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Geometry of Quantum States

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Abstract. In the first part of this work, an attempt of a realistic interpretation of *quantum logic* is presented. Propositions of quantum logic are interpreted as corresponding to certain macroscopic objects called filters; these objects are used to select beams of particles. The problem of representing the propositions as projectors in a Hilbert space is considered and the classical approach to this question due to Birkhoff and von Neumann is criticized as neglecting certain physically important properties of filters. A new approach to this problem is proposed.

The second part of the paper contains a revision of the concept of a state in quantum mechanics. The set of all states of a physical system is considered as an abstract space with a geometry determined by the transition probabilities. The existence of a representation of states by vectors in a Hilbert space is shown to impose strong limitations on the geometric structure of the space of states. Spaces for which this representation does not exist are called non-Hilbertian. Simple examples of non-Hilbertian spaces are given and their possible physical meaning is discussed. The difference between Hilbertian and non-Hilbertian spaces is characterized in terms of measurable quantities.

1. Introduction

One of the fundamental assumptions of quantum mechanics is that quantum states may be represented by vectors in a linear space. For Dirac this assumption was a sort of a guess suggested by the nature of the superposition principle. It leads to representing pure states either by vectors in a Hilbert space or by distributions (i.e., unnormalizable vectors like e.g. plane waves.) The collection of these concepts forms a language in which quantum mechanics describes microphenomena.

It sometimes happens that the physical reality may not be expressed in terms of certain concepts if some applicability conditions do not hold. Thus, e.g., a field of forces cannot be described in terms of a potential if the curl does not vanish. In the Riemannian space the Cartesian coordinates may not be introduced if the space is not flat. The question arises whether the representation of quantum states by vectors does not impose certain limitations on the admissible structure of states.

A well known approach to this problem has been originated by Birkhoff and von Neumann and continued by Piron. It consists in considering the structure of pure states as determined by the structure of the set of *yes-no measurements* (i.e., measurements which can give only two results "yes" and "no"). The yes-no measurements possess certain properties analogous to those of a logical system; for this reason they are called *propositions* and their set is called *quantum logic*. In order to illustrate these properties on a simple model we shall imagine the yes-no measurements as filters which select a certain beam of particles. Each filter absorbs a part of the beam; particles which have passed through it are those for which the result of the measurement was "yes":

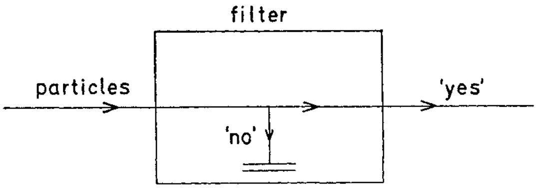

Fig. 1

We shall denote by Q the set of all known filters which can be used to select a certain definite beam of particles (e.g., beam of photons). The symbols 0 and I will stand for special filters: 0 absorbes each particle and I allows each particle pass without being perturbed. We assume that the following relations of equivalence ($\equiv$), inequality ($\leqq$) and orthogonality ($\perp$) exist in Q. The equivalence $a \equiv b$ means that filters a and b act on the beam in the same way. The inequality $a \leqq b$ (to be read: a is contained in b) means that each beam emerging from a passes through b without being absorbed. The orthogonality $a \perp b$ means that every beam emerging from a is completely absorbed by b and vice versa. Clearly $0 \leqq a \leqq \mathrm{I}$ and $a \perp 0$ for any $a \in Q$. One usually assumes that Q has the following structure.

I. The inequality $\leqq$ is a partial ordering relation in Q.

II. For any pair $a, b \in Q$ the subclass of all filters x such that $x \geqq a$ and $x \geqq b$ contains a smallest element. We call this element the *union* of a and b and denote it by $a \cap b$.

For any $a, b \in Q$ the subclass of all filters x such that $x \leqq a$ and $x \leqq b$ contains a greatest element. We call this element the *intersection* of a and b and denote it by $a \cap b$.

III. For any $a \in Q$ the subclass of these $x \in Q$ which are orthogonal to a contains the greatest element called the *complement* of a and denoted by a'. The mapping $a \to a'$ has the following properties

$$(a')' \equiv a \tag{1.1}$$

$$(a \cup b)' \equiv a' \cap b' \,. \tag{1.2}$$

If assumptions I, II, III hold, the analogy between the set Q and a logical system can be established as follows. We call each filter $a \in Q$ a *proposition*. The inequality $a \leqq b$ means "a implies b" and operations $\cup$, $\cap$ and $a \to a'$ mean the alternative, the conjunction and the negation respectively.

Note that "pure states" are closely related to the above concepts. For two filters $a, b \in Q$ we say that b *covers* a if $a \leqq b$ and if no element $x \in Q$ such that $x \not\equiv a$, $x \not\equiv b$ and $a \leqq x \leqq b$ exists. Any filter which covers 0 will be called a minimal filter. We interpret the minimal filter as a device performing the finest selection possible, i.e., a selection which cannot be made "narrower". It is natural to assume that the beam which has passed through a minimal filter is homogeneous, i.e., lacks any detectable internal substructure. A beam of this sort is called *pure*. Now, *pure states* are equivalence classes which correspond to pure beams: we say that two particles are in the same pure state if they belong to the same pure beam. Hence, in the language which we are using pure states correspond simply to minimal filters.

According to the point of view accepted by many authors the logical structure of Q (i.e., the collection of all properties of filters which can be expressed in terms of symbols $\cup, \cap, '$) determines completely the character of quantum laws as well as the mathematical formalism employed in the quantum theory. This philosophy is also a basis of Birkhoff and von Neumann's work. They assume that the applicability of the language of Hilbert spaces in quantum physics depends upon the validity of a certain definite hypothesis about the structure of Q. This hypothesis states that filters can be represented by orthogonal projectors in a Hilbert space so that their logical structure is conserved. In order to quote more exactly the $B - N$ hypothesis we shall introduce some notation. $\mathscr{H}$ will mean a Hilbert space. For two operators of orthogonal projection P_1, P_2 acting in $\mathscr{H}$ we write $P_1 \leqq P_2$ if the subspace into which P_1 projects is contained in the corresponding subspace of the operator P_2. For an arbitrary projector P the complement P' is defined as: $P' = 1 - P$. With these definitions the set of all operators of orthogonal projection in $\mathscr{H}$ is an orthocomplemented lattice isomorphic to the lattice of all closed subspaces of $\mathscr{H}$. The hypothesis of Birkhoff and von Neumann may be reduced to the following:

$(B - N)$ There exists a mapping P of the lattice Q into the lattice of orthogonal projectors in $\mathscr{H}$ such that:

$$P(0) = 0\,, \qquad P(I) = 1 \tag{1.3}$$

$$a \equiv b \Leftrightarrow P(a) = P(b) \tag{1.4}$$

$$a \text{ covers } b \Leftrightarrow P(a) \text{ covers } P(b) \tag{1.5}$$

$$P(a \cap b) = P(a) \cap P(b) \tag{1.6}$$

$$P(a \cap b) = P(a) \cap P(b) \tag{1.7}$$

$$P(a') = P(a') = 1 - P(a) \tag{1.8}$$

Points (1.3 – 8) mean that the structure of "quantum logic" may be adequately represented by projectors in Hilbert space.

