



Ref.TH.3483-CERN

GEOMETRY OF MULTIDIMENSIONAL UNIVERSES

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ABSTRACT

Let G be a compact group of transformation (global symmetry group) of a manifold E (multidimensional Universe) with all orbits of the same type (one stratum). We study G invariant metrics on E and show that there is one-to-one correspondence between those metrics and triples $(g_{\mu\nu}, A_{\mu}^a, h_{\alpha\beta})$, where $g_{\mu\nu}$ is a (pseudo-) Riemannian metric on the space of orbits (space-time), A_{μ}^a is a Yang-Mills field for the gauge group $N|H$, where N is the normalizer of the isotropy group H in G , and $h_{\alpha\beta}$ are certain scalar field characterizing geometry of the orbits (internal spaces). The scalar curvature of E is expressed in terms of the component fields on M . Examples and model building recipes are also given. The results generalize those of non-Abelian Kaluza-Klein theories to the case where internal spaces are not necessarily group manifolds.

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^{**)} Work done in the framework of the Project MRI.1. of the Polish Ministry of Science, Higher Education and Technology.

1. - INTRODUCTION

First special, and later general, theories of relativity invoked a picture of the Universe as being modelled on a four-dimensional space-time manifold. On the other hand, in order to describe regularities of discrete quantum numbers characterizing elementary particles, a concept of "internal" (as opposed to "external", i.e., space-time) symmetry, and with it that of internal space, was introduced. The idea behind what we call a "multidimensional Universe" (denote it by E) is that external and internal spaces are nothing but two aspects of one geometrical entity E , and all elementary forces in nature should be but a reflection of a unique geometry. By some, not yet fully understood, mechanism, certain configurations of a simple, multidimensional field theory are distinguished, and give rise to a "spontaneous compactification" of extra dimensions [see Ref. 1) and references therein] - this idea is at the root of the so-called "dimensional reduction". As a result, the multidimensional Universe splits into a four-dimensional space-time M and a compact internal space S . At the same time, the original simple field(s) on E split(s) into components which are identified with the conventional fields on M , like scalars, tensors, Yang-Mills fields (in supergravity, also spinors), etc. In theories of the Kaluza-Klein type, the "simple theory" is a multidimensional gravitation, and the "distinguished configurations" are those Riemannian metrics on E which admit a given compact group G of isometries. The case where the internal space S is a group manifold itself has been studied in many papers [see, e.g., Refs. 2)-8)] and the geometrical structure of this kind of dimensional reduction is by now well understood. A G -invariant metric g_{AB} splits into a gravitational field $g_{\mu\nu}$ on M , Yang-Mills field A_{μ}^i with G as the gauge group, and scalar fields $h_{ij}(x)$ describing the metric in the internal space (these are generalizations of Jordan-Thierry-Brans-Dicke scalars, [see Refs. 4), 6) and 7)]. Recently, attempts have been made⁹⁾⁻¹³⁾ to generalize these results to a case where S is not necessarily a group manifold, but rather a homogeneous space of the type G/H . No straightforward generalization of the original Kaluza-Klein idea has, however, been obtained [in particular, E of Ref. 12) is the associated bundle so that it hardly even makes sense to consider G invariance] and the construction given in the present paper may be thought of as an alternative to those of Refs. 12) and 13). Homogeneous spaces have also been used [see Ref. 14) and references therein] to provide solutions to supergravity theories (where there are matter fields besides the metric); however, the so-called "Kaluza-Klein ansatz" used in these papers is not, in general, G invariant.

In the present paper we solve the following general problem: what are the most general algebraic and geometric properties of an extended Universe E under the only requirement that G (a compact group) be a group of internal (global) symmetries? In Section 2 we show that E can be written locally as the product $M \times S$ with $S = G/H$, and that a local symmetry group $K = N|H$ arises in a natural way, N being the normalizer of H in G . In Section 3, we characterize all G invariant metrics on E and show that the local symmetry group K is at the same time the gauge group; we prove that there is one-to-one correspondence between G invariant metrics on E and triples $(g_{\mu\nu}, A_{\mu}^{\hat{a}}, h_{\alpha\beta})$, where $g_{\mu\nu}$ is a metric on M , $A_{\mu}^{\hat{a}}$ are Yang-Mills fields corresponding to the gauge group $K = N|H$, and $h_{\alpha\beta}$ are scalar fields. We also express the scalar curvature of E in terms of these fields [formulae (3.5.7) and (3.5.8)]. Examples (see Tables 1 and 2) are given in Section 4. We want to stress the fact that a principal bundle structure, so characteristic in mathematical descriptions of gauge fields, arises automatically and naturally in the present approach - the principal bundle emerges as a specific submanifold of the extended Universe E (see Fig. 2).

Besides its physical aspects and motivations this paper contains, uses, or refers to quite a number of mathematical techniques and results. Most of them are either standard or are simple exercises in differential geometry. However, we believe that the main results of Section 3 are new, and the authors would acknowledge receiving any information on this subject. The reader who are not interested in a "mathematical balast" may get the idea of the present paper by reading the summaries in Section 2.6 and 3.6, and also Section 4 where examples and model building recipes are discussed.

2. - BUNDLE STRUCTURE OF THE EXTENDED SPACE-TIME

2.1 Bundle structure of E

Let G be a compact group of transformations of a manifold E . For each $u \in E$ let $G(u)$ denote the orbit of G through u :

$$G(u) = \{ua : a \in G\}$$

$G(u)$ is then a compact submanifold of E , called also a fibre or an "internal space". When u_1 and u_2 belong to the same orbit then their isotropy groups ("little groups"), denoted by H_{u_1} and H_{u_2} respectively, are conjugate. But isotropy groups associated to points in different orbits need not be conjugate - then E decomposes into "strata". By the "principal orbit theorem"^{15),16)} the stratum consisting of orbits with maximal dimension is an open dense submanifold of E . In our case it is natural to assume from the very beginning

that E , being a model of an extended space-time, consists of one stratum only, i.e., that all isotropy groups H_u ($u \in E$) are mutually conjugate. With the above in mind, we state a theorem that we will comment and explain later in this section:

Let E be a manifold with a right action of a compact Lie group G , and suppose that all isotropy groups H_u ($u \in E$) are conjugate to a standard one, say $H_{u_0} = H$. Let M be the set of all orbits, $G \setminus H$ the coset space of right classes along H , and let N be the normalizer of H in G . Then M is a manifold and $E(M, G \setminus H)$ is an associated bundle with structure group $N|H$.

This statement can be found, without proof, in Ref. 17) (p. 276, Exc. 4.1M) [see also Ref. 18), CH.XII and Ref. 19), p.93]. Because of its importance for us we shall give some more information about several ingredients of the above result.

2.2 The normalizer and the local symmetry group

First of all we shall comment on the definition and meaning of N , the normalizer of H in G . It is defined as the largest subgroup of G in which H is normal [or invariant, or distingué (in French)] or, equivalently,

$$N = \left\{ a \in G \mid aH = Ha \right\} .$$

Since H is normal in N , it is clear that $N|H$ is a group (right and left cosets coincide). What is relevant for us it is that $N|H$ can be identified with the automorphism group of the homogeneous space $G \setminus H$. Here by an automorphism of $G \setminus H$ we mean an invertible mapping $\alpha: G \setminus H \rightarrow G \setminus H$, which commutes with the right action of G :

$$\alpha([a]b) = \alpha([a])b$$

where $[a] = Ha \in G \setminus H$ and $b \in G$.

To see the relation between $N|H$ and the automorphism group of $G \setminus H$, observe that for every $n \in N$ the mapping $\alpha_n: G \setminus H \rightarrow G \setminus H$ defined by

$$\alpha_n([a]) \doteq [na]$$

commutes with the right action of G on $G \setminus H$, and depends on the equivalence class $[n]$ of n in $N|H$ only. Conversely, given an automorphism α of $G \setminus H$ let $n \in G$ be such that $\alpha(H) = Hn$. Then $\alpha(Ha) = \alpha(H)a = Hna$ for all $a \in G$. In particular for $a = h \in H$ we get $Hnh = Hn$, i.e., $n \in N$. It follows that α is of the form α_n , as above. It is precisely because of this identification of elements of $N|H$ with automorphisms of $G \setminus H$ that $N|H$ plays the role of local symmetry group (local gauge group) in our framework.

2.3 Construction of an associated bundle $E(G \setminus H, M)$

Let us consider a principle bundle $P(N|H, M)$ with a base M and structure group $N|H$. We shall discuss here the geometry involved in the constructions of the associated bundle $E(G \setminus H, M)$. The procedure described is nothing but a particular case of the standard construction of an associated bundle from a principal one [see, e.g., Ref. 20), Ch. XVI.14].

