Annali di Matematica (2013) 192:879–900
DOI 10.1007/s10231-012-0250-5

# Geometry of Kaluza–Klein metrics on the sphere $\mathbb{S}^3$

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Received: 18 August 2011 / Accepted: 10 January 2012 / Published online: 22 January 2012


**Abstract** In this paper, we introduce a new family of Riemannian metrics $\tilde{g}_{\lambda\mu\nu}$ on the three-sphere and study its geometric properties, starting from the description of their curvature. Such metrics, which include the standard metric $g_0$ and Berger metrics on $\mathbb{S}^3$ as special cases, are called "of Kaluza–Klein type", because they are induced in a natural way by the corresponding metrics defined on the tangent sphere bundle $T_1\mathbb{S}^2(\kappa)$. Each sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is a homogeneous space, and we obtain a full classification of its homogeneous structures. Moreover, we introduce and study a natural almost contact structure $(\varphi, \xi, \eta)$, for which $(\varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ is a (homogeneous) almost contact metric structure on the three-sphere. Finally, we see that for a suitable family of Kaluza–Klein type metrics $\tilde{g}_{ac}$ on $\mathbb{S}^3$, it is possible to construct a two-parameter family of harmonic morphisms from $(\mathbb{S}^3, \tilde{g}_{ac})$ to $\mathbb{S}^2(\kappa)$.

**Keywords** Kaluza–Klein metrics · Berger metrics · Unit tangent sphere bundles · Homogeneous Riemannian structures · Almost contact metric structures · Harmonic morphisms

**Mathematics Subject Classification (2000)** 53C25 · 53C30 · 53D15 · 53C43

## 1 Introduction

*Berger metrics* are well known in Riemannian geometry. They are defined as the canonical variation $g_\lambda$, $\lambda > 0$, of the standard metric $g_0$ of constant sectional curvature on $\mathbb{S}^3$, obtained deforming $g_0$ along the fibers of the Hopf fibration, that is, putting

$$g_\lambda|_{\xi_1^\perp} = g_0|_{\xi_1^\perp}, \qquad g_\lambda(\xi_1, \cdot) = \lambda\, g_0(\xi_1, \cdot),$$

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where $\xi_1$ denotes the standard Hopf vector field on $\mathbb{S}^3$ and $\xi_1^{\perp}$ is the orthogonal complement of $\xi_1$ with respect to $g_0$. Berger spheres $(\mathbb{S}^3, g_\lambda)$ have been studied under several different points of view. In particular, Berger spheres provide examples of homogeneous almost contact metric three-manifolds (see [20] and references therein).

Denoting by $\theta^1, \theta^2, \theta^3$ the 1-forms dual to a suitable orthonormal frame field $\xi_1, \xi_2, \xi_3$ (see (3.2) below) with respect to $g_0$, an arbitrary Berger metric $g_\lambda$ on $\mathbb{S}^3$ may be written as $g_\lambda = \lambda\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3$. It is then natural to generalize such a construction, allowing deformations of the standard metric $g_0$ not only in the direction of $\xi_1$, but also of $\xi_2$ and $\xi_3$. Thus, we consider on $\mathbb{S}^3$ the three-parameter family of Riemannian metrics of the form

$$\tilde{g}_{\lambda\mu\nu} = \lambda\,\theta^1 \otimes \theta^1 + \mu\,\theta^2 \otimes \theta^2 + \nu\,\theta^3 \otimes \theta^3, \quad \lambda, \mu, \nu > 0.$$

Clearly, all Berger metrics are of the above form. In fact, $g_\lambda = \tilde{g}_{\lambda 11}$. Riemannian metrics $\tilde{g}_{\lambda\mu\nu}$ turn out to be related to a class of well-known Riemannian $g$-natural metrics defined on the unit tangent sphere bundle $T_1\mathbb{S}^2(\kappa)$. For this reason, metrics $\tilde{g}_{\lambda\mu\nu}$ will be called "of Kaluza–Klein type."

The paper is organized in the following way. We shall report in Sect. 2 some basic information on Riemannian $g$-natural metrics on tangent and unit tangent sphere bundles. In Sect. 3, after constructing a covering map from $\mathbb{S}^3(\kappa/4)$ to $T_1\mathbb{S}^2(\kappa)$, we introduce a new family of Riemannian metrics $\tilde{g}_{\lambda\mu\nu}$ on the three-sphere, describing their Levi-Civita connection and curvature. Such metrics, which include the standard metric $g_0$ and Berger metrics on $\mathbb{S}^3$ as special cases, are called "of Kaluza–Klein type," as they are induced in a natural way by the corresponding metrics defined on the tangent sphere bundle $T_1\mathbb{S}^2(\kappa)$. We obtain in Sect. 4 a classification of homogeneous structures on spheres of Kaluza–Klein type, generalizing the results obtained by Gadea and Oubiña [20] on Berger spheres. In Sect. 5, we introduce a natural almost contact structure $(\varphi, \xi, \eta)$, for which $(\varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ is a (homogeneous) almost contact metric structure on the three-sphere. We study several contact metric properties ($H$-contact, Sasakian, and $\eta$-Einstein) on such spheres. In particular, we prove that $\tilde{g}_{\lambda\mu\nu}$ is a critical point of the functional "$I(g)$ = integral of the scalar curvature" restricted to the set of all associated metrics if and only if $(\mathbb{S}^3, \eta, \tilde{g}_{\lambda\mu\nu})$ is $\eta$-Einstein. Finally, in Sect. 6, we see that for a suitable two-parameter family of Kaluza–Klein type metrics $\tilde{g}_{ac}$ on $\mathbb{S}^3$, which includes the standard metric $g_0$ and Berger metrics as special cases, it is possible to construct a corresponding family of harmonic morphisms $h_{ac} : (\mathbb{S}^3, \tilde{g}_{ac}) \to \mathbb{S}^2(\kappa)$.

## 2 Riemannian $g$-natural metrics on the unit tangent sphere bundle

Let $(M, g)$ be an $n$-dimensional Riemannian manifold, $\nabla$ its Levi-Civita connection, and $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ its curvature tensor. At any point $(x, u)$ of its *tangent bundle* $TM$, the tangent space of $TM$ splits into the horizontal and vertical subspaces with respect to $\nabla$:

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)},$$

where $\mathcal{V}_{(x,u)}$ is the kernel of $d\pi_{(x,u)}$ and $\mathcal{H}_{(x,u)}$ is the kernel of the connection map at $(x, u)$. For any vector $X \in M_x$, the *horizontal lift* of $X$ is the unique vector $X^h \in \mathcal{H}_{(x,u)}$, such that $d\pi X^h = X$, where $\pi : TM \to M$ is the canonical projection. The *vertical lift* of a vector $X \in M_x$ to $(x, u) \in TM$ is a vector $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(df) = Xf$, for all functions $f$ on $M$. Here, we consider 1-forms $df$ on $M$ as functions on $TM$ (i.e., $(df)(x, u) = uf$).

The map $X \to X^h$ is an isomorphism between the vector spaces $M_x$ and $\mathcal{H}_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between $M_x$ and $\mathcal{V}_{(x,u)}$. Horizontal and vertical lifts of vector fields on $M$ can be defined in an obvious way and are uniquely defined vector fields on $TM$.

We also recall the definitions of the canonical vertical and the geodesic flow vector fields. The *canonical vertical vector field* on $TM$ is defined, in terms of local coordinates, by $\mathcal{U} = \sum_i u^i \partial/\partial u^i$. For a vector $u = \sum_i u^i (\partial/\partial x^i)_x \in M_x$, we see that $\mathcal{U}_{(x,u)} = \sum_i u^i (\partial/\partial x^i)^v_{(x,u)} = u^v_{(x,u)}$. The *geodesic flow vector field* on $TM$ is given by $\xi_{(x,u)} = \sum_i u^i (\partial/\partial x^i)^h_{(x,u)} = u^h_{(x,u)}$. Both $\mathcal{U}$ and $\xi$ do not depend on the choice of local coordinates and are globally defined on $TM$.

*Riemannian g-natural metrics* form a wide family of Riemannian metrics on $TM$, introduced by Kowalski and Sekizawa in [25]. Such metrics are the image of $g$ under first-order natural operators $D : S^2_+ T^* \rightsquigarrow (S^2 T^*)T$, which transform Riemannian metrics on manifolds into metrics on their tangent bundles, where $S^2_+ T^*$ and $S^2 T^*$ denote the bundle functors of all Riemannian metrics and all symmetric $(0, 2)$-tensors over $n$-manifolds, respectively.

The class of $g$-natural metrics, which depend on six smooth functions from $\mathbb{R}^+$ to $\mathbb{R}$, has been completely described in [5]. Given an arbitrary $g$-natural metric $G$ on the tangent bundle $TM$ of a Riemannian manifold $(M, g)$, there exist six smooth functions $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, such that

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2) g_x(X, Y) + (\beta_1 + \beta_3)(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2) g_x(X, Y) + \beta_2(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2) g_x(X, Y) + \beta_1(r^2) g_x(X, u) g_x(Y, u), \end{cases} \tag{2.1}$$

for every $u, X, Y \in M_x$, where $r^2 = g_x(u, u)$. If $\dim M = 1$, then $\beta_i = 0$ for all $i = 1, 2, 3$. Put

$$\begin{aligned} \phi_i(t) &= \alpha_i(t) + t\beta_i(t), \quad \alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t), \\ \phi(t) &= \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t), \end{aligned} \tag{2.2}$$

for all $t \in \mathbb{R}^+$. Then, a $g$-natural metric $G$ on $TM$ is Riemannian if and only if the following inequalities hold for all $t \in \mathbb{R}^+$:

$$\alpha_1(t) > 0, \qquad \phi_1(t) > 0, \qquad \alpha(t) > 0, \qquad \phi(t) > 0. \tag{2.3}$$

Throughout the paper, when we consider an arbitrary Riemannian $g$-natural metric $G$ on $TM$, we implicitly suppose that $G$ is defined via (2.1) by the functions $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$, satisfying (2.3).

In literature, there are some well-known Riemannian metrics on the tangent sphere bundle, which turn out to be special cases of Riemannian $g$-natural metrics (satisfying (2.3)). In particular:

- the *Sasaki metric* $g_S$ is obtained for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$.
- the *Cheeger–Gromoll metric* $g_{GC}$ [18,28] is obtained when $\alpha_2 = \beta_2 = 0, \alpha_1(t) = \beta_1(t) = -\beta_3(t) = \frac{1}{1+t}$ and $\alpha_3(t) = \frac{t}{1+t}$.
- *metrics of Cheeger-Gromoll type* $h_{m,r}$ [11] are obtained for $\alpha_1(t) = \frac{1}{(1+t)^m}, \alpha_3 = 1 - \alpha_1, \alpha_2 = \beta_2 = 0, \beta_1(t) = -\beta_3(t) = \frac{r}{(1+t)^m}$, where $m \in \mathbb{R}$ and $r \geq 0$.

- *Kaluza–Klein metrics*, as commonly defined on principal bundles (see [35], Subsection I.6 in [21], and Theorem 1, Definition 1 in the next Section), are obtained for $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$.
- The class of *metrics of Kaluza–Klein type*, which includes all examples above, is defined by the geometric condition of orthogonality between horizontal and vertical distributions [16,33]. Thus, a Riemannian $g$-natural metric $G$ is of Kaluza–Klein type if $\alpha_2 = \beta_2 = 0$.