The main effort of Birkhoff and von Neumann was to choose some natural axioms on the structure of quantum logic Q which would imply the existence of the $B - N$ representation. This problem was not completely resolved by Birkhoff and von Neumann. It seems to be resolved today as a result of the work by Piron [7]. With Piron's assumption (weak semi-modularity) one can prove that yes-no measurements may be represented by projectors in a certain unitary space so that (1.3–1.8) hold. Because of the regularity requirements this space must be a Hilbert space over one of three numerical fields: real numbers, complex numbers or quaternions. One thus concludes that, if some "reasonable" assumptions are employed, the language of Hilbert spaces is always applicable to quantum phenomena. An additional conclusion is that any quantum theory may be constructed in the framework of one of the following schemes: either we work with real, complex or quaternionic Hilbert spaces. These results are considered as justifying the use of Hilbert spaces in quantum theories.

The approach of $B - N - P$[1] has, however, the disadvantage of being based on an oversimplified philosophy. It facitly assumes the structure of quantum logic is sufficient in itself to determine the mathematical formalism which should be employed in the quantum theory. This is not true, however. It has been rightly pointed out by Pool [11], Ramsay [12], Gunson [13], and other authors that the "logic" of the physical phenomena is not the only aspect which must be adequately represented by the formalism of the quantum theory. In fact, quantum mechanics is used not so much to reproduce the logical properties of filters but rather to compute transition probabilities, cross section et.c. The probabilistic aspect is unified with the logical aspect in the paper by Gunson who uses new mathematical tools to support the $B - N - P$ old opinion that the only reasonable mathematical schemes to describe quantum phenomena those related to three types of Hilbert spaces.

In this work we make one step more in abandoning the $B - N - P$ approach. We propose to neglect completely the logical properties of filters, as they are of secondary physical importance. We concentrate exclusively on the probabilistic aspect of physica phenomena, which in our approach is represented by "geometric properties" of filters. As the result we obtain a new answer to the question formulated at the beginning: quantum states cannot always be represented by vectors. The physical reality can be too complex in order to fit in any Hilbert space. We shall show how to imagine this sort of reality.

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[1] Birkhoff, von Neumann, Piron.

2. Geometric Properties of Filters

Consider two filters $a, b \in Q$ placed in the sequence:

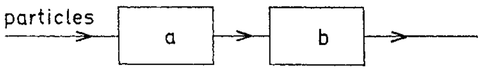

Fig. 2

The intensity of the beam emerging from a is in general diminished under the influence of b. The relative decrease of the intensity is in general undetermined: its value depends upon the specific properties of the beam leaving a. There may exists, however, such special pairs $a, b \in Q$ for which this decrease stays always constant. This happens e.g., for two Nicol prisms placed as it is shown in Fig. 3:

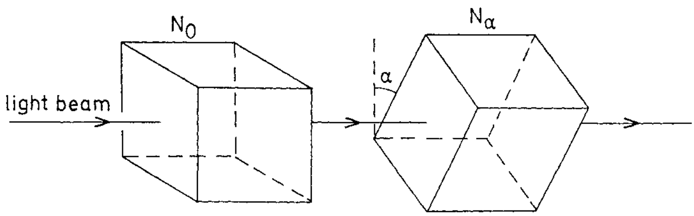

Fig. 3

The decrease caused by N_α of the intensity of the polarized light emerging from the prism N_0 is always $\cos^2 \alpha$ independently of the colour of the light. Situations of this sort will be of particular interest to us.

Definition. We shall say that a definite *coefficient of absorption* exists for a pair of filters $a, b \in Q$ and equals $p(a, b)$ if:

1. for every beam leaving a the relative decrease of the intensity caused by b is $p(a, b)$.

2. If the same is true when we interchange filters, i.e., if for every beam leaving b the relative decrease of the intensity caused by a is $p(a, b)$.

The existence of a definite absorption coefficient for a pair of filters is what we call a *geometric property*. E.g., the orthogonality is a geometric property of filters meaning that the absorption coefficient equals 0.

Let now M be the set of minimal filters in Q. We shall assume that the coefficient of absorption $p(a, b)$ exists for each pair of minimal filters $a, b \in M$. This assumption plays an important role in quantum physics. The orthodox quantum mechanics operates with *pure states* which correspond to minimal filters (see discussion in § 1). The absorption coefficients are then interpreted to be the *probabilities of transitions* between pure states. From now on we shall interpret numbers $p(a, b)$ as fundamental empirical data determining the physical structure of the set of pure states. We shall assume that the whole collection of these numbers establishes a sort of geometry in this set.

We return to the criticism of Birkhoff's and von Neumann's approach. The main objection is that they completely neglected the geometric properties of filters. We would not be, however, so interested in representing filters by projectors in Hilbert space if we did not expect to obtain in this way a correct model of the geometry of transition propabilities. We shall formulate this objection more exactly. Suppose, that filters are represented by projectors in agreement with (1.3)–(1.8), so that each minimal filter (pure state) corresponds to an operator $P(a)$ projecting on a certain one-dimensional subspace $\Psi(a) \subset \mathscr{H}$. By choosing in each subspace $\Psi(a)$ a unit vector $\psi(a)$ we may obtain the conventional representation $a \to \psi(a)$ of pure states by vectors of the unit sphere in $\mathscr{H}$. For each pair of minimal filters $a, b \in M$ we then have two quantities: $p(a, b)$ which expresses the physical relation between minimal filters; $|(\psi(a), \psi(b))|^2$ which characterizes the geometrical relation (angle) between the corresponding vectors in $\mathscr{H}$.

The mapping $a \to P(a)$ is of interest for physics, if the induced mapping $a \to \psi(a)$ obeys the requirement:

$$|(\psi(a), \psi(b))|^2 = p(a, b)\,, \tag{2.1}$$

i.e., if we may reconstruct the "geometry" of absorption coefficients for minimal filters by observing angles between corresponding vectors in $\mathscr{H}$. This assumption is commonly accepted by quantum mechanics under the name of statistical interpretation. This is why we may pretend to obtain correct transition probabilities from a theory which operates with vectors and their scalar products.