In the direct product $P \times G \setminus H$, define the following relation

$$(p, [a]) \sim (p', [a']) \iff \exists [n] \in N|H \text{ so that } p' = p[n], [a'] = [n^{-1}][a]$$

where $p[n]$ is obtained from p by using the right action of $[n] \in N|H$ on the principal bundle P , and $[n^{-1}][a] = [n^{-1}a]$. In other words, $N|H$ acts on the typical fibre $G \setminus H$ by automorphisms, those discussed above: $[a] = \alpha_n([a'])$. It is easy to see that the above relation is an equivalence relation. We shall denote an equivalence class by the symbol $p \cdot [a]$. Let us recall what is the intuitive meaning of this: writing $u = p \cdot [a]$ means that the "geometrical object" u has "co-ordinate" $[a]$ in the "frame" p . Of course, $u = p \cdot [a] = (p[n^{-1}]) \cdot ([n][a])$, so that the group $N|H$ plays the role of the group of transformations of "frames". The space of equivalence classes [quotient of $(P \times G \setminus H)$ by the equivalence relation] is, by definition, the associated fibre bundle $E = E(M, G \setminus H)$ with "geometrical objects" u in the fibre, M in the basis and transition functions valued in $N|H$. The situation is schematically described in Fig. 1.



Fig. 1

In a local trivialization determined by a local cross-section σ (gauge), the element p of P can be represented as $p = (x, [n])$ (i.e., $p = \sigma(x)[n]$), where $x \in M$ and $[n] \in N|H$. The element $u \in E$ can be written as (x, y) , where $y = [a] \in G \setminus H$ and $u = \sigma(x) \cdot [a]$.

2.4 Global action of G on $E(G \setminus H, M)$

Let us ask what are the transformations $\beta: G \setminus H \rightarrow G \setminus H$ of the typical fibre which pass through the equivalence relation defining E to induce transformations of E itself. If $u = (x, y)$ and we want to define $\beta(u)$ as $(x, \beta(y))$, we must check that this definition is gauge independent, i.e., that $p \cdot \beta[a] = p[n^{-1}] \cdot \beta([n][a])$ or $\beta([n][a]) = [n]\beta([a])$. In other words, the left action of the structure group must commute with β . Since the structure group is, in general, non-Abelian, it is clear that β cannot be the left multiplication by $N|H$. But it can well be the right multiplication by an element of G so that $ug = (p \cdot [a])g = (p \cdot [ag])$, since we have shown in Section 2.2 that $N|H$ is precisely the set of all transformations of $G \setminus H$ which commute with such β 's. The situation is now the following: we have a (non-principal) fibre bundle $E(G \setminus H, M)$ with structure group $N|H$, with G acting on E from the right and operating transitively on each fibre, so that the fibres of E coincide with the orbits of G . However, this action is not free; indeed, if $u = p \cdot [a] \in E$ then the isotropy group of u is $H_u = a^{-1}Ha$. In particular, all isotropy subgroups are mutually conjugate. Here and below e denotes the identity of G .

We will now show how P can be identified with a submanifold of E . For each $p \in P$ let $u(p) \in E$ be defined by $u(p) = p \cdot [e]$, i.e., $u(p)$ is the geometrical object uniquely defined by the requirement that it has "co-ordinates" $[e]$ in the "frame" p . It is easy to see that the mapping $p \rightarrow u(p)$ from P into E is an embedding, which also satisfies $u(p[n]) = u(p) \cdot [n]$ for all $p \in P, n \in N$. In particular $u \in E$ is of the form $u = u(p)$ for some $p \in P$ if and only if the isotropy group H_u of u is precisely H .

2.5 Bundle structure of E (cont.)

Let us summarize the discussion given in Sections 2.2, 2.3 and 2.4: given a compact Lie group, G and a subgroup $H \subset G$, we started with a principal bundle $P(N|H, M)$, N being the normalizer of H in G , and showed that on the associated bundle $E(G \setminus H, M)$ the group G operates from the right in a natural way. All isotropy groups of G in E are mutually conjugate and P can be identified with a subset (immersed submanifold) of E on which all isotropy groups are exactly H . With the above construction in mind, it is now easy to understand why the theorem stated in Section 2.1 holds true. Suppose that E is a manifold with right action of G , and in such a way that

all the isotropy groups H_u are conjugate to a standard one $H = H_{u_0}$. Let M be the space of all orbits, and let P be defined as the set of all $u \in E$ such that $H_u = H$:

$$P \doteq \{ p \in E \mid p^H = p \}$$

Observe that if $p \in P$, $a \in G$ then $pa \in P$ if and only if $a \in N$ and, since H acts trivially on the points of P , it is the quotient group $N|H$ which freely acts on the fibres of P . Therefore P has the algebraic structure of the principal bundle with base M and structure group $N|H$. It is also clear that E can be identified with the bundle associated to P via the natural action of $N|H$ on $G \setminus H$. Indeed, we have a natural surjection $P \times G \setminus H \rightarrow E$ given by

$$(p, [a]) \mapsto p \cdot [a] \doteq pa \quad , \quad p \in P \subset E \quad , \quad a \in G$$

Analytically one needs the so-called "slice theorem"¹⁸⁾ (ch. VIII) [see also Ref. 21) for the non-compact case].

The resulting structure is represented in Fig. 2

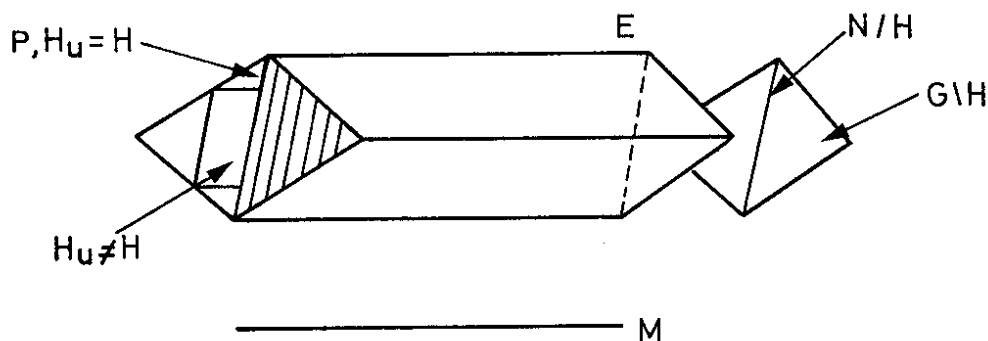


Fig. 2

2.6 Summary

We shall now summarize the main ideas of this section in plain terms. We started with an extended space-time E with a fixed, global, compact, internal symmetry (classification) group G . The orbits of G are internal spaces of E . They are homogeneous spaces of G and, under very mild assumptions, one can locally represent E as a product $M \times G \setminus H$. The group G operates on E just by right translations: if $p = (x, [y])$ is a point in $M \times G \setminus H$ then a group element $a \in G$ transforms p into $(x, [ya])$. Here $[y]$ denotes the right coset $[y] = Hy$, $y \in G$. We have also introduced local transformations $(x, [y]) \rightarrow (x, f(x, [y]))$ characterized by the fact that they commute with the

global ones. We have shown that every local transformation is described by a function $x \rightarrow n(x)$, where $n(x)$ belong to the group $N|H$, N being the normalizer (see the definition in section 2.2) of H in G . In a particular, well understood, case where the internal space is a group manifold (i.e., $H = \{e\}$ and $G|H = G$) the global and local symmetry groups happen to be isomorphic, the first acting on the internal space from the right and the second from the left. If the internal space is a homogeneous space $G|H$ with a non-trivial isotropy group H , then the situation becomes more complicated and the resulting local symmetry group is no longer isomorphic to the global one. Examples are given in Section 4. It will be seen in the next section that the local symmetry group, identified here as $N|H$ for geometrical reasons, will play the role of a gauge group, both kinematically and dynamically.

3. - METRIC AND CURVATURE

3.1 Decomposition of \mathfrak{G}

We first introduce convenient decompositions of the Lie algebra $\mathfrak{G} = \text{Lie}(G)$. To make the discussion simple, both H and G are assumed to be compact, connected Lie groups, with $H \subset G$. Since G is compact we can choose a bi-invariant Riemannian metric on G or, equivalently, an $\text{Ad } G$ invariant p -definite scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ in \mathfrak{G} (if G is simple then $\langle\langle \cdot, \cdot \rangle\rangle$ is unique up to a multiplication by a positive constant). Let $\mathfrak{K} = \text{Lie}(H)$ and denote by \mathfrak{S} the orthogonal complement of \mathfrak{K} in \mathfrak{G} . Since $\langle\langle \cdot, \cdot \rangle\rangle$ is, in particular, $\text{Ad } H$ invariant, we get what is called a reductive pair²²⁾ (Vol.II, Ch.X.2)

$$\mathfrak{G} = \mathfrak{K} + \mathfrak{S} \quad , \quad (\text{Ad } H)(\mathfrak{S}) = \mathfrak{S} \quad (3.1.1)$$

It is then possible to identify \mathfrak{S} , endowed with the above linear action of H , with the vector space tangent to the homogeneous space $S \doteq G|H$ at the origin. For a given homogeneous space, the above reductive decomposition need not be unique, however, this non-uniqueness will not cause any trouble in the following.

