GEOMETRY OF K-TRIVIAL MOISHEZON MANIFOLDS : DECOMPOSITION THEOREM AND HOLOMORPHIC GEOMETRIC STRUCTURES

INDRANIL BISWAS, JUNYAN CAO, SORIN DUMITRESCU, AND HENRI GUENANCIA

Abstract. Let X be a compact complex manifold such that its canonical bundle K_X is numerically trivial. Assume additionally that X is Moishezon or X is Fujiki with dimension at most four. Using the MMP and classical results in foliation theory, we prove a Beauville-Bogomolov type decomposition theorem for X. We deduce that holomorphic geometric structures of affine type on X are in fact locally homogeneous away from an analytic subset of complex codimension at least two, and that they cannot be rigid unless X is an étale quotient of a compact complex torus. Moreover, we establish a characterization of torus quotients using the vanishing of the first two Chern classes which is valid for any compact complex n-folds of algebraic dimension at least $n-1$. Finally, we show that a compact complex manifold with trivial canonical bundle bearing a rigid geometric structure must have infinite fundamental group if either X is Fujiki, X is a threefold, or X is of algebraic dimension at most one.

Contents

1. Introduction

The topic of geometric structures on manifolds, especially the automorphism groups of such structures, is classical. The fundamental works of several leading mathematicians, such as C. F. Gauss, B. Riemann, F. Klein, S. Lie and E. Cartan, created the foundation of this field. In particular, E. Cartan introduced and studied what is now known as Cartan geometry. These are geometric structures infinitesimally modelled on homogeneous spaces

2010 *Mathematics Subject Classification.* 53B35, 32M05, 53A55.
Key words and phrases. Rigid geometric structure; algebraic reduction; principal torus bundle; Moishezon manifold.

[Sh]. These geometric structures are flat when they are actually locally modelled (not just infinitesimally) on homogeneous spaces [Sh].

Important new results pertaining to the partition of a geometric manifold into orbits of local automorphisms of the geometric structure were obtained by Gromov in [Gro] (see also the elegant expository work [DG]). In [Gro] Gromov introduced the *rigid geometric structures* (see Definition 2.1) as a broad class of geometric structure for which a (local) automorphism is completely determined by its finite order jet at any given point. Affine and projective connections on the tangent bundle, pseudo-Riemannian metrics and conformal structures in dimension ≥ 3 are important examples of rigid geometric structures (they are also examples of Cartan geometries). On the other hand, symplectic structures and foliations are not rigid.

Earlier works, [BD1, BD2, BD3, Du1, Du2, BDG], which were inspired by [Gro, DG], aimed to adapt Gromov's ideas and arguments to holomorphic geometric structures on compact complex manifolds. In that vein, the third-named author proved the following theorem:

Theorem 1.1 ([Du2])**.** *Let X be a compact Kähler manifold X with trivial first Chern class bearing a holomorphic geometric structure ϕ of affine type. Then*

(i) ϕ is locally homogeneous.
(ii) If ϕ is rigid, then X is covered by a compact complex torus.

It is important to keep in mind that the proof of the first statement relies in an essential way on the Bochner principle, while the second statement uses the Beauville–Bogomolov decomposition theorem [Bea, Bo].

1.1. **A Beauville-Bogomolov decomposition theorem for Moishezon manifolds.** The goal of the present paper is to generalize Theorem 1.1 to compact complex manifolds with trivial first Chern class that are not necessarily Kähler. A distinguished class of such manifolds is provided by compact Fujiki manifolds; see Section 3.2 for explicit non-Kähler examples. At the moment, there is no Bochner principle available in full generality, and a structural result in the spirit of the Beauville-Bogomolov theorem is unknown as well (see Conjecture 3.5). In short, it is expected that compact Fujiki manifolds with trivial canonical bundle are made up from irreducible Kähler varieties (i.e. Calabi-Yau or symplectic holomorphic) with at most terminal singularities using

- small modifications,
- products,
- finite étale quotients.

Our first result gives a partial answer to the above expectation.

Theorem A. *Let X be a compact Fujiki manifold such that $c_1(X) = 0 \in \mathrm{H}^2(X, \mathbb{R})$. Assume that one of the following holds:*

- *X is Moishezon, or*
- $\dim X \leq 4$.

Then, there exists a finite étale cover $X' \longrightarrow X$ and a decomposition

$$X' \simeq T \times \prod_{i \in I} Y_i \times \prod_{j \in J} Z_j$$

where T is a compact complex torus, the Y_i's are irreducible Calabi-Yau manifolds and the Z_j's are irreducible holomorphic symplectic manifolds.

Moreover, each factor Y_i (respectively, Z_j) in the decomposition is bimeromorphic to a Kähler variety with terminal singularities which is irreducible Calabi-Yau (respectively, irreducible holomorphic symplectic).

We refer to Definition 3.4 and the remarks below it for the definitions of irreducible Calabi-Yau manifolds and irreducible holomorphic symplectic manifolds, which mimic the definition in the singular Kähler case provided in e.g. [GGK, CGGN] and coincide with the usual definitions in the smooth Kähler case.

From the second part in the statement of Theorem A and the properties Kähler ICY and IHS varieties [GGK, CGGN], we deduce

- A Bochner principle for holomorphic tensors on X,
- A polystability result for TX with respect to some movable classes,
- Finiteness results for the linear part of the fundamental group of X,

see Theorem 3.6 and Corollary 3.7.

In the Moishezon case, we provide alternative purely algebraic arguments to study the semistability of the tangent bundle, cf Section 3.1 and Proposition 3.3.

An important application of the Bochner principle is provided by the following partial generalization of the first item of Theorem 1.1 in the case of Moishezon manifolds or Fujiki manifolds with dimension at most four.

Corollary B. *Let X be as in Theorem A. Then there exists a Zariski open subset $U \subset X$, whose complement has complex codimension at least two, such that any holomorphic geometric structure ϕ of affine type on X is locally homogeneous on U.*

As an easy application, we show that if a compact Fujiki manifold X with trivial first Chern class bears a rigid holomorphic geometric structure, then $\pi_1(X)$ is infinite (see Theorem 4.5).

A few words on the proof of Theorem A.

First, X admits a projective/Kähler bimeromorphic model which itself admits a singular minimal model $X_{\min}$ in that same category by results in the Minimal Model Program ([BCHM] and [Dr] in the projective case, and [HP1] and [DHP] in the Kähler case in dimension at most four). Next, an easy application of the negativity lemma shows that $X_{\min}$ has torsion canonical bundle, so that one can apply to $X_{\min}$ the decomposition theorem proved in [HP2] and [BGL] in the projective and Kähler case respectively. From there, one can show that the polystable decomposition of TX induces regular foliations with compact leaves. Using

Reeb stability theorem and the Barlet space of cycles on X, one can then obtain the product structure on X.

If one could prove that a compact Kähler manifold with zero numerical dimension has a minimal model, then Theorem A would be valid for any compact Fujiki manifold with vanishing first Chern class.

1.2. **Uniformization by compact complex tori.** Invoking Theorem A, we give several characterizations of compact complex manifolds covered by a compact complex torus, in part in the spirit of the second item in Theorem 1.1.

Theorem C. *Let X be a compact complex manifold of dimension n such that $c_1(X) = 0 \in \mathrm{H}^2(X, \mathbb{R})$, and denote by $a(X)$ the algebraic dimension of X. Assume that one of the following holds:*

- *$a(X) \geq n-1$ and $c_2(X) = 0 \in H^4(X, \mathbb{R})$, or*
- *X is either Moishezon or Fujiki of $\dim X \leq 4$, and X bears a rigid holomorphic geometric structure of affine type.*

Then there exists a finite étale cover $T \longrightarrow X$ where T is a complex torus.

Let us comment on each case individually.

• If X is Kähler, then this uniformization result is a classical consequence of Yau's solution of the Calabi conjecture [Ya]. This problem has also recently attracted a lot of attention in the singular projective or Kähler setting (see [GKP16] or [CGG]).

In the case where X is Moishezon, then the conclusion of Theorem C follows from the polystability of TX proved in Theorem 3.6 coupled with the results of Demailly-Peternell-Schneider [DPS] on hermitian flat manifolds (see Corollary 3.8). It should be mentioned again that Theorem 3.6 is deeply connected to the recent progress on our understanding of singular Kähler varieties with zero first Chern class [HP2, BGL].

• The case where $a(X) = n-1$ is treated in Corollary 3.13. An important ingredient of the proof is a result of Lin [Lin] which shows that X can be deformed to a Moishezon manifold. In order to reduce our situation to the Moishezon case previously established, we need to prove that the Albanese map of X is étale trivial, which we show to hold for any Fujiki manifold with trivial first Chern class (see Theorem 3.11).

It should be mentioned that while compact Kähler manifolds with numerically trivial canonical bundle are known to admit algebraic approximation [Ca], it is not known whether all Fujiki class $\mathcal{C}$ manifolds with numerically trivial canonical bundle admit algebraic approximations.

• Finally, the case where X is Moishezon or Fujiki of dimension at most four is showed in Theorem 4.1, and partially generalizes the second item in Theorem 1.1. The assumptions are used in order to get a decomposition theorem for X (see Theorem 3.6). A crucial observation is that given a rigid structure on a manifold Y — which need not be simply connected — one can globally extend local Killing vector fields as long as any linear representation $\pi_1(Y) \longrightarrow \mathrm{GL}(N, \mathbb{C})$ is trivial (see Remark 2.5). This observation allows us to apply the finiteness result established in [GGK] for linear representations of the fundamental group of minimal models with vanishing augmented irregularity, see Corollary 3.7.

1.3. **Rigid holomorphic structures and fundamental groups.** Our last result shows that in many instances, compact complex manifolds with trivial canonical bundle bearing a rigid holomorphic geometric structure have infinite fundamental group. The following result is a combination of Corollary 4.5 and Corollary 5.2.

Theorem D. *Let X be a compact complex manifold X with trivial canonical bundle K_X bearing a rigid holomorphic geometric structure of affine type. Assume that X satisfies one of the following assumptions:*

- *X is a Fujiki manifold, or*
- *$\dim X = 3$, or*
- *the algebraic dimension of X is at most one.*

Then the fundamental group of X is infinite.

Theorem D also holds for the holomorphic projective connections, and also for the holomorphic conformal structures, even though these two geometric structures are not of affine type (see Definition 2.1). This is because on manifolds with trivial canonical bundle, these two geometric structures lift to global representatives which are of affine type, namely, a holomorphic affine connection and a holomorphic Riemannian metric respectively (see Proposition 5.3). It may be mentioned that the particular case of holomorphic Riemannian metrics was settled earlier in [BD3]; however, the proof in [BD3] is of very specific nature and it works only in that particular context.

On the other hand, Theorem D does not hold in general for non-affine geometric structures. Indeed, according to Definition 2.1, holomorphic embeddings of a compact complex manifold in complex projective space $\mathbb{CP}^N$ are rigid holomorphic geometric structures (of order 0). So simply connected complex projective manifolds admit rigid holomorphic geometric structures (of order 0) of non-affine type.

To give a more illuminating example of a rigid geometric structure of non-affine type which is not locally homogeneous, recall that a compact complex torus $T^n = \mathbb{C}^n/\Lambda$ (here Λ is a cocompact lattice in $\mathbb{C}^n$), which admits a nontrivial holomorphic map to an elliptic curve, can be endowed with a holomorphic foliation that is not translation invariant [Gh1]. This combined with the standard holomorphic parallelization of the holomorphic tangent bundle of T^n produces a holomorphic *rigid* geometric structure of non-affine type in the sense of Gromov (see Definition 2.1 and Definition 2.2) which is not locally homogeneous.

Also rational homogeneous manifolds X are simply connected and admit holomorphic rigid geometric structures of affine type. Indeed, a basis of $\mathrm{H}^0(X, TX)$ defines such a structure on X (see Definition 2.2 and Remark 4.3). In this case TX is globally generated and K_X is not trivial.

The strategy of the proof of Theorem D is the following.

• In the Fujiki case (Corollary 4.5), this is a consequence of Theorem A (see second remark below the theorem) since the case of non-maximal algebraic dimension had been treated before in [BD2]. As an interesting consequence, we establish Corollary 4.6 asserting that on a compact complex manifold with trivial canonical bundle, any holomorphic rigid geometric structure of affine type ϕ admits non-zero locally Killing vector fields. Such vector fields can be chosen to be global if X is simply connected.

• To address the non-Fujiki case, the key result that we prove is Theorem 5.1, which asserts that the automorphism group of (X, ϕ) contains a maximal abelian Lie subgroup A whose orbits in X are closed and coincide with the fibers of a holomorphic submersion $\pi : X \longrightarrow B$ over a compact simply connected Moishezon manifold with globally generated canonical bundle K_B. Moreover, the fibers of π are compact complex tori and the family π is *not* isotrivial (or equivalently, K_B is not holomorphically trivial and A is non-compact).

The fibrations constructed in Theorem 5.1 cannot exist if X is a compact Kähler Calabi-Yau manifold and we conjecture that they should neither exist in the broader context when M is a compact simply connected complex manifold with trivial canonical bundle (see Remark 5.4).

2. Holomorphic rigid geometric structures

2.1. **Definitions and examples.** In this section we recall the context, definitions as well as some basics about the *rigid geometric structures* in the sense of Gromov [DG, Gro].

To fix notation, consider a complex manifold X of (complex) dimension n. Given any integer $r \geq 0$, associate to it the principal bundle of r–frames $R^r(X) \longrightarrow X$, which is the bundle of r–jets of local holomorphic coordinates on X (i.e., r–jets of *local* biholomorphisms from $\mathbb{C}^n$ to X). It is a holomorphic principal bundle over X with structure group $D^r(\mathbb{C}^n)$ (or simply D^r) which is the group of r–jets, at origin, of local biholomorphisms of $\mathbb{C}^n$ fixing the origin. Notice that D^r is a complex affine algebraic group. Let us now recall a basic definition from [DG, Gro].

Definition 2.1. A *holomorphic geometric structure* (of order r) on the complex manifold X is a holomorphic D^r–equivariant map $\phi : R^r(X) \longrightarrow Z$, with Z being a complex algebraic variety endowed with an algebraic action of the above group D^r. The geometric structure ϕ is said of *affine type* if the complex variety Z is actually affine.

To give examples of holomorphic geometric structures, holomorphic maps from X to a complex algebraic variety Z are evidently holomorphic structures of order 0; they are of affine type if Z is affine. Holomorphic tensors on X are holomorphic geometric structures on X of affine type of order one. Holomorphic affine connections on the holomorphic tangent bundle TX are holomorphic geometric structures of affine type on X of order two [Gro, DG]. Holomorphic fields of planes, holomorphic flags, holomorphic foliations, holomorphic projective connections and holomorphic conformal structures are all holomorphic geometric structures of non-affine type.

A local biholomorphism $f : U \longrightarrow V$ between two open subsets U and V of X is called a *local automorphism* (or *local isometry*) with respect to a holomorphic geometric structure ϕ on X of order r if its natural lift to a map between the corresponding frame bundles

$$f^{(r)} : R^r(U) \longrightarrow R^r(V)$$

takes each fiber of ϕ to a fiber of ϕ. This is the natural notion of local symmetry which coincides with the usual one in each of the examples of geometric structures.

The natural notion of a linearized symmetry is the following.

A (local) holomorphic vector field defined on an open subset $U \subset X$ is called a *Killing vector field*, with respect to ϕ, if its local flow acts on U through local automorphisms (or local isometries).