The insufficiency of the approach of Birkhoff and von Neumann becomes now clear. Even if we establish the existence of a mapping $a \to P(a)$ for which (1.3)–(1.8) but not (2.1) hold, we will obtain a convenient model of "quantum logic" but this model cannot be used to compute transition probabilities, cross sections etc., and hence, it will not be of great use for physics. Such a situation may happen, since (2.1) does not follow from (1.3)–(1.8). Requirements (1.3)–(1.8) imply only that orthogonal minimal filters are represented by orthogonal vectors. But (2.1) is a stronger condition. It may easily happen that many mappings for which (1.3)–(1.8) hold exist but none for which (2.1) holds. This possibility is illustrated by the following example.

Example. Imagine a class of 8 filters $0, I, a, b, c, a', b', c'$ forming a simple orthocomplemented lattice represented in the figure below:

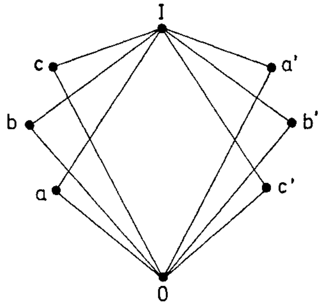

Fig. 4

In this lattice a, b, c, a', b', c' are minimal filters. Suppose that coefficients of absorption for various pairs of them are:

$$p(a, b) = p(a, c) = 1 - \varepsilon\,, \qquad p(b, c) = \frac{1}{2}\,, \tag{2.2}$$

where ε is a small positive number. All remaining absorption coefficients $p(a, b')$, $p(a', b)$, . . . are consistent with the general rule that $p(x, x) = 1$ and $p(x,y) + p(x,y') = 1$ for every x and y. Thus, e.g., $p(a,b') = p(a',b) = \varepsilon$, $p(b',c) = 1/2$, etc.

Obviously there exist many representations of the "quantum logic" given in Fig. 4 such that conditions (1.3)–(1.8) hold. However, there is no representation which would fulfill (2.1). In fact, the assumed values of $p(x, y)$ are in disagreement with the natural structure of a Hilbert space: if ε is small enough we cannot have three vectors $\psi(a)$, $\psi(b)$, $\psi(c)$ in Hilbert space such that $|(\psi(a), \psi(b))|^2 = |(\psi(a), \psi(c))|^2 = 1 - \varepsilon$ and $|(\psi(b), \psi(c))|^2 = 1/2$.

The first pair of the above equalities implies that $\psi(b)$ and $\psi(c)$ are both close to $\psi(a)$, hence $\psi(b)$ must also be close to $\psi(c)$ which contradicts $|(\psi(b), \psi(c))|^2 = 1/2$.

Conclusions. The existence of the mapping considered by Birkhoff and von Neumann is not sufficient to explain the role played by Hilbert spaces in quantum physics. One can imagine situations when this mapping exists but is useless: we can represent states by vectors but we cannot use scalar products to compute transition probabilities. It is also clear from the example given by Fig. 4 that the existence of a mapping for which (2.1) holds must impose a limitation on the "geometry of transition probabilities". The nature of this limitation will be studied in § 5.

3. Space of States

The above idea about the "geometry" of transition probabilities suggests that the traditional concept of the pure state should be revised. The classical version of this concept may be repeated in terms of filters as follows. We first define a pure beam of particles as a beam which has passed through a minimal filter. Then we say that a single particle is in a definite pure state if it belongs to a definite pure beam. In this way pure states correspond to minimal filters. The above definition of the pure state does not differ essentially from the one which was given by Dirac. The minimal filter here plays a similar role as the maximal set of informations in Dirac's definition.

The appearance of the minimal filter in this definition is somewhat inconvenient. The minimality of a filter (or the maximality of the set of information) is not a physical property. It rather reflects our recent state of knowledge. Since the development of physics is not yet complet we may never be sure whether a filter which seems minimal to us will remain so for the physics of the future. Our maximal sets of informations (pure states) may be very incomplete for a physicist of the XXI century. This does not prevent us from operating with pure states with quite good numerical results. This suggests that the demand about the minimality must be redundant in the definition of the pure state. As a matter of fact, only a special implication of this property is needed in practice: this is the existence of a definite absorption coefficient for each pair of minimal filters which allows the definition of a transition probability for each pair of pure states. The absorption coefficient, however, may exist even for pairs of filters which are not minimal. This may serve as the starting point in the generalization of the very notion of the pure state.

We shall first introduce the concept of a *geometric system* of filters. A subset $S \subset Q$ will be called a *system of filters with geometry* or simply a *geometric system* if for each pair of filters $a, b \in S$ a definite absorption coefficient $p(a, b)$ exists. A trivial example of a geometric system is any pair of orthogonal filters. Each system composed of minimal filters is of course a geometric system. The pair of Nicol prisms in Fig. 3 is also a geometric system. It may happen that a certain geometric system $S \subset Q$ is a part of another geometric system $S' \subset Q$: in this case we say that S' is an extension of S. Our experience shows that by extending any geometric system of filters one arrives at a certain maximal geometric system of especially regular properties: this system will be called the *physical space of states.* Filters which belong to it will be called *states.* For each two states the transition probability will be defined as equal to the corresponding absorption coefficient.

The above idea of the physical space of states is closer to practice than the orthodox one. It is not restricted by the philosophical demand that the notion of state should contain all information available about the physical system. On the contrary it may contain only fragmentary information. Various spaces of states related to various special properties of microphenomena may exist in Q. Thus e.g., if we consider the geometric system composed of a number of Nicol prisms and then extend it, we obtain the space which contains all polarization states of light.

Experiments indicate that various physical spaces of states have some common structural properties[2]. In order to represent them conveniently we shall introduce the following abstract space of states which will be called a *probability space.*

Definition. The *probability space* (S, p) is a non-empty set of S (of elements called "states") with a real value function $p(\ ,\ )$ defined on $S \times S$ (and called the "transition probability") such that:

(A) $\quad 0 \leqq p(a, b) \leqq 1 \quad$ and $\quad p(a, b) = 1 \Leftrightarrow a = b$;

(B) $\quad p(a, b) = p(b, a)$;

The third axiom will concern orthogonality in S. Two elements $a, b \in S$ will be called *orthogonal* if $p(a, b) = 0$. A subset $R \subset S$ will be called an *orthogonal system* of elements if each two different elements of R are orthogonal. R will be called a *maximal orthogonal system* or a *basis* if it is not contained in any larger orthogonal system $R' \subset S$. The third axiom is:

(C) For each basis $R \subset S$ and for each $a \in S$:

$$\sum_{r \in R} p(a, r) = 1 \, . \tag{3.1}$$

(The sum on the lefthand side of (3.1) is the upper bound of all finite sums of the form $\sum_{i=1}^{n} p(a, r_i)$ where $r_i \in R$ $(i = 1, \ldots, n)$ and $r_i \neq r_j$ for $i \neq j$.)