With $\mathfrak{N} \doteq \text{Lie}(N)$ being the Lie algebra of the normalizer N of H in G , we decompose \mathfrak{N} into

$$\mathfrak{N} = \mathfrak{K} + \mathfrak{K} \quad (3.1.2)$$

where

$$\mathfrak{K} \doteq \mathfrak{O}_n \mathfrak{S} \quad (3.1.3)$$

is the orthogonal complement of \mathcal{K} in \mathcal{N} . Again we have a reductive pair $(\text{Ad } H)(\mathcal{K}) = \mathcal{K}$, but now, since H is normal in N , we also have $[\mathcal{K}, \mathcal{N}] \subset \mathcal{K}$, what implies $[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}$. Since $\mathcal{K} \cap \mathcal{K} = 0$ we get

$$[\mathcal{K}, \mathcal{K}] = 0 \quad (3.1.4)$$

It is useful at this point to introduce the centralizer (or commutant) of \mathcal{K} in \mathfrak{G} :

$$\mathfrak{z}_H \doteq \left\{ x \in \mathfrak{G} \mid [x, \mathcal{K}] = 0 \right\} \quad (3.1.5)$$

The centralizer contains, in particular, the centre \mathcal{C}_H of \mathcal{K} :

$$\mathcal{C}_H \doteq \mathfrak{z}_H \cap \mathcal{K} \quad (3.1.6)$$

It follows from (3.1.3) and (3.1.4) that $\mathcal{K} \subset \mathcal{S} \cap \mathfrak{z}_H$, and it is easy to see [compare also Ref. 23) (§ 3.2.44, Theorem 2)] that in fact

$$\mathcal{K} = \mathcal{S} \cap \mathfrak{z}_H \quad (3.1.7)$$

In other words \mathcal{K} is composed of those vectors which are tangent to G/H at the origin and are invariant under H . (This remark will be important in Section 4, when we shall count the scalar fields.) Actually \mathcal{K} , being an orthogonal complement (with respect to a bi-invariant metric) of the ideal \mathcal{K} , is itself an ideal and, a fortiori, Lie subalgebra of \mathcal{N} . We can therefore identify, \mathcal{K} with the Lie algebra $\text{Lie}(N|H)$, and for obvious notational reasons we call

$$K \doteq N|H \quad (3.1.8)$$

Finally, we observe that

$$\mathfrak{z}_H = \mathcal{K} + \mathcal{C}_H \quad (3.1.9)$$

Indeed, it follows from (3.1.1)-(3.1.7) that $\mathcal{K} + \mathcal{C}_H \subset \mathfrak{z}_H$, so that it is enough to show that $\mathfrak{z}_H \subset \mathcal{K} + \mathcal{C}_H$. Let $x \in \mathfrak{z}_H$ and decompose $x = x_H + x_S$ with $x_H \in \mathcal{K}$ and $x_S \in \mathcal{S}$. Then, for every $y \in \mathcal{K}$, we have $[x_H, y] = [x, y] - [x_S, y]$, where the first term vanishes (since $x \in \mathfrak{z}_H$), and, by (3.1.1), the second term is in \mathcal{S} . On the other hand, since \mathcal{K} is an algebra, $[x_H, y]$ is in \mathcal{K} , and so $[x_H, y] \in \mathcal{K} \cap \mathcal{S} = 0$. It follows that $x_H \in \mathcal{C}_H$ and

therefore $x_S = x - x_H \in \mathcal{S} \cap \mathfrak{Z}_H = \mathfrak{K}$. In a particular case, if H is semi-simple [i.e., if H contains no direct $U(1)$ factors] then the Lie algebras of $K = N|H$ and of the centralizer of H are isomorphic.

Finally, we introduce \mathcal{L} , the orthogonal complement of \mathcal{N} in \mathcal{G} :

$$\mathcal{G} = \mathcal{N} + \mathcal{L} \quad , \quad (\text{Ad } N)\mathcal{L} = \mathcal{L} \quad . \quad (3.1.10)$$

In particular,

$$\mathcal{S} = \mathfrak{K} + \mathcal{L} \quad , \quad (\text{Ad } H)\mathcal{L} = \mathcal{L} \quad , \quad (3.1.11)$$

and it should be noticed [compare also the discussion in 3.3 Ad i)] that, in fact, \mathcal{L} is orthogonal to \mathfrak{K} with respect to any $\text{Ad } H$ invariant scalar product on \mathcal{S} . This follows from the fact, that according to (3.1.7), the representation of H on \mathcal{L} does not contain the trivial representation.

In practice, in particular in model-building as discussed in Section 4, one often starts with a direct product N of two subgroups H and K of G and the question arises whether N is the normalizer of H in G or not. From Eqs. (3.1.7) and (3.1.2) one can deduce the following criterion: $\mathcal{N} \doteq \mathfrak{K} + \mathfrak{K}$ [$\mathfrak{K}, \mathfrak{K}$] = 0, is the normalizer of \mathfrak{K} in \mathcal{G} if and only if

$$\mathcal{L} \cap \mathfrak{Z}_H = 0 \quad , \quad (3.1.12)$$

where \mathcal{L} is defined as the orthogonal complement of \mathcal{N} in \mathcal{G} . In particular $N = H \cdot K$ is the normalizer of H in G if the adjoint representation of N on \mathcal{L} is faithful and irreducible.

The whole story can be visualized and summarized as follows:

$$\mathcal{G} = \overbrace{\mathfrak{K} \oplus \mathfrak{K} \oplus \mathcal{L}}^{\mathcal{S}}$$

$$\begin{aligned} [\mathfrak{K}, \mathfrak{K}] &\subset \mathfrak{K} \\ [\mathfrak{K}, \mathfrak{K}] &\subset \mathfrak{K} \\ [\mathfrak{K}, \mathfrak{K}] &= 0 \\ [\mathfrak{K}, \mathcal{L}] &\subset \mathcal{L} \\ [\mathfrak{K}, \mathcal{L}] &\subset \mathcal{L} \end{aligned}$$

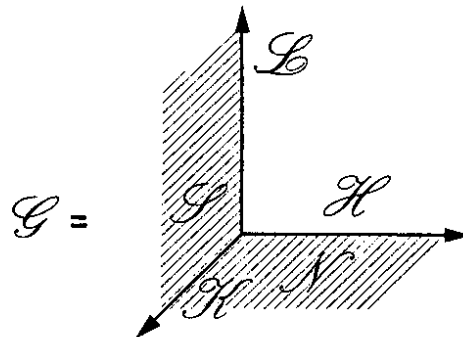


Fig. 3

$\mathfrak{G} = \text{Lie}(G)$
 $\mathfrak{K} = \text{Lie}(H)$ N is locally a direct product of two
 $\mathfrak{N} = \text{Lie}(N)$ normal subgroups: $N = HK$
 $\mathfrak{K} \simeq \text{Lie}(N|H)$
 $S = T_0$ ($S = G \setminus H$)
 $\mathfrak{Z}_H = \text{centralizer of } \mathfrak{K} \text{ in } \mathfrak{G}$
 $C_{\mathfrak{K}} = \text{centre of } \mathfrak{K}$

3.2 Adapted basis

We fix once and for all a basis T_i in \mathfrak{G} , with $[T_i, T_j] = C_{ij}^k T_k$, adapted to the decompositions

$$\mathfrak{g} = \mathfrak{K} \oplus S \quad \text{and} \quad S = \mathfrak{K} \oplus \mathfrak{L} \tag{3.2.1}$$

and introduce notation

$$T_i = \{ T_{\alpha} \in \mathfrak{K}, T_{\alpha} \in S \} \quad \text{and} \quad T_{\alpha} = \{ T_{\alpha} \in \mathfrak{K}, T_{\alpha} \in \mathfrak{L} \} \tag{3.2.2}$$

distinguishing between basis vectors in the different components of \mathfrak{G} . For each $a \in G$ let $\Lambda(a)^i_j$ denote the matrix of the adjoint representation of G :

$$(Ad a) T_i = a T_i a^{-1} = \Lambda(a)^j_i T_j \tag{3.2.3}$$

It should be noticed that for $n \in N$ the matrix $\Lambda(n)$ has the following structure

$$\Lambda(n) = \left[\begin{array}{c|c} \Lambda_{\mathfrak{K}}^{\mathfrak{K}}(n) & \Lambda_{\mathfrak{P}}^{\mathfrak{K}}(n) \\ \hline 0 & \Lambda_{\mathfrak{P}}^{\mathfrak{L}}(n) \end{array} \right] \tag{3.2.4}$$

Indeed, if $n \in N$ then $nHn^{-1} = H$ and, infinitesimally $nT_{\hat{\alpha}}n^{-1} = \Lambda(n)^{\hat{\beta}}_{\hat{\alpha}} T_{\hat{\beta}}$ in particular $\Lambda(n)^{\hat{\beta}}_{\hat{\alpha}} = 0$.

This implies that the submatrices

$$\left(\Lambda(n)^{\mathfrak{L}}_{\mathfrak{P}} \right) \doteq \mathbb{R}(n) \tag{3.2.5}$$

form a representation of N in $S = \mathfrak{K} + \mathfrak{L}$.

