Geometry of multiplicity-free representations of $GL(n)$, visible actions on flag varieties, and triunity

Toshiyuki Kobayashi
Research Institute for Mathematical Sciences,
Kyoto University

Abstract

We analyze the criterion of the multiplicity-free theorem of representations [5, 6] and explain its generalization. The criterion is given by means of geometric conditions on an equivariant holomorphic vector bundle, namely, the "visibility" of the action on a base space (i.e. generic orbits intersecting with a real form) and the multiplicity-free property on a fiber.

Then, several finite dimensional examples are presented to illustrate the general multiplicity-free theorem, in particular, explaining that three multiplicity-free results stem readily from a single geometry in our framework. Furthermore, we prove that an elementary geometric result on Grassmann varieties and a small number of multiplicity-free results give rise to all the cases of multiplicity-free tensor product representations of $GL(n, \mathbb{C})$, for which Stembridge [12] has recently classified by completely different and combinatorial methods.

Keywords and phrases: multiplicity-free representation, branching law, semisimple Lie group, totally real, unitary representation, flag variety, tensor product, visible action
2000*MSC*: primary 22E4; secondary 32A37, 05E15, 20G05.
e-mail address: `toshi@kurims.kyoto-u.ac.jp`

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0 Introduction

Our concern in this paper is with a new geometric aspect on multiplicity-free representations. We shall try to clarify an abstract multiplicity-free theorem (Theorem 1.3) by various examples such as multiplicity-free tensor product representations of $U(n)$, in the framework of "visible actions" on complex manifolds, that is, those actions having a property that all orbits intersect with a fixed totally real submanifold.

We recall that a completely reducible representation π is called **multiplicity-free** if any irreducible representation occurs in π at most once (see Subsection 1.2 for a more general definition). Multiplicity-free representations are a very special class of representations, for which one could expect a simple and detailed study. A priori knowledge of multiplicity-free property of a given representation could give a guidance and encourage in finding its explicit irreducible decomposition. One might also expect that representation theory works effectively in applications to other fields, especially when the representation in consideration is multiplicity-free.

In this article, we study the representation of a group H on the space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections of an H-equivariant holomorphic vector bundle $\mathcal{V} \to D$. We find out geometric conditions on the base space and a typical fiber so that $\mathcal{O}(D, \mathcal{V})$ is multiplicity-free as an H-module (see Subsection 1.1 and Theorem 1.3 for a precise formulation). Loosely, our main assumption consists of the followings:

1) (Base space) Generic H-orbits meet a real form of D ("visible action").

2) (Fiber) The isotropy representation on the fiber is multiplicity-free when restricted to a certain subgroup M. (Here, M is defined by using the H-orbital structure on the base space D.)

Though our multiplicity-free theorem (Theorem 1.3) produces a number of multiplicity-free examples for infinite dimensional representations with D

non-compact ([5, 6]), the applications treated in this paper will be limited to the finite dimensional case with D compact. We shall report in another paper on some further applications of Theorem 1.3 to infinite dimensional representations.

Dealing with concrete examples, we shall explain as explicitly as possible key geometric backgrounds, in which the representations become multiplicity-free by virtue of Theorem 1.3.

For example, *any circle with center at the origin meets the real axis.* Even such a simple geometry (the visibility of the torus action on $\mathbb{C}$) gives rise to the multiplicity-free property of several results such as the tensor product with the symmetric tensor representation $S^k(\mathbb{C}^n)$ (Pieri's rule), the restriction $U(n) \downarrow U(n-1)$, etc.

Then, we also observe that an obvious equivalent expression of group decompositions (e.g. (2.4.1)) leads to non-trivial three different types of multiplicity-free results (which we call "triunity" of multiplicity-free representations; see Subsection 2.4).

Recently, Stembridge [12] has classified those pairs (π_λ, π_ν) of irreducible (finite dimensional) representations of $U(n)$ for which the tensor product $\pi_\lambda \otimes \pi_\nu$ is multiplicity-free. His approach is not geometric, but is combinatorial on a case-by-case basis. By using Theorem 1.3, we find that the same list can be obtained from the multiplicity-free property of very small representations (see Proposition 0.2 below) combined with an elementary geometry on Grassmann varieties given in Proposition 0.1.

Proposition 0.1 (Visible action). *Every H-orbit on D meets $D_{\mathbb{R}}$ in the following cases:*
1) $H = \mathbb{T}^n$, $D = \mathbb{P}^{n-1}\mathbb{C}$ *and* $D_{\mathbb{R}} = \mathbb{P}^{n-1}\mathbb{R}$.
2) $H = U(n_1) \times U(n_2) \times U(n_3)$ $(n = n_1 + n_2 + n_3)$, $D = Gr_p(\mathbb{C}^n)$ *and* $D_{\mathbb{R}} = Gr_p(\mathbb{R}^n)$, *if* $\min(p, n-p) \le 2$ *or* $\min(n_1, n_2, n_3) \le 1$.

This proposition will be explained in (2.2.2) and Theorem 3.1.

Next, let π_λ be the irreducible representation of $U(n)$ with highest weight λ. For example, if we set $\omega_k := (\underbrace{1, \cdots, 1}_{k}, \underbrace{0, \cdots, 0}_{n-k})$, then π_{ω_k} is nothing but the k-th exterior tensor representation of $U(n)$ on $\bigwedge^k(\mathbb{C}^n)$ $(1 \le k \le n)$. Then, we shall see in (3.2.1) and Proposition 3.4.2 the following:

Proposition 0.2. 1) π_{ω_k} $(1 \le k \le n)$ *is multiplicity-free as a* $\mathbb{T}^n$*-module.*
2) $\pi_{2\omega_k}$ $(1 \le k \le n)$ *is multiplicity-free as a* $(U(n_1) \times U(n_2) \times U(n_3))$*-module.*

We shall find that not only multiplicity-free tensor products of $U(n)$ (see Theorem 3.6) but also Kac's multiplicity-free spaces such as $M(n+1, m; \mathbb{C})$ acted by $GL(n, \mathbb{C}) \times GL(m, \mathbb{C})$ (see Theorem 2.7) can be explained in our framework by the same Grassmannian geometry given in Proposition 0.1 (2).

Apart from these applications to representation theory, such geometry itself seems interesting for its own sake, and we shall report a finer structural theorem (a generalized Cartan decomposition) on the double coset space $(U(p) \times U(q)) \backslash U(n)/(U(n_1) \times U(n_2) \times U(n_3))$ in a subsequent paper [7].

Notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$.

Acknowledgement: The results here are an outgrowth of [5], and were obtained while the author stayed at Harvard University during the academic year 2000-2001. He would like to thank sincerely all the people there, and among all, W. Schmid, for the warm and stimulating atmosphere of research. Parts of the results and related topics were presented at MSRI 2001 (the program "Integral Geometry" organized by S. Gindikin, L. Barchini and R. Zierau), NUS 2002 (the IMS program on "Representation Theory of Lie Groups" organized by Eng-Chye Tan and Cheng-Bo Zhu), in the Lorentz center, Leiden 2002 "Representation theory of Lie groups, harmonic analysis on homogeneous spaces and quantization" organized by G. van Dijk and V. F. Molchanov, and in a Paris VI seminar organized by M. Duflo. He would like to thank everyone at these institutes for their hospitality during his stay and also for valuable comments by the participants at various occasions.

1 Geometric conditions for multiplicity-free representations

1.1 Holomorphic bundles and anti-holomorphic maps

Let D be a connected complex manifold, K a (possibly, non-compact) Lie group, and $\varpi : P \to D$ a principal K-bundle. Given a finite dimensional unitary representation (μ, V) of K, we define an associated holomorphic vector bundle $\mathcal{V} := P \times_K V$ over D.

Suppose a group H acts on P from the left, commuting with the right action of K, such that the induced action of H on D is biholomorphic. Then H also acts on the holomorphic vector bundle $\mathcal{V} \to D$, and we form naturally a continuous representation of H on the Fréchet space $\mathcal{O}(D, \mathcal{V})$ consisting of holomorphic sections.

Suppose we are given automorphisms of Lie groups H and K, and a diffeomorphism of P, for which we use the same letter σ, satisfying the following two conditions:

- $\sigma(hpk) = \sigma(h)\sigma(p)\sigma(k) \quad (h \in H, k \in K, p \in P)$. (1.1.1)
- The induced action of σ on D ($\simeq P/K$) is anti-holomorphic. (1.1.2)

1.2 Multiplicity-free representations

Let π be a unitary representation of a group H on a (separable) Hilbert space $\mathcal{H}$, and we write $\mathrm{End}_H(\mathcal{H})$ for the ring of continuous endomorphisms commuting with H. In order to state a general theorem on multiplicity-free representations (see Theorem 1.3 below), we recall:

Definition 1.2. We say $(\pi, \mathcal{H})$ is **multiplicity-free** if the ring $\mathrm{End}_H(\mathcal{H})$ is commutative.

This (abstract) definition makes sense even if $\dim \mathcal{H} = \infty$. Let us observe that the above definition coincides with the usual one in the case $\dim \mathcal{H} < \infty$. In fact, one can write an irreducible decomposition of a finite dimensional representation π as a finite direct sum:

$$\pi \simeq \underbrace{\mu_1 \oplus \mu_1 \oplus \cdots \oplus \mu_1}_{n_1} \oplus \underbrace{\mu_2 \oplus \mu_2 \oplus \cdots \oplus \mu_2}_{n_2} \oplus \cdots \oplus \underbrace{\mu_k \oplus \mu_k \oplus \cdots \oplus \mu_k}_{n_k}.$$

Then, Schur's lemma implies that the ring $\mathrm{End}_H(\mathcal{H})$ is isomorphic to $\bigoplus_{j=1}^{k} M(n_j, \mathbb{C})$. Hence, $\mathrm{End}_H(\mathcal{H})$ is commutative if and only if all the multiplicity $n_j \leq 1$.

1.3 Abstract multiplicity-free theorem

Suppose we are in the setting of Subsection 1.1. For a subset B of $P^\sigma := \{p \in P : \sigma(p) = p\}$, we define the subset $M(B)$ of K by

$$M \equiv M(B) := \{k \in K : bk \in Hb \text{ for any } b \in B\}. \tag{1.3.1}$$

Then it is clear that M is a σ-stable subgroup.

We are interested in when $\mathcal{O}(D, \mathcal{V})$ becomes multiplicity-free. Since $\mathcal{O}(D, \mathcal{V})$ itself is not necessarily unitarizable, we shall consider all possible subrepresentations of $\mathcal{O}(D, \mathcal{V})$ which are unitarizable, and then discuss their multiplicity-free property. Here is our main machinery in finding multiplicity-free representations in both infinite and finite dimensional cases.

Theorem 1.3 (Abstract multiplicity-free theorem). *Retain the setting as above. Assume that there exists a subset B of P^σ satisfying the following four conditions:*

- *HBK contains an interior point of P.* (1.3)(a)
- *The restriction $\mu|_M$ decomposes as a multiplicity-free sum of irreducible representations of M, say,* $\mu|_M \simeq \bigoplus_i \nu^{(i)}$. (1.3)(b)
- *$\mu \circ \sigma$ is isomorphic to μ^* (the contragredient representation of μ) as representations of K.* (1.3)(c)
- *$\nu^{(i)} \circ \sigma$ is isomorphic to $(\nu^{(i)})^*$ as representations of M for every i.* (1.3)(d)

Then, if an (abstract) unitary representation π of H can be embedded H-equivariantly and continuously into $\mathcal{O}(D, \mathcal{V})$, then π is multiplicity-free as an H-module.

The proof of Theorem 1.3 will be given in another paper.

Let us examine the assumptions of Theorem 1.3. In order to get an upper estimate of the multiplicities like Theorem 1.3, it would be natural to require that both base spaces and fibers should be relatively "small", compared to the transformation group H. In this respect, we note:

a) The first assumption (1.3)(a) controls the base space D ($\simeq P/K$). The subset B may be regarded as a set of representatives of (generic) H-orbits on D if we take B as small as possible. (An extremal case is when B consists of finitely many points. This means that there exists an open H-orbit on D.) We note that an H-orbit HpK in D is σ-stable whenever $p \in P^\sigma$. Therefore, the assumption (1.3)(a) implies that generic H-orbits in D are σ-stable because $B \subset P^\sigma$.

b) Another relation of the assumption (1.3)(a) is that generic H-orbits meet D^σ. We shall discuss this point in Section 2 as "visible actions".

c) The second assumption (1.3)(b) is to control the fiber. Loosely, the smaller B is, the larger becomes M and the more likely (1.3)(b) tends to hold.

d) The remaining assumptions (1.3)(c) and (d) are less crucial because they are often automatically fulfilled by an appropriate choice of σ.