We shall prove some simple facts concerning probability spaces. The existence of at least one basis in S follows from Zorn's lemma. We have

Theorem 1. *Let R_1 and R_2 be two bases in S. Then R_1 and R_2 contain the same number of elements, i.e., $\overline{\overline{R}}_1 = \overline{\overline{R}}_2$.*

Proof. Suppose first R_1 is a finite system composed of n elements. Consider transition probabilities $p(r_1, r_2)$ for $r_1 \in R_1$, $r_2 \in R_2$. Because of (C), $\sum_{r_2 \in R_2} p(r_1, r_2) = 1$ for every $r_1 \in R_1$. Hence $\sum_{r_1 \in R_1} \sum_{r_2 \in R_2} p(r_1, r_2) = n$. Since numbers $p(r_1, r_2)$ are non-negative the order of summation here may be interchanged and we obtain $\sum_{r_2 \in R_2} \left(\sum_{r_1 \in R_1} p(r_1, r_2) \right) = n$. Because of (C) this leads to $\sum_{r_2 \in R_2} 1 = n$. This means that R_2 also contains n elements.

Suppose now both R_1 and R_2 are infinite. For each $r_1 \in R_1$ the symbol $R_2(r_1)$ will denote the set of those elements in R_2 which are not orthogonal to r_1, i.e., $R_2(r_1) = \{r_2 \in R_2 : p(r_1, r_2) > 0\}$. Since

$$\sum_{r_2 \in R_2(r_1)} p(r_1, r_2) = \sum_{r_2 \in R_2} p(r_1, r_2) = 1 \, ,$$

[2] See e.g., J. Schwinger [10].

the subset $R_2(r_1)$ must be countable. Since R_1 is a maximal orthogonal system, no element in R_2 orthogonal to all $r_1 \in R_1$ exists. This means that

$$R_2 \subset \bigcup_{r_1 \in R_1} R_2(r_1) .$$

This implies that $\overline{\overline{R}}_2 \leqq \overline{\overline{R}}_1$. Similarly $\overline{\overline{R}}_1 \leqq \overline{\overline{R}}_2$. Hence, $\overline{\overline{R}}_1 = \overline{\overline{R}}_2$ ∎.

If $R \subset S$ is a basis in S the cardinal number $\overline{\overline{R}}$ will be called the *dimension* of S.

It will be of interest to separate from among various subsets of S those which are its subspaces in the sense of the following definition:

Definition. A subset $S' \subset S$ is called a *subspace* of S, if S' with the transition probability defined as the restriction of $p(\ ,\ )$ to $S' \times S'$ is a probability space.

A trivial example of a subspace is an arbitrary orthogonal system of elements in S. Every orthogonal system being itself a subspace may be a basis in various other subspaces. We shall prove that a greatest one among them exists. Let $R' \subset S$ be an orthogonal system in S; then $S(R')$ will denote the subset of those elements $x \in S$ for which:

$$\sum_{r \in R'} p(x, r) = 1 . \tag{3.2}$$

Theorem 2. *The subset $S(R')$ is a subspace of S with the basis R'. Moreover, $S(R')$ contains all other subspaces for which R' is a basis.*

Definition. For any orthogonal system $R' \subset S$ the subspace $S(R')$ will be called a *smooth subspace* spanned by R'.

Proof of Theorem 2. First we shall show that $S(R')$ is indeed a subspace. Axioms (A) and (B) obviously hold. If R' is a basis in S then $S(R') = S$ and (C) also holds. If R' is not a basis, it may be extended to a basis: because of Zorn's lemma there exists a system $R'' \subset S$ such that $R' \cap R''$ is a basis. We shall show that $S(R')$ is the subset of those $x \in S$ which are orthogonal to R''. Indeed, if $x \in S(R')$, the equality (3.2) together with

$$\sum_{r \in R'} p(x, r) + \sum_{r \in R''} p(x, r) = \sum_{r \in R' \cup R''} p(x, r) = 1 \tag{3.3}$$

imply that

$$\sum_{r \in R''} p(x, r) = 0 \tag{3.4}$$

which means that x is orthogonal to all elements of R''. Inversly, if x is orthogonal to all elements of R'', then (3.4) holds and (3.3) implies (3.2).

Suppose now that T is a maximal orthogonal system in $S(R')$. Since all elements of T are orthogonal to R'', then $T \cup R''$ is an orthogonal system of elements in S. Moreoever, no element $x \in S$ orthogonal to all elements of $T \cup R''$ exists. Such an x being orthogonal to R'' would belong to $S(R')$ which is impossible, since $S(R')$ may not contain any x orthogonal to T. Hence, $T \cup R''$ is a maximal orthogonal system in S, and for every $x \in S$ we have:

$$\sum_{r \in T} p(x, r) + \sum_{r \in R''} p(x, r) = \sum_{r \in T \cup R''} p(x, r) = 1 .$$

For $x \in S(R')$ this implies because of (3.4) that $\sum_{r \in T} p(x, r) = 1$. Thus axiom (C) for $S(R')$ holds.

Now let $S' \subset S$ be any subspace for which R' is a basis, then for any $x \in S'$ (3.2) holds and $x \in S(R')$. Hence $S' \subset S(R')$ ∎.

An equivalent definition of the smooth subspace may be obtained as follows. Let $Z \subset S$; then $Z^{\perp}$ will denote the set of all elements in S which are orthogonal to all $z \in Z$. Now let R' be an orthogonal system. Then $S(R') = (R'^{\perp})^{\perp}$. The proof of this statement is quite similar to that of Theorem 2.

If (S_1, p_1) and (S_2, p_2) are two probability spaces, it may happen that they have the same geometric structure. It may also happen that (S_1, p_1) can be considered as a part of (S_2, p_2). These cases correspond to the following concepts of isomorphism and embedding.

Definition. An isomorphism of (S_1, p_1) onto (S_2, p_2) is a reversible mapping $x \to x'$ of S_1 onto S_2 such that $p_1(x, y) = p_2(x', y')$. Two probability spaces $S_1, p_1)$ and (S_2, p_2) are called *isomorphic* if there exists an isomorphism between these spaces. An embedding of (S_1, p_1) into (S_2, p_2) is any injective mapping $x \to x'$ of S_1 in S_2 such that $p_1(x, y) = p_2(x', y')$.

The space (S_1, p_1) may be embedded in (S_2, p_2) if and only if (S_2, p_2) (S_2, p_2) contains a subspace isomorphic with (S_1, p_1).