Next, the *tangent sphere bundle of radius* $r > 0$ over a Riemannian manifold $(M, g)$ is the hypersurface $T_r M = \{(x, u) \in TM : g_x(u, u) = r^2\}$. The tangent space of $T_r M$, at a point $(x, u) \in T_r M$, is given by

$$(T_r M)_{(x,u)} = \{X^h + Y^v : X \in M_x, Y \in \{u\}^\perp \subset M_x\}. \tag{2.4}$$

When $r = 1$, $T_1 M$ is called *the unit tangent (sphere) bundle*.

By definition, *$g$-natural metrics on $T_1 M$* are the restrictions of $g$-natural metrics of $TM$ to its hypersurface $T_1 M$. As proved in [4], every Riemannian $g$-natural metric $\tilde{G}$ on $T_1 M$ is necessarily induced by a Riemannian $g$-natural $G$ on $TM$ of the special form

$$\begin{cases} G_{(x,u)}(X^h, Y^h) = (a + c)\, g_x(X, Y) + \beta\, g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) = b\, g_x(X, Y), \\ G_{(x,u)}(X^v, Y^v) = a\, g_x(X, Y), \end{cases} \tag{2.5}$$

for three real constants $a, b, c$ and a smooth function $\beta : [0, \infty) \to \mathbb{R}$. Such a metric $\tilde{G}$ on $T_1 M$ only depends on the value $d := \beta(1)$ of $\beta$ at 1, and conditions (2.3) for $G$ yield that $\tilde{G}$ is Riemannian if and only if

$$a > 0, \quad \alpha := a(a + c) - b^2 > 0 \quad \text{and} \quad \phi := a(a + c + d) - b^2 > 0. \tag{2.6}$$

Let now $\tilde{G}$ denote an arbitrary Riemannian $g$-natural metric on $T_1 M$. Using the Schmidt's orthonormalization process, a simple calculation yields that the vector field on $TM$ defined by

$$N^G_{(x,u)} = \frac{1}{\sqrt{(a + c + d)\phi}} [-bu^h + (a + c + d)u^v], \tag{2.7}$$

for all $(x, u) \in TM$, is unit normal at any point of $T_1 M$.

One then introduces the *tangential lift* $X^{t_G}$–with respect to $G$–of a vector $X \in M_x$ to $(x, u) \in T_1 M$ as the tangential projection of the vertical lift of $X$ to $(x, u)$ with respect to $N^G$, that is,

$$X^{t_G} = X^v - G_{(x,u)}\left(X^v, N^G_{(x,u)}\right)\, N^G_{(x,u)} = X^v - \sqrt{\frac{\phi}{a + c + d}}\, g_x(X, u)\, N^G_{(x,u)}. \tag{2.8}$$

If $X \in M_x$ is orthogonal to $u$, then $X^{t_G} = X^v$.

The tangent space $(T_1 M)_{(x,u)}$ of $T_1 M$ at $(x, u)$ is spanned by vectors of the form $X^h$ and $Y^{t_G}$, where $X, Y \in M_x$. In particular, $\tilde{\xi} = u^h$ is the *geodesic flow vector field* on $T_1 M$. The Riemannian metric $\tilde{G}$ on $T_1 M$, induced from $G$, is completely determined by formulae

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) \;\; = (a + c)\, g_x(X, Y) + d\, g_x(X, u) g_x(Y, u), \\ \tilde{G}_{(x,u)}(X^h, Y^{t_G}) \;\; = \tilde{G}_{(x,u)}(X^{t_G}, Y^h) = b\, g_x(X, Y), \\ \tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) = a\, g_x(X, Y) - \frac{\phi}{a+c+d}\, g_x(X, u) g_x(Y, u), \end{cases} \tag{2.9}$$

for all $(x, u) \in T_1M$ and $X, Y \in M_x$. It should be noted that, by (2.9), $b = 0$ holds if and only if horizontal and vertical lifts are orthogonal with respect to $\tilde{G}$. Moreover, condition $b = 0$ characterizes metrics on $T_1M$ induced by Riemannian $g$-natural metrics on $TM$ of Kaluza–Klein type.

We also explicitly remark that the Sasaki metric on $T_1M$ is the Riemannian $g$-natural metric of the form (2.9) with $a = 1$ and $b = c = d = 0$. Metrics of Cheeger-Gromoll type $h_{m,r}$ on the tangent bundle $TM$ induce on $T_1M$ the one-parameter family of Riemannian $g$-natural metrics, which does not depend on $r$, defined by $b = d = 0$ and $a = 1/2^m, c = 1 - a$. Metrics of Kaluza–Klein type on the tangent bundle $TM$ induce on $T_1M$ the three-parameter family of Riemannian $g$-natural metrics for which $b = 0$ (and $a, a + c > 0, a + c + d > 0$). Moreover, Kaluza–Klein metrics on the tangent bundle $TM$ induce on $T_1M$ the two-parameter family of Riemannian $g$-natural metrics for which $b = d = 0$ (and $a, a + c > 0$). For metrics of Kaluza–Klein type, (2.9) reduces to

$$\begin{cases} \tilde{G}_{(x,u)}(X^h, Y^h) = (a + c)\, g_x(X, Y) + d\, g(X, u)g(Y, u), \\ \tilde{G}_{(x,u)}(X^h, Y^{t_G}) = \tilde{G}_{(x,u)}(X^{t_G}, Y^h) = 0, \\ \tilde{G}_{(x,u)}(X^{t_G}, Y^{t_G}) = a(g_x(X, Y) - g_x(X, u)g_x(Y, u)). \end{cases} \tag{2.10}$$

The Levi-Civita connection of an arbitrary Riemannian $g$-natural metric on $T_1M$ was calculated in [1]. In the special case of a metric of Kaluza–Klein type, Proposition 5 of [1] yields at once the following.

**Proposition 1** *At $(x, u) \in T_1M$, the Levi-Civita connection $\tilde{\nabla}$ associated to an arbitrary metric $\tilde{G}$ of Kaluza–Klein type, as described in (2.10), is given by*

$$\begin{aligned}
(\tilde{\nabla}_{X^h} Y^h)_{(x,u)} &= (\nabla_X Y)^h_x + \left\{ -\frac{1}{2} R(X_x, Y_x)u - \frac{d}{2a} [g(Y_x, u)\, X_x + g(X_x, u)\, Y_x] \right.\\
&\qquad \left. + \frac{d}{a} g(Y_x, u) g(X_x, u)]u \right\}^{t_G}, \\
(\tilde{\nabla}_{X^h} Y^{t_G})_{(x,u)} &= \left\{ -\frac{a}{2(a+c)} R(Y_x, u)X_x + \frac{d}{2(a+c)} g(X_x, u)\, Y_x \right.\\
&\qquad + \frac{d}{2(a+c)(a+c+d)} [a\, g(R(X_x, u)Y_x, u) + (a + c)\, g(X_x, Y_x) \\
&\qquad \left. - (2(a + c) + d)\, g(X_x, u) g(Y_x, u)]u \right\}^h + (\nabla_X Y)^{t_G}_x, \\
(\tilde{\nabla}_{X^{t_G}} Y^h)_{(x,u)} &= \left\{ -\frac{a}{2(a+c)} R(X_x, u)Y_x + \frac{d}{2(a+c)} g(Y_x, u)\, X_x \right.\\
&\qquad + \frac{d}{2(a+c)(a+c+d)} [a\, g(R(X_x, u)Y_x, u) + (a + c)\, g(X_x, Y_x) \\
&\qquad \left. - (2(a + c) + d)\, g(X_x, u) g(Y_x, u)]u \right\}^h, \\
(\tilde{\nabla}_{X^{t_G}} Y^{t_G})_{(x,u)} &= -g(Y_x, u) X^{t_G}_x,
\end{aligned}$$

*for all $(x, u) \in T_1M$ and $X, Y$ vector fields on $M$.*

With regard to the curvature of Kaluza–Klein metrics described by (2.10) with $d = 0$, from [3], we easily deduce the following.

**Proposition 2** *Let $(M, g)$ be a Riemannian manifold and $\tilde{G}$ be a Kaluza–Klein metric on $T_1M$. Then:*

$$(i)\quad \tilde{R}(X^h, Y^h)Z^h = \left\{R(X,Y)Z + \frac{a}{4(a+c)}\left[R(R(Y,Z)u,u)X - R(R(X,Z)u,u)Y - 2R(R(X,Y)u,u)Z\right]\right\}^h + \left\{\frac{1}{2}(\nabla_Z R)(X,Y)u\right\}^{t_G},$$

$$(ii)\quad \tilde{R}(X^h, Y^{t_G})Z^h = \left\{-\frac{a}{2(a+c)}(\nabla_X R)(Y,u)Z\right\}^h + \left\{\frac{a}{4(a+c)}R(X, R(Y,u)Z)u + \frac{1}{2}R(X,Z)Y\right\}^{t_G},$$

$$(iii)\quad \tilde{R}(X^{t_G}, Y^{t_G})Z^{t_G} = \{g(Y,Z)X - g(X,Z)Y\}^{t_G},$$

*for all $x \in M$, $(x,u) \in T_1M$ and tangent vectors $X, Y, Z \in M_x$, where the operation of tangential lift from $M_x$ to $(x,u) \in T_1M$ is only applied to vectors of $M_x$, which are orthogonal to $u$.*

Let now $\tilde{G}_{ac}$ denote a Kaluza–Klein metric on $T_1\mathbb{S}^2(\kappa)$, determined by the real parameters $a, c$, satisfying $a, a+c > 0$. Let $\tilde{J}$ be the standard complex structure of $\mathbb{S}^2 \equiv \mathbb{CP}^1$. We can consider on $T_1\mathbb{S}^2(\kappa)$ the global $\tilde{G}$-orthogonal frame field $\{(\tilde{J}u)^v_{(x,u)}, (\tilde{J}u)^h_{(x,u)}, u^h_{(x,u)}\}$. Note that $||(\tilde{J}u)^v||^2_{\tilde{G}_{ac}} = a$ and $||(\tilde{J}u)^h||^2_{\tilde{G}_{ac}} = ||u^h||^2_{\tilde{G}_{ac}} = a + c$.

Taking into account (2.10) and Proposition 2, a straightforward calculation yields that the sectional curvatures on $(T_1\mathbb{S}^2(k), \tilde{G}_{ac})$ satisfy

$$K(u^h, (\tilde{J}u)^h) = \frac{1}{a+c}\left(\kappa - \frac{3a\kappa^2}{4(a+c)}\right), \quad K((\tilde{J}u)^v, V) = \frac{a\kappa^2}{4(a+c)^2},$$

for any $V \in \mathrm{Span}(u^h, (\tilde{J}u)^h)$. In particular, we have the following.