3.3 Vertical moving frame

Let ϵ_i be the fundamental vector fields on E corresponding to infinitesimal transformations of E generated by T_i :

$$\epsilon_i(u) = \left. \frac{d}{dt} \left(u e^{tT_i} \right) \right|_{t=0}, \quad u \in E \quad (3.3.1)$$

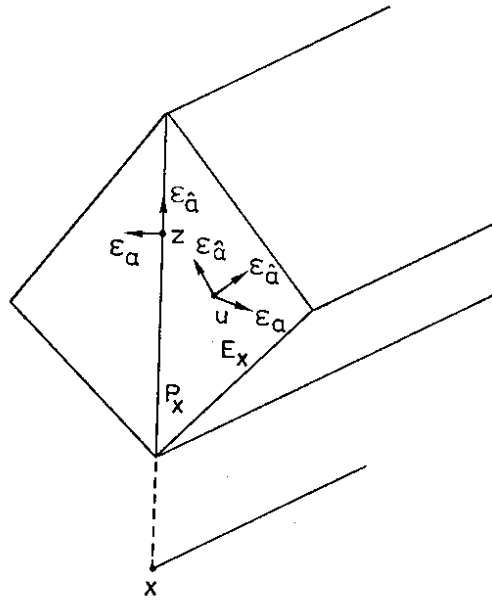
(These vectors are Killing vector fields of every G invariant metric.) Since G acts on E from the right, we have

$$[\epsilon_i, \epsilon_j](u) = C_{ij}^k \epsilon_k(u), \quad u \in E \quad (3.3.2)$$

It should be noticed that, at any given $u \in E$, the family $\epsilon_i(u)$ is over-complete in the vertical tangent space at u . Something special happens, however, on the submanifold $P \subset E$: the vector fields $\epsilon_{\hat{\alpha}}$, included in the family $\{\epsilon_i\}$, vanish on P , whereas the ϵ_{α} -s are linearly independent at every point of P . (We remind the reader that the principal bundle P was defined in 2.5 as the set of all points invariant under H .)

Fig. 4

$$T_i \left\{ \begin{array}{l} T_{\hat{\alpha}} \in \mathcal{K} \longrightarrow \epsilon_{\hat{\alpha}} \\ T_{\hat{a}} \in \mathcal{J} \longrightarrow \epsilon_{\hat{a}} \\ T_a \in \mathcal{L} \longrightarrow \epsilon_a \end{array} \right\} \epsilon_i$$



Now, since the vector fields ϵ_{α} are linearly independent on P , they are also independent in some neighbourhood U of P in E . In particular, the commutator of two vertical fields being again vertical, we have

$$[\epsilon_{\alpha}, \epsilon_{\beta}](u) = \int_{\alpha\beta}^{\gamma} (u) \epsilon_{\gamma}(u), \quad u \in U \quad (3.3.3)$$

where the $f_{\alpha\beta}^{\gamma}$ are the structure functions of the vertical moving frame ϵ_{α} ; notice that they depend generally on the point u at which they are calculated. However, they are constant on P :

$$f_{\alpha\beta}^{\gamma}(P) \equiv C_{\alpha\beta}^{\gamma} \quad \text{for } P \in P \quad (3.3.4)$$

Notice also that the property for ϵ_{α} to constitute a moving frame in the vertical space fails if one goes too far from P : the vectors ϵ_{α} may become linearly dependent there.

From the definition (3.3.1), and taking into account that $\epsilon_{\alpha}(z) = 0$ for $z \in P$, we get the relation

$$\mathcal{E}_{\alpha}(P^n) = R(n)^{\beta}_{\alpha} \mathcal{E}_{\beta}(P)^n, \quad P \in P, \quad n \in N \quad (3.3.5)$$

which will be used in (3.4.4).

For calculation purposes, one also needs the expression of $\epsilon_{\alpha}(f_{\beta\gamma}^{\delta})$ in terms of the structure constants C_{ij}^k of the group G . Indeed, when one wants to compute geometrical quantities [such as, e.g., scalar curvature of a homogeneous space (3.5.8)], one performs the calculation at a point $z \in P$, but what is needed are the values of the structure functions and their derivatives in the directions transversal to P . To obtain such a formula we first observe that, from (3.3.2) and (3.3.3), we have

$$f_{\beta\gamma}^{\delta}(u) \mathcal{E}_{\delta}(u) = C_{\beta\gamma}^i \mathcal{E}_i(u). \quad (3.3.6)$$

Then, taking the commutator of both sides with ϵ_{α} and using (3.3.4), we get

$$\mathcal{E}_{\alpha}(f_{\beta\gamma}^{\delta})(P) = C_{\beta\gamma}^{\hat{\delta}} C_{\alpha\hat{\delta}}^{\delta}, \quad P \in P. \quad (3.3.7)$$

In particular, owing to (3.1.4), we get $\epsilon_{\hat{\alpha}}(f_{\beta\gamma}^{\delta})(P) = 0$ in agreement with (3.3.4).

3.4 Characterization of G invariant metrics on E

Let g now be a G invariant metric on E . We are going to show that it determines and is determined by

- i) a G invariant metric h_x on every fibre E_x of E ;
- ii) a G invariant horizontal distribution $(Z_u)_{u \in E}$ on $E(G \setminus H, M)$ or, equivalently, a principal connection in $P(N|H, M)$;
- iii) a metric γ on M .

A given manifold may not always admit a non-degenerate pseudo-Riemannian metric of given signature. In the following we shall always assume that problems of this type do not arise.

In subparagraphs Ad i), Ad ii), Ad iii) we prove the direct proposition while in Ad iv) we will show the converse.

- Ad i) - g being a metric on E , we know, a fortiori, how to compute the scalar product of two vertical vectors. Since g is G invariant we obtain a G invariant metric h_x on every fibre E_x of E .
- Notice that a co-ordinate representation of h_x is obtained as follows: choose a local cross-section (gauge) of P , $\sigma: M \rightarrow P$ - such a section "marks the origin" on each fibre - then define $h_{\alpha\beta}(x) \doteq g_{\alpha\beta}(\sigma(x))$, where $g_{\alpha\beta}(p) \doteq g_p(\epsilon_\alpha(p), \epsilon_\beta(p))$, $p \in P$, are numerical functions on P . The matrix $h = (h_{\alpha\beta})$, being associated with a G invariant metric on a space isomorphic to $G \setminus H$ is automatically $\text{Ad } H$ invariant²²⁾ (Vol. II, Ch. X, Prop. 3.1), and therefore satisfies the constraints.

$$R(a)^T h R(a) = h, \quad a \in H, \quad (3.4.1)$$

where $R(a)$, $a \in H \subset N$, is given by (3.2.5).

- According to (3.4.1) the matrix $h(x)$ defines an $\text{Ad } H$ invariant scalar product in \mathcal{S} . In (3.1.11) we have defined a splitting of \mathcal{S} into the direct sum of two subspaces \mathcal{K} and \mathcal{L} using an auxiliary bi-invariant metric on G . we now realize that \mathcal{K} and \mathcal{L} are also orthogonal with respect to the scalar product induced by $h_{\alpha\beta}(x)$. Indeed, \mathcal{K} and \mathcal{L} carry disjoint representations of H and therefore they are orthogonal with respect to every $\text{Ad } H$ invariant scalar product [see, e.g., Ref. 24), Ch. VIII, § 3]

$$h = \begin{pmatrix} h_{\hat{a}\hat{b}} & 0 \\ 0 & h_{ab} \end{pmatrix} \quad (3.4.2)$$

Owing to the formulae (3.2.5) and (3.1.4) we also have

$$R(a) = \begin{pmatrix} \delta_{\hat{e}}^{\hat{a}} & 0 \\ 0 & R_{\hat{e}}^{\hat{a}} \end{pmatrix} \quad (3.4.3)$$

so that the constraints (3.4.1) are effective on h_{ab} only.

- The field $x \rightarrow h_x$ of G invariant metrics on fibres of E can be identified with a cross-section of a bundle associated to P . Actually, by (3.3.5) we find

$$g_{\alpha\beta}(P^n) = R(n)_{\alpha}^{\alpha'} R(n)_{\beta'}^{\beta} g_{\alpha'\beta'}(P), \quad n \in N \quad (3.4.4)$$

and

$$g_{\alpha\beta}(P) = R(h)_{\alpha}^{\alpha'} R(h)_{\beta'}^{\beta} g_{\alpha'\beta'}(P), \quad h \in H \quad (3.4.5)$$

so that the numerical functions $g_{\alpha\beta}(z)$ on P can indeed be identified with sections of an appropriate associated bundle²²⁾ (Vol. I, Ch. II, Example 5.2). The relations (3.4.4) and (3.4.5) define the type of transformations of these scalar fields under $N|H$ gauge transformations.

- Notice that it is enough to know the functions $h_{\alpha\beta}(x) = g_{\alpha\beta}(\sigma(x))$, where $\sigma: M \rightarrow P$ is a local gauge. Using G invariance and the constraints (3.4.1) $h_{\alpha\beta}$ can be then unambiguously propagated all over the bundle E .

Ad ii) For every $u \in E$ let V_u denote the internal, or vertical, tangent space, i.e., subspace of $T_u(E)$ consisting of all vectors tangent to the fibre at u . Let Z_u be the orthogonal complement of V_u in $T_u(E)$ with respect to our given G invariant metric g . Since g is G invariant we have $(Z_u)a = Z_{ua}$, $a \in G$, i.e., we have a G invariant horizontal distribution on E . In order to show that the distribution determines a principal connection, we must prove that, at $p \in P$, Z_p is tangent to P . This follows by observing that Z_p is orthogonal to the vectors $\varepsilon_a(p)$ which span the orthogonal complement to P at p . Now, $(Z_p)_{p \in P}$ is an N , and therefore also an $N|H$ invariant horizontal distribution on P , i.e., a principal connection²²⁾ (Vol. I, Ch.II.1).