Relevant results were previously given in some special settings; in [1] for the trivial line bundle case, and in [5, 6] for the general line bundle case. The novelty here is to find out the conditions (1.3)(b), (c) and (d), by which the multiplicity becomes still free for the general vector bundle case. We note that these conditions are automatically satisfied in the trivial line bundle case. The generalization to the vector bundle case here enables us to handle some delicate examples of finite dimensional representations, as we shall see in Section 3.

1.4 Propagation of multiplicity-free property

Putting a special emphasis on the assumption (1.3)(b), we may regard Theorem 1.3 as a **propagation theorem of multiplicity-free property** (or an induced theorem of multiplicity-free property) from a smaller representation $(\mu|_M, V)$ of a smaller group M to a larger representation $(\pi, \mathcal{H})$ of a larger group H. We note that $\mathcal{H}$ can be infinite dimensional, while V is finite dimensional.

As an example, we shall see in Section 3 that all multiplicity-free tensor product representations of $U(n)$ can be obtained as a propagation of the multiplicity-free property of very small representations given in Proposition 0.2 (and the obvious one dimensional cases).

2 Triunity and visible actions

In this section, we shall illustrate Theorem 1.3 by elementary examples such as a toral action on $\mathbb{C}$.

We shall see that a single geometry leads to three different multiplicity-free results (**triunity**) in our framework. This fact reflects the obvious three equivalent conditions on group structure (see (2.4.1)).

2.1 Three examples of multiplicity-free representations

We start with well-known examples of multiplicity-free decompositions.

For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$, we put $|\lambda| := \sum_{j=1}^n \lambda_j$, and write $\mathbb{C}_\lambda$ for the one dimensional representation of the n-torus $\mathbb{T}^n$. Furthermore, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, we denote by $\pi_\lambda \equiv \pi_\lambda^{U(n)}$ the irreducible representation of $U(n)$ with highest weight λ. For example, $\pi^{U(n)}_{(k,0,\ldots,0)}$ is realized as the k-th symmetric tensor representation $S^k(\mathbb{C}^n)$ $(k \in \mathbb{N})$, and $\pi^{U(n)}_{\omega_k} \equiv \pi^{U(n)}_{(1,\cdots,1,0,\cdots,0)}$ is realized as $\bigwedge^k(\mathbb{C}^n)$ $(0 \leq k \leq n)$.

Here are some explicit formulas of decompositions:

(Weight decomposition)
$$\pi^{U(n)}_{(k,0,\dots,0)}|_{\mathbb{T}^n} \simeq \bigoplus_{\substack{\mu\in\mathbb{N}^n \\ |\mu|=k}} \mathbb{C}_{(\mu_1,\cdots,\mu_n)}. \tag{2.1.1}$$

($U(n) \downarrow U(n-1)$)
$$\pi^{U(n)}_{\lambda}|_{U(n-1)} \simeq \bigoplus_{\substack{\mu\in\mathbb{Z}^{n-1} \\ \lambda_1\geq\mu_1\geq\lambda_2\geq\cdots\geq\mu_{n-1}\geq\lambda_n}} \pi^{U(n-1)}_{(\mu_1,\cdots,\mu_{n-1})}. \tag{2.1.2}$$

(Tensor product)
$$\pi^{U(n)}_{\lambda} \otimes \pi^{U(n)}_{(k,0,\dots,0)} \simeq \bigoplus_{\substack{\mu_1\geq\lambda_1\geq\cdots\geq\mu_n\geq\lambda_n \\ |\mu-\lambda|=k}} \pi^{U(n)}_{(\mu_1,\cdots,\mu_n)} \quad \text{(Pieri's rule)}. \tag{2.1.3}$$

As the explicit formulas show, all of the above decompositions are multiplicity-free. The multiplicity-free property itself in each case can be also shown a priori without explicit computations. We shall explain and compare two methods of proving (abstract) multiplicity-free property of the above examples — one is a new approach based on Theorem 1.3 (see Subsection 2.4) and the other is a well-established approach by using Borel subgroups (see Subsection 2.5). The geometry involved is apparently different (see Problem 2.6).

2.2 Torus action

Let us consider the natural action of a one dimensional toral subgroup $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ on $\mathbb{C}$. Then, an obvious observation is:

every $\mathbb{T}$*-orbit on* $\mathbb{C}$ *meets* $\mathbb{R}$.

Likewise, the natural action of an n-torus $\mathbb{T}^n$ on $\mathbb{C}^n$ has the following property:

$$\textit{every } \mathbb{T}^n\textit{-orbit on } \mathbb{C}^n \textit{ meets } \mathbb{R}^n, \tag{2.2.1}$$

and also on the projective space $\mathbb{P}^{n-1}\mathbb{C}$:

(Geometry)
$$\textit{every } \mathbb{T}^n\textit{-orbit on } \mathbb{P}^{n-1}\mathbb{C} \textit{ meets } \mathbb{P}^{n-1}\mathbb{R}. \tag{2.2.2}$$

In turn, the geometric property (2.2.2) can be interpreted as the following decomposition of the unitary group $G := U(n)$.

(Group structure)
$$G = TG^{\sigma}L. \tag{2.2.3}$$

Here, $T := \mathbb{T}^n$ (maximal toral subgroup of G), $L := U(1) \times U(n-1)$, and σ is an automorphism of G given by $\sigma(g) = \overline{g}$ (complex conjugation) so that $G^\sigma = O(n)$.

Proof of (2.2.3). We identify the homogeneous space G/L with $\mathbb{P}^{n-1}\mathbb{C}$. Likewise, $G^\sigma/L^\sigma \simeq \mathbb{P}^{n-1}\mathbb{R}$. Then, the geometry (2.2.2) means that any T-orbit on G/L contains a representative coming from G^σ, that is, $TG^\sigma L = G$. □

2.3 Visible actions

Definition 2.3 (Visible action). Suppose a Lie group H acts holomorphically on a complex manifold D. We say the action is **visible** if there exists a totally real manifold $D_\mathbb{R}$ such that every H-orbit meets $D_\mathbb{R}$.

For a connected D, the action is **generically visible** if there exists an H-invariant open subset D' of D such that the action on D' is visible.

Example 2.3.1. The (standard) action of $\mathbb{T}^n$ on $\mathbb{P}^{n-1}\mathbb{C}$ is visible in light of (2.2.2).

Example 2.3.2. In the setting of Theorem 1.3, suppose that D^σ is a totally real submanifold of D. Then, it follows from the condition (1.3)(a) that the action of H on $D \simeq P/K$ is generically visible.