**Proposition 3** *If $(M, g) = (\mathbb{S}^2(\kappa), g_0)$ and $\tilde{G}_{ac}$ is a Kaluza–Klein metric on $T_1\mathbb{S}^2$, then $(T_1\mathbb{S}^2, \tilde{G}_{ac})$ has constant sectional curvature $\tilde{K}$ if and only if $\kappa a = (a+c)$. In this case, $\tilde{K} = 1/(4a) = \kappa/(4(a+c))$.*

## 3 Metrics of Kaluza–Klein type on $\mathbb{S}^3$

We start with the description of a covering map from $\mathbb{S}^3(\kappa/4)$ to $T_1\mathbb{S}^2(\kappa)$ in terms of quaternions, where $\mathbb{S}^n(c)$ denotes the standard sphere of constant sectional curvature $c$. Consider the quaternions algebra $\mathbb{H} = \{q = a_1 + a_2 i + a_3 j + a_4 k : a_1, a_2, a_3, a_4 \in \mathbb{R}\}$. Then, the unit sphere is given by $\mathbb{S}^3(1) = \{q \in \mathbb{H} : ||q|| = 1\}$. For any $q \in \mathbb{S}^3(1)$, the map $\varphi_q(z) := \bar{q}zq$ defines an orthogonal transformation of $\mathbb{H}$, which leaves invariant $\mathbb{R}^3 = \{q \in \mathbb{H} : q = a_2 i + a_3 j + a_4 k\}$. More precisely, the map

$$\Phi : \mathbb{S}^3(1) \to SO(3),\ q \mapsto \varphi_q,$$

describes $\mathbb{S}^3(1)$ as the universal covering of $SO(3)$. Explicitly, for any $q = (a_1+a_2 i+a_3 j+a_4 k) \in \mathbb{S}^3(1)$, one has

$$\varphi_q(i) = (a_1^2 + a_2^2 - a_3^2 - a_4^2)i + 2(a_2a_3 - a_1a_4)j + 2(a_1a_3 + a_2a_4)k,$$
$$\varphi_q(j) = 2(a_1a_4 + a_2a_3)i + (a_1^2 + a_3^2 - a_2^2 - a_4^2)j + +2(a_3a_4 - a_1a_2)k,$$
$$\varphi_q(k) = 2(a_2a_4 - a_1a_3)i + 2(a_1a_2 + a_3a_4)j + (a_1^2 + a_4^2 - a_2^2 - a_3^2)k.$$

Then, if we put $z_1 = (a_1 + ia_2)$, $z_2 = (a_3 + ia_4) \in \mathbb{C}$, the matrix of $SO(3)$ corresponding to $\varphi_q$ is given by

$$A_q = \begin{pmatrix} |z_1|^2 - |z_2|^2 & 2\mathrm{Im}(z_1z_2) & -2\mathrm{Re}(z_1z_2) \\ 2\mathrm{Im}(z_1\bar{z}_2) & \mathrm{Re}(\bar{z}_1^2 + z_2^2) & \mathrm{Im}(z_1^2 - \bar{z}_2^2) \\ 2\mathrm{Re}(z_1\bar{z}_2) & \mathrm{Im}(\bar{z}_1^2 + z_2^2) & \mathrm{Re}({z_1}^2 - \bar{z}_2^2) \end{pmatrix}$$

and we put $\Phi(q) = A_q$ for any $q \in \mathbb{S}^3(1)$. On the other hand, since
$$T_1\mathbb{S}^2(\kappa) = \{(x, u) \in \mathbb{R}^3 \times \mathbb{R}^3 : x \in \mathbb{S}^2(\kappa),\ u \perp x,\ ||u|| = 1\},$$
we can consider the diffeomorphism

$$\psi : T_1\mathbb{S}^2(\kappa) \to SO(3),\ \ (x, u) \mapsto (\sqrt{\kappa}\, x, \sqrt{\kappa}\, u \wedge x, u),$$

and the inverse diffeomorphism

$$\psi^{-1} : SO(3) \to T_1\mathbb{S}^2(\kappa),\ A = (c_1\, c_2\, c_3) \mapsto (x, u) = \left(\frac{1}{\sqrt{\kappa}} c_1, c_3\right),$$

where the $c_i$ denote the columns of $A \in SO(3)$. Then, introducing the homothety

$$\tau : \mathbb{S}^3(\kappa/4) \to \mathbb{S}^3(1),\ p \mapsto \frac{\sqrt{\kappa}}{2} p,$$

we have the covering map

$$F = \psi^{-1} \circ \Phi \circ \tau : \mathbb{S}^3(\kappa/4) \to T_1\mathbb{S}^2(\kappa),\ \ p = \frac{2}{\sqrt{\kappa}} q \mapsto \left(\frac{1}{\sqrt{\kappa}} \varphi_q(i), \varphi_q(k)\right). \tag{3.1}$$

More explicitly, if $p = (z_1, z_2)$, we have

$$F(z_1, z_2) = \left(\frac{\sqrt{\kappa}}{4} \left(|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2\right), \frac{\kappa}{4} \left(-2\mathrm{Re}(z_1z_2), z_1^2 - \bar{z}_2^2\right)\right).$$

We remark that the covering map $F$ was also constructed in [10] by a different approach. Consider now the unit vector fields $\{\xi_1, \xi_2, \xi_3\}$ on $\mathbb{S}^3(\kappa/4)$ defined, at any point $p = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3(\kappa/4)$, by

$$\begin{cases} \xi_1(p) = \frac{\sqrt{\kappa}}{2} ip = \frac{\sqrt{\kappa}}{2}(-x_2, x_1, -x_4, x_3), \\ \xi_2(p) = \frac{\sqrt{\kappa}}{2} jp = \frac{\sqrt{\kappa}}{2}(-x_3, x_4, x_1, -x_2), \\ \xi_3(p) = \frac{\sqrt{\kappa}}{2} kp = \frac{\sqrt{\kappa}}{2}(-x_4, -x_3, x_2, x_1). \end{cases} \tag{3.2}$$

The differential of the covering map $F : \mathbb{S}^3(\kappa/4) \to T_1\mathbb{S}^2(\kappa)$ defined by (3.1) was calculated in [10], proving the following formulae:

$$F_*\xi_1 = -\sqrt{\kappa}\, (\tilde{J}u)^v, \qquad F_*\xi_2 = u^h, \qquad F_*\xi_3 = (\tilde{J}u)^h, \quad \text{where} \quad u = \varphi_q(k). \tag{3.3}$$

Now, we consider on the standard three-sphere $(\mathbb{S}^3(\kappa/4), g_0)$ the 1-forms $\theta^1, \theta^2, \theta^3$ dual, with respect to $g_0$, to the unit vector fields $\{\xi_1, \xi_2, \xi_3\}$ on $\mathbb{S}^3(\kappa/4)$ defined by (3.2). Then, for any arbitrary choice of three real constants $\lambda, \mu, \nu > 0$, we can introduce a corresponding Riemannian metric on $\mathbb{S}^3$, given by

$$\tilde{g}_{\lambda\mu\nu} = \lambda\, \theta^1 \otimes \theta^1 + \mu\, \theta^2 \otimes \theta^2 + \nu\, \theta^3 \otimes \theta^3. \tag{3.4}$$

Let $\tilde{G}$ denote an arbitrary Riemannian $g$-natural metric of Kaluza–Klein type on $T_1\mathbb{S}^2(\kappa)$, determined by three real parameters $a, c, d$, satisfying $a, a+c, a+c+d > 0$. If we put

$$X_1 = (\tilde{J}u)^v, \qquad X_2 = (\tilde{J}u)^h, \qquad X_3 = u^h,$$

then $\{X_1, X_2, X_3\}$ is a global $\tilde{G}$-orthogonal frame field on $T_1\mathbb{S}^2(\kappa)$, with

$$||X_1||^2_{\tilde{G}} = a, \qquad ||X_2||^2_{\tilde{G}} = a+c, \qquad ||X_3||^2_{\tilde{G}} = a+c+d.$$

Let now $\eta^i$ denote the 1-forms $\tilde{G}$-dual of $X_i$, $i = 1, 2, 3$. Then, from (2.10), we get

$$\tilde{G} = \frac{1}{a}\,\eta^1 \otimes \eta^1 + \frac{1}{a+c}\,\eta^2 \otimes \eta^2 + \frac{1}{a+c+d}\,\eta^3 \otimes \eta^3$$

and the map $F$ described in (3.1) determines a corresponding Riemannian metric $F^*\tilde{G}$ on $\mathbb{S}^3$, given by

$$\tilde{g} = \frac{1}{a}\,(F^*\eta^1) \otimes (F^*\eta^1) + \frac{1}{a+c}\,(F^*\eta^2) \otimes (F^*\eta^2) + \frac{1}{a+c+d}\,(F^*\eta^3) \otimes (F^*\eta^3).$$

Now, (3.3) implies at once

$$F^*\eta^1 = -\sqrt{k}\,a\,\theta^1, \quad F^*\eta^2 = (a+c)\,\theta^3, \quad F^*\eta^3 = (a+c+d)\,\theta^2$$

and so,

$$\tilde{g} = ka\,\theta^1 \otimes \theta^1 + (a+c+d)\,\theta^2 \otimes \theta^2 + (a+c)\,\theta^3 \otimes \theta^3.$$

Thus, $\tilde{g}$ is exactly of the form (3.4). More precisely, $\tilde{g}$ is determined by three real parameters $\lambda, \mu, \nu > 0$, given by $\lambda = ka$, $\mu = a+c+d$, $\nu = a+c$. So, we proved the following.

**Theorem 1** *The covering map $F$ establishes a one-to-one correspondence between Riemannian metrics $\tilde{g}_{\lambda\mu\nu}$ on $\mathbb{S}^3$ of the form* (3.4) *and metrics $\tilde{G}$ of Kaluza–Klein type on $T_1\mathbb{S}^2(\kappa)$ of the form* (2.9)*, defined by parameters*

$$a = \lambda/\kappa, \quad b = 0, \quad c = \nu - \lambda/\kappa \quad \textit{and} \quad d = \mu - \nu.$$

Note that if $\lambda = \mu = \nu = 1$, then $(a + c =)a\kappa = 1$ and $d = 0$. So, by Proposition 2, $(T^1S^2(\kappa), \tilde{G})$ has constant sectional curvature $\kappa/4$. Correspondingly, $(\mathbb{S}^3, \tilde{g}) = (\mathbb{S}^3(\kappa/4), g_0)$.

Next, Berger metrics on $\mathbb{S}^3(\kappa/4)$ are of the form (3.4) with $\mu = \nu = 1$. Therefore, they correspond via $F$ to Kaluza–Klein metrics on $T_1\mathbb{S}^2(k)$, satisfying $a + c = 1$ and $d = 0$. Thus, we get the following result, which corresponds to Theorem 1.1 of [10].

**Corollary 1** (i) *The standard metric $g_0$ on $\mathbb{S}^3(\kappa/4)$, obtained for $\lambda = \mu = \nu = 1$, corresponds to the metric $\tilde{G}$ on $T_1\mathbb{S}^2(k)$ defined by $a = 1/\kappa$, $b = d = 0$, $c = 1 - 1/\kappa$.*

(ii) *The Berger metrics on $\mathbb{S}^3(\kappa/4)$, obtained for $\mu = \nu = 1$, correspond to metrics $\tilde{G}_a$ on $T_1\mathbb{S}^2(k)$ defined by $a = \lambda/\kappa$, $b = d = 0$, $c = 1 - \lambda/\kappa$.*

*Both metrics $\tilde{G}$ and $\tilde{G}_a$ are of Cheeger-Gromoll type.*

The above results justify the following.

**Definition 1** A Riemannian metric $g$ on $\mathbb{S}^3$ is said to be *of Kaluza–Klein type* if there exist three real constants $\lambda, \mu, \nu > 0$, such that $g = \tilde{g}_{\lambda\mu\nu}$ is described by (3.4). In particular, $g$ is said to be *a Kaluza–Klein metric* if $\mu = \nu$. By a *sphere of Kaluza–Klein type*, we shall mean the sphere $\mathbb{S}^3$ equipped with any Riemannian metric $\tilde{g}_{\lambda\mu\nu}$ described by (3.4).