The group $\mathrm{Aut}(X, \phi)$ of all global automorphisms (isometries) of (X, ϕ) is a complex Lie subgroup of the group $\mathrm{Aut}(X)$ of biholomorphisms of X. Its unique maximal connected subgroup $\mathrm{Aut}_0(X, \phi)$ is a complex Lie subgroup of the unique maximal connected subgroup $\mathrm{Aut}_0(X)$ of the group of biholomorphisms of X. The Lie algebra of $\mathrm{Aut}_0(X, \phi)$ is the finite dimensional Lie algebra consisting of all globally defined holomorphic Killing vector fields with respect to ϕ.

Let s be a nonnegative integer. The s–jet of the geometric structure ϕ of order r is the geometric structure of order $(r + s)$ on X defined by the map

$$\phi^{(s)} : R^{r+s}(X) \longrightarrow Z^{(s)} \tag{2.1}$$

given by ϕ, where $Z^{(s)}$ is the variety of s–jets of holomorphic maps from $\mathbb{C}^n$ to Z. We note that $Z^{(s)}$ is naturally endowed with an algebraic action of D^{r+s} by pre-composition [Ben, DG, Gro]; recall that $R^{r+s}(X)$ is a principal D^{r+s}–bundle over X. For these actions of D^{r+s} on $Z^{(s)}$ and $R^{r+s}(X)$, the above map $\phi^{(s)}$ is D^{r+s}–equivariant.

The $(r + s)$–jet of a local biholomorphism of X is called an *isometric jet* of order s (with respect to ϕ) if its canonical lift to $R^{r+s}(X)$ takes any fiber of the map $\phi^{(s)}$ in (2.1) to some fiber of $\phi^{(s)}$. This is the natural definition of an isometry of order s [Ben, DG, Gro].

Definition 2.2. A holomorphic geometric structure ϕ is called *rigid* of order l (in the sense of Gromov) if any $(l + 1)$–isometric jet of ϕ is uniquely determined by its underlying l–jet. In other words, the forgetful map from $(l + 1)$–isometric jets l jets is injective.

Holomorphic affine connections are rigid of order one in the sense of Gromov (see [Ben, DG, Gro]). This is because any local biholomorphism, that fixes a point and preserves a connection, is actually linearized in local exponential coordinates around the fixed point, which means that such local biholomorphisms are completely determined just by their differential at the fixed point.

Holomorphic Riemannian metrics are rigid holomorphic geometric structures of order one. Holomorphic conformal structures for dimension at least three, as well as holomorphic projective connections, are rigid holomorphic geometric structures of order two. The holomorphic symplectic structures, and the holomorphic foliations, are examples of non-rigid geometric structures [Ben, DG, Gro].

The orbits, in X, of local isometries of ϕ are locally closed, and moreover the holomorphic tangent space to a given orbit space at any point of the orbit is spanned by the local holomorphic Killing vector fields [Ben, DG, Gro]. The sheaf of local Killing vector fields of a rigid holomorphic geometric structure is locally constant. Its stalk at any point is a finite dimensional Lie algebra which is known as the *Killing algebra* of ϕ.

2.2. **Two important results.** In this section, we would like to recall two results about holomorphic geometric structures that we will use extensively in this article.

To state the next result, recall that a complex manifold X satisfies the Bochner principle if any holomorphic tensor field on X vanishing at one point is vanishing identically. For instance, a compact Kähler manifold X with $c_1(X) = 0$ satisfies the Bochner principle, but other classes of examples exist. The following result was proved by the third-named author in [Du2, Lemma 3.2]:

Lemma 2.3 ([Du2])**.** *Let X be a complex manifold satisfying the Bochner principle. Then any holomorphic geometric structure of algebraic affine type on X is locally homogeneous.*

The next result is an extendibility type result for local Killing fields relative to a rigid geometric structure on a simply connected manifold. It was first proved by Nomizu, [No], in the case Killing vector fields for real analytic Riemannian metrics and then extended to any rigid geometric structure by [Am, DG, Gro].

Theorem 2.4 ([No, Am, DG, Gro])**.** *Let X be a complex manifold bearing a rigid holomorphic geometric structure ϕ and let ξ be a local Killing field for ϕ. If X is simply connected, then ξ extends to a global holomorphic vector field on X.*

Remark 2.5. The arguments in the proof of Theorem 2.4 actually show that one can replace the assumption that $\pi_1(X)$ is trivial by the weaker property that any complex linear representation $\pi_1(X) \longrightarrow \mathrm{GL}(n, \mathbb{C})$ is trivial, where $n = \dim X$.

2.3. **Orbits of isometries and algebraic reduction.** Recall that the *algebraic dimension $a(X)$* of a compact complex manifold X is the degree of transcendence over $\mathbb{C}$ of the field of meromorphic functions $\mathcal{M}(X)$ on X. It is known that $a(X) \in \{0, \ldots, \dim X\}$ [Ue]. By definition, two bimeromorphic compact complex manifolds have the same algebraic dimension.

Compact complex manifolds X with maximal algebraic dimension ($a(X) = \dim X$) are called Moishezon manifolds [Ue, p. 26, Definition 3.5]. It is the class of manifolds for which the meromorphic functions separate points in general position. Moishezon studied them in

[Mo] and proved that each of them is birational to some smooth complex projective manifold [Mo], [Ue, p. 26, Theorem 3.6].

More generally, a compact complex manifold is said to be in *Fujiki class* $\mathcal{C}$ (or a *Fujiki manifold* for short) if it is the meromorphic image of a compact Kähler manifold. A basic result of Varouchas says that a compact complex manifold belongs to Fujiki class $\mathcal{C}$ if and only if it is bimeromorphic to a compact Kähler manifold [Va, Section IV.3]. Manifolds lying in Fujiki class $\mathcal{C}$ share many of the features of compact Kähler manifolds.

The following classical result is known as the *algebraic reduction* theorem.

Theorem 2.6 ([Ue, p. 25, Definition 3.3], [Ue, p. 26, Proposition 3.4])**.** *Let X be a compact connected complex manifold of dimension n and algebraic dimension $a(X) = n - d$. Then there exists a bi-meromorphic modification*

$$\Psi : \widetilde{X} \longrightarrow X$$

and a holomorphic surjective map

$$t : \widetilde{X} \longrightarrow V ,$$

with connected fibers, where V is a $(n-d)$–dimensional projective manifold, such that

$$t^*(\mathcal{M}(V)) = \Psi^*(\mathcal{M}(X)) \tag{2.2}$$

as subspaces of $\mathcal{M}(\widetilde{X})$.

Consider the meromorphic fibration

$$\pi_{red} : X \longrightarrow V \tag{2.3}$$

given by $t \circ \Psi^{-1}$ in Theorem 2.6; it is called the algebraic reduction of X. If the algebraic dimension of X is zero, then the target of this algebraic reduction is a point. Note that for manifolds with maximal algebraic dimension Theorem 2.6 is equivalent to the earlier mentioned theorem of Moishezon.

Theorem 2.1 in [Du1] asserts that the fibers of the algebraic reduction of X are contained in the orbits of local isometries of ϕ (see also Theorem 3 in [Du3]). In the special case where X is simply connected, the following result is a direct consequence of Theorem 2.1 in [Du1].

Theorem 2.7. *Let X be a compact complex simply connected manifold equipped with a holomorphic rigid geometric structure ϕ. Then there exists*

- *an open dense subset $U \subset X$ whose complement $X \setminus U$ is an analytic subspace of X and π_{red} (see (2.3)) is defined on U, and*
- *a connected complex abelian Lie subgroup A of the automorphisms group $\mathrm{Aut}(X, \phi)$ of (X, ϕ),*

such that the fiber of π_{red} through any $z \in U$ is contained in the A–orbit of z. Moreover, A is a maximal connected abelian subgroup in $\mathrm{Aut}(X, \phi)$. Furthermore, A coincides with the identity component of the automorphism group $\mathrm{Aut}(X, \phi')$ of the rigid geometric structure ϕ' which is constructed as the juxtaposition of ϕ with a basis of the vector subspace of $\mathrm{H}^0(X, TX)$ spanned by the globally defined holomorphic Killing vector fields for ϕ.

Proof. Given the extendibility result Theorem 2.4, Theorem 2.1 in [Du1] (see also Theorem 3 in [Du3]) implies that there is a open subset $U \subset X$ as in the statement of the theorem such that the fiber of π_{red} through any $z \in U$ is contained in the $\mathrm{Aut}_0(X, \phi)$–orbit of z.

Fix a basis $\{X_1, \cdots, X_k\} \subset \mathrm{H}^0(X, TX)$ of the Lie algebra of $\mathrm{Aut}_0(X, \phi)$. Let ϕ' denote the holomorphic rigid geometric structure on X obtained by juxtaposing ϕ with this family of holomorphic vector fields $\{X_1, \cdots, X_k\}$.

Denote by $A = \mathrm{Aut}_0(X, \phi')$ the connected component, containing the identity element, of the automorphism group $\mathrm{Aut}(X, \phi')$ of ϕ'. By construction, A is a maximal connected abelian subgroup in $\mathrm{Aut}(X, \phi)$. □

The following proposition is proved using Theorem 2.4.

Proposition 2.8. *Let X be a compact complex manifold with trivial canonical bundle. If X admits a holomorphic rigid geometric structure which is locally homogeneous on an open dense subset, then the fundamental group of X is infinite.*

Proof. Let ϕ be a holomorphic rigid geometric structure on X satisfying the condition of being locally homogeneous on an open dense subset U. To prove the proposition by contradiction, assume that the fundamental group of X is finite. Replace X by its universal cover, and also replace ϕ by its pull-back through this finite covering map. Note that this pull-back of ϕ is again locally homogeneous on the inverse image of U (which is also an open dense subset). Therefore, we can assume that X is simply connected and compact, and ϕ is locally homogeneous on an open dense subset.

Denote by n the complex dimension of X. Since ϕ is locally homogeneous on an open dense subset of X, we may choose local holomorphic Killing vector fields $X_1, \cdots, X_n$ that span TX over a nonempty open subset of X. By Theorem 2.4, the vector fields X_i extend as global holomorphic sections of TX on X. When contracted by $X_1 \bigwedge \cdots \bigwedge X_n$, a nontrivial holomorphic section $\omega \in \mathrm{H}^0(X, K_X)$ defines a holomorphic function on X. This function must be constant, by the maximum principle, and it is nonzero at the points where the vector fields $X_1, \cdots, X_n$ are linearly independent. Consequently, this constant function is nonzero, which in turn implies that the vector fields X_i are linearly independent at every point of X. Hence the vector fields $X_1, \cdots, X_n$ span TX on entire X.

In particular, the holomorphic tangent bundle of X is trivial, in other words, X is a parallelizable manifold, and hence it is biholomorphic to the quotient of a complex Lie group G by a cocompact lattice of it [Wa]. Consequently, the fundamental group of X is infinite, which contradicts the fact that X is simply connected. □

The following classical lemma will be useful in the proof of Theorem D; its proof can be found in [BD2, Lemma 2.5].

Lemma 2.9. *Let X be a compact complex manifold with trivial canonical bundle K_X, and let A be a connected complex Lie group acting on it through complex automorphisms. Then* *the choice of any* $\omega \in \mathrm{H}^0(X, K_X) \setminus \{0\}$ *produces a smooth finite A–invariant measure on* X.

3. Moishezon manifolds with vanishing first Chern class

Recall that the holomorphic tangent bundle of a compact Kähler manifold, with numerically trivial canonical bundle, is polystable, this being an immediate consequence of Calabi's conjecture proved by Yau [Ya]. The aim in this section is to prove a similar result in the set-up of Moishezon manifolds.

3.1. **Semistability of the tangent bundle.** The main result from this section, Proposition 3.3, states a strong semistability property for the tangent bundle of Moishezon manifolds with trivial first Chern class. Its proof is purely algebraic and relies on a deep result of Campana and Păun [CP] recalled below. In § 3.2, we will prove a strengthened result using (independent) analytic methods, but that only applies to *some* movable classes unlike Proposition 3.3.

Theorem 3.1 ([CP, Theorem 1.1])**.** *Let X be a projective manifold, and let $\mathcal{F} \subset TX$ be a holomorphic foliation. Let $\alpha \in \mathrm{Mov}_{\mathrm{NS}}(X)$ be a movable class such that the minimal slope $\mu_{\alpha,\min}(\mathcal{F}) > 0$, i.e., for any quotient $\mathcal{F} \longrightarrow \mathcal{G}$, the slope of $\mathcal{G}$ with respect to α is strictly positive. Then $\mathcal{F}$ is algebraically integrable and a general leaf of it is rationally connected.*

Recall that if X is a compact Fujiki manifold of dimension n, then the movable cone $\mathrm{Mov}(X) \subset H^{n-1,n-1}_{\mathrm{BC}}(X, \mathbb{R})$ is defined to be the closed convex cone generated by all the Bott-Chern classes of the form $[f_*(\omega_1 \wedge \cdots \wedge \omega_{n-1})]_{\mathrm{BC}}$ where $f : Y \longrightarrow X$ ranges over all Kähler modifications of X and $\omega_1, \cdots, \omega_{n-1}$ range over all Kähler metrics in Y.

When X is Moishezon, one can further define the cone of movable curves $\mathrm{Mov}_{\mathrm{NS}}(X) \subset \mathrm{N}_1(X)_{\mathbb{R}}$ as the closed convex cone generated by the numerical classes of curves of the form $[f_*(A_1 \cap \cdots \cap A_{n-1})]$ where $f : Y \longrightarrow X$ ranges over all projective modifications of X and $A_1, \cdots, A_{n-1}$ range over all ample divisors on Y.

The following proposition shows that Theorem 3.1 continues to holds when X is Moishezon.

Proposition 3.2. *Theorem 3.1 remains true under the weaker assumption that X is a Moishezon manifold.*

Proof. Let $f : X' \longrightarrow X$ be a projective modification. Let $df : TX' \longrightarrow f^*TX$ be the differential of f. For any foliation $\mathcal{F} \subset TX$, we have the integrable subsheaf

$$\mathcal{F}' := (df)^{-1}(f^*\mathcal{F}) \subset TX'.$$

Take α as in the statement of Theorem 3.1. Then $f^*\alpha \in \mathrm{N}_1(X')_{\mathbb{R}}$ defined by $(D \cdot f^*\alpha) := (f_*D \cdot \alpha)$ for any $D \in \mathrm{N}^1(X)$ is a movable class on X'. For any subsheaf $\mathcal{E}' \subset \mathcal{F}'$, if $\mathcal{E} \subset \mathcal{F}$ is its image under df, then we have $\mu_\alpha(\mathcal{E}) = \mu_{f^*\alpha}(\mathcal{E}')$, because $f^*\mathcal{E}$ and $\mathcal{E}'$ are isomorphic outside of the exceptional divisor of f. This shows that $\mathcal{F}'$ satisfies the assumptions of

Theorem 3.1. Therefore, the leaves of $\mathcal{F}'$ are algebraic, and a general leaf of it is rationally connected.

If $x \in X$ is a general point, the leaf F_x of $\mathcal{F}$ through x is the image under f of the leaf $F'_x \subset X'$ of $\mathcal{F}'$ through the point $f^{-1}(x)$. In particular, F_x is open in its Zariski closure $\overline{F}_x \subset f(\overline{F'_x})$, which is rationally connected. This concludes the proof of the proposition. □

Now we will prove the semistability of TX by using Proposition 3.2.