Further examples will be given in Proposition 2.8, Theorem 3.1, and Example 3.1.2.

2.4 Triunity — simplest examples

Next, we rewrite (2.2.3) in the following three different assertions on the decomposition of a group G (or $G \times G$), of which the equivalence is obvious:

$$TG^\sigma L = G \Leftrightarrow LG^\sigma T = G \Leftrightarrow \operatorname{diag}(G)(G^\sigma \times G^\sigma)(T \times L) = G \times G. \quad (2.4.1)$$

Correspondingly, Theorem 1.3 gives a proof of three different types of (abstract) multiplicity-free results that we have observed in (2.1.1), (2.1.2) and (2.1.3), respectively.

Example 2.4 (Triunity). 1) (Weight multiplicity-free) For any $k \in \mathbb{N}$, $S^k(\mathbb{C}^n)$ is multiplicity-free as a $\mathbb{T}^n$-module.
2) $(U(n) \downarrow U(n-1))$ For any $\pi \in \widehat{U(n)}$, the restriction $\pi|_{U(n-1)}$ decomposes

with multiplicity free.

3) (Tensor product) For any $\pi \in \widehat{U(n)}$ and for any $k \in \mathbb{N}$, the tensor product $\pi \otimes S^k(\mathbb{C}^n)$ decomposes with multiplicity free.

Sketch of proof. 1) Set $(H, B, K, P) := (T, G^\sigma, L, G)$.
2) Set $(H, B, K, P) := (L, G^\sigma, T, G)$.
3) Set $(H, B, K, P) := (\mathrm{diag}(G), G^\sigma \times G^\sigma, T \times L, G \times G)$.

Accordingly, the representations $S^k(\mathbb{C}^n)$, π, and $\pi \otimes S^k(\mathbb{C}^n)$ are realized on the space of holomorphic sections of some holomorphic line bundles over $P/K \simeq G/L$, G/T, and $(G \times G)/(T \times L)$, respectively, by the Borel-Weil theorem. Since σ is given by $\sigma(g) = \overline{g}$, the induced action of σ on P/K is anti-holomorphic. Now, let us apply Theorem 1.3 with $\dim V = 1$. Then, the assumption (1.3)(a) is fulfilled in each case, as we saw the equivalent form in (2.4.1). The other assumptions (1.3)(b)$\sim$(d) are verified easily. Hence, all the statement of Example 2.4 follows from Theorem 1.3. $\square$

2.5 Open orbits of Borel subgroups

The point of the approach in Subsection 2.4 is that such an elementary geometry (2.2.2) gives rise to three different (easy but non-trivial) multiplicity-free results simultaneously without computations of representations.

Example 2.4 can be verified also by another geometry, that is, by the existence of an open orbit of a Borel subgroup. For this, we recall a well-known fact:

Fact 2.5. Suppose $\mathcal{V} \to D$ be a holomorphic line bundle over a connected complex manifold D, on which a complex reductive Lie group $H_\mathbb{C}$ acts equivariantly. If D is a **spherical variety** (this means that a Borel subgroup of $H_\mathbb{C}$ acts on D with an open dense orbit), then any irreducible (finite dimensional, holomorphic) representation of $H_\mathbb{C}$ occurs in $\mathcal{O}(D, \mathcal{V})$ at most once.

Thus, the three statements of Example 2.4 are proved also by the following assertions, respectively.

- *The complex torus* $(\mathbb{C}^\times)^n$ *admits an open orbit on* $\mathbb{P}^{n-1}\mathbb{C}$. (2.5.1)
- *A Borel subgroup of* $GL(n-1,\mathbb{C})$ *admits an open orbit on the full flag variety* $\mathcal{B}_n$ *of* $GL(n,\mathbb{C})$. (2.5.2)
- *A Borel subgroup of* $GL(n,\mathbb{C})$ *admits an open orbit on* $\mathcal{B}_n \times \mathbb{P}^{n-1}\mathbb{C}$ *under the diagonal action.* (2.5.3)

The assertions (2.5.2) and (2.5.3) may not be so obvious as (2.5.1) (or (2.2.2)), but can be verified by straightforward computation on Lie algebras.

2.6 Visible actions versus spherical varieties

So far, we have seen that two different arguments on geometry, namely, visible actions (2.2.2) and spherical varieties ((2.5.1) $\sim$ (2.5.3)) lead to the same representation theoretic conclusions (Example 2.4). We raise the following problem:

Problem 2.6. Suppose a complex reductive Lie group $H_{\mathbb{C}}$ acts holomorphically on a complex manifold D. Are the following two conditions equivalent?
i) (visible actions) There exists a real form H of $H_{\mathbb{C}}$ such that the action of H on D is visible.
ii) (spherical variety) There exists an open orbit of a Borel subgroup of $H_{\mathbb{C}}$ on D.

2.7 Multiplicity-free spaces

We end this section with Kac's example of multiplicity-free spaces as another application of the action of $\mathbb{T}^n$ on $\mathbb{C}^n$, as stated in Subsection 2.1.

A complex vector representation D of H is sometimes referred as a **multiplicity-free space** if the space $\mathbb{C}[D]$ of polynomials on D splits into an algebraic direct sum of irreducible representations of H. For example, $M(n,m;\mathbb{C})$ is a multiplicity-free space of $GL(n,\mathbb{C}) \times GL(m,\mathbb{C})$, as is also known as the "GL_n-GL_m duality" (see [3], Subsection 2.1). More strongly, the following theorem holds.

Theorem 2.7 (Kac, [4]). $M(n,m;\mathbb{C}) \oplus \mathbb{C}^m$ *are multiplicity-free spaces of* $GL(n,\mathbb{C}) \times GL(m,\mathbb{C})$ *in both cases* (2.7.1) *and* (2.7.2).

Here, we let $H_{\mathbb{C}} := GL(n, \mathbb{C}) \times GL(m, \mathbb{C})$ *act on* $D := M(n, m; \mathbb{C}) \oplus \mathbb{C}^m$ *in the following two ways: For* $g = (g_1, g_2) \in H_{\mathbb{C}}$,

$$D \to D, \qquad (A, b) \mapsto (g_1 A g_2^{-1}, b g_2^{-1}), \tag{2.7.1}$$

$$D \to D, \qquad (A, b) \mapsto (g_1 A g_2^{-1}, b\, {}^t g_2). \tag{2.7.2}$$

In the next subsection, we shall give a new proof of Theorem 2.7 by using Theorem 1.3, and an elementary example of visible actions (see Proposition 2.8, which reduces essentially to (2.2.1)).