Let now $\tilde{g}_{\lambda\mu\nu}$ be an arbitrary Riemannian metric of Kaluza–Klein type on $\mathbb{S}^3$. We know from the proof of Theorem 1 that $F : (\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu}) \to (T_1\mathbb{S}^2(\kappa), \tilde{G})$ is a local isometry, where $\tilde{G}$ is the $g$-natural metric of Kaluza–Klein type determined by parameters $a, c, d$, such that $\lambda = \kappa a,\ \mu = a + c + d,\ \nu = a + c$. Then, starting from Proposition 1, it is easy to describe the Levi-Civita connection $\nabla$ of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$. In particular, for the global field $\{\xi_1, \xi_2, \xi_3\}$ orthogonal with respect to $\tilde{g}_{\lambda\mu\nu}$, we obtain:

$$
\begin{aligned}
&\nabla_{\xi_1}\xi_1 = 0, && \nabla_{\xi_2}\xi_1 = \tfrac{\sqrt{\kappa}(\lambda-\mu+\nu)}{2\nu}\,\xi_3, && \nabla_{\xi_3}\xi_1 = -\tfrac{\sqrt{\kappa}(\lambda+\mu-\nu)}{2\mu}\,\xi_2,\\
&\nabla_{\xi_1}\xi_2 = \tfrac{\sqrt{\kappa}(\lambda-\mu-\nu)}{2\nu}\,\xi_3, && \nabla_{\xi_2}\xi_2 = 0, && \nabla_{\xi_3}\xi_2 = \tfrac{\sqrt{\kappa}(\lambda+\mu-\nu)}{2\lambda}\,\xi_1,\\
&\nabla_{\xi_1}\xi_3 = -\tfrac{\sqrt{\kappa}(\lambda-\mu-\nu)}{2\mu}\,\xi_2, && \nabla_{\xi_2}\xi_3 = -\tfrac{\sqrt{\kappa}(\lambda-\mu+\nu)}{2\lambda}\,\xi_1, && \nabla_{\xi_3}\xi_3 = 0.
\end{aligned}
\tag{3.5}
$$

It is well known that a vector field $V$ is Killing if and only if $\nabla V$ is skew-symmetric. In particular, from (3.5), we deduce at once that $\xi_1$ *(respectively,* $\xi_2, \xi_3$*) is a Killing vector field if and only if* $\mu = \nu$ *(respectively,* $\lambda = \nu, \lambda = \mu$*).*

We shall now describe the curvature of an arbitrary sphere of Kaluza–Klein type $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$. By (3.4), we have that

$$
e_1 := \frac{1}{\sqrt{\lambda}}\,\xi_1, \qquad e_2 := \frac{1}{\sqrt{\mu}}\,\xi_2, \qquad e_3 := \frac{1}{\sqrt{\nu}}\,\xi_3 \tag{3.6}
$$

is a global frame field on $\mathbb{S}^3$, orthonormal with respect to $\tilde{g}_{\lambda\mu\nu}$. Then, the covariant derivatives $\nabla_{e_i}e_j$ can be deduced at once from (3.5). With regard to the curvature components with respect to $\{e_1, e_2, e_3\}$, a standard calculation then gives

$$
\begin{aligned}
R_{1212} &= \frac{\kappa}{4}\left[\frac{(\lambda-\mu)^2-\nu^2}{\lambda\mu\nu} + 2\frac{\lambda+\mu-\nu}{\lambda\mu}\right], && R_{1213} = 0,\\
R_{1313} &= \frac{\kappa}{4}\left[\frac{(\lambda-\nu)^2-\mu^2}{\lambda\mu\nu} + 2\frac{\lambda-\mu+\nu}{\lambda\nu}\right], && R_{1223} = 0,\\
R_{2323} &= \frac{\kappa}{4}\left[\frac{(\nu-\mu)^2-\lambda^2}{\lambda\mu\nu} - 2\frac{\lambda-\mu-\nu}{\mu\nu}\right], && R_{1323} = 0.
\end{aligned}
\tag{3.7}
$$

Notice that the curvature of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ could also be deduced from the description of the curvature of $(T_1M, \tilde{G})$ given in [2], using the fact that $F : (\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu}) \to (T_1\mathbb{S}^2(\kappa), \tilde{G}_{acd})$ is an isometry when $\lambda = \kappa a$, $\mu = a + c + d$ and $\nu = a + c$.

Next, formulae (3.7) yield at once that the components of the Ricci tensor $\varrho$ with respect to $\{e_i\}$ are given by

$$
\begin{aligned}
&\varrho_{11} = \frac{\kappa\left[\lambda^2-(\mu-\nu)^2\right]}{2\lambda\mu\nu}, && \varrho_{22} = \frac{\kappa\left[\mu^2-(\lambda-\nu)^2\right]}{2\lambda\mu\nu}, && \varrho_{33} = \frac{\kappa\left[\nu^2-(\lambda-\mu)^2\right]}{2\lambda\mu\nu},\\
&\varrho_{12} = 0, && \varrho_{13} = 0, && \varrho_{23} = 0.
\end{aligned}
\tag{3.8}
$$

In particular, $\varrho_{ii}, i = 1, 2, 3$, are exactly the Ricci eigenvalues of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$, with corresponding eigenvectors $e_i$. A direct calculation then proves the following.

**Theorem 2** *Let* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *denote an arbitrary three-sphere of Kaluza–Klein type. Then, its Ricci curvature is described by* (3.8). *In particular:*

(i) $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *has three distinct Ricci eigenvalues if and only if* $\lambda, \mu, \nu$ *satisfy restrictions*

$$\lambda \neq \mu \neq \nu \neq \lambda, \quad \lambda \neq \mu + \nu, \quad \mu \neq \lambda + \nu, \quad \nu \neq \lambda + \mu. \tag{3.9}$$

(ii) $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *has two distinct Ricci eigenvalues if and only if either* $(ii)_a$ *exactly two of* $\lambda, \mu, \nu$ *coincide; or* $(ii)_b$ *one of them is the sum of the remaining two.*

(iii) *the three Ricci eigenvalues of* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *coincide (and* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *has constant sectional curvature) if and only if* $\lambda = \mu = \nu$.

Berger metrics are included in case $(ii)_a$ of Proposition 2. Also the metrics corresponding to case $(ii)_b$ have a special geometrical meaning. In fact, by (3.8), if either $\lambda = \mu + \nu$, $\mu = \lambda + \nu$ or $\nu = \lambda + \mu$, then two Ricci eigenvalues vanish.

A Riemannian manifold $(M, g)$ is said to be an *Ivanov-Petrova manifold* (shortly, an *IP manifold*) if its skew-symmetric curvature operator

$$R(\pi) = |g(X, X)g(Y, Y) - g(X, Y)^2|^{-1/2} R(X, Y),$$

where $\pi = \mathrm{Span}(X, Y)$, has constant eigenvalues on $G^+(2, n)$, the Grassmannian of all oriented 2-planes. $IP$ manifolds of dimension $n \geq 4$ are completely classified. A three-dimensional Riemannian manifold is $IP$ if and only if either it has constant curvature, or its Ricci tensor has rank 1 (see for example [15] and references therein). By Proposition 2, we then have at once the following.

**Corollary 2** *A sphere* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *of Kaluza–Klein type is an* $IP$ *manifold if and only if either the three parameters* $\lambda, \mu, \nu$ *coincide, or one of them is the sum of the remaining two.*

## 4 Homogeneity properties of spheres of Kaluza–Klein type

First of all, we remark that (3.5) implies

$$[\xi_1, \xi_2] = -\sqrt{\kappa}\, \xi_3, \qquad [\xi_2, \xi_3] = -\sqrt{\kappa}\, \xi_1, \qquad [\xi_3, \xi_1] = -\sqrt{\kappa}\, \xi_2.$$

Hence, Proposition 1.9 of [34] yields the following.

**Theorem 3** *Any sphere of Kaluza–Klein type* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *has a Lie group structure, unique up to isomorphisms, such that the vector fields* $\xi_1, \xi_2, \xi_3$ *are left invariant. In particular,* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *is a homogeneous space.*

More precisely, the signs of the coefficients in the Lie brackets above, together with the classification given in [26], yield that $\tilde{g}_{\lambda\mu\nu}$ corresponds to a left invariant Riemannian metric on $SU(2)$, as it could be expected.

We shall now study homogeneous structures on spheres of Kaluza–Klein type. We first recall some basic facts about homogeneous structures. For a detailed and systematic study, we refer to [34]. We start with the following.

**Definition 2** [34] Let $(M, g)$ be a Riemannian manifold. A *(Riemannian) homogeneous structure* on $(M, g)$ is a tensor field $T$ of type $(1, 2)$ on $M$, such that the connection $\tilde{\nabla} = \nabla - T$ satisfies

$$\tilde{\nabla} g = 0, \qquad \tilde{\nabla} R = 0, \qquad \tilde{\nabla} T = 0. \tag{4.1}$$

The geometric meaning of the existence of a homogeneous structure is explained by the renowned Theorem of Ambrose and Singer, which may be stated in the following way.

**Theorem 4** [7,34] *A connected, simply connected and complete Riemannian manifold* $(M, g)$ *is homogeneous if and only if it admits a homogeneous structure.*

Each homogeneous structure on a connected, simply connected and complete Riemannian manifold $(M, g)$ gives a representation of $M$ as quotient space $G/H$, where $G$ is a Lie group of isometries acting transitively on $M$. More precisely, $\tilde{\nabla} = \nabla - T$ turns out to be the canonical connection [24], associated to the corresponding reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of $G$.

It is well known that two different homogeneous structures $T_1$ and $T_2$ on $(M, g)$ give rise to either different decompositions $\mathfrak{g} = \mathfrak{m}_1 \oplus \mathfrak{h}_1 = \mathfrak{m}_2 \oplus \mathfrak{h}_2$ of the same Lie algebra $\mathfrak{g}$, or to representations of $(M, g)$ corresponding to some nonisomorphic Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. In particular, the following special case was pointed out by the first author in [14], in more general pseudo-Riemannian settings.

Suppose that $(M, g)$ is a connected, simply connected and complete Riemannian manifold, admitting a global orthonormal frame field $\{e_1, ..., e_n\}$ and some real constants $\gamma_{ij}^k$, satisfying $\nabla_{e_i} e_j = \sum_k \gamma_{ij}^k e_k$ for all $i, j$. Then, $M$ has a Lie group structure, unique up to isomorphisms, such that $e_i$ are left invariant vector fields and $g$ is left invariant. In fact, one can then define a special homogeneous structure $T$ on $(M, g)$, putting

$$T_{e_i} := \frac{1}{2} \sum_{jk} \gamma_{ij}^k \, e_j \wedge e_k, \tag{4.2}$$

for all $i$, where $e_j \wedge e_k(X) = g(e_j, X)e_k - g(e_k, X)e_j$, and the above conclusion easily follows from Proposition 1.9 of [34]. It must be noted that if $T$ satisfies (4.2), then $\tilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j - T(e_i, e_j) = 0$ for all indices $i, j$.

The above comments on homogeneous structures, though brief, show that it is a natural problem to classify all homogeneous structures of a given homogeneous space. Homogeneous structures on Berger spheres were obtained in [20]. We shall now obtain homogeneous structures on spheres of Kaluza–Klein type, proving the following.