Proposition 3.3. *Let X be a Moishezon manifold with $c_1(X) = 0$. Let $\alpha \in \mathrm{Mov}_{\mathrm{NS}}(X)$ be a movable class. Then the following two hold:*

1. *(1) The tangent bundle TX is semistable with respect to α.*
2. *(2) Take $\alpha \in \mathrm{Mov}_{\mathrm{NS}}(X)^\circ$, and consider a filtration of TX such that the successive quotients are torsionfree and stable of same slope with respect to α:*

$$0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_r = TX$$

(from the first statement it follows that such a filtration exists). Then $c_1(\mathcal{F}_{i+1}/\mathcal{F}_i) = 0$ for every $0 \leq i \leq r-1$.

Proof. Assume that TX is not semistable with respect to α. Take $\mathcal{F}_1 \subset TX$ to be the maximal semistable subsheaf, meaning the first nonzero term in the Harder–Narasimhan filtration of TX with respect to α. Then

$$c_1(\mathcal{F}_1) \cdot \alpha > \frac{\mathrm{rank}\mathcal{F}_1}{n} c_1(TX) \cdot \alpha = 0.$$

We first show that $\mathcal{F}_1$ is a foliation, following [Pe]. There is a natural map given by Lie bracket

$$\bigwedge^2 \mathcal{F}_1 \longrightarrow [\mathcal{F}_1, \mathcal{F}_1] \longrightarrow TX/\mathcal{F}_1 \tag{3.1}$$

which is $\mathcal{O}_X$–linear. Since $\mathcal{F}_1$ is semistable, it follows that $\bigwedge^2 \mathcal{F}_1$ is semistable with respect to α; note that

$$\mu_\alpha(\bigwedge^2 \mathcal{F}_1) = 2\mu_\alpha(\mathcal{F}_1) > \mu_\alpha(\mathcal{F}_1).$$

Therefore, we have

$$\mu_\alpha(\bigwedge^2 \mathcal{F}_1) > \mu_{\alpha,\max}(TX/\mathcal{F}_1),$$

and hence the image of the homomorphism $\bigwedge^2 \mathcal{F}_1 \longrightarrow TX/\mathcal{F}_1$ in (3.1) is zero. Consequently, we have $[\mathcal{F}_1, \mathcal{F}_1] \subset \mathcal{F}_1$, and hence $\mathcal{F}_1$ is a foliation.

Proposition 3.2 says that $\mathcal{F}_1$ is algebraically integrable and a general leaf of it is rationally connected. In particular, X is uniruled. On the other hand, if $\pi : X' \longrightarrow X$ is a modification such that X' is projective, then $c_1(K_{X'})$ is effective, and hence X' can't be uniruled for trivial reasons. In view of this contradiction we conclude that TX is semistable with respect to α.

Proof of the second statement of the proposition: We will prove using induction that $c_1(\mathcal{F}_{i+1}/\mathcal{F}_i) = 0$ for all $0 \leq i \leq r-1$.

The induction hypothesis says that $c_1(\mathcal{F}_{i+1}/\mathcal{F}_i) = 0$ for every $0 \leq i < k$. By the first part of the proposition we know that

$$\mu_\alpha(\mathcal{F}_{i+1}/\mathcal{F}_i) = 0 \tag{3.2}$$

for every $0 \leq i \leq r-1$. Take $\beta \in \mathrm{H}^{n-1,n-1}(X) \bigcap \mathrm{H}^{2n-2}(X, \mathbb{Q})$ close to α. Since α is in the interior of the movable cone, it follows that β is still movable. By applying the first part of the proposition to β it is deduced that TX is semistable with respect to β. Then $\mu_\beta(\mathcal{F}_{k+1}) \leq 0$. This combined with the induction hypothesis gives that

$$\mu_\beta(\mathcal{F}_{k+1}/\mathcal{F}_k) \leq 0.$$

Since this holds for every β close to α, using (3.2) it follows that $c_1(\mathcal{F}_{k+1}/\mathcal{F}_k) = 0$. □

3.2. **Fujiki manifolds with vanishing first Chern class.** In light of the Beauville-Bogomolov decomposition theorem for Kähler manifolds and its generalization to singular varieties ([Dr2, GGK, HP2, BGL]), it is natural to introduce the following definition.

Definition 3.4. Let X be a compact Fujiki manifold of dimension n such that K_X is linearly trivial. One says that X is an

- irreducible Calabi-Yau manifold (ICY) if for any finite étale cover $f : Y \longrightarrow X$ and any integer $0 < p < n$, one has $H^0(Y, \Omega_Y^p) = \{0\}$.
- irreducible holomorphic symplectic manifold (IHS) if there exists a holomorphic symplectic form $\sigma \in H^0(X, \Omega_X^2)$ such that for any finite étale cover $f : Y \longrightarrow X$, one has $\bigoplus_{p=0}^n H^0(Y, \Omega_Y^p) \simeq \mathbb{C}[f^*\sigma]$ as $\mathbb{C}$-algebras.

Let us make a few remarks about the above definition.

• If X is Kähler, an ICY or IHS manifold is automatically simply connected (as follows from the Beauville-Bogomolov decomposition theorem), hence our definitions are consistent with the ones existing in the Kähler case already. We expect that the same holds in the Fujiki setting, but we do not know how to prove it. We refer to Corollary 3.7 for a partial result in that direction.

• Definition 3.4 can be extended to the case where X has at most canonical singularities, by replacing the sheaf of holomorphic forms Ω_X^p by that of reflexive holomorphic forms $\Omega_X^{[p]}$ and replacing étale covers by quasi-étale covers. We refer to [CGGN, Definition 6.11] for details.

• If X is a compact Fujiki manifold with $c_1(X) = 0 \in H^2(X, \mathbb{R})$, then K_X is holomorphically torsion by [To, Theorem 1.5]. In particular, a finite étale cover of X has trivial canonical bundle.

• A Fujiki manifold X which is ICY is automatically Moishezon. Indeed, any Kähler modification X' satisfies $h^0(X', \Omega_{X'}^2) = 0$ hence it is projective by Kodaira's theorem.

We will now give standard examples of non-Kähler ICY and IHS manifolds. They are (small) modifications of Kähler ICY and IHS varieties with at most terminal singularities, and it is expected that they are all obtained in this fashion, see Conjecture 3.5 below and the evidence for it provided by Theorem 3.6 in the Moishezon and low-dimensional cases.

Irreducible Calabi-Yau threefolds. A rich source of examples comes out of the ordinary double point $V_n := \{\sum_{i=0}^n z_i^2 = 0\} \subset \mathbb{C}^{n+1}$ for $n \geq 2$. The singularity $(V_n, 0)$ is isolated, Gorenstein (as hypersurface singularity). If $n = 2$, $V_2 \simeq \mathbb{C}^2/\{\pm 1\}$ is a quotient singularity, but for $n \geq 3$, it is not anymore the case as one can see using Schlessinger's theorem asserting that isolated quotient singularities of codimension at least three are rigid.

The ODP can be seen as a cone over a smooth projective quadric Q_n. In particular, one can resolve the singular point by blowing up the origin once. Let $\pi : \widetilde{V}_n \longrightarrow V_n$ be the blow-up map. Then $K_{\widetilde{V}_n} = \pi^* K_{V_n} + (n-2)Q_{n-1}$, hence V_n is canonical and terminal if $n \geq 3$.

If $n = 3$, small resolutions exist. It can be seen in many different ways. The quickest way is to rewrite $V_3 \simeq \{xy = zt\} \subset \mathbb{C}^4$ and consider the graph $\widehat{V}_3 \subset V_3 \times \mathbb{P}^1$ of the meromorphic function $\frac{x}{z} = \frac{t}{y}$. It can be described explicitly as

$$\widehat{V}_3 := \{((x,y,z,t),[u:v]) \in \mathbb{C}^4 \times \mathbb{P}^1; xy = zt, xv = zu, yv = tu\},$$

and the exceptional locus is simply $\{0\} \times \mathbb{P}^1$. Choosing the meromorphic function $\frac{x}{t} = \frac{z}{y}$ would have provided another small resolution. One recognizes the blow up of the two Weil (non Cartier) divisors $(x = z = 0)$ and $(y = t = 0)$.

Alternatively, consider the blow up of the origin $\widetilde{V}_3$. It is isomorphic to the restriction $\mathbb{L}|_{Q_2}$ of the total space $\mathbb{L}$ of $\mathcal{O}_{\mathbb{P}^3}(-1)$ to the projective quadric $Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The normal bundle of the zero section $\mathbb{P}^3 \subset \mathbb{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^3}(-1)$ hence the normal bundle of $Q_2 \subset \widetilde{V}_3$ is isomorphic to $\mathcal{O}_{Q_2}(-1)$ hence its restriction to any ruling of the quadric is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$. By the Nakano-Fujiki criterion, one can contract either family of $\mathbb{P}^1$s to a smooth space $\widehat{V}^3$, hence get a small resolution with exceptional locus isomorphic to $\mathbb{P}^1$. Contracting the other family of $\mathbb{P}^1$s provides another small resolution.

If $f : X \xrightarrow{2:1} \mathbb{P}^3$ is a double cover of the projective space ramified over a surface B, then $K_X = f^*(K_{\mathbb{P}^3} + \frac{1}{2}B)$ is trivial if and only if $\deg(B) = 8$. In order to create singular examples, one will consider octic surfaces B with s isolated singularities locally isomorphic to V_2. The resulting X will then have s isolated singularities locally isomorphic to V_3, which we can resolve by the procedure explained above, which glues globally. We therefore get a small resolution $\pi : \widehat{X} \longrightarrow X$ with trivial canonical bundle (actually there are 2^s such resolutions). Such varieties X and $\widehat{X}$ have been extensively studied by Clemens [Cle].

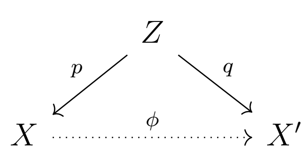
We claim that such an $\widehat{X}$ is an ICY threefold. Indeed, $\widehat{X}$ is simply connected [Cle, Corollary 1.19], and for $p \in \{1,2\}$, one has successively

$$\begin{aligned} H^0(\widehat{X}, \Omega^p_{\widehat{X}}) &\simeq H^0(X, \Omega_X^{[p]}) \\ &\simeq H^p(X, \mathcal{O}_X) \\ &\simeq H^p(\mathbb{P}^3, f_*\mathcal{O}_X) \\ &\simeq H^p(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-4)) \end{aligned}$$

which is indeed zero. The first two identities are consequences of [GKKP, Theorem 1.4] and [GKP11, Proposition 6.9], the third one comes from Leray spectral sequence given that $R^q f_* \mathcal{O}_X = 0$ if $q > 0$ since f is finite. The fourth one is a classic result for cyclic covers, see for example [KM, 2.50]. Alternatively, if one does not want to rely on the simple connectedness of $\widehat{X}$, one can use the decomposition theorem for X [HP2]. Indeed, since X has isolated non quotient singularity, no finite quasi-étale cover of X can split a torus; hence X is irreducible. Since it is of odd dimension, it must be an ICY variety, hence so is $\widehat{X}$ which follows from [GKKP, Theorem 1.4] given that quasi-étale covers of X are in one-to-one correspondence with étale covers of $\widehat{X}$ since $\widehat{X} \longrightarrow X$ is small.

Finally, one can find examples of such manifolds $\widehat{X}$ that are not projective algebraic, or equivalently Kähler. A great reference for that topic, which includes a survey of [Cle] is the PhD thesis of J. Werner that has recently been translated into English in [W]. Given any B as before, it follows from Clemens results [W, Theorem 4.3] that one can find *one* small resolution $\widehat{X} \longrightarrow X$ such that the exceptional curves $C_1, \dots, C_s$ satisfy a non-trivial relation $\sum_{i=1}^s m_i [C_i] = 0$ in $H_2(\widehat{X}, \mathbb{Q})$ and $m_i \geq 0$, which prevents $\widehat{X}$ from being Kähler. More precisely, the result cited guarantees that there exists a non-trivial relation between the $[C_i]$ without the information on the sign of the m_i, but then changing the small resolution of each ODP has the effect of switching the sign of m_i. Now, by choosing B to be certain specific Chmutov octic, one can see that *no* small resolution of the resulting double solid X is Kähler, see [W, § 6].

Irreducible holomorphic symplectic manifolds. The key construction here is the so-called Mukai flop. Let us briefly recall what it is and refer to e.g. [GHJ, Example 21.7] for more details. Let X be an $2n$-dimensional Kähler (irreducible) holomorphic symplectic manifold containing a submanifold $P \subset X$ isomorphic to $\mathbb{P}^n$. The existence of a holomorphic symplectic form imposes $N_{P|X} \simeq \Omega^1_P$ and one can then see that the projective bundle $\mathbb{P}(\Omega^1_P) \longrightarrow P$ is isomorphic to the first projection of the incidence variety $D \subset \mathbb{P}^n \times (\mathbb{P}^n)^\vee$; i.e. $D = \{(x, H) | x \in H\}$. In particular, if $Z \longrightarrow X$ is the blow-up of P, then the exceptional divisor is isomorphic to D and, moreover, the restriction of the normal $N_{D|Z}$ to any fiber of $D \longrightarrow (\mathbb{P}^n)^\vee$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ as one can see by using the adjunction formula twice (once for $D \subset P^n \times (\mathbb{P}^n)^\vee$ and once for $D \subset Z$) . Using again Nakano-Fujiki's criterion, one can contract all such fibers in Z to a obtain a smooth manifold X'



which can then easily be shown to be holomorphic symplectic.

The important property of the Mukai flop X' of X is as follows. Set $P' := q(D) \simeq \mathbb{P}^n$. For dimensional reasons, there exists an isomorphism $H^2(X, \mathbb{R}) \longrightarrow H^2(X', \mathbb{R})$. Let $\omega \in H^2(X, \mathbb{R})$ and let $\omega' \in H^2(X', \mathbb{R})$ be the corresponding class. We have $p^*\omega = q^*\omega' + a[D]$ for some $a \in \mathbb{R}$. Now, if $\ell \subset P$ and $H \subset \mathbb{P}^n$ is an hyperplane containing ℓ, then $\widetilde{\ell} := \ell \times \{H\} \subset D$

is a curve such that $p_*\widetilde{\ell} = \ell, q_*\widetilde{\ell} = 0$ as cycles and $\mathcal{O}_Z(D)|_{\widetilde{\ell}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$. In particular, $\omega\cdot\ell = p^*\omega\cdot\widetilde{\ell} = -a$. One can do the same operation with a line $\ell' \subset P'$ and find $\omega'\cdot\ell' = a$ so that

$$\omega\cdot\ell = -\omega'\cdot\ell'. \tag{3.3}$$

We now give a general construction of non-Kähler IHS manifolds obtained as the Mukai flop of certain projective IHS manifolds as described in [GHJ, Example 21.9]. Assume that X admits two *disjoint* submanifolds $P_1, P_2 \simeq \mathbb{P}^n$ and let X' be the Mukai flop of P_1. The map $X \dashrightarrow X'$ is therefore isomorphic near P_2. Assume further that the Picard number of X is two and that there exists a non-trivial morphism $X \longrightarrow Y$ to a projective variety Y contracting both P_1 and P_2. If ℓ_i is a line on P_i, then $\mathbb{R}\ell_1 = \mathbb{R}\ell_2 \in H^{4n-2}(X, \mathbb{R})$ since $\rho(X) = 2$. If α is a Kähler class on X, we have $\alpha\cdot\ell_i > 0$ for $i \in \{1, 2\}$, hence

$$\ell_1 = \lambda\ell_2 \quad \text{for some } \lambda > 0. \tag{3.4}$$

If X' were to admit a Kähler class ω', then the corresponding class $\omega \in H^2(X, \mathbb{R})$ would satisfy

$$\omega\cdot\ell_1 = -\omega'\cdot\ell_1' < 0 \quad \text{and} \quad \omega\cdot\ell_2 = \omega'\cdot\ell_2 > 0$$

(as follows from (3.3)), which contradicts (3.4). Hence X' cannot be Kähler. Explicit examples of IHS manifolds X satisfying the requirements have been constructed by Yoshioka and Namikawa; see references in [GHJ, Example 21.9].