2.8 Geometry of Kac's examples and Triunity

Retain the notation as in Section 2.7. For the proof of Theorem 2.7, we need:

Proposition 2.8. *Let* $H := U(n) \times U(m)$*. In both cases* (2.7.1) *and* (2.7.2)*, every* H*-orbit on* $D = M(n+1, m; \mathbb{C})$ *meets* $D_{\mathbb{R}} := M(n, m; \mathbb{R}) \oplus \mathbb{R}^m$*. That is, the* H*-action on* D *is visible (see Definition 2.3).*

Proof. Let E_{ij} be the matrix unit, and we set

$$\mathfrak{a} := \sum_{i=1}^{\min(m,n)} \mathbb{R} E_{ii}.$$

What follows below is a proof of a stronger statement, namely, every H-orbit on D meets $\mathfrak{a} \oplus \mathbb{R}^m$.

First, it follows from a theory of normal forms in linear algebra that any element of $M(n, m; \mathbb{C})$ can be transformed into $\mathfrak{a}$ by the action of H.

Second, take an arbitrary element $(A, b) \in D$. Then, as far as the H-orbit through (A, b) is concerned, we may and do assume that $A \in \mathfrak{a}$. Now, we define a subgroup T of $H = U(n) \times U(m)$ by

$$\{(\mathrm{diag}(t_1, \cdots, t_m, (1, \cdots, 1)), \mathrm{diag}(t_1, \cdots, t_m)) : t_j \in \mathbb{T}\}.$$

Then T is isomorphic to an m-torus $\mathbb{T}^m$, stabilizes the element A, and acts on $\mathbb{C}^m$ as rotations, so that every H-orbit through (A, b) $(\in \mathfrak{a} \oplus \mathbb{C}^m)$ has an intersection with $A \oplus \mathbb{R}^m$, as we saw in (2.2.1) (what we have used here is again the geometry that any circle with center at the origin meets the real axis). Hence, every H-orbit meets $\mathfrak{a} \oplus \mathbb{R}^m$. ☐

Since $M(n+1, m; \mathbb{C})$ is embedded in the Grassmann variety $Gr_{n+1}(\mathbb{C}^{n+m+1})$ as an open dense set (a Bruhat cell), Proposition 2.8 implies that the action of $U(n) \times U(m)$ on $Gr_{n+1}(\mathbb{C}^{n+m+1})$ is generically visible (in fact, it is visible). In turn, we obtain the following three multiplicity-free properties for representations of $U(n)$ as corollaries of Theorem 1.3:

• (Theorem 3.3) $\pi_\nu \in \widehat{U(n)}$ is multiplicity-free, when restricted to $U(p) \times U(q)$ for any p and q with $p+q=n$ if ν is of the form $\nu = (x, \cdots, x, y, \cdots, y, z, \cdots, z) \in \mathbb{Z}^n$ for some $x \geq y \geq z$ such that at least one of x, y or z appears at most once.

• (Theorem 3.4) $\pi_\lambda \in \widehat{U(n)}$ is multiplicity-free when restricted to $(U(n_1) \times U(n_2) \times 1)$ for any n_1 and n_2 with $n_1 + n_2 = n - 1$ if λ is of the form $\lambda = (\underbrace{a, \cdots, a}_{p}, \underbrace{b, \cdots, b}_{q}) \in \mathbb{Z}^n$ such that $a \geq b$ and $p + q = n$.

• (Theorem 3.6) $\pi_\lambda \otimes \pi_\nu$ is multiplicity-free if λ and ν are of the above forms.

These three results may be regarded as a part of triunity arising from the equivalence (2.4.1). To see this more systematically, we shall explain in the next section, a generalization of the visibility of the action of $U(n) \times U(m)$ on the Grassmann variety $Gr_{n+1}(\mathbb{C}^{n+m+1})$ to a more general setting in Theorem 3.1, and then we shall give some applications to representation theory by using Theorem 1.3.

3 Multiplicity-free representations of $U(n)$

In this section, various multiplicity-free results on finite dimensional representations of $U(n)$ (or equivalently, rational representations of $GL(n, \mathbb{C})$) will be provided in the framework of our abstract multiplicity-free theorem (Theorem 1.3). Relevant elementary Grassmannian geometry is also discussed.

3.1 Visible actions on Grassmann varieties

We start with a geometric background that will lead to multiplicity-free tensor product representations of $U(n)$.

Let $n_1 + n_2 + n_3 = p + q = n$. We consider naturally embedded subgroups $L := U(n_1) \times U(n_2) \times U(n_3)$ and $H := U(p) \times U(q)$ in $G := U(n)$. We define an automorphism σ of G by $\sigma(g) := \overline{g}$, the complex conjugate of

$g \in G$. Then, the fixed point subgroup G^σ is nothing but the orthogonal group $O(n)$.

Let $Gr_p(\mathbb{C}^n)$ be the Grassmann variety, and $\mathcal{B}_{n_1,n_1+n_2}(\mathbb{C}^n)$ the generalized flag variety consisting of pairs (F_1, F_2) of vector spaces of dimensions n_1, n_1+n_2, respectively, in $\mathbb{C}^n$. Similarly, the real Grassmann variety $Gr_p(\mathbb{R}^n)$ and $\mathcal{B}_{n_1,n_1+n_2}(\mathbb{R}^n)$ are defined. We note that $Gr_p(\mathbb{C}^n) \simeq \mathcal{B}_{0,p}(\mathbb{C}^n) \simeq \mathcal{B}_{p,n}(\mathbb{C}^n)$.

Theorem 3.1 (see [7]). *Let $p+q = n_1+n_2+n_3 = n$. The the following five conditions are equivalent:*
i) *Any orbit of $(U(n_1) \times U(n_2) \times U(n_3))$ on $Gr_p(\mathbb{C}^n)$ meets $Gr_p(\mathbb{R}^n)$.*
i)′ *Any orbit of $(U(p) \times U(q))$ on $\mathcal{B}_{n_1,n_1+n_2}(\mathbb{C}^n)$ meets $\mathcal{B}_{n_1,n_1+n_2}(\mathbb{R}^n)$.*
ii) *$G = LG^\sigma H$.*
ii)′ *$G = HG^\sigma L$.*
iii) *$\min(p,q) \le 2$ or $\min(n_1,n_2,n_3) \le 1$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) holds because the homogeneous space G/H is isomorphic to $Gr_p(\mathbb{C}^n)$, and G^σ/H^σ to $Gr_p(\mathbb{R}^n)$. Similarly, (i)′ $\Leftrightarrow$ (ii)′ holds. Since the equivalence (ii)$\Leftrightarrow$ (ii)′ is obvious, all of (i), (i)′, (ii) and (ii)′ are equivalent.