**Theorem 5** *Let* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *denote an arbitrary sphere of Kaluza–Klein type.*

(i) *If* $\lambda, \mu, \nu$ *satisfy* (3.9), *then* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *only admits one homogeneous structure, given by*

$$T = \gamma_{12}^3 \, \theta^1 \otimes (\theta^2 \wedge \theta^3) + \gamma_{21}^3 \, \theta^2 \otimes (\theta^1 \wedge \theta^3) + \gamma_{31}^2 \, \theta^3 \otimes (\theta^1 \wedge \theta^2).$$

*This homogeneous structure corresponds to the Lie group structure of* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

(ii) *When* $\lambda = \mu + \nu$, *the sphere* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *admits a one-parameter family of homogeneous structures, given by*

$$T = t\theta^1 \otimes (\theta^2 \wedge \theta^3) + \sqrt{\frac{k}{\mu\nu(\mu + \nu)}} \left(\nu\theta^2 \otimes (\theta^1 \wedge \theta^3) - \mu\theta^3 \otimes (\theta^1 \wedge \theta^2)\right), \quad t \in \mathbb{R}.$$

*For* $t = 0$, $T$ *is the homogeneous structure corresponding to the Lie group structure of* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

*Homogeneous structures of* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *when either* $\mu = \lambda + \nu$ *or* $\nu = \lambda + \mu$ *can be easily deduced from this case, interchanging the coefficients* $\lambda, \mu, \nu$ *and the indices* 1, 2, 3 *in a suitable way.*

(iii) *When* $\lambda \neq \mu = \nu$, *the homogeneous structures on* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *are given by*

$$T = t\,\theta^1 \otimes (\theta^2 \wedge \theta^3) + \frac{1}{2\mu}\sqrt{\frac{\kappa}{\lambda}}\,\lambda\left(\theta^2 \otimes (\theta^1 \wedge \theta^3) - \theta^3 \otimes (\theta^1 \wedge \theta^2)\right), \qquad t \in \mathbb{R}.$$

*For* $t = \frac{1}{2\mu}\sqrt{\frac{\kappa}{\lambda}}(\lambda - 2\mu)$, $T$ *is the homogeneous structure corresponding to the Lie group structure of* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$.

*Homogeneous structures of* $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ *when either* $\mu \neq \lambda = \nu$ *or* $\nu \neq \lambda = \mu$ *easily follow from this case.*

*Proof* Let $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ be an arbitrary sphere of Kaluza–Klein type. We consider the corresponding global orthonormal frame field $\{e_1, e_2, e_3\}$ described by (3.6) and put $\nabla_{e_i} e_j = \sum_k \gamma_{ij}^k e_k$ for all indices $i, j$. By (3.5) and (3.6), $\gamma_{ij}^k$ is constant for all $i, j, k$. A homogeneous structure $T$ on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is uniquely determined by its components $T_{ij}^k$ with respect to $\{e_i\}$, defined by

$$T(e_i, e_j) := \sum_k T_{ij}^k\, e_k,$$

for all indices $i, j = 1, 2, 3$. We explicitly remark that the case of the special structure described in (4.2) occurs when $T_{ij}^k = \gamma_{ij}^k$ for all $i, j, k$. As $\{e_i\}$ is orthonormal, the Ambrose-Singer equations (4.1) are equivalent to the following system:

$$\begin{cases} T_{ij}^k + T_{ik}^j = 0, & i, j, k = 1, 2, 3, \\ \nabla_i \varrho_{jk} = -T_{ij}^r \varrho_{rk} - T_{ik}^r \varrho_{jr}, & i, j, k = 1, 2, 3, \\ T_{jk}^r T_{ir}^s - T_{ik}^r T_{jr}^s - T_{ij}^r T_{rk}^s = e_i(T_{jk}^s) + \gamma_{ir}^s T_{jk}^r - \gamma_{ij}^r T_{rk}^s - \gamma_{ik}^r T_{jr}^s, & i, j, k, s = 1, 2, 3. \end{cases} \tag{4.3}$$

Notice that in dimension three, the curvature is completely determined by the Ricci tensor. For this reason, in (4.3), we replaced the second Ambrose-Singer equation $\tilde{\nabla} R = 0$ by the equivalent condition $\tilde{\nabla}\varrho = 0$.

Now, using formulae (3.5) and (3.8), we easily find that the components of $\nabla\varrho$ satisfy

$$\nabla_i \varrho_{jk} = \gamma_{ij}^k (\varrho_{jj} - \varrho_{kk}), \tag{4.4}$$

for all indices $i, j, k$. On the other hand, we know from (3.8) that $\varrho_{ij} = 0$ whenever $i \neq j$. Henceforth, the second equation in (4.3) becomes

$$\nabla_i \varrho_{jk} = T_{ij}^k (\varrho_{jj} - \varrho_{kk}). \tag{4.5}$$

Comparing (4.4) with (4.5), we see that if $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ has three distinct Ricci eigenvalues, then $T_{ij}^k = \gamma_{ij}^k$ for all $i, j, k$. Thus, this sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza–Klein type only admits one homogeneous structure, namely, the one corresponding to the Lie group structure of the manifold. By Proposition 2, $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ has three distinct Ricci eigenvalues if and only if the coefficients $\lambda, \mu, \nu$ satisfy restrictions (3.9). Considering the $(0,3)$-tensor $T(X, Y, Z) := g(T(X, Y), Z)$ corresponding to $T$, we then obtain case *(i)* of Theorem 5.

We are now left with the case when the Ricci eigenvalues of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ are not all distinct. Obviously, if $\varrho_{11} = \varrho_{22} = \varrho_{33}$, then $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ has constant sectional curvature. By Proposition 2, this case only occurs when $\lambda = \mu = \nu$, that is, when $\tilde{g}_{\lambda\mu\nu} = \lambda g_0$. Homogeneous structures on the canonical sphere $(\mathbb{S}^3, g_0)$ were already classified in Theorem 3.1 of [20].

Hence, we only have to consider the case when a sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza–Klein type has exactly two distinct Ricci eigenvalues. Without loss of generality, we can assume

$\varrho_{11} \neq \varrho_{22} = \varrho_{33}$, which, by (3.8), holds if and only if either $\lambda \neq \mu = \nu$ or $\lambda = \mu + \nu$. We treat these two cases separately.

**First case:** $\lambda = \mu + \nu$. By (3.5) and (3.6), the Levi-Civita connection of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is now completely determined by

$$\gamma_{21}^3 = -\gamma_{23}^1 = \sqrt{\frac{k\,\nu}{\mu(\mu+\nu)}}, \qquad \gamma_{31}^2 = -\gamma_{32}^1 = -\sqrt{\frac{k\,\mu}{\nu(\mu+\nu)}}, \qquad \gamma_{ij}^k = 0 \quad \text{otherwise},$$

and (3.8), (4.4) yield that the only nonvanishing components of $\varrho$ and $\nabla\varrho$ are, respectively, given by

$$\varrho_{11} = \frac{2\kappa}{\mu+\nu} \quad \text{and} \quad \nabla_2\varrho_{13} = \nabla_2\varrho_{31} = \frac{2\kappa}{\mu+\nu}\gamma_{21}^3, \quad \nabla_3\varrho_{12} = \nabla_3\varrho_{21} = \frac{2\kappa}{\mu+\nu}\gamma_{31}^2.$$

Applying the first two equations of (4.3), formulae above easily yield that when $\lambda = \mu+\nu$, the components of an arbitrary homogeneous structure $T$ on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ satisfy

$$\begin{cases} T_{ik}^j = -T_{ij}^k, \\ T_{11}^2 = T_{11}^3 = T_{21}^2 = T_{31}^3 = 0, \\ T_{31}^2 = \gamma_{31}^2, \quad T_{21}^3 = \gamma_{21}^3 \end{cases} \tag{4.6}$$

and so, $T$ depends on three unknown functions $T_{12}^3, T_{22}^3, T_{32}^3$. We now take $(i,j,k,s) = (2,2,3,2)$ in the third equation of (4.3). Using (4.6), we find

$$0 = T_{22}^3(\gamma_{31}^2 - \gamma_{21}^3) = -\sqrt{\tfrac{k}{\mu\nu(\mu+\nu)}}(\mu+\nu)\,T_{22}^3 = -\sqrt{\tfrac{k}{\mu\nu(\mu+\nu)}}\lambda\,T_{22}^3$$

and so, $T_{22}^3 = 0$. Next, we take $(i,j,k,s) = (3,2,3,2)$ in the third equation of (4.3). Taking into account (4.6) and $T_{22}^3 = 0$, we obtain $(T_{32}^3)^2 = 0$, that is, $T_{32}^3 = 0$. Finally, using $T_{22}^3 = T_{32}^3 = 0$ and (4.6), we take $(i,j,k,s) = (1,1,3,2), (2,1,2,3), (3,1,3,2)$ in the third equation of (4.3) and we get $e_1(T_{12}^3) = 0$, $e_2(T_{12}^3) = 0$ and $e_3(T_{12}^3) = 0$, respectively. Henceforth, $T_{12}^3$ is a real constant. It is now easy to check that the third equation of (4.3) is satisfied for any choice of indices $i,j,k,s$. Summarizing, when $\lambda = \mu+\nu$, an arbitrary homogeneous structure $T$ on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is completely determined by its components

$$T_{12}^3 = -T_{13}^2 = t, \quad T_{21}^3 = -T_{23}^1 = \gamma_{21}^3, \quad T_{31}^2 = -T_{32}^1 = \gamma_{31}^2, \quad T_{ij}^k = 0 \quad \text{otherwise},$$

where $t$ is an arbitrary real constant. This gives case *(ii)* of Theorem 5.

**Second case:** $\lambda \neq \mu = \nu$. Proceeding as in the previous case, we first calculate the components of the Levi-Civita connection, the Ricci tensor and its covariant derivative with respect to $\{e_i\}$, obtaining

$$\gamma_{12}^3 = -\gamma_{23}^1 = \frac{1}{2\mu}\sqrt{\frac{\kappa}{\lambda}}(\lambda - 2\mu), \quad \gamma_{21}^3 = -\gamma_{23}^1 = -\gamma_{31}^2 = \gamma_{32}^1 = \frac{1}{2\mu}\sqrt{\frac{\kappa}{\lambda}}\lambda,$$

$$\gamma_{ij}^k = 0 \quad \text{otherwise},$$

$$\varrho_{11} = \frac{\kappa\lambda}{2\mu^2}, \qquad \varrho_{22} = \varrho_{33} = \frac{\kappa(2\mu-\lambda)}{2\mu^2}, \qquad \varrho_{ij} = 0 \quad \text{otherwise},$$

and

$$\nabla_2\varrho_{13} = \nabla_2\varrho_{31} = \frac{\kappa(\lambda-\mu)}{\mu^2}\,\gamma^3_{21}, \quad \nabla_3\varrho_{12} = \nabla_3\varrho_{21} = \frac{\kappa(\lambda-\mu)}{\mu^2}\,\gamma^2_{31},$$
$$\nabla_i\varrho_{jk} = 0 \quad \text{otherwise.}$$

The formulae above imply that the first two equations of (4.3) give again restrictions (4.6). Therefore, even when $\lambda \neq \mu = \nu$, the components of an arbitrary homogeneous structure $T$ on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ may be expressed by means of three unknown functions $T^3_{12}$, $T^3_{22}$, $T^3_{32}$. Taking into account (4.6), a long but straightforward calculation yields that the third equation of (4.3) is now equivalent to the following system of partial differential equations:

$$\begin{cases} e_1(T^3_{12}) = 0, & e_2(T^3_{12}) = 0, & e_3(T^3_{12}) = 0, \\ e_1(T^3_{22}) = -T^3_{32}(T^3_{12} - \gamma^3_{12}), & e_2(T^3_{22}) = -T^3_{22}T^3_{32}, & e_3(T^3_{22}) = -(T^3_{32})^2, \\ e_1(T^3_{32}) = T^3_{22}(T^3_{12} - \gamma^3_{12}), & e_2(T^3_{22}) = (T^3_{22})^2, & e_3(T^3_{22}) = T^3_{22}T^3_{32}. \end{cases} \tag{4.7}$$

More precisely, equations in the first row of (4.7) are obtained from the third equation of (4.3) taking $(i, j, k, s) = (1, 1, 3, 2), (2, 1, 2, 3), (3, 1, 3, 2)$, the ones in the second row taking $(i, j, k, s) = (1, 2, 3, 2), (2, 2, 2, 3), (3, 2, 3, 2)$, and the ones in the third row taking $(i, j, k, s) = (1, 3, 3, 2), (2, 3, 2, 3), (3, 3, 3, 2)$, respectively.