Ad iii) The scalar product $\gamma_x(v,w)$ of two vectors, tangent to M at a point x , is obtained as follows: choose an arbitrary point u in the orbit labelled by x , and let v^* and w^* be the vectors in Z_u which project onto v and w respectively. Then define $\gamma_x(v,w) \doteq g_u(v^*,w^*)$. The result is independent of the choice u on the orbit because of G invariance of the metric g .

Ad iv) Conversely, it is easy to see that given a metric γ on M , principal connection $(Z_p)_{p \in P}$, and G invariant metrics h_x on the fibres of E , one constructs a G invariant metric on E . Indeed, given $\xi, \eta \in T_u E$ let $p \in P$ be such that $u = pa$ for some $a \in G$, and let $\pi(\xi)$ and $\pi(\eta)$ denote the projections of ξ and η on M . Denote by ξ^* and η^* the horizontal lifts of $\pi(\xi)$ and $\pi(\eta)$ to Z_p . Then the vectors $\xi - \xi^*a$ and $\eta - \eta^*a$ are vertical, and the scalar product of ξ and η in E can be defined as

$$g_u(\xi, \eta) = \gamma_x(\pi(\xi), \pi(\eta)) + h_x(\xi - \xi^*a, \eta - \eta^*a) \quad (3.4.6)$$

3.5 Curvature

Let x^μ be a co-ordinate system on M , and let e_μ be the horizontal lift of the vector fields ∂_μ from M to E . The vectors $e_A = (e_\mu, \epsilon_\alpha)$ then form a basis at every point $u \in U$, where U is an appropriate neighbourhood of P . By their very definition the vector fields e_μ are orthogonal to ϵ_α , and the metric takes the form

$$g_{AB} = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & g_{\alpha\beta}(x,y) \end{pmatrix}, \quad (3.5.1)$$

where $(x,y) \in M \times G/H$ is a local product representation of E in a given gauge $\sigma: M \rightarrow P$. For $p = (x, \sigma(x) = g) \in P$, the connection form can be written

$$\omega(p) = \hat{g}^{-1} A_\mu^{\hat{a}}(x) dx^\mu + \hat{g}^{-1} dg \quad \text{with} \quad A_\mu^{\hat{a}}(x) = A_\mu^{\hat{a}}(x) T_{\hat{a}}$$

and the horizontal lift $e_\mu(p)$ can be written

$$e_\mu(p) = \left(\partial_\mu \right)_p - A_\mu^{\hat{a}}(x, g) \epsilon_{\hat{a}}(g) \quad (3.5.2)$$

where

$$A_\mu(x, g) = g^{-1} A_\mu(x) g \quad \text{and} \quad A_\mu(x, g) = A_\mu^{\hat{a}}(x, g) T_{\hat{a}}$$

Therefore the inverse metric in P can be written as

$$g^{-1} = \gamma^{\mu\nu}(x) \left(\partial_\mu - A_\mu^{\hat{a}}(x, g) \varepsilon_{\hat{a}}(g) \right) \otimes \left(\partial_\nu - A_\nu^{\hat{b}}(x, g) \varepsilon_{\hat{b}}(g) \right) + h^{\hat{a}\hat{b}}(x, g) \left(\varepsilon_{\hat{a}}(g) \otimes \varepsilon_{\hat{b}}(g) \right)$$

It is enlightening - although unnecessary - to express g^{-1} in P in terms of the vector fields $e_{\hat{a}}(g) = g^{-1} \varepsilon_{\hat{a}}(g) g$ (which are only defined through the choice of a gauge, and satisfy $[\varepsilon_{\hat{a}}, \varepsilon_{\hat{b}}](g) = 0$). These vector fields $e_{\hat{a}}$ can be thought of as right invariant vector fields in the copy of $K = N|H$ above x . Then one obtains on P

$$g^{-1} = \gamma^{\mu\nu}(x) \left(\partial_\mu - A_\mu^{\hat{a}}(x) e_{\hat{a}}(g) \right) \otimes \left(\partial_\nu - A_\nu^{\hat{b}}(x) e_{\hat{b}}(g) \right) + h^{\hat{a}\hat{b}}(x) \left(e_{\hat{a}}(g) \otimes e_{\hat{b}}(g) \right)$$

This writing clearly exhibits the G invariance (but destroys the explicit gauge invariance).

The commutation relations of the basis e_A are

$$[e_\mu, e_\nu] = \int_{\mu\nu}^\alpha \varepsilon_\alpha \quad (3.5.3)$$

$$[e_\mu, \varepsilon_\alpha] = 0 \quad (3.5.4)$$

$$[\varepsilon_\alpha, \varepsilon_\beta] = \int_{\alpha\beta}^\gamma \varepsilon_\gamma \quad (3.5.5)$$

with

$$\int_{\mu\nu}^{\hat{a}}(P) = -F_{\mu\nu}^{\hat{a}}(P) \quad , \quad P \in P \quad , \quad (3.5.6)$$

where $F_{\mu\nu}^{\hat{a}}$ are the components of the curvature two-form of the connection D_μ . Information about the structure functions $f_{\alpha\beta}^\gamma$ on P is given by (3.3.4) and (3.3.7). These are the necessary ingredients for calculation of the scalar curvature R of the Levi-Civita connection of g_{AB} . Taking also into account the fact that ε_α are Killing vector fields for $g_{\alpha\beta}$, the result is

$$R(E) = R(M) + R(G \setminus H) - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu}_a - \frac{1}{4} R^{\alpha\beta} R^{\gamma\delta} \left(D_\mu R_{\alpha\gamma} D^\mu R_{\beta\delta} + D_\mu R_{\alpha\beta} D^\mu R_{\gamma\delta} \right) - \nabla_\mu \left(R^{\alpha\beta} D_\mu R_{\alpha\beta} \right) \quad (3.5.7)$$

$R(M)$ is the scalar curvature of the Levi-Civita connection ∇_μ of $g_{\mu\nu}$, D_μ and $F_{\mu\nu}^a$ are respectively the covariant derivative and the curvature of the principal connection A_μ^a , and $R(G \setminus H)$ is given by

$$R(G \setminus H) = h^{\alpha\beta} \left(\frac{1}{2} C_{\alpha\beta}^\gamma C_{\beta\gamma}^\alpha - \frac{1}{4} h^{\alpha'\gamma'} h_{\gamma'\beta'} C_{\alpha\beta}^{\gamma'} C_{\alpha'\beta'}^{\gamma'} + C_{\alpha\beta}^{\gamma'} C_{\beta\gamma'}^\alpha - C_{\alpha\beta}^\alpha C_{\gamma\beta'}^\alpha \right) \quad (3.5.8)$$

The indices μ, ν are lowered with the metric $g_{\mu\nu}$, while the indices α, β (respectively a, b) are lowered with the help of the matrix $h_{\alpha\beta}$ (respectively h_{ab}) given by (3.4.2). When G is unimodular (in particular when G is compact), the last term in (3.5.8) vanishes. If H is trivial ($H = \{e\}$) the third term of (3.5.8) vanishes and if $G \setminus H$ is a symmetric space ($[\mathfrak{S}, \mathfrak{S}] \subset \mathfrak{K}$) the first two terms of (3.5.8) vanish. The last term in (3.5.7) gives rise to a total derivative in the Lagrangian $R \sqrt{g(M)}$ and can be neglected. While varying this Lagrangian the constraints (3.4.1) are to be taken into account.

Expression (3.5.4) agrees with the scalar curvature given in Ref. 4) where the group case is treated (H trivial) although it is not written in the same way; however, we believe that there is an erroneous sign inside the kinetic term of the scalar fields in Ref. 13).

3.6 Summary

We summarize the results given in this section. We have considered an extended space-time E having a local product structure $E = M \times G \setminus H$, with global symmetry group G acting on E from the right. We have shown that there is one-to-one correspondence between G invariant Riemannian metrics g_{AB} on E and triples $(g_{\mu\nu}, A_\mu^{\hat{a}}, h_{\alpha\beta})$, where $g_{\mu\nu}$ is a Riemannian metric on M , $A_\mu^{\hat{a}}$ are gauge potentials corresponding to the local symmetry group $N|H$, and $h_{\alpha\beta}$ are certain scalar fields determining an $\text{Ad } H$ invariant metric on $\mathfrak{G} \setminus \mathfrak{K}$. The curvature $R(E)$ of the metric g_{AB} splits into

- a) $R(M)$ - the curvature of space-time metric $g_{\mu\nu}$,
- b) $R(G \setminus H)$ - the curvature of the G invariant metric on $G \setminus H$ which gives the potential term for scalars $h_{\alpha\beta}$,
- c) Yang-Mills Lagrangian of $A_\mu^{\hat{a}}$,
- d) kinetic term for the scalars $h_{\alpha\beta}$,

[see the formulae (3.5.7) and (3.5.8)]. When varying the scalars $h_{\alpha\beta}$ the constraints (3.4.1) have to be taken into account. To make the discussion given in this paper as simple as possible, we have considered Riemannian metrics instead of vielbeins. In many cases, especially when spinors are taken into account, the vielbein formalism is a necessity. In such a case we would have $h(x) = \delta_{\alpha\beta} \psi^\alpha(x) \otimes \psi^\beta(x)$, with $\psi^\alpha(x)$ being the soldering forms (vielbein) on the internal space at x . Taking into account the fact that ϵ_α can be thought of as an orthogonal basis on $G \setminus H$ with respect to a fixed metric $\langle\langle \cdot, \cdot \rangle\rangle$, induced by a bi-invariant metric on G , it is natural to write $\psi^\alpha(x) = \theta^\alpha(x) + \phi^\alpha(x)$, where θ^α are the duals of ϵ_α . The $\{\psi^\alpha\}$ constitute a set of $s = \dim(G \setminus H)$ one-forms, orthogonal with respect to h , and their deviations ϕ^α play the role of Higgs fields.

