) + \frac{r}{3}(Xr) + \frac{Xr}{2}(-3r + 6\mu) + \frac{r}{2}(-3Yr) \right] [Y - \eta(Y)\xi] \\ &\quad + \left[ \frac{r}{3}(3r + 5\lambda - \mu) + \frac{r}{2}(-3r + 6\mu) \right] [\nabla_X Y - \eta(\nabla_X Y)\xi]. \end{aligned} \quad (5.31)$$

Also, we obtain

$$\nabla_{[X, Y]} Dr = \left[ \frac{r}{3}(3r + 5\lambda - \mu) + \frac{r}{2}(-3r + 6\mu) \right] [\nabla_X Y - \nabla_Y X - g(\nabla_X Y, \xi)\xi + g(\nabla_Y X, \xi)\xi]. \quad (5.32)$$

Now

$$\begin{aligned} g(R(X, Y)Dr, W) &= \left[ \frac{Xr}{3}(3r + 5\lambda - \mu) + \frac{r}{3}(Xr) + \frac{Xr}{2}(-3r + 6\mu) - 3(Xr)\frac{r}{2} \right] [g(Y, W) - \eta(Y)\eta(W)] \\ &\quad - \left[ \frac{Yr}{3}(3r + 5\lambda - \mu) + \frac{r}{3}(Yr) + \frac{Yr}{2}(-3r + 6\mu) + \frac{r}{2}(-3Yr) \right] [g(X, W) - \eta(X)\eta(W)]. \end{aligned} \quad (5.33)$$

Now, we set  $X = W = e_i$  in the previous equation to yield

$$S(Y, Dr) = -\left[\frac{Yr}{3}(3r + 5\lambda - \mu) + \frac{r}{3}(Yr) + \frac{Yr}{2}(-3r + 6\mu) - 3(Yr)\left(\frac{r}{2} + 1\right)\right]. \quad (5.34)$$

So, it can be written as

$$S(Y, Dr) = kg(Y, Dr),$$

where  $k$  is a scalar function.

Hence, if  $r \neq 0$ , then the manifold is an Einstein manifold, and if  $r = 0$ , then the manifold is Ricci-flat.  $\square$

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