A well known example of a probability space is obtained by considering the set $S_{\mathscr{H}}$ of all one-dimentional subspaces (rays) in a certain Hilbert space $\mathscr{H}$ ($\mathscr{H}$ is assumed to be a Hilbert space over the field of real numbers, complex numbers or quaternions). For Ψ_1, Ψ_2 being two rays in $\mathscr{H}$ we define $p(\Psi_1, \Psi_2)$ as:

$$p(\Psi_1, \Psi_2) = |(\psi_1, \psi_2)|^2 , \tag{3.5}$$

where ψ_1, ψ_2 are unit vectors, $\psi_1 \in \Psi_1, \psi_2 \in \Psi_2$. The set $S_{\mathscr{H}}$ with the transition probability (3.5) is a probability space; we shall call this space "Hilbertian". As can be easily seen, smooth subspaces of $S_{\mathscr{H}}$ correspond to closed vector subspaces of $\mathscr{H}$. However, the general probability subspaces of $S_{\mathscr{H}}$ correspond to certain subsets of $\mathscr{H}$ which in general are not linear subspaces.

Although axioms (A), (B), (C) are derived by generalizing properties of Hilbertian spaces, not every probability space must be Hilbertian.

A large class of probability spaces which may not be embedded in Hilbertian spaces exists; examples of these structures will be given in § 4.

We may now assign a definite meaning to the question whether Hilbert spaces are an appropriate tool for quantum physics. This question is related not so much to the structure of quantum logic but rather to the geometric properties of filters. We have to do in practice with certain classes of filters: inside of them there exist various geometric systems. These systems form various probability spaces. Their geometric structure may not be determined a priori by any theory but it should be studied by the experiment. The problem is: can each of these spaces be embedded in a certain Hilbertian space? In § 5 we show how this question can be answered by measuring the transition probabilities.

4. Two-Dimensional Spaces. The Possibility of Non-Hilbertian Structures. Geometric Interpretation of the Superposition Principle

We shall illustrate concepts of § 3 by considering two-dimensional spaces. Let (S, p) be a 2-dimensional probability space; then to each $a \in S$ there corresponds exactly one element $a' \in S$ which is orthogonal to a. Indeed, for every $a \in S$ there exists at least one a' orthogonal to a; otherwise a itself would be a maximal orthogonal system in S in contradiction to the assumption that S is 2-dimensional. Let $\tilde{a}'$ be another element orthogonal to a. Since $\{a, a'\}$ is a basis, (3.1) implies: $1 = p(\tilde{a}', a) + p(\tilde{a}', a') = p(\tilde{a}', a')$, and $\tilde{a}' = a'$ because of (A). This proves the uniqueness of a'. In consequence $(a')' = a$; hence, the mapping $a \to a'$ is an involution in S. This mapping is also an isomorphism, because for every $a, b \in S$ relations $p(a, b) + p(a, b') = 1$ and $p(a, b') + p(a', b') = 1$ imply that $p(a', b') = p(a, b)$. Thus, 2-dim. spaces are spaces with an isomorphic involution.

A simple class of these spaces may be constructed as follows. Consider a sphere of radius 1/2 in n-dimentional Euclidean space $\mathbb{R}^n$. Let S be the set of all points of the $n-1$-dimentional surface of this sphere. We shall define the transition probability on $S \times S$ as follows: for two points $a, b \in S$ the number $p(a, b)$ is the square of the distance between a and the antipode of b (see Fig. below):

Axiom (A) obviously holds, and since the distance between a and the antipode of b is the same as between b and the antipode of a, (B) is also valid. For each $a \in S$ there exists a unique element a', orthogonal to a: a' is the antipode of a. For each pair $\{a, a'\}$ and for every $b \in S$ Pythagoras' theorem implies that $p(a, b) + p(a', b) = 1$, hence (C) holds. Thus we have to do here with a certain 2-dimentional probability space in which each pair of points $\{a, a'\}$ is a basis and the involution $a \to a'$ is the reflection with respect to the center of the sphere. We shall denote this space by $S(2, n)$. All subspaces of $S(2, n)$ may be easily determined. 1-dimensional subspaces are one-element subsets. 2-dimentional subspaces are all subsets which are involution invariant (i.e.,

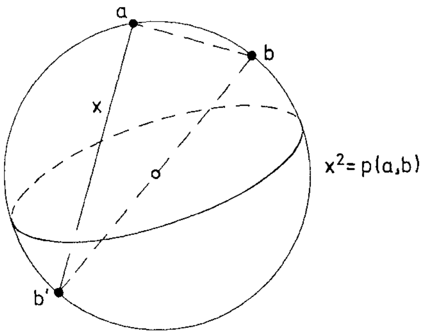

Fig. 5

symmetric with respect to the center of the sphere in $\mathbb{R}^n$). Each one dimentional subspace is smooth, but the only two-dimensional smooth subspaces is the whole space $S(2, n)$.

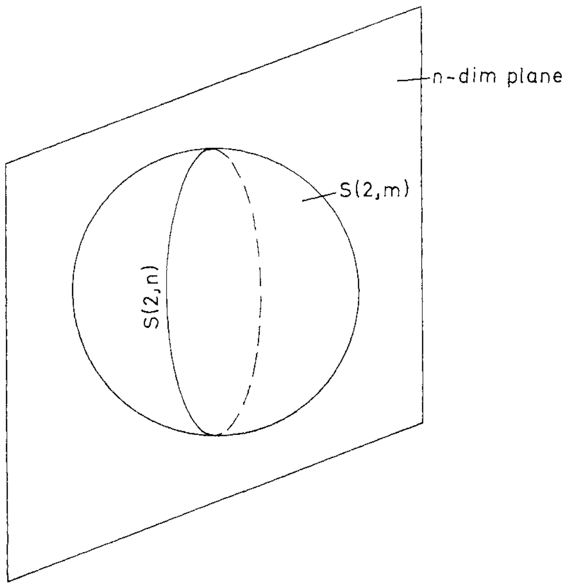

Fig. 6

Spaces $S(2, n)$ form an "increasing" family, i.e., for $n < m$ the space $S(2, n)$ may be embedded in $S(2, m)$. In fact, $S(2, m)$ possesses many subspaces isomorphic with $S(2, n)$. They are determined by all n-dimensional planes in $\mathbb{R}^m$ passing through the center of $S(2, m)$ sphere:

For $n < m$, $S(2, m)$ may, however, not be embedded in $S(2, n)$ since it is too extensive; $S(2, n)$ does not have any subspace isomorphic with $S(2, m)$.