The first row of equations in (4.7) is equivalent to requiring that $T^3_{12}$ is a real constant. We shall now prove that the remaining equations in (4.7) yield $T^3_{22} = T^3_{32} = 0$. In fact, the above description of the Levi-Civita connection of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ easily implies

$$[e_1, e_2] = -\sqrt{\frac{\kappa}{\lambda}}\,e_3, \qquad [e_2, e_3] = -\sqrt{\frac{\kappa}{\lambda}}\,\frac{\lambda}{\mu}\,e_1, \qquad [e_3, e_1] = -\sqrt{\frac{\kappa}{\lambda}}\,e_2. \tag{4.8}$$

We now put $u := T^3_{22}$, $v := T^3_{32}$ and calculate the Lie brackets $[e_i, e_j](u)$, $[e_i, e_j](v)$ both using (4.7) and (4.8). Taking into account the constancy of both $T^3_{12}$ and $\gamma^3_{12}$, comparison between the corresponding Lie brackets gives

$$\begin{cases} (T^3_{12} - \gamma^3_{12})u^2 = \sqrt{\frac{\kappa}{\lambda}}\,u^2, \\ (T^3_{12} - \gamma^3_{12})v^2 = \sqrt{\frac{\kappa}{\lambda}}\,v^2, \\ (T^3_{12} - \gamma^3_{12})uv = \sqrt{\frac{\kappa}{\lambda}}\,uv, \\ u(u^2+v^2) = -(T^3_{12} - \gamma^3_{12})\sqrt{\frac{\kappa}{\lambda}}\,\frac{\lambda}{\mu}\,u, \\ v(u^2+v^2) = -(T^3_{12} - \gamma^3_{12})\sqrt{\frac{\kappa}{\lambda}}\,\frac{\lambda}{\mu}\,v. \end{cases} \tag{4.9}$$

Now, system (4.9) easily implies $u = v = 0$. In fact, if we assume $u \neq 0$ (respectively, $v \neq 0$), then the first and fourth (respectively, second and fifth) equations of (4.9) yield $u^2 + v^2 = -(\kappa/\mu) < 0$, which cannot occur. So, $T^3_{22} = T^3_{32} = 0$, while $T^3_{12}$ is an arbitrary real constant. Thus, we obtain case *(iii)* of Theorem 5 and this ends the proof □

*Remark 1* Case (i) of Theorem 5 emphasizes the fact that not all metrics of Kaluza–Klein type admit as many homogeneous structures as Berger metrics. In fact, each Berger sphere admits a one-parameter family of homogeneous structures [20]. On the other hand, it is evident that conditions (3.9) are incompatible with Berger metrics and, more in general, with Kaluza–Klein metrics. Case (ii) of Theorem 5 extends to Kaluza–Klein metrics the classification of homogeneous structures on the Berger spheres obtained in [20].

We end this Section by characterizing naturally reductive spaces among spheres of Kaluza–Klein type. We recall that, as proved in [6], a three-dimensional (simply connected) homogeneous Riemannian manifold $(M, g)$ is *naturally reductive* if and only if its Ricci tensor is *cyclic-parallel*, that is, when

$$(\nabla_X \varrho)(Y, Z) + (\nabla_Y \varrho)(Z, X) + (\nabla_Z \varrho)(X, Y) = 0, \tag{4.10}$$

for all vector fields $X, Y, Z$ tangent to $M$. Using the description of the Ricci tensor of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ and its covariant derivative given in formulae (3.8) and (4.4) respectively, a straightforward calculation permits to check when (4.10) is satisfied for a metric $\tilde{g}_{\lambda\mu\nu}$ of Kaluza–Klein type.

We find that this is never the case when $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ has three distinct eigenvalues, coherently with the results of [6]. On the other hand, if $\mu = \nu$, then all spheres $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ are naturally reductive. Finally, if $\lambda = \mu + \nu$, then $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is naturally reductive if and only if $\mu = \nu$. Thus, taking into account Definition 1, we have the following.

**Theorem 6** *A sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza–Klein type is naturally reductive if and only if at least two among parameters $\lambda, \mu, \nu$ coincide. In particular, if $\tilde{g}_{\lambda\mu\nu}$ is a Kaluza–Klein metric, then $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is naturally reductive.*

## 5 Almost contact metric geometry on spheres of Kaluza–Klein type

We first recall some basic facts about almost contact and contact metric manifolds, referring to [13] for further information. An *almost contact structure* on a $(2n+1)$-dimensional smooth manifold $M$ is a triple $(\varphi, \xi, \eta)$, where $\varphi$ is a $(1, 1)$-tensor, $\xi$ a global vector field and $\eta$ a 1-form, such that

$$\eta(\xi) = 1, \qquad \varphi^2 = -Id + \eta \otimes \xi. \tag{5.1}$$

Then, $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$ and $\varphi$ has rank $2n$. The one-form $\eta$ is said to be a *contact form* if $\eta \wedge (d\eta)^n \neq 0$. A Riemannian metric $g$ on $M$ is called *compatible* with the almost contact structure $(\varphi, \xi, \eta)$ if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{5.2}$$

Given an almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$, one considers $M^{2n+1} \times \mathbb{R}$ and the almost complex structure defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right).$$

The almost contact structure $(\varphi, \xi, \eta)$ is said to be *normal* if and only if the almost complex structure $J$ is integrable. A necessary and sufficient condition for integrability of $J$ is the vanishing of its Nijenhuis tensor, which, expressed in terms of the Nijenhuis tensor of $\varphi$, gives

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0. \tag{5.3}$$

Next, consider the 2-form $\Phi(X, Y) := g(X, \varphi Y)$. A normal almost contact metric manifold is called *quasi-Sasakian* if $d\Phi = 0$ [12]. If

$$\Phi(X, Y) = (d\eta)(X, Y), \tag{5.4}$$

then $\eta$ is said to be a *contact form* on $M$, $\xi$ the *Reeb vector field*, $g$ an *associated metric*, and $(M, \eta, g)$ (or $(M, \varphi, \xi, \eta, g)$) is called a *contact metric manifold*. We recall that the set $\mathcal{A}(\eta)$, of all associated metrics of a given contact form $\eta$, is infinite dimensional (see p. 37 of [13]). Moreover, each associated metric has the same volume element $v_g = \frac{(-1)^n}{2^n n!}\eta \wedge (\mathrm{d}\eta)^n$.

A contact metric manifold is said to be *K-contact* if $\xi$ is a Killing vector field, or equivalently, the tensor $h := \mathcal{L}_\xi \varphi$ vanishes. A *Sasakian manifold* is a normal contact metric manifold. Any Sasakian manifold is $K$-contact, and the converse holds for three-dimensional contact metric manifolds.

In [32], the second author introduced and studied *H-contact manifolds*. These are contact metric manifolds whose Reeb vector field $\xi$ is a critical point for the energy functional restricted to the space $\mathfrak{X}^1(M)$ of all unit vector fields on $(M, g)$, considered as smooth maps from $(M, g)$ into its unit tangent sphere bundle $T_1 M$ equipped with the Sasaki metric. It was proved in [32] that $(M, \varphi, \xi, \eta, g)$ is $H$-contact if and only if $\xi$ is an eigenvector of the Ricci operator. In particular, Sasakian and $K$-contact manifolds are $H$-contact.

We now consider an arbitrary Riemannian metric $\tilde{g}_{\lambda\mu\nu}$ of Kaluza–Klein type on the three-sphere $\mathbb{S}^3$, as described in (3.4). Let $\{e_1, e_2, e_3\}$ be the global orthonormal frame field on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ given by (3.6), and $\{\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3\}$ the basis of one-forms dual to $\{e_1, e_2, e_3\}$ with respect to $\tilde{g}_{\lambda\mu\nu}$. We put

$$\eta := \bar{\theta}^1, \qquad \xi := e_1, \qquad \varphi := \bar{\theta}^3 \otimes e_2 - \bar{\theta}^2 \otimes e_3. \tag{5.5}$$

Then, a straightforward calculation proves that conditions (5.1) are fulfilled and so (5.5) defines an almost contact structure on $\mathbb{S}^3$. Moreover, such structure and $\tilde{g}_{\lambda\mu\nu}$ satisfy (5.2). Hence, we have the following.

**Proposition 4** *Any sphere of Kaluza–Klein type $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is an almost contact metric manifold, whose almost contact structure $(\varphi, \xi, \eta)$ is described by* (5.5).

We explicitly remark that, by (5.2) and (5.5), $\{e_2, e_3 = -\varphi e_2\}$ is an orthonormal frame field on the contact distribution Ker $\eta$. Next, from (3.5) and (3.6), we easily get

$$[e_1, e_2] = -\sqrt{\frac{\kappa\nu}{\lambda\mu}}\, e_3, \qquad [e_2, e_3] = -\sqrt{\frac{\kappa\lambda}{\mu\nu}}\, e_1, \qquad [e_3, e_1] = -\sqrt{\frac{\kappa\mu}{\lambda\nu}}\, e_2. \tag{5.6}$$

We can now calculate $d\eta$ and we find

$$d\eta(\xi, \cdot) = 0, \qquad d\eta(e_2, e_3) = -d\eta(e_3, e_2) = \frac{1}{2}\sqrt{\frac{\kappa\lambda}{\mu\nu}} > 0,$$

which easily implies $\eta \wedge d\eta \neq 0$ and

$$(d\eta)(X, Y) = \frac{1}{2}\sqrt{\frac{\kappa\lambda}{\mu\nu}}\, \Phi(X, Y), \tag{5.7}$$

for all $X, Y$ tangent to $M$. Thus, (5.4) holds if and only if $\kappa\lambda = 4\mu\nu$. Moreover, taking into account the description of the Ricci tensor of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ we gave in (3.8), we see that $\xi$ is a Ricci eigenvector. Hence, we get the following.

**Theorem 7** *The one-form $\eta$ of the almost contact metric manifold $(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ is always a contact form. In particular, it is a contact metric manifold if and only if $\kappa\lambda = 4\mu\nu$. In this case, $(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ is also H-contact.*

By Theorem 7, for any real constant $\kappa > 0$, we find a two-parameter family of ($H$-)contact structures of Kaluza–Klein type on $\mathbb{S}^3$.

The above description of $d\eta$ also permits to easily calculate condition (5.3), characterizing normal almost contact structures. We also recall that an *$\alpha$-Sasakian manifold* is an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, satisfying

$$(\nabla_X \varphi)Y = \alpha\big(g(X, Y)\xi - \eta(Y)X\big), \tag{5.8}$$

for all tangent vector fields $X, Y$, where $\alpha$ is a real constant. 1-Sasakian manifolds are Sasakian manifolds. Using (3.5), (3.6), and (5.5), a direct calculation gives $\nabla_\xi \varphi = 0$ and

$$\begin{aligned}
&(\nabla_{e_2}\varphi)\xi = -\tfrac{\sqrt{\kappa}(\lambda-\mu+\nu)}{2\sqrt{\lambda\mu\nu}}e_2, \quad (\nabla_{e_2}\varphi)e_2 = \tfrac{\sqrt{\kappa}(\lambda-\mu+\nu)}{2\sqrt{\lambda\mu\nu}}\xi, \quad (\nabla_{e_2}\varphi)e_3 = 0,\\
&(\nabla_{e_3}\varphi)\xi = -\tfrac{\sqrt{\kappa}(\lambda+\mu-\nu)}{2\sqrt{\lambda\mu\nu}}e_3, \quad (\nabla_{e_3}\varphi)e_2 = 0, \qquad\qquad\quad (\nabla_{e_3}\varphi)e_3 = \tfrac{\sqrt{\kappa}(\lambda+\mu-\nu)}{2\sqrt{\lambda\mu\nu}}\xi.
\end{aligned}$$

Then, also taking into account Eq. (5.7) and Theorem 7, we obtain the following.