The implication (ii)′ $\Leftarrow$ (iii) follows from a main result in [7], where we constructed explicitly a subset $B \subset G^\sigma$ such that $G = LBH$ under the assumption (iii).

The implication (i) $\Rightarrow$ (iii) will not be used logically in this paper. An elementary proof based on linear algebra can be found in [7]. Here, we give an alternative proof by using Theorem 1.3. If the condition (i) were the case, then the same argument of Theorem 3.6 would show that the tensor product representations $\pi_\lambda \otimes \pi_\nu$ were multiplicity-free for any λ and ν of the form (3.6.1) and (3.6.2) (for any a, b, x, y, z with the notation therein). This contradicts to the fact due to Stembridge (see Remark 3.6.4) that $\pi_\lambda \otimes \pi_\nu$ is not multiplicity-free if neither (λ, ν) nor (ν, λ) satisfies the condition in Theorem 3.6. □

Remark 3.1.1. One of (therefore, all of) the conditions in Theorem 3.1 is also equivalent to:
vi) *The direct product manifold $Gr_p(\mathbb{C}^n) \times \mathcal{B}_{n_1,n_1+n_2}(\mathbb{C}^n)$ is a spherical variety of $GL(n,\mathbb{C})$ under the diagonal action.*

See Littelmann [10] for the statement (vi) in the case $n_3 = 0$. (We note that complex reductive Lie groups other than $GL(n,\mathbb{C})$ are also treated in [10].)

We pin down a special case of Theorem 3.1 by putting $n_3 = 0$:

Example 3.1.2. The standard action of $U(n_1) \times U(n - n_1)$ on $Gr_p(\mathbb{C}^n)$ is visible (Definition 2.3) for any n_1 and p such that $1 \leq n_1, p \leq n$.

This geometry leads to a multiplicity-free theorem of the branching law of $\pi_\nu^{U(n)}$ when restricted to $U(p) \times U(q)$ if ν is a rectangular shape ($n_3 = 0$ in (3.3.1)). See Theorem 3.6 and Remark 3.6.2.

In the following three subsections, we shall consider the restriction of representations of $U(n)$ with respect to standard subgroups. We shall see that the above geometry is used to prove some of multiplicity-free results.

3.2 $U(n) \downarrow \mathbb{T}^n$

First, consider the restriction of $\pi_\nu^{U(n)}$ ($\equiv \pi_\nu$) to a maximal toral subgroup $\mathbb{T}^n$ of $G = U(n)$.

We have seen in Example 2.4 (1) (or in (2.1.1)) that the k-th symmetric tensor representation $S^k(\mathbb{C}^n)$ is weight multiplicity-free, namely, the restriction $\pi_{(k,0,\cdots,0)}|_{\mathbb{T}^n}$ is multiplicity-free as a $\mathbb{T}^n$-module for any $k \in \mathbb{N}$.

The exterior tensor representation $\pi_{\omega_k}^{U(n)}$ on $\bigwedge^k(\mathbb{C}^n)$ $(1 \leq k \leq n)$ is also weight multiplicity-free, as one sees the following branching law:

$$\pi_{\omega_k}^{U(n)}|_{\mathbb{T}^n} \simeq \bigoplus_{\substack{\mu_i \in \{0,1\}\ (i=1,\cdots,n) \\ \mu_1+\cdots+\mu_n=k}} \mathbb{C}_{(\mu_1,\cdots,\mu_n)}. \tag{3.2.1}$$

Conversely, it is known that all of irreducible representations of $U(n)$ that are weight multiplicity-free are either $S^k(\mathbb{C}^n)$ $(k \in \mathbb{N})$ or $\bigwedge^k(\mathbb{C}^n)$ $(1 \leq k \leq n)$ up to one dimensional character ([3], Theorem 4.6.3).

3.3 $U(p + q) \downarrow (U(p) \times U(q))$

Next, we consider the restriction of $\pi_\nu \in \widehat{U(n)}$ to the subgroup $H = (U(p) \times U(q))$, where $n = p + q$. It turns out that Theorem 1.3 yields the following multiplicity-free result as an outcome of the Grassmannian geometry given in Theorem 3.1.

Theorem 3.3 **($U(p+q)\downarrow(U(p)\times U(q))$).** *The irreducible representation $\pi_\nu^{U(n)}$ decomposes as a multiplicity-free sum of irreducible representations when restricted to the subgroup $U(p)\times U(q)$, if one of the following three conditions is satisfied:*
1) $\min(p,q)=1$ *(and ν is arbitrary).*
2) $\min(p,q)=2$ *and ν is of the form*

$$(\underbrace{x,\cdots,x}_{n_1},\underbrace{y,\cdots,y}_{n_2},\underbrace{z,\cdots,z}_{n_3}), \tag{3.3.1}$$

where $x\geq y\geq z$ and $n_1+n_2+n_3=n$.
3) $\min(p,q)\geq 3$ *and ν is of the form* (3.3.1) *satisfying*

$$\min(x-y,y-z,n_1,n_2,n_3)\leq 1. \tag{3.3.2}$$

The converse also holds ([12], see Remark 3.6.4). An example of explicit branching laws will be given in Lemma 3.4.3 in the case $(x,y,z)=(2,1,0)$.

Proof. The statement (1) has been already explained in Example 2.4(2), where we attributed its reasoning to the visibility of the action of $\mathbb{T}^n$ on $\mathbb{P}^{n-1}\mathbb{C}$.

Likewise, Theorem 1.3 leads to the statement (2) from the Grassmannian geometry given in Theorem 3.1.

Let us prove the statement (3). One could prove a part of it (namely, under the assumption $\min(n_1,n_2,n_3)=1$) by using Theorem 3.1 again. However, Theorem 3.1 does not cover the case where $\min(x-y,y-z)\leq 1$. So, we shall give a proof by using a slightly different setting (still in the framework of Theorem 1.3). For this, we set

$$(H,P,K,\mu):=(U(p)\times U(q),U(n),U(n_1)\times U(n_2+n_3),\pi^{U(n_1)}_{(x,\dots,x)}\boxtimes\pi^{U(n_2+n_3)}_{(y,\dots,y,z,\dots,z)}).$$

Since both (P,H) and (P,K) are symmetric pairs, it follows from a Cartan decomposition (see Hoogenboom [2] or [7]) that there exists a compact torus B of $O(n)$ with dimension $l=\min(p,q,n_1,n_2+n_3)$ such that $HBK=P$. Then the subgroup $M\equiv M(B)$ (recall (1.3.1) for the definition) is of the form:

$$M(B)\simeq\begin{cases}\mathbb{T}^l\times U(p-l)\times U(q-l) & (l=\min(n_1,n_2))\\ \mathbb{T}^l\times U(n_1-l)\times U(n_2-l) & (l=\min(p,q))\end{cases} \tag{3.3.3}$$

because $M(B)$ coincides with the centralizer $Z_{H\cap K}(B)$ of B in $H \cap K$.