3.3. **The decomposition theorem.** We would like to propose the following statement on the structure of Fujiki manifolds with vanishing first Chern class, generalizing the well-known Beauville-Bogomolov decomposition theorem.

Conjecture 3.5 (Decomposition Conjecture)**.** *Let X be a compact Fujiki manifold such that $c_1(X) = 0 \in \mathrm{H}^2(X, \mathbb{R})$. Then there exists a finite étale cover $X' \longrightarrow X$ such that*

$$X' \simeq T \times \prod_{i\in I} Y_i \times \prod_{j\in J} Z_j$$

where

- *(i) T is a complex torus,*
- *(ii) For all $i \in I$, Y_i an irreducible Calabi-Yau manifold,*
- *(iii) For all $j \in J$, Z_j is an irreducible holomorphic symplectic manifold.*

Moreover, there exist bimeromorphic maps $Y_i \dashrightarrow \widehat{Y}_i$ (respectively, $Z_j \dashrightarrow \widehat{Z}_j$), isomorphic in codimension one, such that $\widehat{Y}_i$ is a projective ICY variety with terminal singularities (respectively, Z_j is a Kähler IHS variety with terminal singularities).

The goal of this section is to establish the above conjecture assuming either that X is Moishezon or that it satisfies $\dim X \leq 4$, which will enable us to refine the statements obtained in § 3.1 above.

Theorem 3.6. *Let X be a Fujiki manifold such that $0 = c_1(X) \in \mathrm{H}^2(X, \mathbb{R})$. Assume that one of the following holds:*

- *X is Moishezon, or*
- $\dim X \leq 4$.

Then the Decomposition Conjecture holds for X.

Moreover, there exists a Zariski open set $U \subset X$, whose complement $X \setminus U$ has codimension at least two, satisfying the condition that there is an incomplete Ricci-flat Kähler metric ω on U for which the following two hold:

(*i*) *(Bochner principle). For every holomorphic tensor $\sigma \in \mathrm{H}^0(X, TX^{\otimes p} \otimes T^*X^{\otimes q})$ the restriction $\sigma\big|_U$ is parallel with respect to ω.*

(*ii*) *(Polystability). There exists a subset $\Lambda \subset \mathrm{Mov}(X)$ with non-empty interior such that TX is polystable with respect to λ. More precisely, there is a holomorphic decomposition of the tangent bundle*

$$TX = \bigoplus_{i=1}^{\ell} F_i \tag{3.5}$$

satisfying the conditions that F_i is locally free,

$$c_1(F_i) = 0 \in \mathrm{H}^2(X, \mathbb{R})$$

and F_i is stable with respect to any $\lambda \in \Lambda$. Additionally, over the subset U, the decomposition in (3.5) is orthogonal and $F_i\big|_U$ is parallel with respect to ω. Finally, one can take $\Lambda \supset \mathrm{Mov}_{\mathrm{NS}}(X)^\circ$ if X is Moishezon.

Proof. We proceed in several steps.

Step 1. Existence of a good minimal model.

Let $X' \longrightarrow X$ be a modification such that X' is Kähler. Since K_X is torsion, we know that $\kappa(X') = 0$ and the numerical dimension $\mathrm{nd}(K_{X'}) = 0$. We have two cases:

- If X' is projective (i.e., if X is Moishezon), then we can apply [BCHM] and in particular [Dr, Corollary 3.4] to find a birational model $X' \dashrightarrow X_{\min}$ with terminal singularities and torsion canonical bundle $K_{X_{\min}}$.
- If X' is merely Kähler but $\dim X \leq 4$, then if follows from [HP1, Theorem 1.1] in dimension three and [DHP, Theorem 1.1] in dimension four that there exists a bimeromorphic model $X' \dashrightarrow X_{\min}$ with terminal singularities and nef canonical bundle $K_{X_{\min}}$.

Let us resolve the indeterminacies of $\phi : X \dashrightarrow X_{\min}$ as follows

$$\begin{array}{ccc} & Z & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \overset{\phi}{\dashrightarrow} & X_{\min} \end{array} \tag{3.6}$$

where Z is smooth Kähler.

We claim that the map ϕ is an isomorphism in codimension one and that $K_{X_{\min}}$ is torsion. Indeed, write

$$K_Z = p^* K_X + \sum a_i E_i \quad \text{and} \quad K_Z = q^* K_{X_{\min}} + \sum b_j F_j,$$

where $\sum E_i$ (respectively, $\sum F_j$) is the exceptional locus of p (respectively, q). As X and $X_{\min}$ are terminal, we have $a_i,\, b_j > 0$ for all indices $i,\, j$. Set $D := \sum a_i E_i - \sum b_j F_j = q^* K_{X_{\min}} - p^* K_X$. Then D is p-nef and $p_* D \leq 0$, so that by the negativity lemma (see e.g. [Wan, Lemma 1.3]), we actually have $D \leq 0$. Similarly, $-D$ is q-nef and $q_*(-D) \leq 0$ so $-D \leq 0$. All in all, $D = 0$. This implies that $q^* K_{X_{\min}} = p^* K_X$ is torsion, hence so is $K_{X_{\min}}$. Moreover, since that coefficients a_i, b_j are positive, then we also get $\sum E_i = \sum F_j$ which implies that ϕ is isomorphic in codimension one. Our claim follows.

In the following, we denote by $U \subset X$ the maximal Zariski open subset over which ϕ is defined and is an isomorphism. By the above, the codimensions of $X \setminus U$ and $X_{\min} \setminus \phi(U)$ are at least two.

Step 2. Bochner principle.

Let $\alpha \in H^2(X, \mathbb{R})$ be Kähler class on $X_{\min}$. By [EGZ, Pa], there exists a positive current

$$\omega_{\min} \in \alpha$$

with bounded potentials on $X_{\min}$ whose restriction to the regular locus $X_{\min}^{\text{reg}}$ of $X_{\min}$ is a genuine Kähler Ricci flat metric with finite volume equal to α^n. Define

$$\omega := \phi^*(\omega_{\min}|_{\phi(U)}),$$

where ϕ is as in (3.6); it is a Kähler Ricci flat metric on U. Since $\int_U \omega^n < +\infty$, the Kähler manifold (U, ω) is incomplete unless $X = U$ is already Kähler (see [GGK, Proposition 4.2]).

By [CGGN, Theorem A] every reflexive holomorphic tensor on $X_{\min}^{\text{reg}}$ is parallel with respect to $\omega_{\min}$. Since U and $\phi(U)$ have complements of codimension at least two, there are natural isomorphisms

$$\mathrm{H}^0(U,\, TX^{\otimes a} \otimes T^*X^{\otimes b}) \simeq \mathrm{H}^0(X,\, TX^{\otimes a} \otimes T^*X^{\otimes b})$$

and

$$\mathrm{H}^0(\phi(U),\, TX_{\min}^{\otimes a} \otimes T^*X_{\min}^{\otimes b}) \simeq \mathrm{H}^0(X_{\min}^{\text{reg}},\, TX_{\min}^{\otimes a} \otimes T^*X_{\min}^{\otimes b})$$

for all integers $a,\, b \geq 0$. Statement (i) follows immediately.

Step 3. Polystability of TX.

The tangent bundle $TX_{\min}$ of $X_{\min}$ holomorphically decomposes as

$$T_{X_{\min}} = \bigoplus_{i=1}^{r} F_i^{\min},$$

where each $F_i^{\min}$ is α-stable with zero first Chern class and its restriction to the regular locus of $X_{\min}$ is a parallel subbundle with respect to $\omega_{\min}$ (see [GGK, Proposition D] and [CGGN, Remark 3.5]). Using ϕ, we get a similar holomorphic decomposition

$$T_X = \bigoplus_{i=1}^{r} F_i$$

valid over U for some parallel subbundles $F_i \subset T_U$ with respect to ω. Taking the saturation of F_i in T_X, we get reflexive subsheaves — still denoted by F_i — and a homomorphism $\bigoplus_{i=1}^{r} F_i \longrightarrow T_X$ which is an isomorphism over U, hence it is an isomorphism everywhere.

Next, we check that $c_1(F_i) = 0$. This is easy to see because

$$p^* c_1(F_i) = q^* c_1(F_i^{\min}) + \sum c_i E_i = \sum c_i E_i$$

for some $c_i \in \mathbb{Z}$ (see (3.6) for p, q). Recall that the divisors E_i are exceptional for both p and q. In particular, for any $\gamma \in \mathrm{H}^{2n-2}(X)$ we have

$$c_1(F_i) \cdot \gamma = p^* c_1(F_i) \cdot p^* \gamma = \sum c_i\, E_i \cdot p^* \gamma = 0$$

using the projection formula.

The same arguments show that F_i is stable with respect to

$$\beta := p_*(q^* \alpha^{n-1}) \in \mathrm{H}^{n-1,n-1}(X) \cap \mathrm{H}^{2n-2}(X, \mathbb{R}).$$

The class β is movable for being a limit of the classes $p_*((q^*\alpha + \varepsilon\gamma)^{n-1})$ as $\varepsilon \to 0$, where $\gamma \in H^2(Z, \mathbb{R})$ is a fixed Kähler class. Finally, [GKP, Remark 3.5] shows that the set $\Lambda \subset \mathrm{Mov}(X)$ of movable classes with respect to which F_i is stable has non-empty interior for every $i = 1, \cdots, r$. This together with the condition $c_1(F_i) = 0$ implies that TX is polystable with respect to any $\lambda \in \Lambda$.

It remains to see that if X is Moishezon, then for any $\lambda \in \mathrm{Mov}_{\mathrm{NS}}(X)^\circ$ and any index i, F_i is λ-stable. We know from Proposition 3.3 that F_i is λ-semistable. Arguing by contradiction, one can consider the maximal destabilizing sheaf $G_i \subset F_i$. It is a proper, λ-stable subsheaf with $\mu_\lambda(G_i) = 0$. Now, given any $\alpha \in \mathrm{N}_1(X)_{\mathbb{R}}$, we have $\lambda + \varepsilon\alpha \in \mathrm{Mov}_{\mathrm{NS}}(X)$ for $0 \le \varepsilon \ll 1$, hence $\mu_{\lambda+\varepsilon\alpha}(G_i) \le 0$ by Proposition 3.3. This implies that $c_1(G_i) \cdot \alpha \le 0$. Since this holds for any $\alpha \in \mathrm{N}_1(X)_{\mathbb{R}}$, we deduce that $c_1(G_i) = 0 \in \mathrm{N}^1(X)_{\mathbb{R}}$. This is a contradiction with the fact that F_i is stable with respect to some movable class and satisfies $c_1(F_i) = 0$, hence it shows the claim.

Step 4. The splitting of X.

The validity of Bochner principle implies that the Albanese map of our manifold X is étale trivial (see proof of [CGGN, Theorem 4.1] or Theorem 3.11 below). This implies that a finite étale cover $X' \longrightarrow X$ splits as a product $X' \simeq T \times Y$ where T is a complex torus and Y is a compact Fujiki manifold Y such that K_Y is trivial and $H^0(Y', \Omega^1_{Y'}) = 0$ for any finite étale cover $Y' \longrightarrow Y$. Up to replacing X with Y, one can assume that the augmented irregularity of X vanishes. In particular, the augmented irregularity of $X_{\min}$ vanishes as well. Using [HP2] or [BGL], one can find a finite quasi-étale cover $g : X'_{\min} \longrightarrow X_{\min}$ which further decomposes as a product

$$X'_{\min} \simeq \prod_{i \in I} Z_i$$

where the Z_i are irreducible singular ICY or IHS varieties in the sense of *loc. cit.*

Set $V := \phi(U) \subset X_{\min}^{\rm reg}$. The restriction $g|_{g^{-1}(V)}$ is étale and induces via ϕ a finite étale cover $U' \longrightarrow U$ that extends to a finite étale cover $f : X' \longrightarrow X$ for some compact manifold X, which follows from [DeGr, Theorem 3.4] and purity of branch locus. We therefore have the following diagram

$$\begin{array}{ccc} X' & \overset{\psi}{\dashrightarrow} & X'_{\min} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \overset{\phi}{\dashrightarrow} & X_{\min} \end{array} \tag{3.7}$$

From now on, we replace X with X'. By induction, one can assume that $X_{\min} \simeq Y \times Z$ where Y and Z are compact Kähler with terminal singularities of respective dimensions k and ℓ, not necessarily irreducible (i.e., they can be a product of lower dimensional ICY or IHS varieties). On X, we have an induced splitting of the tangent bundle into regular foliations $T_X = \mathcal{F} \oplus \mathcal{G}$.

Next, we will show that $\mathcal{F}$ has smooth compact leaves with finite holonomy. It is clear that a general leaf is compact, hence smooth since $\mathcal{F}$ is regular. Indeed, if $x \in U$, then the leaf F_x of $\mathcal{F}$ though x is included in the strict transform by ϕ of the leaf $Y \times \{z\}$ where $\phi(x) = (y, z)$. Now we want to show that if F is a general (compact) leaf of $\mathcal{F}$, then the volume $\int_F \omega^k$ of F with respect to a fixed hermitian metric ω is bounded uniformly independently of F, which will show that all leaves of $\mathcal{F}$ are compact by Bishop theorem; smoothness will then follow from the regularity of $\mathcal{F}$.

In order to show the volume bound, let W be a Kähler desingularization of the graph of ϕ, with projections $p : W \longrightarrow X$ and $q : W \longrightarrow Y \times Z$. Set

$$r := {\rm pr}_Z \circ q : W \longrightarrow Z$$

so that the cycles $W_z := r^{-1}(z)$ are homologous (and smooth) for z general. When $z \in Z$ varies (say in a dense open subset of Z), the varieties $F_z := p(W_z)$ form a family of smooth leaves of $\mathcal{F}$ sweeping out a dense open set of X. Moreover, p induces a bimeromorphic map between W_z and F_z. In particular, we have $\int_{F_z} \omega^k = \int_{W_z} p^*\omega^k$. Pick a Kähler metric ω_W such that $\omega_W \geq p^*\omega$. Then we have $\int_{F_z} \omega^k \leq \int_{W_z} \omega_W^k$ which is independent of z (general) since the W_z are homologous and ω_W is closed.

Finiteness of the holonomy can be showed as follows. Let $x \in X$ and let F_x be the leaf of $\mathcal{F}$ though x. Choose a transversal S to $\mathcal{F}$ at x (e.g. a neighborhood of x in the leaf of $\mathcal{G}$ through x). Choose a chart near x given by the unit polydisk where the leaves of $\mathcal{F}$ are given by the affine subspaces $(z_1 = a_1, \ldots, z_\ell = a_\ell)$, $a \in \mathbb{D}^\ell$, and where $\omega \geq C^{-1}\omega_{\rm eucl}$. Then a leaf of F can hit S at most $C^n \int_F \omega^k$ times, and that number is bounded above independently of F.