**Theorem 8** *For the almost contact structure $(\varphi, \xi, \eta)$ on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$, described in (5.5), the following properties are equivalent:*

(i) $(\varphi, \xi, \eta)$ *is normal;*
(ii) $(\varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ *is quasi-Sasakian.*
(iii) $(\varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ *is $\alpha$-Sasakian. In this case, $\alpha = \frac{\sqrt{\kappa\lambda}}{2\mu}$.*
(iv) $\mu = \nu$*, that is, $\tilde{g}_{\lambda\mu\nu}$ is a Kaluza–Klein metric.*

*In particular, $(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ is Sasakian if and only if $\mu = \nu$ and $\kappa\lambda = 4\mu^2$.*

By Theorem 8, there exists a one-parameter family of Sasakian Kaluza–Klein structures on the three-sphere. The special case of the standard Sasakian structure of the canonical sphere $(\mathbb{S}^3(1), g_0)$ is obtained when $(\lambda = \nu =)\mu = \kappa/4$.

As proved in Theorem 5.2 of [20], all Berger metrics, equipped with their natural almost contact structure, are $\alpha$-Sasakian. Theorem 8 extends this result and characterizes Kaluza–Klein spheres as spheres of Kaluza–Klein type that carry a natural $\alpha$-Sasakian structure.

An almost contact manifold $(M, \varphi, \xi, \eta)$ is said to be *homogeneous* if there exists a connected Lie group $G$ of diffeomorphisms acting transitively on $M$ and leaving $\eta$ invariant. If a Riemannian metric $g$ satisfies (5.2) and $G$ is a group of isometries, then $(M, \eta, g)$ is said to be a *homogeneous almost contact metric manifold*. Following [23] (see also [20]), this is equivalent to requiring that there exists a homogeneous structure $T$ on $(M, g)$, such that the corresponding canonical connection $\tilde{\nabla} = \nabla - T$ satisfies $\tilde{\nabla}\varphi = 0$ (and so, $\tilde{\nabla}\eta = 0$ and $\tilde{\nabla}\xi = 0$).

In the case of the almost contact structure (5.5) on a sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza–Klein type, such condition is obviously satisfied. In fact, by Theorem 3, each sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ is a Lie group, and vector fields $e_1, e_2, e_3$ are left invariant. Henceforth, by (5.5), $\varphi$ is also left invariant, that is, $\tilde{\nabla}\varphi = 0$, where $\tilde{\nabla}$ is the canonical connection associated to the special homogeneous structure corresponding to the Lie group structure of $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$. Thus, we proved the following.

**Theorem 9** *Every sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza–Klein type, equipped with the almost contact structure $(\varphi, \xi, \eta)$ described in (5.5), is a homogeneous almost contact metric manifold. If $\kappa\lambda = 4\mu\nu$, then $(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ is a homogeneous contact metric manifold.*

The second author studied simply connected homogeneous contact metric three-manifolds [31], showing that such manifolds are three-dimensional Lie groups equipped with a left invariant contact metric structure and completely classifying these Lie groups. The above explicit description of homogeneous contact metric structures on $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ can be considered as complementary to the classification obtained in [31].

We recall that an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be *$\eta$-Einstein* if its Ricci tensor $\varrho$ is of the form

$$\varrho = Ag + B\eta \otimes \eta, \tag{5.9}$$

where $A, B$ are two smooth functions on $M$. Using (3.8) and (5.5), we can easily solve Eq. (5.9) for an arbitrary sphere $(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ of Kaluza–Klein type. We obtain the following.

**Proposition 5** *A sphere $(\mathbb{S}^3, \tilde{g}_{\lambda\mu\nu})$ of Kaluza–Klein type, equipped with the almost contact structure $(\varphi, \xi, \eta)$ described in (5.5), is $\eta$-Einstein if and only if one of the following conditions holds:*

(i) $\lambda = \mu + \nu$. *In this case,* $\varrho = \dfrac{2\kappa}{\lambda}\eta \otimes \eta$.
(ii) $\mu = \nu$. *In this case,* $\varrho = \dfrac{\kappa(2\mu - \lambda)}{2\mu^2}\tilde{g}_{\lambda\mu\nu} + \dfrac{\kappa(\lambda - \mu)}{2\mu^2}\eta \otimes \eta$.

*In particular, all Berger spheres, equipped with their natural almost contact structures, are $\eta$-Einstein.*

Next, given a compact orientable manifold $M$, a Riemannian metric $g$ on $M$ is a critical point of the integral of the scalar curvature, $I(g) = \int_M r\, v_g$, defining a functional on the set $\mathcal{M}_1$ of all Riemannian metrics of the same total volume, if and only if $g$ is an Einstein metric. This famous result, due to Hilbert (1915), may be found for example in [29]. Now, let $(M, \eta)$ be a compact contact three-manifold. In [30], the second author considered the functional $I(g)$ restricted to the set $\mathcal{A}(\eta)$ of all associated metrics and found a weaker critical point condition. In fact, if $(M, \eta)$ is a compact contact three-manifold, then $g \in \mathcal{A}(\eta)$ is a critical point of the functional $I$ restricted to $\mathcal{A}(\eta)$ if and only if $\nabla_\xi h = 0$.

We now consider an arbitrary almost contact metric structure $(\varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ on $\mathbb{S}^3$. Using (5.6), it is easily seen that $h = \mathcal{L}_\xi \varphi$ is determined by

$$h(\xi) = 0, \qquad h(e_2) = \sqrt{\frac{\kappa\lambda}{4\mu\nu}}\frac{\nu - \mu}{\lambda}e_2, \qquad h(e_3) = -\sqrt{\frac{\kappa\lambda}{4\mu\nu}}\frac{\nu - \mu}{\lambda}e_3.$$

We can now use the above formulae and (3.5) to calculate $\nabla_\xi h$. We find that $\nabla_\xi h = 0$ if and only if either $\mu = \nu$ (and so, $h = 0$), or $\lambda = \mu + \nu$. In particular, if we restrict to the contact metric case and take into account Proposition 5, the characterization proved in [30] leads to the following.

**Theorem 10** *Consider on $\mathbb{S}^3$ an arbitrary contact metric structure $(\varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$, that is, $(\varphi, \xi, \eta)$ is described by (5.5) and $\kappa\lambda = 4\mu\nu$. Then, $\tilde{g}_{\lambda\mu\nu}$ is a critical point of the functional $I$ restricted to $\mathcal{A}(\eta)$ if and only if $(\mathbb{S}^3, \varphi, \xi, \eta, \tilde{g}_{\lambda\mu\nu})$ is $\eta$-Einstein.*

*Remark 2* All Berger spheres are homogeneous almost contact metric manifolds (see Theorem 5.2 in [20]). Theorem 9 extends this result to arbitrary spheres of Kaluza–Klein type, providing a three-parameter family of three-dimensional homogeneous almost contact metric manifolds. Our study of almost contact metric geometry of spheres of Kaluza–Klein clarifies which properties of Berger spheres are specific and which ones hold for broader families of metrics on $\mathbb{S}^3$.

We also point out the fact that for a metric of Kaluza–Klein type on $\mathbb{S}^3$, the roles of Hopf vector fields $\xi_1, \xi_2, \xi_3$ are perfectly interchanging. Hence, the construction and results of this Section can be promptly adapted to introduce and study some almost contact and contact metric structures on $\mathbb{S}^3$, whose Reeb vector field is collinear to either $\xi_2$ or $\xi_3$.

## 6 Harmonic morphisms from Kaluza–Klein spheres into $\mathbb{S}^2$

We briefly recall the definition of harmonic maps and morphisms. A (smooth) map $f : (M', g') \to (M, g)$ between two Riemannian manifolds is said to be *harmonic* if $f$ is a critical point of the energy functional $\mathcal{E}(f, \Omega) := \frac{1}{2}\int_\Omega ||df||^2 dv_{g'}$, for any compact domain $\Omega \subset M'$. Harmonic maps have been characterized as maps whose *tension field* $\tau(f) = \mathrm{tr}\nabla df$ vanishes.

A map $\varphi : (M', g') \to (M, g)$ is a *harmonic morphism* if it pulls back (local) harmonic functions to harmonic functions, that is, for any open set $U$ of $M$ with $\varphi^{-1}(U) \neq \emptyset$ and any harmonic function $f$ on $(U, g|_U)$, the map $f \circ \varphi$ is a harmonic function on $(\varphi^{-1}(U), g'|_{\varphi^{-1}(U)})$. A fundamental characterization states that *a smooth map is a harmonic morphism if and only if it is harmonic and horizontally weakly conformal* [19,22]. In general, it is not easy to construct examples of harmonic morphisms, since they give rise to an overdetermined, nonlinear system of partial differential equations. We may refer to the monograph [9] for a survey on harmonic morphisms.

Consider now $(TM, G)$ and $(T_1M, \tilde{G})$ equipped with arbitrary Riemannian $g$-natural metrics. In [17], the present authors studied necessary and sufficient conditions for the canonical projections $\pi : (TM, G) \to (M, g)$ and $\pi_1 : (T_1M, \tilde{G}) \to (M, g)$ to be harmonic morphisms. In particular, we proved the following result.