From now, assume $n_2 = 1$ (or $n_3 = 1$) or $y - z = 1$ (other cases are similar). Then, $\mu|_{U(n_2+n_3)} = \pi^{U(n_2+n_3)}_{(y,\cdots,y,z,\cdots,z)}$ is the $(y-z)$-th symmetric tensor representation $S^{y-z}(\mathbb{C}^{n_2+n_3})$ if $n_2 = 1$ (or its dual if $n_3 = 1$) or the exterior tensor representation $\bigwedge^{n_2}(\mathbb{C}^{n_2+n_3})$ if $y - z = 1$ up to a one dimensional character. In any case, the restriction $\mu|_{M(B)}$ is multiplicity-free because $\pi^{U(n_2+n_3)}_{(y,\cdots,y,z,\cdots,z)}$ is weight multiplicity-free (see Subsection 3.2) and because $\dim \pi^{U(n_1)}_{(x,\dots,x)} = 1$. Hence, all of the assumptions of Theorem 1.3 are verified.

Since the representation π_ν is realized in the space of holomorphic sections of the vector bundle $P \times_K \mu$ over the Grassmann variety $P/K \simeq Gr_{n_1}(\mathbb{C}^n)$ by the Borel-Weil theorem, Theorem 1.3 implies that the restriction $\pi_\nu|_H$ is multiplicity-free. □

3.4 $U(n) \downarrow (U(n_1) \times U(n_2) \times U(n_3))$

Third, we consider the restriction to the direct product subgroup $U(n_1) \times U(n_2) \times U(n_3)$ of $U(n) = U(n_1 + n_2 + n_3)$.

Theorem 3.4 ($U(n) \downarrow (U(n_1) \times U(n_2) \times U(n_3))$). *Suppose λ is of the form*

$$\lambda = (\underbrace{a, \dots, a}_{p}, \underbrace{b, \dots, b}_{q})$$

for some p, q such that $p + q = n$ and $a, b \in \mathbb{Z}$ with $a \geq b$.

Then the irreducible representation $\pi^{U(n)}_\lambda$ decomposes as a multiplicity-free sum of irreducible representations when restricted to the subgroup $U(n_1) \times U(n_2) \times U(n_3)$ if one of the following three conditions is satisfied:
1) $a - b \leq 2$ *(and p, q, n_1, n_2, n_3 are arbitrary).*
2) $\min(p, q) \leq 2$ *(and a, b, n_1, n_2, n_3 are arbitrary).*
3) $\min(n_1, n_2, n_3) \leq 1$ *(and a, b, p, q are arbitrary).*

Proof. The statement (1) is obvious when $a - b = 1$ because it is already weight multiplicity-free (see (3.2.1)). The statement (1) with $a - b = 2$ follows from a direct computation for $(a, b) = (2, 0)$ (see Proposition 3.4.2 below). The statements (2) and (3) are a consequence of Theorem 3.1 (visible actions on Grassmann varieties). □

Remark 3.4.1. As we have seen in the proof, Theorem 3.3 and the statements (2) and (3) of Theorem 3.4 are proved simultaneously from the same geometric result given in Theorem 3.1. This is a part of triunity, of which the counterpart in geometry is the equivalence (i) $\Leftrightarrow$ (i)$'$ in Theorem 3.1. One more multiplicity-free result (tensor product representations) will be given in Theorem 3.6 based on the same geometry (the remaining part of triunity in this case).

For the statement (1) of Theorem 3.4, we pin down the branching law for $(a, b) = (2, 0)$:

Proposition 3.4.2 **(**$U(n) \downarrow (U(n_1) \times U(n_2) \times U(n_3))$**).** *Let* $1 \le p \le n = n_1 + n_2 + n_3$.

$$\pi_{2\omega_p}^{U(n)} \simeq \bigoplus_{\substack{p_1,p_2,p_3,q_1,q_2,q_3 \ge 0 \\ p_i+q_i \le n_i \ (i=1,2,3) \\ 2q_i \le q_1+q_2+q_3 \ (i=1,2,3) \\ 2(p_1+p_2+p_3)+(q_1+q_2+q_3)=2p}} \pi_{\omega_{p_1}+\omega_{p_1+q_1}}^{U(n_1)} \otimes \pi_{\omega_{p_2}+\omega_{p_2+q_2}}^{U(n_2)} \otimes \pi_{\omega_{p_3}+\omega_{p_3+q_3}}^{U(n_3)}.$$

In particular, $\pi_{2\omega_p}^{U(n)}$ $(1 \le p \le n)$ is multiplicity-free when restricted to the subgroup $(U(n_1) \times U(n_2) \times U(n_3))$ for any partition (n_1, n_2, n_3) of n.

It is interesting to observe that the condition $2q_i \le q_1+q_2+q_3$ $(i = 1, 2, 3)$ is nothing but the triangular inequality:

$$q_1 \le q_2 + q_3, \quad q_2 \le q_3 + q_1, \quad q_3 \le q_1 + q_2.$$

Proof of Proposition 3.4.2. Use twice the following branching law, which in turn is obtained in an elementary way by the Littlewood-Richardson rule. □

Lemma 3.4.3 **(**$U(n_1+n_2) \downarrow (U(n_1) \times U(n_2))$**).** *Suppose* $p + q \le n_1 + n_2$. *Then*

$$\pi_{\omega_p+\omega_{p+q}}^{U(n_1+n_2)}|_{U(n_1)\times U(n_2)} \simeq \bigoplus_{\substack{p_1,p_2,q_1,q_2 \ge 0 \\ p_1+q_1 \le n_1,\ p_2+q_2 \le n_2 \\ |q_1-q_2| \le q \le q_1+q_2 \\ 2(p_1+p_2)+(q_1+q_2)=2p+q}} \pi_{\omega_{p_1}+\omega_{p_1+q_1}}^{U(n_1)} \otimes \pi_{\omega_{p_2}+\omega_{p_2+q_2}}^{U(n_2)}.$$

We note that this decomposition is also multiplicity-free, corresponding to the case $(x, y, z) = (2, 1, 0)$ in Theorem 3.3.