Using the holomorphic version of Reeb stability theorem (see for example [HW, Proposition 2.5]) it is concluded that $\mathcal{F}$ induces a holomorphic map $X \longrightarrow \mathcal{C}_k(X)$, where $\mathcal{C}_k(X)$ is the Barlet space of (compact) cycles of X of dimension k. The map is defined by associating to x the cycle $|G_x| \cdot F_x$ where F_x is the leaf of $\mathcal{F}$ though x and G_x is the holonomy group of F_x, i.e., the image of $\pi_1(F_x) \longrightarrow \mathrm{Diff}(S, x)$ which is finite by what we explained above. We let $C_{\mathcal{F}}$ be the image of the map $X \longrightarrow \mathcal{C}_k(X)$. This is a compact complex variety and one can check that it has only quotient singularities and that it has dimension ℓ (see [HW, Theorem 2.4]). We now consider the product map

$$f : X \longrightarrow C_{\mathcal{F}} \times C_{\mathcal{G}}.$$

From the local description of the foliations provided by Reeb stability theorem, it follows that df_x is an isomorphism for x general. As a consequence, $f(X)$ is compact of dimension n, hence f is surjective. Since $\mathcal{F}$ and $\mathcal{G}$ are transverse with smooth compact leaves, a leaf of $\mathcal{F}$ can intersect a leaf of $\mathcal{G}$ only finitely many times, hence f is finite. If the degree of f were greater than one, we could find two points $x, x' \in U$ such that $f(x) = f(x')$. That is, the leaves F_x and G_x would intersect again at the point x'. This is a contradiction with the fact that over U via ϕ, F_x and G_x correspond to $Y \times \{z\}$ and $Z \times \{y\}$ where $\phi(x) = (y, z)$. In particular, their only intersection point on $Y \times Z$ is the point (y, z). In conclusion, f is finite and generically $1:1$. Since the source and target of f are normal, Zariski's main theorem implies that f is isomorphic.

It follows from the construction that $\mathcal{F}$ is isomorphic to $\mathrm{pr}_1^* T_{C_{\mathcal{G}}}$ (and similarly for $\mathcal{G}$) via f. Moreover, we have seen that a (general) leaf of $\mathcal{F}$, which isomorphic to a copy of $C_{\mathcal{G}}$ via f, is bimeromorphic to a copy of Y and that map is actually isomorphic in codimension one. After iterating the construction, one will split $X \simeq \prod_{j \in J} X_j$ where X_j is bimeromorphic and isomorphic in codimension one to a possibly singular ICY or IHS Kähler variety.

It is then straightforward to check that this implies that X_j itself is an ICY or IHS manifold in the sense of Definition 3.4. The theorem is proved. □

Recall that if X is a compact complex manifold, the augmented irregularity of X is defined as

$$\widetilde{q}(X) := \sup\{q(Y) \,\big|\, Y \longrightarrow X \text{ finite étale}\} \in \mathbb{N} \cup \{\infty\},$$

where $q(\bullet) = h^0(\bullet, \Omega^1_\bullet)$ is the usual irregularity. Using Theorem 3.6, the results in [GGK] and [BGL] yield the following

Corollary 3.7. *Let X be a compact Fujiki manifold such that $c_1(X) = 0 \in \mathrm{H}^2(X, \mathbb{R})$ and $\widetilde{q}(X) = 0$. Assume either that X is Moishezon or that $\dim X \leq 4$.*

Then $\pi_1(X)$ does not admit any finite-dimensional representation with infinite image (over any field). Moreover, for each $k \in \mathbb{N}$, $\pi_1(X)$ admits only finitely many k-dimensional complex representations up to conjugation.

Proof. In view of the proof of Theorem 3.6, there exists a bimeromorphic map $\phi : X \dashrightarrow X_{\min}$ where $X_{\min}$ is a Kähler variety with klt singularities, zero first Chern class, and zero augmented irregularity. The last property comes from the fact that ϕ is isomorphic in codimension one, hence there is a one-to-one correspondence between finite étale covers of X and finite quasi-étale covers of $X_{\min}$.

If Z is a desingularization of the graph of ϕ, it follows from Takayama's result [Ta] that the maps $p : Z \longrightarrow X$ and $q : Z \longrightarrow X_{\min}$ induce isomorphic maps at the level of fundamental groups

$$p_* : \pi_1(Z) \xrightarrow{\simeq} \pi_1(X) \quad \text{and} \quad q_* : \pi_1(Z) \xrightarrow{\simeq} \pi_1(X_{\min}) \tag{3.8}$$

The corollary is an now an application of [GGK, Theorem I] in the projective case and the combination of Theorem A and Corollary 3.10 in [BGL] in the Kähler case. □

Corollary 3.8. *Let X be a compact Fujiki manifold such that $0 = c_1(X) \in \mathrm{H}^2(X, \mathbb{R})$ and $0 = c_2(X) \in \mathrm{H}^4(X, \mathbb{R})$. Assume either that X is Moishezon or that $\dim X \leq 4$. Then X admits a finite étale cover $T \longrightarrow X$ where T is complex torus.*

Proof. We use the notation from Theorem 3.6. Consider the vector bundle

$$p^*TX = \bigoplus_{i=1}^{r} p^*F_i,$$

where each p^*F_i has vanishing first Chern class. By the arguments in the proof of Theorem 3.6, each p^*F_i is stable with respect $q^*\alpha$. Since stability is an open condition, p^*F_i is also stable with respect to a Kähler class θ on X'. As the first Chern class of p^*F_i vanishes, the Bogomolov–Gieseker's inequality says that

$$c_2(p^*F_i) \cdot \theta^{n-2} \geq 0.$$

As $c_2(p^*TX) = \sum c_2(p^*F_i) = 0$, we conclude that $c_2(p^*F_i) \cdot \theta^{n-2} = 0$ for every i. By Simpson's correspondence, p^*F_i is hermitian flat, and hence p^*TX is hermitian flat. This implies that $TX\big|_U$ is unitary flat as well, and therefore using fact that the complement $X \setminus U \subset X$ is of codimension at least two it follows that TX too is unitary flat. By [DPS], X is Kähler and it follows that X admits a finite étale cover by a torus. □

Remark 3.9. The proof of the corollary above shows that one can weaken the assumption $0 = c_2(X) \in \mathrm{H}^4(X, \mathbb{R})$ and replace it by the existence of a modification $f : Y \longrightarrow X$ such that Y admits a Kähler form ω satisfying $c_2(X) \cdot f_*[\omega^{n-2}] = 0$.

Remark 3.10. A compact complex manifold in Fujiki class $\mathcal{C}$ bearing a holomorphic affine connection has vanishing Chern classes [At, Theorem 4 on p. 192–193]. Therefore Corollary 3.8 implies that any compact complex Moishezon manifold admitting a holomorphic affine connection also admits an étale covering by an abelian variety.

3.4. **Étale triviality of the Albanese map.** We now prove the étale triviality of the Albanese map for complex manifolds in Fujiki class $\mathcal{C}$ with numerically trivial canonical bundle, following the strategy of [Fu, Lemma 6.1] (see also [CGGN, Theorem 4.1]).

Theorem 3.11. *Let X be a compact complex manifold in Fujiki class $\mathcal{C}$ such that $c_1(X) = 0 \in \mathrm{H}^2(X, \mathbb{R})$. Then after taking some finite étale cover, there is a decomposition*

$$X = T \times Z,$$

where T is a torus, and $h^{1,0}(Z') = 0$ for any finite étale cover $Z' \longrightarrow Z$.

Proof. Since the canonical bundle K_X is torsion [To, Theorem 1.5], we can assume, by replacing X with a finite unramified covering of it, that X is a compact complex manifold in the Fujiki class such that its canonical bundle K_X is holomorphically trivial.

The key point is to prove that the canonical pairing on X

$$H^0(X,\, T_X) \times H^0(X,\, \Omega_X) \longrightarrow H^0(X,\, \mathcal{O}_X) \simeq \mathbb{C}$$

given by the natural contraction is a perfect pairing (i.e., it is non-degenerate). We first prove that $H^0(X,\, T_X)$ and $H^0(X,\, \Omega_X)$ have the same complex dimension. This follows from Serre duality, Hodge symmetry and triviality of K_X:

$$h^0(T_X) \,=\, h^n(K_X \otimes \Omega_X) \,=\, h^n(\Omega_X) \,=\, h^1(K_X) \,=\, h^1(\mathcal{O}_X) \,=\, h^0(\Omega_X).$$

Let $\text{Aut}_0(X)$ be the connected component containing the identity element of the group of holomorphic automorphisms of X. Its Lie algebra is $H^0(X,\, T_X)$. Let $\text{alb} \,:\, X \longrightarrow A(X)$ be the Albanese map. This map is equivariant with respect to the Jacobi homomorphism

$$\rho \,:\, \text{Aut}_0(X) \longrightarrow T(X)$$

with $T(X)$ being the compact complex torus given by the connected component of the group of holomorphic automorphisms of the Albanese manifold $A(X)$. The differential of ρ is a homomorphism of Lie algebras

$$d\rho \,:\, H^0(X,\, T_X) \longrightarrow H^0(X,\, \Omega_X)^*,$$

The kernel of $d\rho$ is the Lie subalgebra $\mathfrak{l} \subset H^0(X,\, T_X)$ consisting of holomorphic vector fields on X that are tangent to the fibers of the map alb; they are characterized as follows (see [Fu, Proposition 6.7]):

$$\mathfrak{l} \,=\, \{V \,\in\, H^0(X,\, T_X) \;\mid\; \theta(V) \,=\, 0 \;\forall\; \theta \,\in\, H^0(X,\, \Omega_X)\}.$$

If $\mathfrak{l} \,\neq\, \{0\}$, then X is bimeromorphic to a unirational manifold [Fu, Proposition 5.10]. In this case K_X does not admit nontrivial holomorphic sections (also [Fu, Corollary 5.11]). Since K_X is holomorphically trivial, we conclude that $\mathfrak{l} \,=\, \{0\}$. Consequently $d\rho$ is injective.

Since $\dim H^0(X,\, T_X) \,=\, \dim H^0(X,\, \Omega_X)$, we conclude that the injective homomorphism $d\rho$ is an isomorphism. As $T(X)$ is connected, and $d\rho$ is surjective, it follows that ρ is surjective as well.

Moreover, the kernel of the Jacobi homomorphism ρ is known to be a linear algebraic group [Fu, Corollary 5.8]. In particular, kernel(ρ) has only finitely many connected components. Since the Lie algebra of kernel(ρ) is trivial, it is a finite group. Consequently, $\text{Aut}_0(X)$ is a compact complex Lie group, and $\rho \,:\, \text{Aut}_0(X) \longrightarrow T(X)$ is a finite covering map. This map factors though a complex Lie group homomorphism $p : \text{Aut}_0(X) \longrightarrow A(X)$ satisfying

$$\text{alb}(\phi(x)) \,=\, \text{alb}(x) + p(\phi), \quad \forall\; x \,\in\, X. \tag{3.9}$$

It is now easy to conclude that alb is an étale trivial fiber bundle onto $A(X)$ (see for example [Fu, Lemma 6.1] or [CGGN, Theorem 4.1]). We recall the argument here for the reader's convenience. Consider the finite étale base change $X \times_{A(X)} \mathrm{Aut}_0(X) \longrightarrow \mathrm{Aut}_0(X)$ by p and let $Z = \mathrm{alb}^{-1}(0)$. By (3.9), the map

$$\begin{aligned} Z \times \mathrm{Aut}_0(X) &\longrightarrow X \times_{A(X)} \mathrm{Aut}_0(X) \\ (z, \phi) &\longmapsto (\phi(z), \phi) \end{aligned}$$

is well-defined, isomorphic and commutes with the projections to the factor $\mathrm{Aut}_0(X)$. This implies our claim on alb, and one can check the connectedness of F by observing that the finite factor in the Stein factorization of alb is étale and has a section by the universal property of the Albanese map.

Finally, the Albanese map of $X \times_{A(X)} \mathrm{Aut}_0(X) \simeq Z \times \mathrm{Aut}_0(X)$ coincides with the projection to the second factor. In particular, we have $h^{1,0}(Z) = 0$. Now, if $Z' \longrightarrow Z$ is an étale cover such that $h^{1,0}(Z') \neq 0$, we repeat the construction on Z' to split off another torus factor after a further étale cover of Z', and the process stops after finitely many steps. □

Let X be a compact manifold in Fujiki class $\mathcal{C}$ with trivial canonical bundle. It is said to admit an *algebraic approximation* if there is a small deformation $\mathcal{X} \longrightarrow \Delta$ of X, with $X = X_0$ being the central fiber, and a sequence of points $t_i \in \Delta$, $i \in \mathbb{N}$, converging to 0, such that all the fibers $\mathcal{X}_{t_i}$ are Moishezon.

Lemma 3.12. *Let X be a compact complex manifold in Fujiki class $\mathcal{C}$ such that $c_1(X) = 0 \in \mathrm{H}^2(X, \mathbb{R})$ and $c_2(X) = 0 \in \mathrm{H}^4(X, \mathbb{R})$. Assume that X admits an algebraic approximation. Then X admits a finite unramified covering by a compact complex torus.*

Proof. By Theorem 3.11, up to a finite étale cover, we have a decomposition $X = T \times Z$ where $h^{1,0}(Z) = 0$ and T is a compact complex torus. The given condition that $c_2(X) = 0$ implies that $c_2(Z) = 0$. Let $\mathcal{X} \longrightarrow \Delta$ be an algebraic approximation of X and let $t_0 \in \Delta$ such that X_{t_0} is Moishezon. Since $c_2(X) = 0$, it follows that $c_2(X_{t_0}) = 0$.

Corollary 3.8 applies to the Moishezon manifolds X_{t_0} to show that a finite étale cover of X_{t_0} is an abelian variety. Since $\mathcal{X} \longrightarrow \Delta$ is C^∞-trivial, one can extend the latter cover to a finite étale cover of the family $\mathcal{X} \longrightarrow \Delta$. This enables us to assume that X_{t_0} is torus. Since $t \longmapsto b_1(X_t)$ is constant and both X and X_{t_0} are Fujiki, it follows that $h^{1,0}(X) = h^{1,0}(X_{t_0}) = \dim X$. Since $X = T \times Z$ and $h^{1,0}(Z) = 0$, we get $\dim T = \dim X$ so that $X = T$ is a compact complex torus. □

Corollary 3.13. *Let X be a compact complex manifold in Fujiki class $\mathcal{C}$ such that $c_1(X) = 0 \in \mathrm{H}^2(X, \mathbb{R})$ and $c_2(X) = 0 \in \mathrm{H}^4(X, \mathbb{R})$. Assume that X has algebraic dimension $a(X) = \dim X - 1$. Then X admits a finite étale cover $T \longrightarrow X$ where T is a compact complex torus.*

Proof. This is a direct consequence of Lemma 3.12 and of the approximation result [Lin, Corollary 1.4] stating that a Fujiki class $\mathcal{C}$ manifold X of algebraic dimension $a(X) = \dim X - 1$ admits an algebraic approximation. □

Remark 3.14. While compact Kähler manifolds with numerically trivial canonical bundle are known to admit algebraic approximation [Ca], it is not known whether all Fujiki class $\mathcal{C}$

manifolds with numerically trivial canonical bundle admit algebraic approximations. Despite the unobstructedness of the Kuranishi space ([Po, Theorem 1.2] and [ACRT, Theorem 3.3]), a major stumbling block in adapting the arguments in [Ca] is the fact that Fujiki class $\mathcal{C}$ manifolds are not stable under deformations, already in dimension three [Cam].

4. Geometric structures on Moishezon manifolds

The aim in this section is to prove the following theorem on holomorphic geometric structures on Fujiki manifolds with numerically trivial canonical bundle that are either Moishezon or of dimension no greater than four.