4. - COMMENTS AND EXAMPLES

4.1 Counting the number of scalar fields

The set of all G invariant metrics which can be defined on a given homogeneous space $S = G \setminus H$ is itself a (connected) manifold, which we shall call $R(G;S)$. The reader should be aware of the fact that a given manifold S may admit several homogeneous structures [for example, the homogeneous spaces $SO(8)/SO(7)$, $SU(4)/SU(3)$, $Spin\ 7/G_2$, $USp(2)/USp(1)$ are all diffeomorphic to the standard seven-sphere S^7]; therefore we stress the fact that, in the following, we will consider a manifold S with a given homogeneous structure $G \setminus H$, and we shall consider those metrics on S which are invariant with respect to the action of G on S . Both G and H are assumed to be compact and connected.

It is well known²²⁾ that G invariant metrics on $G \setminus H$ are in one-to-one correspondence with $Ad\ H$ invariant bilinear symmetric forms on the tangent space S at the origin of $S = G \setminus H$. Indeed, owing to the transitivity of G action, one can transport such a scalar product from the origin to any point of S , and the transport is unambiguous because of the assumed $Ad\ H$ invariance. In order to find the dimension d of the manifold $R(G,S)$, one has therefore to decompose the representation $Ad\ H$ on the vector space S into irreducible ones:

$$S = \bigoplus_i (V_i \otimes R^{n_i}) \quad (4.1.1)$$

where the index i runs over unequivalent irreducible representations of H on V_i , and r_i is the multiplicity with which V_i occurs in S . Since the dimension d of $R(G;S)$ is equal to the dimension of the space of symmetric operators on S commuting with the representation $\text{Ad } H$, it follows that

$$d = \sum_i \frac{r_i(r_i+1)}{2} \quad (4.1.2)$$

From the formula (3.1.7) it follows that the dimension k of the gauge group K coincides with r_0 , where $i = 0$ denotes the trivial representation of H

$$k = \dim K = r_0 \quad (4.1.3)$$

It is sometimes natural to restrict the attention to G invariant metrics on S with a fixed volume element. We shall denote by d_0 the dimension of the manifold of conformal equivalence classes of G invariant metrics on $G \backslash H$ ($d_0 = d - 1$).

In order to find out the decomposition (4.1.1) one can use the tables²⁵⁾ - in practice one looks at the branching rule of $\text{Ad } G$ into $N = HK$. However, one has to remember that what we need are decompositions into real-irreducible representations, while the tables (and most papers on the subject) give the branching rules in terms of complex-irreducible ones. Special care has to be taken if H or K is one of the groups $SU(n)$, $\text{Spin}(4n+2)$ or E_6 . Indeed, these groups admit some representations (ρ) which are not self-conjugate. In such cases ρ and $\bar{\rho}$ will appear simultaneously in the reduction of the adjoint representation of G , and one has to collect together such pairs to build \mathbb{R} -irreducible representations.

4.2 A class of almost trivial examples

a) $S = G/\{e\}$, i.e., $H = \{e\}$. Now S itself is a group, and the number of scalars is the number of right invariant metrics on G . The isotropy group $H = \{e\}$ is trivial and its irreducible representations are one-dimensional. We have $\mathcal{S} = \mathcal{G} = \mathcal{N} = \mathcal{K}$ and $\mathcal{K} = \mathcal{L} = 0$. The number of gauge potentials (i.e., the dimension of the gauge group) is $k = g = \dim(G)$. The number of scalars is $d = g(g+1)/2$. In that case $d_0 = \dim SL(g)/SO(g)$.

b) $G = G_1 \times G_1$, $H = \text{diag } G = \{(a,a) | a \in G_1\}$. The homogeneous space $S = G \backslash H$ can be naturally identified with G_1 , the action of G on S being given by $x \rightarrow a^{-1}xb$, $(a,b) \in G_1 \times G_1$. In particular H acts on S by $x \rightarrow a^{-1}xa$ so that the number of scalars is equal to the number of bi-invariant metrics on G_1 . This can be determined by decomposing the adjoint representation of G into irreducible representations and applying the formula (4.1.2). The gauge group $K = N|H$ is easily seen to be isomorphic to the centre of G_1 . In particular if G_1 is simple then $d = 1$ and $k = 0$.

c) $S = G \setminus H$ is an irreducible symmetric space. In that case we have a reductive decomposition $\mathfrak{g} = \mathfrak{K} + \mathfrak{S}$ with $[\mathfrak{S}, \mathfrak{S}] \subset \mathfrak{K}$, and the adjoint representation of $\text{Ad } H$ on \mathfrak{S} is irreducible. It follows that $\mathfrak{K} = 0$ and $\mathfrak{S} = \mathfrak{L}$ so that the gauge group is at most discrete ($k = 0$) and only one scalar field is present ($d = 1$). All irreducible symmetric spaces have been thoroughly studied and classified²⁶⁾. Example: $S^7 = \text{SO}(8) \setminus \text{SO}(7)$ is a symmetric space admitting, up to a scale, only one $\text{SO}(8)$ invariant metric.

d) $S = G \setminus H$ is an isotropy-irreducible homogeneous space. The cases discussed in c) fall into this category but there are many isotropy-irreducible homogeneous spaces which are not symmetric. They are classified in Ref. 27). Here again $d = 1$ and $N|H$ are discrete. Example: $S^7 = \text{Spin } 7 \setminus G_2$ is a simply connected isotropy-irreducible, but non-symmetric space. It admits, up to a a scale, only one Spin 7 invariant metric [n.b. the same as the one in b)²⁸⁾].

Another example of this type: $G = \text{Spin } 8$, $H = \text{SU}(3)/Z_3$, $S = G \setminus H$ is isotropy irreducible (not symmetric) with $N \simeq H$ and a discrete gauge group K . The decomposition of $\text{Ad } G$ into real irreducible representations of H reads: $28 = 8 \oplus [10 + \overline{10}]$. Notice (see the end remark of Section 4.1) that $10 + \overline{10}$ is to be understood as \mathbb{R} -irreducible and therefore the number of scalars (i.e., Spin 8 invariant metrics on S) is $d = 1$ (and not $1 + 1$). The 8 in the decomposition is of course \mathfrak{K} itself.

e) $S = G \setminus H$ is normal space. Any metric on S which is induced by a bi-invariant metric on G is G -invariant. Such metrics on S are called normal. In a sense normal metrics are the "most natural" G -invariant metrics on S . A homogeneous space $S = G \setminus H$ is called normal if every G invariant metric on S is normal. Notice that isotropy irreducible spaces are normal, but they are not the only spaces of normal type²⁹⁾.

4.2 Model building

To build a model one has to choose a global group G together with two subgroups H and K so that $N = H \times K$ is the normalizer of H in G . Then $S = G \setminus H$ is the internal space and $K = N|H$ is the gauge group. We can notice that S itself admits a principal bundle structure with basis $L = G \setminus N$ and structure group $K = N|H$ ¹⁸⁾ (Ch. XII) schematically



Fig. 5

Given H and K it is natural to restrict oneself to those cases where the representation of $N = H \cdot K$ on \mathfrak{G}/\mathcal{N} is faithful and irreducible. As it was mentioned in Section 3.1, N is then automatically the normalizer of H in G . Symmetric irreducible spaces are listed in Refs. 26) and 30) and the non-symmetric ones can be found in Ref. 27). When one goes through these lists one realizes that N is often a simple Lie group. In such a case either $H = N$ (in which case the gauge group K is trivial) or $H = \{e\}$ (modulo discrete groups) so that S is a group manifold itself $S = N = K = G$. $G \setminus N$ simply connected with non-simple N are very rare and are all listed in Tables 1 and 2. Table 1 is extracted from Ref. 30) and Table 2 is extracted from Ref. 27). We also give the reduction χ (over the reals) of $\text{Ad } G$ with respect to $N = H \cdot K$. Examples with reducible representations of N on \mathfrak{L} can be obtained by taking products of irreducible ones.

Let us now analyze in some detail several cases from the tables.