**Theorem 11** [17] *Let $(M, g)$ be a Riemannian manifold of dimension $n > 1$ and $(T_1M, \tilde{G})$ its unit tangent bundle, equipped with an arbitrary Riemannian g-natural metric $\tilde{G}$. Then, the canonical projection $\pi_1 : (T_1M, \tilde{G}) \to (M, g)$ is a harmonic morphism if and only if $\tilde{G}$ is a Kaluza–Klein metric.*

Next, we recall the description of the Hopf map in terms of quaternions. Using the notations introduced in Sect. 3, the Hopf map can be described as follows:

$$h : \mathbb{S}^3(1) \to \mathbb{S}^2(1), \quad q \mapsto \varphi_q(i) = \left(2z_1\bar{z}_2, |z_1|^2 - |z_2|^2\right).$$

Generalizing the above construction to the three-sphere $\mathbb{S}^3(\kappa/4)$ and the two-sphere $\mathbb{S}^2(\kappa)$, the Hopf map is given by

$$\tilde{h} : \mathbb{S}^3(\kappa/4) \to \mathbb{S}^2(\kappa), \quad p \mapsto \frac{1}{\sqrt{\kappa}}\varphi_q(i),$$

where $q = \frac{\sqrt{\kappa}}{2}p \in \mathbb{S}^3(1)$. Finally, denoting by $\pi_1 : T_1\mathbb{S}^2(\kappa) \to \mathbb{S}^2(\kappa)$ the canonical projection, the Hopf map is explicitly given by

$$\begin{aligned}\tilde{h} = \pi_1 \circ F : \mathbb{S}^3(\kappa/4) &\to \mathbb{S}^2(\kappa)\\ (z_1, z_2) &\mapsto \frac{\sqrt{\kappa}}{4}\left(2z_1\bar{z}_2, |z_1|^2 - |z_2|^2\right).\end{aligned}$$

Let now $\tilde{G}_{ac}$ denote a Kaluza–Klein metric on $T_1\mathbb{S}^2(\kappa)$, determined by two real parameters $a$ and $c$ satisfying $a, a+c > 0$. We put $\mathcal{V} := \mathrm{Span}((\tilde{J}u)^v)$ and $\mathcal{H} := \mathrm{Span}(u^h, (\tilde{J}u)^h)$, and denote by $\tilde{G}_a$ the Kaluza–Klein metric of constant sectional curvature $K = 1/(4a)$, that is, by Proposition 3 and (2.10),

$$\begin{cases} (\tilde{G}_a)_{(x,u)}(X^h, Y^h) &= \kappa a\, g_x(X, Y), \\ (\tilde{G}_a)_{(x,u)}(X^h, Y^{t_G}) &= (\tilde{G}_a)_{(x,u)}(X^{t_G}, Y^h) = 0, \\ (\tilde{G}_a)_{(x,u)}(X^{t_G}, Y^{t_G}) &= a(g_x(X, Y) - g_x(X, u) g_x(Y, u)). \end{cases} \tag{6.1}$$

Then, formulae (2.10) (with $d = 0$) and (6.1) easily imply that the metric $\tilde{G}_{ac}$ satisfies

$$\tilde{G}_{ac} = \kappa a\, (\tilde{G}_{1/\kappa})_{|\mathcal{V}} + (a+c)(\tilde{G}_{1/\kappa})_{|\mathcal{H}}.$$

We denote by $\tilde{g}_{ac} = F^*\tilde{G}_{ac}$ the Riemannian metric on the sphere $\mathbb{S}^3$, corresponding to the Kaluza–Klein metric $\tilde{G}_{ac}$ on $T_1\mathbb{S}^2(\kappa)$. Then, $\tilde{g}_a = F^*\tilde{G}_a$ has constant sectional curvature $K = 1/(4a)$. In particular, $\tilde{g}_{1/\kappa}$ has constant sectional curvature $\kappa/4$. For any real constant $\lambda > 0$, we put $\bar{a} = \lambda/\kappa$ and $\bar{c} = 1 - (\lambda/\kappa)$. As $F_*\xi_1 = -\sqrt{k}(\tilde{J}u)^v$, where $\xi_1$ is the Hopf vector field, we have

$$\bar{g}_\lambda := F^*(\tilde{G}_{\bar{a}\bar{c}}) = \lambda(\tilde{g}_{1/\kappa})_{|\mathcal{V}'} + (\tilde{g}_{1/\kappa})_{\mathcal{H}'},$$

where $\mathcal{V}' = \mathrm{Span}(\xi_1)$ and $\mathcal{H}' = \xi_1^{\perp}$. Thus, $\bar{g}_\lambda$ is obtained by deforming the metric $\tilde{g}_{1/\kappa}$, of constant sectional curvature $\kappa/4$, in the direction of the Hopf vector field, by a real constant $\lambda > 0$. Therefore, metrics of the form $\bar{g}_\lambda$ are exactly the Berger metrics on $\mathbb{S}^3(\frac{\kappa}{4})$.

Now, the map

$$F : (\mathbb{S}^3, \tilde{g}_{ac}) \to (T_1\mathbb{S}^2(\kappa), \tilde{G}_{ac})$$

is a Riemannian covering and so, a harmonic morphism. In fact, more in general, any Riemannian submersion with discrete fibers is a harmonic morphism [9]. On the other hand, by Theorem 11, the canonical projection $\pi_1 : (T_1\mathbb{S}^2(\kappa), \tilde{G}_{ac}) \to (\mathbb{S}^2(\kappa), g_0)$ is a harmonic morphism for any Kaluza–Klein metric $\tilde{G}_{ac}$. As the composition of harmonic morphisms is again a harmonic morphism, the map

$$\tilde{h}_{ac} = \pi_1 \circ F : (\mathbb{S}^3, \tilde{g}_{ac}) \to (\mathbb{S}^2(\kappa), g_0),$$

is a harmonic morphism. Hence, we proved the following.

**Theorem 12** *Let $\tilde{g}_{ac}$ denote a Kaluza–Klein metric on $\mathbb{S}^3$. Then, the maps*

$$\tilde{h}_{ac} : (\mathbb{S}^3, \tilde{g}_{ac}) \to (\mathbb{S}^2(\kappa), g_0)$$

*form a two-parameter family of harmonic morphisms. In particular:*

(i) *if $a = 1/\kappa$ and $c = 1 - 1/\kappa$, then the corresponding harmonic morphism $\tilde{h}_{ac}$ is the Hopf map $\tilde{h} : \mathbb{S}^3(\kappa/4) \to \mathbb{S}^2(\kappa)$;*
(ii) *if $a = (\lambda/\kappa) > 0$ and $c = 1 - (\lambda/\kappa)$, then the corresponding maps $\tilde{h}_\lambda$ are harmonic morphisms defined on Berger spheres.*

It is worthwhile to compare the results of Theorem 12 with the rigidity result about harmonic morphisms $\mathbb{S}^3(1) \to M^2$ proved by Baird and Wood [8]. In fact, they proved that any harmonic morphism from $\mathbb{S}^3(1)$ to a Riemannian surface $M^2$ is essentially the simplest possible one, as it coincides with the standard Hopf fibration $h : \mathbb{S}^3(1) \to \mathbb{S}^2(4)$ after an orthogonal change of coordinates.

We also recall that Montaldo and Ratto [27] used a different generalization of the Hopf construction to obtain a family of harmonic morphisms from a 5-dimensional manifold with singularities onto the Euclidean 2-sphere $\mathbb{S}^2$.

## References


1. Abbassi, M.T.K., Calvaruso, G.: $g$-natural contact metrics on unit tangent sphere bundles. Monatsh. Math. **151**, 89–109 (2006)
2. Abbassi, M.T.K., Calvaruso, G.: The curvature tensor of $g$-natural metrics on unit tangent sphere bundles. Int. J. Contemp. Math. Sci. **3**, 245–258 (2008)
3. Abbassi, M.T.K., Calvaruso, G.: Curvature properties of $g$-natural contact metric structures on unit tangent sphere bundles. Beiträge Algebra Geom. **50**, 155–178 (2009)
4. Abbassi, M.T.K., Kowalski, O.: Naturality of homogeneous metrics on Stiefel manifolds $SO(m+1)/SO(m-1)$. Differ. Geom. Appl. **28**, 131–139 (2010)
5. Abbassi, M.T.K., Sarih, M.: On some hereditary properties of Riemannian $g$-natural metrics on tangent bundles of Riemannian manifolds. Differ. Geom. Appl. **22**, 19–47 (2005)
6. Abbena, E., Garbiero, S., Vanhecke, L.: Einstein-like metrics on three-dimensional Riemannian homogeneous manifolds. Simon Stevin **66**, 173–182 (1992)
7. Ambrose, W., Singer, I.M.: On homogeneous Riemannian manifolds. Duke Math. J. **25**, 647–669 (1958)
8. Baird, P., Wood, J.C.: Bernstein theorems for harmonic morphisms from $\mathbb{R}^3$ and $\mathbb{S}^3$. Math. Ann. **280**, 579–603 (1988)
9. Baird, P., Wood, J.C.: Harmonic Morphisms between Riemannian Manifolds, London Mathematical Society Monographs (N.S.), vol. 29. Oxford University Press (2003)
10. Benyounes, M., Loubeau, E., Nishikawa, S.: Generalized Cheeger-Gromoll metrics and the Hopf map. Differ. Geom. Appl. **39**, 187–213 (2011)
11. Benyounes, M., Loubeau, E., Wood, C.M.: Harmonic sections of Riemannian vector bundles, and metrics of Cheeger-Gromoll type. Differ. Geom. Appl. **25**, 322–334 (2007)
12. Blair, D.E.: The theory of quasi-Sasakian structures. J. Differ. Geom. **1**, 331–345 (1967)
13. Blair, D.E.: Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Mathematics, vol. 203. Birkhäuser, Boston (2002)
14. Calvaruso, G.: Homogeneous structures on three-dimensional Lorentzian manifolds. J. Geom. Phys. **57**, 1279–1291 (Addendum: J. Geom. Phys. **58**, 291–292) (2007)
15. Calvaruso, G.: Three-dimensional Ivanov-Petrova manifolds. J. Math. Phys. **50**, 063509, 1–12 (2009)
16. Calvaruso, G., Perrone, D.: Homogeneous and $H$-contact unit tangent sphere bundles. Aust. J. Math. **88**, 323–337 (2010)
17. Calvaruso, G., Perrone, D.: Harmonic morphisms and Riemannian geometry of tangent bundles. Ann. Global Anal. Geom. **39**, 187–213 (2011)
18. Cheeger. Gromoll, J., Gromoll, D.: On the structure of complete manifolds of non negative curvature. Ann. Math. **96**, 413–443 (1972)
19. Fuglede, B.: Harmonic morphisms between Riemannian manifolds. Ann. Inst. Fourier (Grenoble) **28**, 107–144 (1978)
20. Gadea, P.M., Oubiña, J.A.: Homogeneous Riemannian structures on Berger 3-spheres. Proc. Edinburgh Math. Soc. **48**, 375–387 (2005)
21. Hermann, R.: Yang-Mills, Kaluza–Klein, and the Einstein Program, Interdisciplinary. Math., vol. XIX. Math. Sci. Press, Brookline (1978)
22. Ishihara, T.: A mapping of Riemannian manifolds which preserves harmonic functions. J. Math. Kyoto Univ. **19**, 215–229 (1979)
23. Kiričenko, V.F.: On homogeneous Riemannian spaces with invariant tensor structures. Sov. Math. Dokl. **21**, 734–737 (1980)
24. Kobayashi, S., Nomizu, K.: Foundations of Differential Geometry I, II. Interscience Publ., New York (1963, 1969)
25. Kowalski, O., Sekizawa, M.: Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles—a classification. Bull. Tokyo Gakugei Univ. **40**, 1–29 (1988)
26. Milnor, J.: Curvature of left invariant metrics on Lie groups. Adv. Math. **21**, 293–329 (1976)
27. Montaldo, S., Ratto, A.: A generalisation of the Hopf construction and harmonic morphisms into $\mathbb{S}^2$. Ann. Mat. Pura Appl. **189**, 605–613 (2010)
28. Musso, E., Tricerri, F.: Riemannian metrics on tangent bundles. Ann. Mat. Pura Appl. **150**, 1–19 (1988)
29. Nagano, T.: A problem on the existence of a Einstein metric. J. Math. Soc. Jpn. **19**, 30–31 (1967)

30. Perrone, D.: Torsion and critical metrics on contact three-manifolds. Kodai Math. J. **13**, 88–100 (1990)
31. Perrone, D.: Homogeneous contact Riemannian three-manifolds. Illinois J. Math. **42**, 243–256 (1998)
32. Perrone, D.: Contact metric manifolds whose characteristic vector field is a harmonic vector field. Differ. Geom. Appl. **20**, 367–378 (2004)
33. Perrone, D.: Minimality, harmonicity and CR geometry for Reeb vector fields. Int. J. Math. **21**, 1189–1218 (2010)
34. Tricerri, F., Vanhecke, L.: Homogeneous Structures on Riemannian Manifolds. London Math. Soc. Lecture Notes 83. Cambridge University Press (1983)
35. Wood, C.M.: An existence theorem for harmonic sections. Manuscripta Math. **68**, 69–75 (1990)