Theorem 4.1. *Let X be a compact Fujiki manifold such that $c_1(X) \in \mathrm{H}^2(X, \mathbb{R})$ vanishes. Assume either that X is Moishezon or that $\dim X \leq 4$.*

- *(i) There exists a Zariski open set $U \subset X$, whose complement has complex codimension at least two, such that any holomorphic geometric structure ϕ of affine type on X is locally homogeneous on U.*
- *(ii) If there exists a rigid holomorphic geometric structure ϕ of affine type on X, then there exists a finite étale cover $T \longrightarrow X$, where T is a complex torus. The pull-back of ϕ on T is translation invariant.*

Remark 4.2. If Conjecture 3.5 were to hold, then the proof of Theorem 4.1 below would apply to show that its conclusions are valid for any compact Fujiki manifold with trivial first Chern class. A weaker but unconditional result will be proven in that direction (see Corollary 4.5).

Proof. Let U be the Zariski open set defined in the statement of Theorem 3.6, then the first statement follows from Lemma 2.3 and the Bochner principle proved in Theorem 3.6.

Next assume that ϕ is rigid. From Theorem 3.6 we know that X satisfy the Decomposition conjecture. Hence one can find a finite étale cover $X' = T \times Y \longrightarrow X$ where Y has zero augmented irregularity and is either Moishezon or has dimension at most four. Since the tangent bundle of T is trivial, ϕ induces a rigid holomorphic geometric structure ϕ_Y of affine type on Y as explained in the proof of [Du2, Theorem 2]. By the first part, ϕ_Y is locally homogeneous on a Zariski open set of Y.

We claim that there exists $g : Y' \longrightarrow Y$ a finite étale cover such that any complex linear representation of $\rho : \pi_1(Y') \longrightarrow \mathrm{GL}(n, \mathbb{C})$ is trivial, where $n = \dim Y$. Indeed, by Corollary 3.7 there are only finitely many classes $\{[\rho_i] \,\big|\, i \in I\}$ of such representations up to conjugation and each ρ_i has finite image. The sought cover g then corresponds to the subgroup of finite index of $\pi_1(Y)$ obtained as the intersection $\cap_I \ker(\rho_i)$. Up to replacing Y with Y', we can therefore assume that $\pi_1(Y)$ has no non-trivial complex representation of dimension n. Moreover, up to applying Theorem 3.6 again, one can assume that $Y = \prod_{i \in I} Y_i$ is a product of ICY and IHS manifolds.

In view of Remark 2.5, non-trivial local Killing fields for the rigid structure ϕ_Y (which exist since ϕ_Y is locally homogeneous on a non-empty Zariski open set of Y) can be extended globally to Y. In particular, we find that $H^0(Y, T_Y) \neq 0$. Now, if we set $n_i = \dim Y_i$, then the identity $K_{Y_i} \simeq \mathcal{O}_{Y_i}$ implies that $H^0(Y_i, T_{Y_i}) \simeq H^0(Y_i, \Omega^{n_i-1}_{Y_i})$, and the latter space is zero by the very definition of ICY and IHS manifolds. In particular, we must have $H^0(Y, T_Y) = 0$, which provides the expected contradiction. □

Remark 4.3. In Theorem 4.1 the condition on triviality of $c_1(X)$ is essential. To illustrate this, consider the complex projective line equipped with two holomorphic nonzero global vector fields v_1 and v_2 such that the two divisors $\text{div}(v_1)$ and $\text{div}(v_2)$ are actually disjoint. The geometric structure ϕ on the complex projective line given by v_1, v_2 is a holomorphic rigid geometric structure of affine type. Note that ϕ is not locally homogeneous on any nonempty open subset of the projective line. Indeed, the quotient v_1/v_2, which is a non-constant meromorphic function, is actually a scalar invariant of ϕ; it is not a constant on any nonempty open subset of the projective line.

The following is derived using Theorem 4.1.

Corollary 4.4. *Let $\mathcal{T}$ be a compact complex torus. Let $\pi : X \longrightarrow B$ be a holomorphic principal $\mathcal{T}$–bundle over a compact Moishezon manifold B with numerically trivial canonical bundle K_B. Then there exists a $\mathcal{T}$-invariant Zariski open set $U \subset X$, whose complement has complex codimension at least two, such that any holomorphic geometric structure ϕ of affine type on X is locally homogeneous on U. If such a ϕ is moreover rigid, then the fundamental group of X must be infinite.*

Proof. Theorem 1.5 of [To] shows that there exists a positive integer l such that $K_B^{\otimes l}$ is holomorphically trivial. We consider the index one cover $B' \longrightarrow B$; this is a finite unramified cover of B such that $K_{B'}$ is trivial. For convenience of the reader, we recall its construction. Fix a nonzero holomorphic section $\sigma : B \longrightarrow K_B^{\otimes l}$, take a connected component $B' \subset K_B$ of the inverse image of $\sigma(B)$ for the map $K_B \longrightarrow K_B^{\otimes l}$ defined by $v \longmapsto v^{\otimes l}$. The induced map $B' \longrightarrow B$ is the cover we are looking for. Up to replacing X with $X \times_B B'$, we may assume that K_B is trivial.

We claim that K_X is trivial. Indeed, since X is a principal $\mathcal{T}$-bundle, any trivialization Ω_T of K_T glues to a trivialization $\Omega_{X/B}$ of $K_{X/B}$. If Ω_B is a trivialization of K_B, then $\Omega_{X/B} \wedge \Omega_B$ yields a trivialization of K_X.

Consider the Zariski open dense subset $U \subset B$ defined in the statement of Theorem 3.6. Also consider the Zariski open set $\pi^{-1}(U) \subset X$. We will prove that any holomorphic geometric structure of affine type on X is locally homogeneous on $\pi^{-1}(U)$.

To prove this by contradiction, assume that there exists a holomorphic geometric structure of affine type on X which is not locally homogeneous on $\pi^{-1}(U)$. By Lemma 2.3, there exist a point $x \in \pi^{-1}(U)$, integers $a, b \geq 0$ and a nontrivial holomorphic section

$$\eta \in \mathrm{H}^0(X, (TX)^{\otimes a} \otimes (T^*X)^{\otimes b}) \setminus \{0\} \tag{4.1}$$

such that $\eta(x) = 0$; note that this condition implies that $a + b > 0$.

Since K_X is trivial, contraction by TX of a fixed nonzero holomorphic section of K_X produces a holomorphic isomorphism $TX \simeq \Lambda^{n-1}(T^*X)$ where $n = \dim_{\mathbb{C}} X$. Therefore, the section η in (4.1) produces

$$s \in \mathrm{H}^0(X, (T^*X)^{\otimes r}) \setminus \{0\},$$

where $r = (n-1)a + b$, such that $s(x) = 0$.

Then the proof of Theorem 1.2 in [BD2] shows that s is the pull-back of a holomorphic section $t \in \mathrm{H}^0(B, (T^*B)^{\otimes r})$; the condition that $s(x) = 0$ implies that $t(\pi(x)) = 0$. Since $\pi(x) \in U$, this is in contradiction with the Bochner principle proved in Theorem 3.6.

Finally, assume that ϕ is rigid. As in the proof of Theorem 4.1, since ϕ is locally homogeneous on an open dense subset of X, Proposition 2.8 shows that the fundamental group of X is infinite. □

As a consequence of Theorem 4.1 we have the following:

Corollary 4.5. *Let X be a compact complex manifold in Fujiki class $\mathcal{C}$ such that $c_1(TX) \in \mathrm{H}^2(X, \mathbb{R})$ vanishes. If X bears a holomorphic rigid geometric structure of affine type, then the fundamental group of X must be infinite.*

Note that the conclusion of Corollary 4.5 is weaker than that of Theorem 4.1 (ii), but it applies to any Fujiki manifold (see Remark 4.2).

Proof. This follows from Theorem 4.1 if X is a Moishezon manifold. Assume that X is not a Moishezon manifold. Since the algebraic dimension of X is not maximal, Theorem 4.2 of [BD2] implies that the fundamental group of X is infinite. □

Another consequence of Theorem 4.1 is the following:

Corollary 4.6. *Let X be a compact complex manifold with $c_1(X) = 0 \in H^2(X, \mathbb{R})$. Then any holomorphic rigid geometric structure of affine type on X admits a non-trivial Lie algebra of (local) Killing vector fields.*

Proof. To prove by contradiction, assume that the Lie algebra of Killing vector fields for ϕ is trivial. Then Theorem 2.1 of [Du1] (see also Theorem 3 of [Du2]) implies that the fibers of the algebraic reduction of X have dimension zero, meaning X is Moishezon. But Theorem 4.1 says that ϕ is locally homogeneous on an open dense subset of X. Consequently, the Lie algebra of Killing vector fields is transitive on an open dense subset in X, in particular, the Lie algebra of Killing vector fields is non-trivial. In view of this contradiction the proof is complete. □

5. Automorphism group and fibrations by complex tori

This section is devoted to generalizing Corollary 4.5 to other classes of manifolds, namely compact complex manifolds with algebraic dimension at most one and trivial canonical bundle and compact complex threefolds with trivial canonical bundle (see Corollary 5.2). The combination of Corollary 4.5, and Corollary 5.2 yields Theorem D.

The following statement is the main result of this section, from which we will easily derive Corollary 5.2.

Theorem 5.1. *Let X be a compact simply connected complex manifold with trivial canonical bundle K_X. Assume that X bears a holomorphic rigid geometric structure ϕ of affine type. Then the following statements hold:*

- *(i) There exists a holomorphic submersion $\pi : X \longrightarrow B$ to a simply connected Moishezon manifold B with globally generated canonical bundle K_B such that the fibers of π are complex tori.*
- *(ii) The fibration π is not isotrivial. Equivalently, K_B is not trivial.*
- *(iii) There exists a maximal connected abelian subgroup A of the automorphism group $\mathrm{Aut(X,}\ \phi)$ whose orbits coincide with the fibers of π. Moreover, A is noncompact and its (real) maximal compact subgroup K acts freely and transitively on the fibers of π (hence X is a C^∞ principal K–bundle over B).*

Proof. Take a simply connected compact complex manifold X endowed with a rigid holomorphic geometric structure ϕ of affine type. As done in the proof of Theorem 2.7, consider the Lie subalgebra of $\mathrm{H}^0(X,\, TX)$ corresponding to the subgroup $\mathrm{Aut}_0(X,\, \phi) \subset \mathrm{Aut}_0(X)$, and fix a basis

$$\{X_1,\, \cdots,\, X_k\} \subset \mathrm{Lie}(\mathrm{Aut}_0(X,\, \phi)) \subset \mathrm{H}^0(X,\, TX)$$

of this subalgebra. Note that the latter subalgebra is non-trivial (i.e., $k \geq 1$) by Corollary 4.6 coupled with Theorem 2.4.

Let $\phi' = (\phi,\, X_1,\, \cdots,\, X_k)$ be the rigid geometric structure on X; its automorphism group is denoted by $\mathrm{Aut}(X,\, \phi')$. From Theorem 2.7 we know that the maximal connected subgroup $A := \mathrm{Aut}_0(X,\, \phi')$ of $\mathrm{Aut}(X,\, \phi')$ is abelian. Since A preserves a smooth measure on X (see Lemma 2.9) its orbits are compact and they coincide with the orbits of the maximal compact (real) subgroup

$$K \subset A \tag{5.1}$$

(see Section 3.7 in [Gro] and Section 3.5.4 in [DG]). It should be mentioned that the proof in [Gro, DG] first shows that the stabilizer $A(x)$ of any point $x \in X$ for the action of A on X is an algebraic group, and hence the stabilizer has only finitely many connected components; then their proof uses the main result of [Mon] which asserts that for a homogeneous space $A/A(x)$ with a finite A–invariant measure, if $A(x)$ has only finitely many connected components, then $A/A(x)$ is compact and furthermore the action of the maximal compact (real) subgroup K of A on $A/A(x)$ is transitive.

Step 1. All A–orbits in X have the same dimension.

Arguing by contradiction, assume that there exists an A–orbit

$$\mathbf{O} \subset X$$

whose dimension is strictly less than the maximum of the dimensions of the A-orbits in X. Since A and K (see (5.1)) orbits are the same, from the above property of $\mathbf{O}$ it follows immediately that the stabilizer $K_o \subset K$ of a point $o \in \mathbf{O}$ for the action of K is a real Lie subgroup of positive dimension. The maximal connected subgroup $K_o^0 \subset K_o$ is a compact connected abelian group, and hence we have

$$K_o^0 = (S^1)^l \tag{5.2}$$

for some positive integer l. The action of K_o^0 linearizes locally (on some neighborhood of o in X), meaning it is locally isomorphic to the linear action of K_o^0 on T_oX given by the differential of the action of K_o^0 on X. The group K being abelian, the action of K_o^0 on the tangent space of the orbit $T_o\mathbf{O}$ is actually trivial; indeed, the action of K_o^0 on $T_o\mathbf{O}$ is conjugate to the restriction, to K_o^0, of the adjoint action of K on the quotient $\text{Lie}(K)/\text{Lie}(K_o^0)$ of Lie algebras. Moreover, K_o^0 preserves the orthogonal complement

$$V_o = (T_o\mathbf{O})^\perp \subset T_oX \tag{5.3}$$

with respect to any K_o^0–invariant Hermitian form on T_oX. Fix a K_o^0–invariant Hermitian form on T_oX.

Denote by A_o the stabilizer of $o \in \mathbf{O} \subset X$ for the action of A on X. It is a complex Lie subgroup of A; in fact A_o is identified with a complex algebraic subgroup of D^{r+s} for some integer s [Gro, DG]. Denote by C the smallest connected complex Lie subgroup of A containing K_o^0; the Lie algebra of C is the image of $\text{Lie}(K_o^0) \oplus \sqrt{-1}\text{Lie}(K_o^0)$ in $\text{Lie}(A)$. Since the K_o^0–action on T_oX is linearizable, the C-action on T_oX is linearizable as well (in fact, it is linearizable with respect to the same coordinates). More precisely, C is isomorphic to $(\mathbb{C}^*)^l$ acting on V_o diagonally (see (5.2) and (5.3)); in other words, V_o splits as a direct sum of complex lines

$$V_o = L_1 \oplus \cdots \oplus L_p, \tag{5.4}$$

such that on each direct summand L_i the group C acts through a character defined by

$$(t_1, \cdots, t_l) \longmapsto t_1^{n_{1,i}} \cdot t_2^{n_{2,i}} \cdot \ldots \cdot t_l^{n_{l,i}} \tag{5.5}$$

for every $(t_1, \cdots, t_l) \in (\mathbb{C}^*)^l$. Note that C acts trivially on $T_o\mathbf{O}$, because K_o^0 acts trivially on $T_o\mathbf{O}$.

Assume that the character of C in (5.5) corresponding to an eigenline L_i in (5.4) is not trivial, in other words, $n_{j,i} \neq 0$ for some $1 \leq j \leq l$ (see (5.5)). Then $L_i \setminus \{0\} \subset V_o$ is a C–orbit; the origin $0 \in V_o$ is an accumulation point of this orbit. From this it follows that for any open neighborhood $U \subset X$ of o on which the K_o^0–action is linearizable, there are C–orbits (and hence A–orbits) in $U \setminus (U \bigcap \mathbf{O})$ for which $o \in U \bigcap \mathbf{O}$ is an accumulation point.

But this contradicts the fact that the orbits of A in X are locally closed (in fact they are compact). Therefore, we conclude that $n_{j,i} = 0$ for all $1 \leq j \leq l$ and $1 \leq i \leq p$ (see (5.5)).

Consequently, the K_o^0–linear action on T_oX must be trivial, and hence the K_o^0–action is trivial on the open neighborhood of o where the action is linearized. Using analyticity, this implies that the K_o^0–action on X is trivial. This is a contradiction because l is (5.2) is positive.