- a) $G = E_8$, $N = (E_6 \times SU(3))/Z_3$, $H = E_6$
 $S = G \setminus H$ is of dimension 170, $L = G \setminus N$ is isotropy irreducible but not symmetric. The connected component of the identity of the gauge group is $K = N \setminus H = SU(3)/Z_3$. The reduction of $\text{Ad } G$ with respect to real irreducible representations of $N = H \cdot K$ is

$$[248] = \underbrace{[78 \otimes 1]}_{\mathfrak{R}} + \underbrace{[1 \otimes 8]}_{\mathfrak{K}} + \underbrace{[27 \otimes 3 + \bar{27} \otimes \bar{3}]}_{\mathfrak{L}}$$

The reduction of the subspace $S = K + \mathfrak{L}$ with respect to H is

$$S = 8[1] + 3[27 + \bar{27}]$$

The dimension of the space of G invariant metrics on S is therefore

$$d = \frac{8 \times 9}{2} + \frac{3 \times 4}{2} = 42.$$

- b) $G = SO(10)$, $N = SU(5) \times U(1)$, $H = U(1)$
 $S = G \setminus H$ is of dimension 20, $L = G \setminus N$ is a Hermitian symmetric space. The gauge group is $SU(5)$, and the reduction of $\text{Ad } G$ with respect to $N = H \cdot K$ reads

$$[45] = [1 \otimes 1] + [1 \otimes 24] + [1 \otimes 10 + \bar{1} \otimes \bar{10}]$$

The reduction of S with respect to H is

$$S = 24[1] + 10[1 + \bar{1}]$$

(Observe that in the real domain an Abelian group may have two-dimensional irreducible representations). Therefore

$$d = \frac{24 \times 25}{2} + \frac{10 \times 11}{2} = 355 .$$

- c) $G = U(q+1, \mathbb{H})$, $N = U(q, \mathbb{H}) \times SU(2)$, $H = U(q, \mathbb{H}) (= USp(q, \mathbb{C}))$,
 $S = G \setminus H = S^{4q+3}$ (spheres), $L = G \setminus N = \mathbb{H}P^q$ is a quaternionic projective space ($G \setminus N$ is symmetric). The gauge group is $SU(2)$. The reduction of $Ad G$ with respect to $H \cdot K$ reads

$$[(q+1)(2q+3)] = [q(2q+1) \otimes 1] + [1 \otimes 3] + [2q \otimes 2]$$

and the reduction of S with respect to H is

$$S = 3[1] + 1[4q]$$

Therefore

$$d = \frac{3 \times 4}{2} + \frac{1 \times 2}{2} = 7 .$$

Notice that for $q = 1$ ($U(2, \mathbb{H}) \simeq SO(5)$) we obtain the $SU(2)$ foliation of S^7 over $\mathbb{H}P^1 \simeq S^4$, i.e., the usual $k = 1$ instanton bundle^{31), 32)}

- d) $G = SU(4)$, $N = SU(3) \times U(1)$, $H = SU(3)$
 $S = G \setminus H = S^7$, $L = SU(4) \setminus SU(3) \times U(1)$ is an irreducible Hermitian symmetric space. The gauge group is $U(1)$ and the reduction of $Ad G$ with respect to $H \cdot K$ is

$$[15] = [1 \otimes 8] + [1 \otimes 1] + [3 \otimes 1 + \bar{3} \otimes \bar{1}]$$

The H -reduction of S reads,

$$S = 1 + 1[3 + \bar{3}]$$

and therefore

$$d = 2 .$$

4.3 Comments

Here we collect various comments, remarks and information which are not used in the present paper but are closely related to the subject

* on the sign of scalar curvature r in the internal space S *

If S is a compact, non-Abelian group G then, according to Ref. 33):

- there exist right invariant metrics on S so that $r < 0$, $r = 0$, $r > 0$;
- for a bi-invariant metric on G , r is positive

If S is a homogeneous space $G \backslash H$ with G compact non-Abelian then, according to Ref. 34)

- If S is of normal type then every G invariant metric on S has $r > 0$;
- if S is not normal then there exist G invariant metrics on S with $r < 0$, $r = 0$ and $r > 0$.

* on the (un)boudness of the scalar curvature of the internal space *

If the scalar curvature of E is considered as a possible Lagrangian then, writing formally "Lagrangian = kinetic term - potential", it is clear that the scalar curvature $r(x)$ of the internal space at x is to be interpreted as "minus the potential". It is natural to freeze the volume of the internal space and to allow only for squashing deformations of S . The space of all G invariant metrics on S with a fixed volume is denoted by $R_0(G;S)$, and has the dimension $d_0 = d - 1$. Except in very special cases [for example $G = SO(3)$ or $G = SL(2,R)$ non-compact] the scalar curvature r considered as a function on R_0 is still unbounded from above³⁵⁾. For homogeneous spaces little is known, see however Ref. 36). An interesting possibility is to allow for solvable (non-compact) groups, in which case r as a function on R_0 is always non-positive³⁵⁾ (§ 6) [see also Ref. 6) for a discussion of "flat" groups].

* on the critical points of the scalar curvature in S *

Consider the space of Riemannian metrics of a given fixed volume form dv on a compact manifold S . It is known³⁷⁾ that the critical points of the functional

$$A[g] = \int_S r[g] d\sigma$$

are precisely the Einstein metrics on S . If $S = G \backslash H$ is a homogeneous space then $r[g]$ is constant on S and $A[g] = r[g] \cdot V_0$. It is shown in Ref. 35) that every compact simple Lie group G , except for $SO(3)$ and possibly G_2 and $USp(2n + 1)$ admits at least two conformally inequivalent right invariant Einstein metrics. Notice that these critical points are usually saddle points of the functional A on R_0 , and not always local extrema. The results of Ref. 35) have been extended to a class of homogeneous spaces in Ref. 38). A classification of homogeneous spaces admitting Einstein metrics and the structure of moduli of such metrics is a difficult problem, still unsolved. For a recent account see Refs. 39) and 40). [Notice that an isotropy irreducible homogeneous space is always an Einstein space with respect to its unique invariant metric²⁷⁾.] It is clear that these problems are relevant for a semi-classical approximation of a quantum theory containing the scalars $h_{\alpha\beta}$. Notice that making an expansion

of $h_{\alpha\beta}$ around some non-trivial critical point (Einstein metric on S) amounts to give a "non-zero expectation value" to the Higgs fields $\phi_\alpha(x)$ so that, because of the minimal coupling of the gauge fields of $N|H$ to the scalars, we have a spontaneous symmetry breaking and some of the gauge fields may acquire masses.

* The case when L is an irreducible symmetric space *

The scalar curvature $R(G|H)$ can be computed using (3.5.8) but it can be also obtained via the standard Kaluza-Klein construction applied to S considered as a principal bundle over L with structure group K (Fig. 5). If L is isotropy irreducible, the $(1 + \frac{\ell(\ell+1)}{2})$ parameter family of G invariant metrics on S is given by

$$h_{ab} = \mu^2 \eta_{ab} \quad , \quad h_{\hat{a}\hat{e}} = \frac{4}{\hat{a}} \frac{4}{\hat{e}} \eta_{\hat{a}\hat{e}} \quad ,$$

where $-\eta_{ij}$ is the Killing form on G . In particular, when $L = G|N$ is symmetric, $\mu^2 = 1$ and $\psi_{\hat{a}}^{\hat{a}'} = t \delta_{\hat{a}}^{\hat{a}'}$ then (3.5.7) gives

$$R(G|H) = R(L) + R(K) - \frac{1}{4} C_{ab}^{\hat{c}} C^{\hat{c}ab}$$

where $R(L) = \ell/2$, $R(K) = ck/4t^2$ and $C_{ab}^{\hat{c}}$ can be considered as the curvature of the canonical connection on S . In the above formula $k = \dim K$, $\ell = \dim L$ and c is defined by $-\eta_{\hat{a}\hat{b}} = c^{-1} \times$ (the Killing form of K) or, equivalently $c = \text{index (Ad } K)/\text{index (Ad } G)$. Notice that by replacing $h_{\alpha\beta}$ by $(t^2)^{-k/s} h_{\alpha\beta}$, $s = \dim S$, we obtain one-parameter family of metrics with a fixed volume, the scalar curvature being given by

$$R(G|H; t) = t^{2k/s} \left[\frac{\ell}{2} + \frac{ck}{4t^2} + \frac{k}{4} (c-1) t^2 \right]$$