It will be of interest to identify some of the above $S(2, n)$ spaces with two-dimensional Hilbertian spaces. Consider first the two-dimensional real Hilbert space $\mathscr{H}(2, \mathbb{R})$. We shall show that the corresponding probability space $S_{\mathscr{H}(2,\mathbb{R})}$ is isomorphic with $S(2, 2)$. The unit sphere in $\mathscr{H}(2, \mathbb{R})$ is the circle in two-dimensional x, y-plane. Since two vectors with opposite directions correspond to the same ray in $\mathscr{H}(2, \mathbb{R})$, rays may be represented by half of this circle. For two unit vectors ψ_1, ψ_2 representing two rays the transition probability $p(\psi_1, \psi_2)$ is defined as the cosine square of the angle between ψ_1 and ψ_2. The isomorphism between the above space $S_{\mathscr{H}(2,\mathbb{R})}$ and $S(2, 2)$ is established by the stereographic projection:

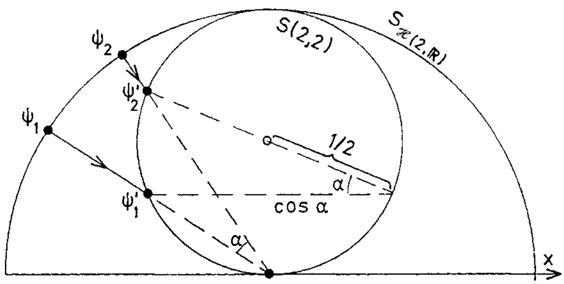

Fig. 7

Now let $\mathscr{H}(2, \mathbb{C})$ be the 2-dimensional Hilbert space over the field of complex numbers. Each vector $\psi \in \mathscr{H}(2, \mathbb{C})$ may be represented as $\left\|\begin{matrix}\zeta_1\\ \zeta_2\end{matrix}\right\|$ where ζ_1, ζ_2 are two complex numbers. Rays in $\mathscr{H}(2, \mathbb{C})$ may be represented by vectors $\left\|\begin{matrix}x\\ y+iz\end{matrix}\right\|$ where x, y, z are real numbers, $x^2+y^2+z^2=1$ and $v \geqq 0$. Hence rays in $\mathscr{H}(2, \mathbb{C})$ correspond to points $P = (x, y, z)$ of the surface of a unit hemi-sphere in three-dimensional Euclidean space. (The exception is the ray $\left\|\begin{matrix}0\\ \zeta_2\end{matrix}\right\|$ which corresponds to the great circle $x = 0$). By defining the transition probability $p(P_1, P_2) = (x_1x_2 + y_1y_2 + z_1z_2)^2 + (y_1z_2 - y_2z_1)^2$ for two points $P_1 = (x_1, y_1, z_1)$, $P_2(x_1, x_2, z_2)$ located on the surface of the hemi-sphere and by identifying all points on the circle $x = 0$ we obtain the correct model of the space $S_{\mathscr{H}(2,\mathbb{C})}$. This space is isomorphic with $S(2, 3)$ and the isomorphism is again established by a stereographic projection:

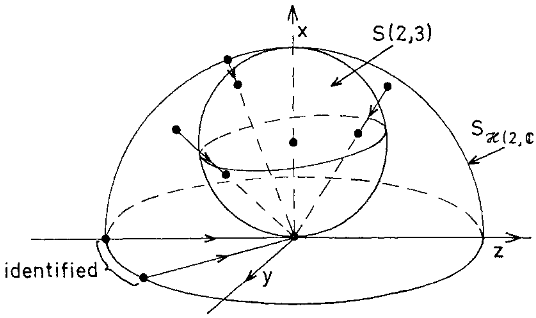

Fig. 8

For a similar reason, the probability space $S_{\mathscr{H}(2,\mathbb{K})}$ of rays in 2-dimensional Hilbert space over the field of quaternions is isomorphic with $S(2, 5)$ space. Thus, we identified $S(2, 2)$, $S(2, 3)$ and $S(2, 5)$ with the three basic types of two-dimensional Hilbertian spaces. The intermediate $S(2, 4)$ is isomorphic with a certain substructure of $S(2, 5)$ but it may be embedded in neither $S(2, 2)$ nor $S(2, 3)$.

It follows also that $S(2, 4)$ cannot be embedded in any real or complex Hilbertian space independently on its dimension. Indeed let $\mathscr{H}(\mathbb{C})$ be a complex Hilbert space and suppose that $S_{\mathscr{H}(\mathbb{C})}$ contains a subspace isomorphic with $S(2, 4)$. Because of Theorem 2 this subspace would be contained in a certain 2-dimensional smooth subspace of $S_{\mathscr{H}(\mathbb{C})}$. But all two-dimensional smooth subspaces of $S_{\mathscr{H}(\mathbb{C})}$ are of the type $S(2, 3)$ and may not contain $S(2, 4)$. For a similar reason spaces $S(2, n)$ for $n > 5$ may not be embedded in real complex or quaternionic Hilbertian spaces.

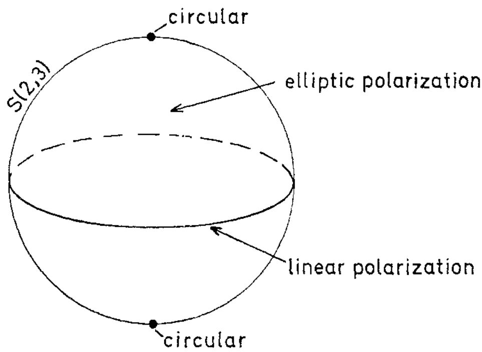

Fig. 9

For all known two-dimensional physical spaces of states (like e.g., spaces of polarization states of photons or electrons) the problem whether they are complex Hilbertian spaces reduced then to the question whether they represent structures isomorphic with $S(2, 3)$. This may be answered only experimentally. For the space of polarization states of light the answer seems to be positive. First, it was established using maximum available accuracy that states of linear polarization (Nicol prisms) form a geometric system isomorphic with $S(2, 2)$. It was also established that if we add to them circular and elliptic polarization states the structure isomorphic with $S(2, 3)$ is obtained (see Fig. 9).

The introduction of Pauli's spinors into electron theory is based on the implicit assumption that the geometry of polarization states of electrons is also of type $S(2, 3)$. This was not verified so directly as for the polarization states of light. However, the successful development of the spinorial theory of electrons seems to indicate that this geometry at least is not far from $S(2, 3)$.

The above considerations clarify the physical meaning of the problem raised by FINKELSTEIN, JAUCH, SCHIMONOWICH and SPEISER [8], whether the quaternionic Hilbert space should not be used by quantum theory instead of the usual complex Hilbert space. Now we can imagine what type of a situation would force a physicist to admit that the complex Hilbert space is insufficient to give an adequate theory, and to suggest that the quaternionic Hilbert space may be appropriate. This would happen e.g., if someone discovered a geometric system of filters of the structure $S(2, 5)$. The discovery of a system of filters with the geometry of type, e.g., $S(2, 6)$ would, however, make the use of any Hilbert spaces compeltely impossible.