Therefore, we conclude that all A–orbits in X have the same dimension.

Step 2. The action of K on X is free.

Take any point $x_0 \in X$; let $K(x_0) \subset K$ be the connected component, containing the identity element, of the stabilizer of x_0 for the action of K on X. Since $K(x_0)$ is compact, its action linearizes on a neighborhood of x_0 in X. For any $k \in K(x_0)$, the differential $dk(x_0)$ acts trivially on $T_{x_0}(Kx_0) = T_{x_0}(Ax_0)$; indeed, as mentioned earlier, this action is induced by the restriction, to $K(x_0)$, of the adjoint representation of K, and this adjoint action is trivial because K is abelian. Since $K(x_0)$ is compact, the orthogonal complement of $T_{x_0}(Kx_0) = T_{x_0}(Ax_0) \subset T_{x_0}X$, for a $K(x_0)$–invariant Hermitian form on $T_{x_0}X$, is actually $K(x_0)$–invariant.

On the other hand, the differential $dk(x_0)$ acts trivially on $T_{x_0}X/T_{x_0}(Ax_0)$ because the action of any element of A (in particular of k) preserves each A–orbit; in other words, k acts trivially on the space of A–orbits.

This implies that the linear action of $K(x_0)$ on $T_{x_0}X$ is trivial. From this it follows that the action of $K(x_0)$ on X is trivial; indeed, the action of $K(x_0)$ is linearizable, so on any open neighborhood of $x_0 \in X$ on which the action of $K(x_0)$ is linearizable, the action of $K(x_0)$ is trivial, because the action of $K(x_0)$ on $T_{x_0}X$ is trivial. In view of this, since the A–action, and therefore the K–action, on X is faithful, we now conclude that the group $K(x_0)$ is trivial. Consequently, the action of K on X is actually free.

Step 3. Fibers of $X \longrightarrow X/K$ are tori.

It now follows that X, equipped with the action of K, has the structure of a real principal K–bundle over the smooth real manifold $B = X/K$. The K–orbits are complex submanifolds because they are also A–orbits and A is a complex Lie group acting holomorphically on X. This implies that B is also a complex manifold, and the projection

$$\pi : X \longrightarrow X/K = B \tag{5.6}$$

is in fact a holomorphic submersion.

Since A is abelian, and any A–orbit Ax_0 is identified with $A/(A(x_0))$ with $A(x_0)$ being the stabilizer of x_0 for the action of A on X, the holomorphic tangent bundle $T(Ax_0)$ is holomorphically trivial. In fact, a holomorphic trivialization of $T(Ax_0)$ is gotten by fixing a basis of the abelian Lie algebra $\text{Lie}(A)/\text{Lie}(A(x_0))$ (the adjoint representation of A is trivial because A is abelian). As every A–orbit is compact with its holomorphic tangent bundle trivialized by commuting holomorphic vector fields, we conclude that every A–orbit in X is a compact complex torus [Wa].

Step 4. K_B is globally generated.

First, recall that since the fibers of π in (5.6) are connected and X is simply connected, the base B is simply connected as well.

Denote by m the complex dimension of the fibers of π. Take a point

$$b \in B. \tag{5.7}$$

Choose a family of holomorphic vector fields on X belonging to the Lie algebra of A

$$(X_1, \cdots, X_m)$$

(they are chosen from the fundamental vector fields for the A–action) satisfying the condition that their restrictions to $\pi^{-1}(b)$ span the tangent bundle $T(\pi^{-1}(b))$. This implies that the locus of all $b' \in B$ such that the restrictions of $(X_1, \cdots, X_m)$ to $\pi^{-1}(b')$ span $T(\pi^{-1}(b'))$ is an open dense subset of B whose complement is a closed complex analytic subspace.

Let

$$\omega \in \mathrm{H}^0(X, K_X) \setminus \{0\} \tag{5.8}$$

be a trivializing section of K_X. Then the holomorphic form

$$\omega' := i_{X_1} \circ i_{X_2} \circ \cdots \circ i_{X_m}\omega \in \mathrm{H}^0(X, \Omega_X^{n-m}) \setminus \{0\}, \tag{5.9}$$

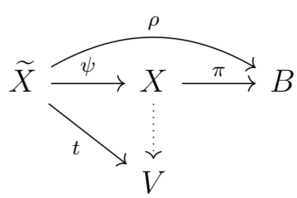
where $n = \dim_{\mathbb{C}} X$, satisfies the equation

$$\omega' = \pi^*\widehat{\omega} \tag{5.10}$$

for some $\widehat{\omega} \in \mathrm{H}^0(B, K_B) \setminus \{0\}$, with π being the projection in (5.6). Note that $\widehat{\omega}(b') \neq 0$ for all $b' \in B$ such that the restrictions of $(X_1, \cdots, X_m)$ to $\pi^{-1}(b')$ span $T(\pi^{-1}(b'))$; in particular, $\widehat{\omega}(b) \neq 0$ where b is the point in (5.7). Now moving b over B we conclude that K_B is generated by its global holomorphic sections.

Step 5. B is Moishezon.

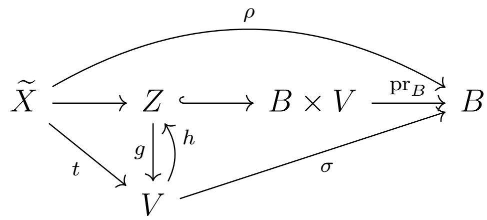
Using the notation of Theorem 2.6, consider the algebraic reduction $t : \widetilde{X} \longrightarrow V$ and set $\rho := \pi \circ \psi : \widetilde{X} \longrightarrow B$. We have a diagram as follows.



The maps ρ and t are proper, surjective with connected fibers, and by Theorem 2.7, ρ contracts every fiber of t. It is then classical that ρ factors through t, that is, there exists a map $\sigma : V \longrightarrow B$ such that $\rho = \sigma \circ t$. Clearly, σ is surjective; since V is Moishezon, so is B by Theorem 2 of [Mo].

Let us briefly recall how σ is constructed, for the reader's convenience. Consider the image Z of $\rho \times t : \widetilde{X} \longrightarrow B \times V$ and the map $g : Z \longrightarrow V$ induced by the projection $\mathrm{pr}_V : B \times V \longrightarrow V$. Let $\nu : Z^\nu \longrightarrow Z$ be the normalization of Z. Next, g is surjective, proper and by assumption, one has $|g^{-1}(v)| = 1$ for any $v \in V$. By Zariski main theorem, the map $g^\nu : Z^\nu \longrightarrow V$ is an isomorphism; set $h := \nu \circ (g^\nu)^{-1} : V \longrightarrow Z$. If one defines

$\sigma : V \longrightarrow B$ by $\sigma =: \mathrm{pr}_B \circ h$, then we have $\rho = \sigma \circ t$ as desired, and have the following commutative diagram:



This completes the proof of (i).

Step 6. K_B **is not trivial and** π **is not isotrivial.**

Argue by contradiction and assume that K_B is trivial. We will prove that the group K has a complex (torus) structure such that the quotient map π in (5.6) makes X a holomorphic principal K–bundle over B. Since X is simply connected and B is Moishezon by the previous step, this will contradict Corollary 4.4.

Going back to the proof, our assumption implies that the holomorphic section $\widehat{\omega} \in \mathrm{H}^0(B,\, K_B) \setminus \{0\}$ constructed in (5.10) does not vanish at any point, and hence $\widehat{\omega}$ trivializes K_B holomorphically. In view of (5.9) and (5.10), this implies that the family of holomorphic vector fields

$$\{X_1,\, \cdots,\, X_m\}$$

in (5.9) satisfies the condition that $\{X_1(x),\, \cdots,\, X_m(x)\} \subset T_xX$ are linearly independent for every $x \in X$. Consequently, the natural evaluation map

$$\beta \ :\ \bigoplus_{j=1}^{m} \mathbb{C} \cdot X_j \ \longrightarrow\ T_{X/B}\,, \tag{5.11}$$

where $T_{X/B} \subset TX$ is the relative holomorphic tangent bundle for π in (5.6), is a holomorphic isomorphism. For any $1 \leq j \leq m$, let $X'_j \in \mathrm{Lie}(A)$ be the element corresponding to the vector field X_j. Let

$$\mathcal{H} \ \subset\ \mathrm{Lie}(A) \tag{5.12}$$

be the complex subspace generated by $\{X'_1,\, \cdots,\, X'_m\}$. Let

$$\mathrm{Lie}(K)_{\mathbb{C}} \ \subset\ \mathrm{Lie}(A)$$

be the complex subspace generated by $\mathrm{Lie}(K) \subset \mathrm{Lie}(A)$.

For any $v \in \mathrm{Lie}(K)_{\mathbb{C}}$, there are complex valued functions $f^v_1,\, \ldots,\, f^v_m$ on B such that the holomorphic vector field v' on X corresponding to v satisfies the equation

$$v'(b) \ =\ \sum_{j=1}^{m} f^v_j(b) \cdot X'_j(b)$$

for all $b \in B$. Indeed, this follows immediately from the fact that β in (5.11) is an isomorphism. The functions $f^v_1,\, \ldots,\, f^v_m$ are evidently holomorphic, and hence they are constants. This implies that

$$\mathrm{Lie}(K)_{\mathbb{C}} \subset \mathcal{H}\,, \tag{5.13}$$

where $\mathcal{H}$ is the subspace in (5.12). On the other hand,

$$\dim_{\mathbb{C}} \mathcal{H} = m \leq \dim_{\mathbb{C}} \mathrm{Lie}(K)_{\mathbb{C}} ,$$

because $\dim_{\mathbb{R}} \mathrm{Lie}(K) = 2m$. This and (5.13) together imply that $\mathcal{H} = \mathrm{Lie}(K)_{\mathbb{C}}$. But this implies that $\mathrm{Lie}(K)_{\mathbb{C}} = \mathrm{Lie}(K)$, because $2 \cdot \dim_{\mathbb{C}} \mathcal{H} = 2 \cdot \dim_{\mathbb{C}} \mathrm{Lie}(K)_{\mathbb{C}} = \dim_{\mathbb{R}} \mathrm{Lie}(K)$.

Since $\mathrm{Lie}(K) = \mathrm{Lie}(K)_{\mathbb{C}}$, we conclude that K is a complex Lie subgroup of A, in particular, K is a compact complex torus. Therefore, the projection π in (5.6) makes X a *holomorphic principal* K–bundle over B.

Let us now show that π is not isotrivial arguing by contradiction. The fibers of π are isomorphic as complex manifolds to a fixed torus T, and by a fundamental result of Fischer and Grauert π is a holomorphic bundle over B. Since any trivialization of K_T is preserved by $\mathrm{Aut}_0(T)$ and $\mathrm{Aut}(T)/\mathrm{Aut}_0(T)$ is finite, there exists $p \geq 1$ such that $pK_{X/B}$ is trivial. As K_X is trivial, this implies that pK_B is trivial, hence K_B is trivial since it is globally generated (alternatively, one can use the simple connectedness of X to go from $K_{X/B}$ torsion to $K_{X/B}$ trivial). This a contradiction with what we have just proved.

The combination of our arguments in the current Step 6 show that, in particular, the isotriviality of π is equivalent to the triviality of K_B. Note that this was also proved by Campana, Oguiso and Peternell, see [COP, Theorem 3.1]. This completes the proof of item (ii).

Finally, one can now easily see that A is non-compact. Indeed, arguing by contradiction, A coincides with its maximal compact subgroup K; so A is a compact complex torus. We have seen that K acts freely and transitively on the fibers of π. Therefore, π defines an principal A–bundle. This implies that X is a holomorphic principal compact torus bundle over B. This leads to a contradiction as in the proof of item (ii). This completes the proof of the theorem. □

As a by-product of the proof of Theorem 5.1 we obtain another proof of Corollary 4.5.

Another proof of Corollary 4.5. Take a compact, simply connected complex manifold X in Fujiki class $\mathcal{C}$ with trivial canonical class and admitting a holomorphic rigid geometric structure. Theorem 5.1 shows that a maximal connected abelian subgroup A of Aut(X, ϕ) has closed orbits in X, and these orbits coincide with the fibers of the fibration π defined in the proof of item (i). Theorem 2.3 in [GW] (see also [Ho] for the particular case of $\mathbb{C}$–actions) proves that there is a compact *complex* torus K in A such that the A–orbits coincide with the K–orbits. Moreover, the induced K–action on X is free and the quotient map $X \longrightarrow X/K$ gives a holomorphic principal K–bundle. This implies that the fibration π is isotrivial, so it is a holomorphic principal bundle. This is in contradiction with item (ii) in Theorem 5.1. □

The following is an easy application of Theorem 5.1.

Corollary 5.2. *Let X be a compact complex manifold with trivial canonical bundle bearing a rigid holomorphic geometric structure. Assume that one of the following holds:*

- *The dimension of X is at most three, or*
- *The algebraic dimension of X is at most one.*

Then the fundamental group of X is infinite.

Proof. To prove by contradiction, assume that the fundamental group of X is finite. Replacing X by it universal cover we will assume that X is simply connected.

Case $a(X) \leq 1$.

Denote by A the maximal abelian subgroup of the automorphism group $\text{Aut}(X, \phi)$ in the statement of Theorem 2.7. Then Theorem 2.7 shows that A acts on X with orbits containing the fibers of the algebraic reduction of X. Since the algebraic dimension of X is at most one, the dimension of the A–orbits is at least $\dim X - 1$. If the action of A on X has an open orbit, the geometric structure ϕ is locally homogeneous and Proposition 2.8 furnishes a contradiction.

Let us now consider the case where the dimension of the A-orbits in X is $\dim X - 1$. Theorem 5.1 constructs a holomorphic submersion $\pi : X \longrightarrow B$ over a compact simply connected complex manifold B with globally generated canonical bundle K_B such that the fibers of π coincide with the A–orbits. Since the A-orbits in X coincide with the fibers of π, we have $\dim B = 1$. Therefore, B is a compact Riemann surface. This is a contradiction because the canonical bundle of the only simply connected compact Riemann surface $\mathbb{CP}^1$ is not globally generated.

Case $\dim X \leq 3$.

Give the previously treated case and Theorem 4.1, we only need to address the case where the threefold X has algebraic dimension two.

Then the A–orbits have dimension at least one. If the orbits have dimension two or three, we conclude as in the proof of the previous case. If the dimension of the A–orbits in X is one, then the fibration $\pi : X \longrightarrow B$ constructed in Theorem 5.1 is an elliptic fibration; its fibers are elliptic curves. Since the base B is simply connected the map from B to the moduli space of elliptic curves lifts to a holomorphic map from B to the upper-half plane (the Teichmüller space of elliptic curves). Since B is compact, it is a constant map. This implies that π is isotrivial, which contradicts (ii) in Theorem 5.1. This finishes the proof. □

The following proposition is a consequence of Corollary 5.2.

Proposition 5.3. *Let ϕ be a holomorphic projective connection on a compact complex manifold X with trivial canonical bundle. Then the following two hold:*

(i) *X admits a holomorphic affine connection ∇ which is projectively isomorphic to ϕ.*
(ii) *If X has algebraic dimension at most one, or X is a threefold, then the fundamental group of X is infinite.*

Proof. (i). Since $K_X = \mathcal{O}_X$, there is a (global) holomorphic torsionfree affine connection projectively equivalent to the projective connection ϕ [BD4, p. 7449, Lemma 5.6].