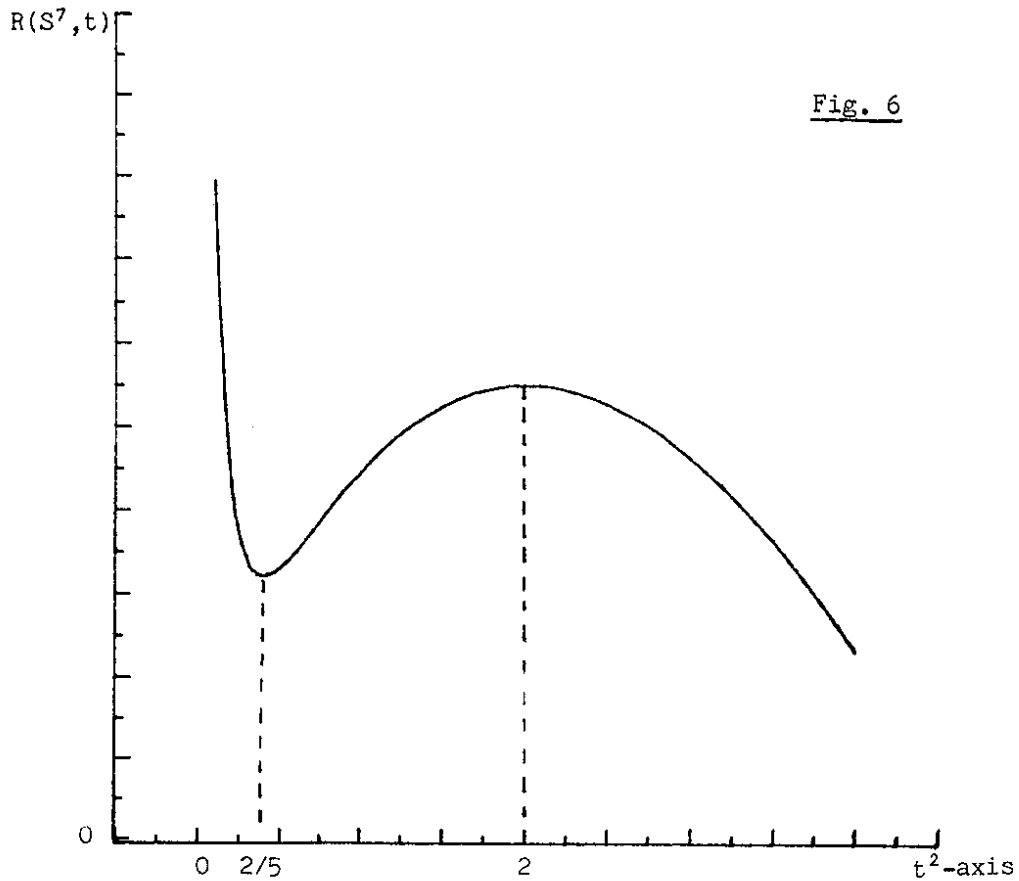
In particular for $S^7 = \text{USp}(2)|\text{SU}(2)$ we get, using $c = 2/3$

$$R(S^7; t) = t^{6/7} \left[2 + \frac{1}{2t^2} - \frac{t^2}{4} \right]$$

$$\dot{R}(S^7; t) = -\frac{5}{7} t^{-15/7} (t^2 - 2)(t^2 - 2/5)$$

The condition $\dot{R} = 0$ is a necessary condition for obtaining an Einstein metric. The two Einstein metrics on S^7 corresponding to $t^2 = 2$ and $t^2 = 2/5$ have been found in Ref. 38) by computing the Ricci tensor. Notice that $t^2 = 2$ is the $\text{SO}(8)$ invariant metric and $t^2 = 2/5$ has symmetry $\text{USp}(2) \times \text{SU}(2)$. Notice finally that $t^2 = 1$ - the normal metric on $\text{USp}(2)|\text{SU}(2)$ - is not Einstein (which should not be surprising since $\text{USp}(2)|\text{SU}(2)$ is not isotropy irreducible).

The scalar curvature $R(S^7;t)$ as a function of t is given in Fig. 6. The $t^2 = 2/5$ Einstein metric on S^7 have recently been used for providing a new vacuum for $d = 11$ supergravity⁴¹⁾.



* Spinors *

Introducing spinor fields in the extended space-time E is an obvious necessity (and can be achieved, for example, by replacing multidimensional gravitation by supergravity). By studying the Dirac equation on E , one would be led naturally to a system of spinor fields coupled to gauge fields and to the scalar fields. To make a link with the previous comment, notice that zero modes of the Dirac operator on the typical compact internal space S (endowed with a G invariant metric) exist only if the scalar curvature is negative.

ACKNOWLEDGEMENTS

We wish to thank M. Duff, B. Morel, C. Orzalesi, L. Palla and P. Sorba for comments and clarifying remarks. Special thanks are due to Professor R. Stora for his continuous interest, encouragement and discussions. One of us (A.J.) is also grateful to Professor R. Haag for many illuminating discussions concerning the structure of multidimensional unified field theories. Thanks are also due to the Theory Division of CERN for its kind hospitality.

TABLE CAPTIONS

Table 1

Simply connected irreducible symmetric spaces $L = G \backslash N$ where N is not a simple Lie group. In column \mathfrak{L} , the real irreducible representation $\text{Ad}(N)$ on \mathfrak{L} is expressed in terms of reducible complex representations. Decompositions not appearing in this column should be computed for given values of p, q (or n).

Table 2

Same as in 1 but $L = G \backslash N$ are simply connected irreducible and isotropy-irreducible (but not symmetric) spaces where N is not a simple Lie group.

Table 3

In order to simplify the reading of Tables 1 and 2, Table 3 recalls the usual Cartan classification of Lie groups.

Table 1

G	N	Remarks	ℓ
SU(4)	SO(4)		$3 \otimes 3$
SU(p+q)	$S(U_p \times U_q)$	$p \geq 2, q \geq 1$	
SO(p+q)	$SO(p) \times SO(q)$	$p \geq 2, q \geq 2$	
SO(4+q)	$SO(4) \times SO(q)$	$p = 4, q \geq 2$	
SO(2n)	U(n)	$n \geq 3$	
USp(n)	U(n)	$n \geq 1$	
USp(p+q)	$USp(p) \times USp(q)$	$p \geq 1, q \geq 1$	
E_6	$SU(6) \times SU(2)$		$20 \otimes 2$
E_6	$SO(10) \times U(1)$		$45 \otimes 1 + (16 + \overline{16}) \otimes 1 + 1 \otimes 1$
E_7	$SO(12) \times SU(2)$		$32 \otimes 2 + \overline{32} \otimes 2$
E_7	$E_6 \times U(1)$		$27 \otimes 1 + \overline{27} \otimes 1$
E_8	$E_7 \times SU(2)$		$56 \otimes 2$
F_4	$USp(3) \times SU(2)$		$14 \otimes 2$
G_2	SO(4)	(2 cases)	$4 \otimes 2$

In this table, $USp(n)$ denotes $USp(n, \mathbb{C}) = U(n, \mathbb{H})$ and $S(U(p) \times U(q)) \doteq (U(p) \times U(q)) \cap SU(p+q)$. Notice the following local isomorphism $SO(4) \sim SU(2) \times SU(2)$
 $S(U_p \times U_q) \sim SU(p) \times SU(q) \times U(1)$

Table 2

G	N	Remarks	ℓ
$SU(pq)/Z_m$	$\{SU(p)/Z_p\} \times \{SU(q)/Z_q\}$	$p \geq q \geq 2, pq > 4$ $m = \text{l.c.m.}(p, q)$	
F_4	$SO(3) \times G_2$		$5 \otimes 7$
F_4	$\{SU(3) \times SU(3)\}/Z_3$		$3 \otimes 6 + \bar{3} \otimes \bar{6}$
E_6/Z_3	$\{SU(3)/Z_3\} \times G_2$		$8 \otimes 14$
E_6/Z_3	$\frac{SU(3) \times SU(3) \times SU(3)}{Z_3 \times Z_3}$		$3 \otimes 3 \otimes 3 + \bar{3} \otimes \bar{3} \otimes \bar{3}$
E_7/Z_2	$\{USp(3)/Z_2\} \times G_2$		$14 \otimes 14$
E_7/Z_2	$SO(3) \times F_4$		$3 \otimes 26$
E_7/Z_2	$\{SU(3) \times SU(6)\}/Z_6$		$\bar{3} \otimes 15 + 3 \otimes \bar{15}$
E_8	$G_2 \times F_4$		$14 \otimes 52$
E_8	$\{SU(3) \times E_6\}/Z_3$		$3 \otimes 27 + \bar{3} \otimes \bar{27}$

Table 3

G	dimension	Centre	Complex extension
$A_m = SU(m+1, \mathbb{C})$	$m(m+2)$	Z_{m+1}	$S\mathfrak{l}(m+1, \mathbb{C})$
$B_m = Spin(2m+1, \mathbb{R})$	$m(2m+1)$	Z_2	$Spin(2m+1, \mathbb{C})$
$C_m = U(m, \mathbb{H})$	$m(2m+1)$	Z_2	$Sp(2m, \mathbb{C})$
$D_m = spin(2m, \mathbb{R})$	$m(2m-1)$	Z_4 if $m = 2\ell+1$ $Z_2 \times Z_2$ if $m = 2\ell$	$Spin(2m, \mathbb{C})$
G_2	14	1	
F_4	52	1	
E_6	78	Z_3	
E_7	133	Z_2	
E_8	248	1	

G: simple, compact, simply connected real Lie group.

The usual isomorphisms are:

$A_1 = B_1 = C_1, B_2 = C_2, A_3 = D_3, D_2 = A_1 \times A_1$

Usual notations are:

$U(m, \mathbb{H}) = USp(m, \mathbb{C}) = Sp(2m, \mathbb{C}) \cap SU(2m, \mathbb{C})$

Notice that $Spin(n)$ is the two-fold covering of $SO(n)$.

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