Spaces $S(2, n)$ for $n > 5$ are non-Hilbertian since they are too extensive. Spaces with a geometry arbitrarily close to $S(2, 2)$, $S(2, 3)$ or $S(2, 5)$ but non-Hilbertian also exist. This may easily be illustrated for the real Hilbertian space $S(2, 2)$.

We shall consider $S(2, 2)$ as embedded in $S(2, n)$ $(n > 5)$. Then $S(2, 2)$ is a great circle on the $S(2, n)$ surface:

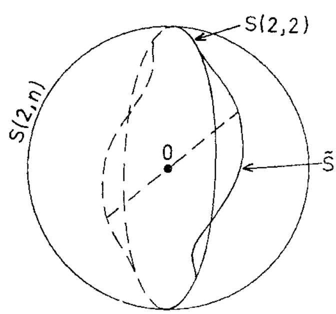

Fig. 10

An arbitrary subset of $S(2, n)$ symmetric with respect to the center 0 of the sphere is a subspace of $S(2, n)$ and satisfies all axioms (A), (B), (C). As this subset a curve $\tilde{S}$ symmetric with respect to 0 which does not determine any 5-dimensional plane in $\mathbb{R}^n$ will now be chosen. In this way we obtain a space which may not be embedded in any Hilbertian space. The geometry of this space may be, however, arbitrarily close to this of $S(2, 2)$. Quite similarly we can imagine spaces with geometry close to $S(2, 3)$ or $S(2, 5)$ but non-Hilbertian.

The possibility of continuous deformations of Hilbertian spaces allows the hypothesis that the geometry of certain physical spaces of states do not have to be necessarily constant but may depend upon external influences; a hypothesis of this type would be in the spirit of Finkelstein, Jauch, Schimonowich and Speiser idea about the conditional quantum logic (see [8]).

In the example in Fig. 10 the initial homogeneity of the Hilbertian space $S(2, 2)$ is destroyed by the deformation. This is not a necessary consequence of the deformation, however. There are probability spaces which are non-Hilbertian but possess all symmetries of e.g., $S(2, 3)$. A simple example may be obtained as follows.

Imagine a sphere with volume 2 in 3-dimensional Euclidean space. Let $T(2, 3)$ be the set of its hemi-spheres. For two hemi-spheres r, s the transition probability $p(r, s)$ will be defined as the volume of the common part $r \cap s$:

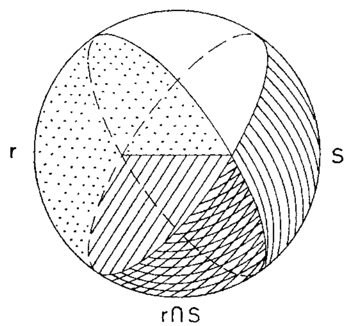

Fig. 11

The validity of (A) and (B) is obvious. Two hemi-spheres are orthogonal if $r \cup s$ is the whole sphere. Hence, for each $s \in T(2, 3)$, s' means the complementary hemi-sphere. Furthermore, for each two hemi-spheres r and s we have $p(s, r) + p(s', r) = \mathrm{vol}(s \cap r) + \mathrm{vol}(s' \cap r) = \mathrm{vol}(r) = 1$, hence (C) holds and $T(2, 3)$ is a probability space. The above space has the same group of symmetries as $S(2, 3)$ but a different geometry. Indeed, consider an arbitrary $s_0 \in T(2, 3)$ and consider all states $s \in T(2, 3)$ such that $p(s, s_0) = 1/2$. If s_1, s_2, s_3 are any three of these states then either $p(s_1, s_2) + p(s_2, s_3) = 1 + p(s_1, s_3)$ or $p(s_1, s_3) + p(s_3, s_2) = 1 + p(s_1, s_2)$ or $p(s_2, s_1) + p(s_1, s_3) = 1 + p(s_2, s_3)$. A similar statement would not be true for $S(2, 3)$. Hence, both spaces are not isomorphic. As can also be shown $T(2, 3)$ cannot be embedded in any Hilbertian space. A simple story about the discovery of a non-Hilbertian quantum phenomenon may be told now.

Drop of Non-Hilbertian Quantum Liquid

... Someone looked at a small spherical glass bubble: inside there was a drop of liquid. The drop occupied exactly half of the bubble in the shape of a hemi-sphere. He was able to introduce inside a thin, flat partition dividing the interior of the buble into two equal volumes. He tried to do this so that the drop would become split. However, the drop exhibited a quantum behaviour: instead of being divided into two parts the drop jumped and occupied the space on only one side of the partition. He repeated the attempt obtaining a similar result. He began to observe this phenomenon and discovered that each time the partition is introduced the drop chooses a certain side with a definite probability. This probability depends upon the angle between the partition and the initial surface of the drop. If the drop occupied a hemi-sphere s and the partition forces it to choose between two hemi-spheres r and r' the probabilities of transition into r and r' are proportional to volumes of $s \cap r$ and $s \cap r'$ ... He was struck by the analogy between positions of the drop and quantum states and between the partition and the macroscopic measuring aparatus. He wanted to formulate the quantum theory of this phenomenon, but he realized that he could not use Hilbert spaces: the space of states of the drop was not Hilbertian ....

It may seem somewhat restrictive that we base this discussion on examples of two-dimensional spaces. They have, however, a special significance for the general case since they are related with the superposition principle. In orthodox quantum theory, the superposition principle is a law which to each pair of states assigns a certain subspace of their superpositions. This subspace is a priori assumed to be a two-dimensional complex Hilbertian space; this is how we arrive to a general Hilbertian space representing the physical space of states. This assumption is not necessary, however. If we reject it, we obtain the following generalization of the superposition principle: the superposition principle is any law determining the structure of all 2-dimensional subspaces of a certain physical space of states. For instance, the orthodox superposition principle typical for complex Hilbertian spaces is: *each two-dimensional subspace of a physical space of states is of type* $S(2, 3)$. If $S(2, n)$ is substituted in place of $S(2, 3)$ a sequence of non-equivalent superposition principles can be obtained; among which those corresponding to real and quaternionic Hilbertian spaces ($n = 2, 5$). For $n = 4$ and $n > 5$ these principles do not correspond to any of the three principal types of Hilbertian spaces; it would be interesting to find in which spaces these superposition principles could be valid.

The consideration of probability spaces with two-dimensional subspaces deformed as Fig. 10 shows may also be interesting. It seems natural to expect that they may represent a sort of non-linearity in physics. Usually speaking about the non-linear quantum mechanics we assume the representation of states by vectors (wave functions). We reject only the assumption about the linearity of time evolution. Spaces with non-Hilbertian geometry could correspond to a more basic type of non-linearity concerning the superposition principle and making the very representation of states by vectors in linear spaces impossible.