3.5 Multiplicity-free tensor product

Let $\mathfrak{g}_{\mathbb{C}}$ be a (general) complex reductive Lie algebra. We take a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$, and fix a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. For a dominant integral weight $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$, we denote by π_λ the irreducible finite dimensional representation of $\mathfrak{g}_{\mathbb{C}}$ with highest weight λ. Let $\mathfrak{l}_{\mathbb{C}}$ be a Levi subalgebra containing $\mathfrak{t}_{\mathbb{C}}$. The following theorem also fits nicely into the framework of Theorem 1.3:

Theorem 3.5. *The tensor product representation $\pi_\lambda \otimes \pi_\nu$ decomposes as a multiplicity-free sum of representations of $\mathfrak{g}_{\mathbb{C}}$ if both* (3.5)(a) *and* (b) *are satisfied:*
(3.5)(a) *λ vanishes on $\mathfrak{t}_{\mathbb{C}} \cap [\mathfrak{l}_{\mathbb{C}}, \mathfrak{l}_{\mathbb{C}}]$.*
(3.5)(b) *π_ν decomposes with multiplicity-free when restricted to $\mathfrak{l}_{\mathbb{C}}$.*

Sketch of proof. We may and do assume that $\mathfrak{g}_{\mathbb{C}}$ is a semisimple Lie algebra. Let G be a simply connected compact Lie group and L a connected subgroup such that their Lie algebras are real forms of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{l}_{\mathbb{C}}$, respectively. We set

$$(P, H, K, B, \mu) = (G \times G, \mathrm{diag}(G), L \times G, \{e\} \times \{e\}, \mathbb{C}_\lambda \otimes \pi_\nu).$$

Here, $\mathbb{C}_\lambda$ denotes the one dimensional representation of L with differential λ. We note that the tensor product representation $\pi_\lambda \otimes \pi_\nu$ is realized on the space $\mathcal{O}(G/L, G \times_L \mathbb{C}_\lambda) \otimes \pi_\nu \simeq \mathcal{O}(P/K, P \times_K \mu)$. Thus, the proof of Theorem 3.5 will be complete if all assumptions of Theorem 1.3 are shown.

Obviously, we have $HBK = P$. Hence the assumption (1.3)(a) holds.

It is straightforward to see $M = \mathrm{diag}(L)$ (recall (1.3.1) for the definition of M). As $\pi_\nu|_L$ is multiplicity-free by the assumption (3.5)(b), so is $\mu|_M$ because $\mathbb{C}_\lambda$ is one dimensional. Hence, the assumption (1.3)(b) holds.

The remaining assumptions (1.3)(c) and (d) are automatically fulfilled by taking a suitable involutive automorphism σ of G so that $\mathrm{rank}\, G/G^\sigma = \mathrm{rank}\, G$ (e.g. $\sigma(g) = \overline{g}$ for $g \in G = U(n)$). Hence, Theorem 3.5 follows from Theorem 1.3. □

Let us consider a special case $\mathfrak{l}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}$. Then, the condition (3.5)(a) is automatically satisfied. Hence, we obtain a new proof of the following well-known fact:

Corollary 3.5.1. *Let F and π be irreducible finite dimensional representations of $\mathfrak{g}_{\mathbb{C}}$. If F is weight multiplicity-free, then the tensor product representation $\pi \otimes F$ decomposes with multiplicity free.*

Example 3.5.2. As we saw in Example 2.4(3), $\pi \otimes S^k(\mathbb{C}^n)$ is multiplicity-free for any k and $\pi \in \widehat{U(n)}$.

3.6 Multiplicity-free tensor product of $U(n)$

In this subsection, we consider the tensor product of two irreducible representation π_λ and π_ν of $U(n)$ with highest weights $\lambda, \nu \in \mathbb{Z}^n$, respectively. We shall assume that λ is of the form

$$\lambda = (\underbrace{a, \dots, a}_{p}, \underbrace{b, \dots, b}_{q}) \tag{3.6.1}$$

for some (p, q) such that $p + q = n$ and for some $a, b \in \mathbb{Z}$ with $a \geq b$.

Theorem 3.6. *The tensor product representation $\pi_\lambda \otimes \pi_\nu$ is multiplicity-free as a $U(n)$-module, if one of the following three conditions is satisfied.*
1) $\min(a - b, p, q) = 1$ *(and ν is arbitrary).*
2) $\min(a - b, p, q) = 2$ *and ν is of the form*

$$(\underbrace{x, \cdots, x}_{n_1}, \underbrace{y, \cdots, y}_{n_2}, \underbrace{z, \cdots, z}_{n_3}), \tag{3.6.2}$$

where $x \geq y \geq z$ and $n_1 + n_2 + n_3 = n$.
3) $\min(a - b, p, q) \geq 3$ *and ν is of the form* (3.6.2) *satisfying*

$$\min(x - y, y - z, n_1, n_2, n_3) = 1. \tag{3.6.3}$$

Proof. This theorem follows from Theorems 3.3 and 3.5. For example, to see the statement (3), we note that the condition (3.5)(a) is satisfied by setting $L := U(p) \times U(q)$ if λ is of the form (3.6.1). On the other hand, the condition (3.5) (b) is satisfied, that is, $\pi_\nu|_L$ is multiplicity-free if ν satisfies (3.6.3) because of Theorem 3.3. Hence, Theorem 3.5 implies that $\pi_\lambda \otimes \pi_\nu$ is multiplicity-free. $\square$

Example 3.6.1. The case $q = 1$ corresponds to Example 2.4 (3) assured by Pieri's rule.

Remark 3.6.2. The multiplicity-free property for the case $\min(n_1, n_2, n_3) = 0$ was noticed previously by Kostant, and can be also read from the list by Littelmann [10] on spherical varieties.

Remark 3.6.3. For some special cases, explicit branching laws are also found by Okada [11] and Krattenthaler [9] by combinatorial methods.

Remark 3.6.4. Recently, Stembridge [12] gave a different and combinatorial proof of Theorem 3.6. Furthermore, he proved that the above description exhausts all the cases of multiplicity-free tensor products of irreducible representations of $U(n)$ up to a switch of factor.

Further applications including infinite dimensional cases and a proof of Theorem 1.3 will be given in subsequent papers.

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