This can also be seen as a direct consequence of the results in [Gu] and [KO], which is explained below. There exists a holomorphic affine connection representing the projective connection ϕ if and only if the cocycle (3.2) in [KO] (defined as $d\log(\Delta_{ij})$, where $\{\Delta_{ij}\}$ is the 1-cocycle of the canonical bundle K_X) vanishes in $\mathrm{H}^1(X, \Omega_X)$; see the explicit formula (3.6) in [KO, p. 78–79]. This condition is satisfied if and only if the canonical bundle K_X admits a holomorphic affine connection (this coincides with the vanishing condition of the Atiyah class for K_X [At, p. 195, Theorem 5]; see also [Gu, p. 96–97] for an alternative approach). In the particular case where X is compact and Kähler, this is equivalent to the vanishing of the real first Chern class of X. In any case, the above condition is automatically satisfied when, as it happens in our situation, K_X is trivial.

(ii). Since a holomorphic affine structure is rigid geometric of affine type, this is indeed a particular case of Corollary 5.2. □

Remark 5.4. The last argument in the proof of Corollary 5.2 (involving the Teichmüller space) generalizes to the case where the fibers of the fibration π are polarized abelian varieties. Hence non-isotrivial fibrations π as in Theorem 5.1 do not exist if X is a projective manifold with trivial canonical bundle. The fibrations constructed in Theorem 5.1 neither exist when X is a compact Kähler Calabi-Yau manifold (see Theorem 3.1 in [TZ] or Section 4 in [Ber]). We conjecture that they do not exist if X is a compact simply connected complex manifold with trivial canonical bundle. This conjecture implies that compact simply connected complex manifolds with trivial canonical bundle do not admit holomorphic rigid geometric structures.

Acknowledgements

The authors are grateful to Frédéric Campana for very helpful comments and, in particular, for his suggestion to use Theorem 4.9 of [CP] in the proof of Proposition 3.3. The authors thank also Uri Bader, Stéphane Druel, Hsueh-Yung Lin, Laurent Meersseman, Mihnea Popa and Valentino Tosatti for very helpful discussions.

The first and third named authors thank ICTS Bangalore and IISc Bangalore for hospitality. The third-named author wishes to thank T.I.F.R. Mumbai for hospitality. This work has been supported by the French government through the UCAJEDI Investments in the Future project managed by the National Research Agency (ANR) with the reference number ANR2152IDEX201. The first-named author is partially supported by a J. C. Bose Fellowship. The second and fourth-named authors acknowledge the support of the French Agence Nationale de la Recherche (ANR) under reference ANR-21-CE40-0010.

References

[Am] A. M. Amores, Vector fields of a finite type G-structure, *Jour. Diff. Geom.* **14** (1980), 1–6.

[ACRT] B. Anthes, A. Cattaneo, S. Rollenske and A. Tomassini, $\partial\bar{\partial}$-Complex Symplectic and Calabi-Yau manifolds: Albanese map, deformations and period maps, *Annals of Global Analysyis and Geometry* **54** (2018), 377–398.

[At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.

[Bea] A. Beauville, Variétés kähleriennes dont la première classe de Chern est nulle, *Jour. Diff. Geom.* **18** (1983), 755–782.

[Ben] Y. Benoist, Orbites des structures rigides (d'après M. Gromov), *Progr. Math.* **145** (1997), 1–17.

[Ber] B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, *Ann. of Math.* **169** (2009), 531–560.

[BCHM] C. Birkar, P. Cascini, C.D. Hacon and J. McKernan, Existence of minimal models for varieties of log general type, *J. Amer. Math. Soc.* **23** (2010), 405–468.

[Bl] A. Blanchard, Sur les variétés analytiques complexes, *Ann. Sci. École. Nor. Sup.* **73** (1956), 157–202.

[Bo] F. A. Bogomolov, Kähler manifolds with trivial canonical class, *Izv. Akad. Nauk. SSSR* **38** (1974), 11–21; English translation in *Math. USSR Izv.* **8** (1974), 9–20.

[BD1] I. Biswas and S. Dumitrescu, Holomorphic Affine Connections on Non-Kähler manifolds, *Internat. J. Math.* **27** (2016), no. 11.

[BD2] I. Biswas and S. Dumitrescu, Fujiki class $\mathcal{C}$ and holomorphic geometric structures, *Internat. J. Math.* **31** (2020), no. 5.

[BD3] I. Biswas and S. Dumitrescu, Holomorphic Riemannian metric and fundamental group, *Bull. Soc. Math. Fr.* **147** (2019), 455–468.

[BD4] I. Biswas and S. Dumitrescu, Branched holomorphic Cartan geometries and Calabi-Yau manifolds, *Int. Math. Res. Not.* **23** (2019), 7428–7458.

[BDG] I. Biswas, S. Dumitrescu and H. Guenancia, A Bochner principle and its applications to Fujiki class $\mathcal{C}$ manifolds with vanishing first Chern class, *Comm. Contemp. Math.* **22** (2020), no. 6.

[BM] I. Biswas and B. McKay, Holomorphic Cartan geometries, Calabi–Yau manifolds and rational curves, *Diff. Geom. Appl.* **28** (2010), 102–106.

[Br] V. Brînzanescu, *Holomorphic vector bundles over compact complex surfaces*, Lectures Notes in Mathematics, 1624, Editors A. Dold, F. Takens, Springer, Berlin Heidelberg, 1996.

[BDPP] S. Boucksom, J.-P. Demailly, M. Paun and T.Peternell, The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, *Jour. Alg. Geom.* **22** (2013), 201–248.

[BGL] B. Bakker, H. Guenancia and C. Lehn, Algebraic approximation and the decomposition theorem for Kähler Calabi-Yau varieties, *Invent. Math.* **228** (2022), 1255–1308.

[CC] F. Campana and B. Claudon, Abelianity conjecture for special compact Kähler 3-folds. (English) Zbl 1337.32029 *Proc. Edinb. Math. Soc. II* **57** (2014) 55–78.

[CP] F. Campana and M. Paun, Foliations with positive slopes and birational stability of orbifold cotangent bundle, *Publ. Math. Inst. Hautes Études Sci.* **129** (2019), 1–49.

[Ca] J. Cao, On the approximation of Kähler manifolds by algebraic varieties, *Math. Ann.* **363** (2015), 393–422.

[Cl] B. Claudon, Positivité du cotangent logarithmique et conjecture de Shafarevich-Viehweg (d'après Campana, Paun, Taji,$\cdots$), *Séminaire Bourbaki*, **1105** (2015), 1–34.

[Cle] H. Clemens, Double solids, *Adv. Math.* **47** (1983), 107–230.

[CGG] B. Claudon, P. Graf and H. Guenancia, Numerical characterization of complex torus quotients, *Comment. Math. Helv.* **4** (2022), 769–799.

[CGGN] B. Claudon, P. Graf, H. Guenancia and P. Naumann, Kähler spaces with first zero Chern class: Bochner principle, Albanese map and fundamental groups, *J. Reine Angew. Math.* **786** (2022), 245–275.

[Cam] F. Campana, The class $\mathcal{C}$ is not stable by small deformations, *Math. Ann.* **290** (1991), 19–30.

[COP] F. Catanese, K. Oguiso and T. Peternell, On volume-preserving complex structures on real tori, *Kyoto J. Math.* **50** (2010), 753–775.

[DG] G. D'Ambra and M. Gromov, *Lectures on transformations groups: geometry and dynamics*, Surveys in Differential Geometry, Cambridge MA, (1991).

[DeGr] G. Dethloff and H. Grauert, Seminormal complex spaces, *Several complex variables VII*, Encyclopaedia Math. Sci. **74** (1994), 183–220.

[DHP] O. Das, C. Hacon and M. Păun, On the 4-dimensional minimal model program for Kähler varieties, arXiv preprint: 2205.12205.

[DPS] J.-P. Demailly, Th. Peternell and M. Schneider, Compact complex manifolds with numerically effective tangent bundles, *J. Algebr. Geom.* **3** (1994), 295–345.

[Dr] S. Druel, Quelques remarques sur la décomposition de Zariski divisorielle sur les variétés dont la première classe de Chern est nulle, *Math. Zeit.* **267** (2011), 413–423.

[Dr2] S. Druel, A decomposition theorem for singular spaces with trivial canonical class of dimension at most five, *Invent. Math.* **211** 1 (2018), 245-296.

[Du1] S. Dumitrescu, Meromorphic almost rigid geometric structures, *Geometry, Rigidity and Group Actions*, Editors Benson Farb and David Fisher, Chicago Lectures in Mathematics Series, 32–58.

[Du2] S. Dumitrescu, Structures géométriques holomorphes sur les variétés complexes compactes, *Ann. Scient. Ec. Norm. Sup* **34** (2001), 557–571.

[Du3] S. Dumitrescu, Killing fields of holomorphic Cartan geometries, *Monatsh. Math.* **161** (2010), 145–154.

[EGZ] P. Eyssidieux, V. Guedj and A. Zeriahi, Singular Kähler-Einstein metrics, *J. Amer. Math. Soc.* **22** (2009), 607–639

[Fu] A. Fujiki, On automorphism group on Kähler manifolds, *Invent. Math.* **44** (1978), 225–258.

[GHJ] M. Gross; D. Huybrechts and D. Joyce, *Calabi-Yau manifolds and related geometries*, Lectures at a summer school in Nordfjordeid, Norway, June 2001, Universitext Berlin Springer (2003), 239 p.

[GGK] D. Greb, H. Guenancia and S. Kebekus, Klt varieties with trivial canonical class: holonomy, differential forms, and fundamental groups, *Geom. Topol.* **23** (2019), 2051–2124.

[GKKP] D. Greb, S. Kebekus, S. Kovács and T. Peternell, Differential forms on log canonical spaces, *Publ. Math. Inst. Hautes Études Sci.* **114** (2011), 87–169.

[GKP] D. Greb, S. Kebekus and T. Peternell, Movable curves and semistable sheaves, *Int. Math. Res. Not. IMRN* **2** (2016), 536–570.

[GKP11] D. Greb, S. Kebekus and T. Peternell, Singular spaces with trivial canonical class, from "Minimal models and extremal rays" (J Kollár, O Fujino, S Mukai, N Nakayama, editors), *Adv. Stud. Pure Math.* **70**, Math. Soc. Japan, Tokyo (2016), 67–113

[GKP16] D. Greb, S. Kebekus and T. Peternell, Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties, *Duke Math. J.* **165** (2016), 1965–2004.

[GW] C. Gellhaus and T. Wurzbacher, Holomorphic group actions with many compact orbits, *Comment. Math. Helv.* **64** (1989), 639–649.

[Gh1] E. Ghys, Feuilletages holomorphes de codimension un sur les espaces homogènes complexes, *Ann. Fac. Sci. Math. Toulouse* **5** (1996), 493–519.

[Gro] M. Gromov, Rigid transfomations groups, Editors D. Bernard and Y. Choquet Bruhat, *Géométrie Différentielle*, **33**, Hermann, (1988), 65–139.

[Gu] R. Gunning, *On uniformization of complex manifolds: the role of connections*, *Math. Notes*, **22**, Princeton University Press, Princeton, 1966.

[Ha] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Hi] H. Hironaka, Bimeromorphic smoothing of a complex-analytic space, *Math. Inst. Warwick Univ.*, England, (1971).

[Ho] H. Holmann, Analytische periodische Strömungen auf kompakten komplexen Räumen, *Comment. Math. Helv.* **52** (1977), 251–257.

[HP2] A. Höring and Th. Peternell, Algebraic integrability of foliations with numerically trivial canonical bundle, *Invent. Math.* **216** (2019), 395–419.

[HP1] A. Höring and Th. Peternell, Minimal models for Kähler threefolds, *Invent. Math.* **203** (2016), 217–264.

[HW] J.-M. Hwang and E. Viehweg, Characteristic foliation on a hypersurface of general type in a projective symplectic manifold, *Compos. Math.* **146** 2 (2010), 497–506.

[KM] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics 134. Cambridge: Cambridge University Press 254 p. (1998).

[Ko] S. Kobayashi, *Differential geometry of complex vector bundles*, Publications of the Mathematical Society of Japan, 15. Kanô Memorial Lectures, 5, Princeton University Press, Princeton, NJ; Princeton University Press, Princeton, NJ, 1987.

[KO] S. Kobayashi and T. Ochiai, Holomorphic projective structures on compacty complex surfaces, *Math. Ann.* **249** (1980), 75–94.

[Lin] H.-Y. Lin, Algebraic approximations of compact Kähler manifolds of algebraic codimension 1, *Duke Math. Journal* **169** (2020), 2767–2826.

[Mo] B. Moishezon, On n dimensional compact varieties with n independent meromorphic functions, *Amer. Math. Soc. Transl.* **63** (1967), 51–77.

[Mon] D. Montgomery, Simply connected homogeneous spaces, *Proc. Amer. Math. Soc.*, **I** (1950), 467–469.

[No] K. Nomizu, On local and global existence of Killing vector fields, *Ann. of Math.* **72** (1960), 105–120.

[Pa] M. Păun, Regularity properties of the degenerate Monge–Ampère equations on compact Kähler manifolds, *Chin. Ann. Math. Ser. B* **29** (2008), 623–630.

[Pe] Th. Peternell, *Subsheaves in the tangent bundle: integrability, stability and positivity*, Lectures given at the "School on vanishing theorems and effective results in algebraic geometry", Trieste, 2000.

[Po] D. Popovici, Holomorphic deformations of balanced Calabi-Yau $\delta\bar{\delta}$-manifolds, *Ann. Instit. Fourier* **69** (2019), 673–728.

[Sh] R. Sharpe, *Differential Geometry: Cartan's generalization of Klein's Erlangen program*, New York: Springer, 1996.

[Ta] S. Takayama, Local simple connectedness of resolutions of log-terminal singularities, *Internat. J. Math.* **14** (2003), 825–836.

[To] V. Tosatti, Non-Kähler Calabi-Yau manifolds, *Contemp. Math.* **644** (2015), 261–277.

[TZ] V. Tosatti and Y. Zhang, Triviality of fibered Calabi-Yau manifolds without singular fibers, *Math. Res. Lett.* **23** (2014), 905–918.

[Ue] K. Ueno, *Classification theory of algebraic varieties and compact complet spaces*, Notes written in collaboration with P. Cherenack, Lecture Notes in Mathematics, Vol. 439, Springer-Verlag, Berlin-New York, 1975.

[Va] J. Varouchas, Kähler spaces and proper open morphisms, *Math. Ann.* **283** (1989), 13–52.

[W] J. Werner, Small resolutions of special three-dimensional varieties, translated by Simon Venter and Nicolar Addington, arXiv preprint: 2208.01383.

[Wan] J. Wang, On the Iitaka Conjecture $C_{n,m}$ conjecture for Kähler Fibre Spaces, *Ann. Fac. Sci. Toulouse Math.* **30** (2021), 813–897.

[Wa] H.-C. Wang, Complex Parallelisable manifolds, *Proc. Amer. Math. Soc.* **5** (1954), 771–776.

[Ya] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.

Mathematics Department, Shiv Nadar University, NH91, Tehsil Dadri, Greater Noida, Uttar Pradesh 201314, India

Email address: indranil.biswas@snu.edu.in, indranil29@gmail.com

Université Côte d'Azur, CNRS, Laboratoire J.-A. Dieudonné, Parc Valrose, F-06108 Nice Cedex 2, France

Email address: junyan.cao@unice.fr

Université Côte d'Azur, CNRS, Laboratoire J.-A. Dieudonné, Parc Valrose, F-06108 Nice Cedex 2, France

Email address: dumitres@unice.fr

Institut de Mathématiques de Toulouse; UMR 5219, Université de Toulouse; CNRS, UPS, 118 route de Narbonne, F-31062 Toulouse, France

Email address: henri.guenancia@math.cnrs.fr