It should be stressed at this moment that in the framework of our axioms non-Hilbertian spaces of arbitrary dimension are possible. A simple example of n-dimensional non-Hilbertian space can be constructed like a $T(2, 3)$ space. Consider any set X with measure $\mu(X) = n$. Now, call a "state" a measurable subset $s \subset X$ with measure $\mu(s) = 1$. For two states $s_1, s_2 \subset X$ the transition probability is $p(s_1, s_2) = \mu(s_1 \cap s_2)$:

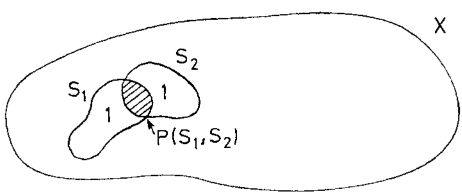

Fig. 12

We obtain in this way an example of n-dimensional probability space with a non-Hilbertian geometry. As is clear from work by GUNSON this example can appear in our scheme since we do not insist on reproducing in our approach the lattice structure assumed by orthodox quantum mechanics.

5. Numerical criteria

We rarely know the structure of a whole physical space of states. In practice we rather deal with its finite substructures. The question arises whether it is possible to recognize the non-Hilbertian type of a space only by observing the properties of finite systems of filters in it. The problem is non-trivial in general. It may be, however, easily resolved for two-dimensional spaces. Any two-dimensional space can be embedded in a real, complex, or quaternionic Hilbert space if and only if it may be embedded in $S(2, 2)$, $S(2, 3)$ or $S(2, 5)$ correspondingly.

We shall examine below the possibility of embedding in $S(2, 3)$ since this space is of the most interest to recent theory. For 2-dimensional spaces composed of 4 elements no specific condition exists: they may always be embedded in $S(2, 3)$:

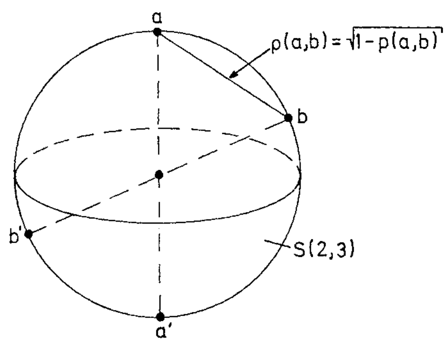

Fig. 13

The space composed of 6 elements $\{a, b, c, a', b', c'\}$ is the simplest one for which values of transition probabilities may not allow the embedding in $S(2, 3)$. In fact, if two elements x, y belong to $S(2, 3)$ their distance in $\mathbb{R}^3$ is determined by the transition probability:

$$\varrho(x, y) := \sqrt{p(x, y')} = \sqrt{1 - p(x, y)} .$$

This leads to the following conditions on three numbers $\varrho(a, b)$, $\varrho(a, c)$, $\varrho(b, c)$:

$$\left.\begin{aligned} \varrho(a, b) &\leqq \varrho(a, c) + \varrho(c, b) \\ \varrho(a, c) &\leqq \varrho(a, b) + \varrho(b, c) \\ \varrho(b, c) &\leqq \varrho(b, a) + \varrho(a, c) \end{aligned}\right\} \tag{5.1}$$

and

$$r(a, b, c) \leqq \frac{1}{2} \tag{5.2}$$

where $r(a, b, c)$ is the radius of the circle determined by the triangle of sides $\varrho(a, b)$ $\varrho(b, c)$ $\varrho(a, c)$:

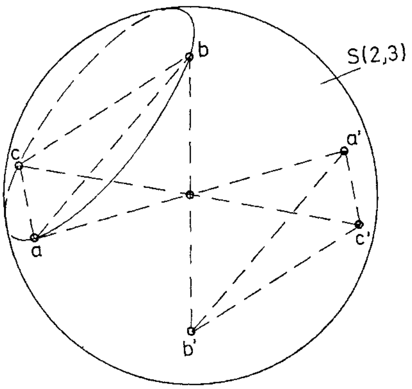

Fig. 14

Each 6-element space for which (5.1) and (5.2) hold may be embedded in $S(2, 3)$.

Example. Consider 6 states of polarization of light: a, a' being two complementary circular polarizations, b, b' and c, c' being two pairs of complementary linear polarizations. Let b and c correspond to polarization planes forming an angle $\pi/4$. Then: $p(a, b) = p(a, c) = p(b, c) = 1/2$, and all remaining transition probabilities are consistent with (A), (B), (C). We can easily show that the above 6 polarization states may be embedded in the complex Hilbertian space: $\varrho(a, b) = \varrho(b, c) = \varrho(a, c) = \frac{1}{\sqrt{2}}$ and $r(a, b, c) = \frac{1}{\sqrt{6}} < \frac{1}{2}$. It is of interest to consider a more general hypothetical 6-element system of filters $\{a, a', b, b', c, c'\}$ with absorption coefficients: $p(a, b) = p(b, c) = p(a, c) = p$ $(0 < p < 1)$, and all remaining coefficients consistent with (A), (B), (C), e.g., $p(a, b') = 1 - p$, etc. Since $r(a, b, c) = \sqrt{\frac{1-p}{3}}$ the above system may form a part of a Hilbertian space for $p \geqq 1/4$ but not for $p < 1/4$. It appears that $p = 1/4$ is a critical value: below this value the system is impossible from the point of view of present quantum theory. It is possible, however, that the value $p = 1/4$ is not truely significant but only that the present theory is artificial.

The space composed of 8 elements $\{a, b, c, d, a', b', c', d'\}$ may be embedded in $S(2, 3)$ if: (1) conditions (5.1) and (5.2) hold for each three elements, (2) the radius of the sphere determined by the tetra-hedron with sides $\varrho(a, b)$, $\varrho(a, c)$, $\varrho(a, d)$, $\varrho(b, c)$, $\varrho(b, d)$, $\varrho(c, d)$, where $\varrho(x, y) = \sqrt{1 - p(x, y)}$, is $1/2$:

$$R(a, b, c, d) = \frac{1}{2}. \tag{5.3}$$

The meaning of this condition is clear from the picture below:

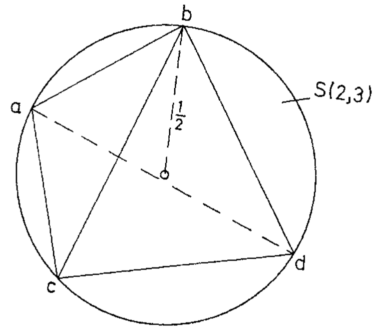

Fig. 15

Thus for an 8-element system the question whether it is Hilbertian or not requires subtle measurements. A positive answer may never be final since even a small correction of empirical values of $p(x, y)$ can negate it.

Example. The space $\{a, b, c, d, a', b', c', d'\}$ with transition probabilities $p(a, b) = p(a, c) = p(a, d) = p(b, c) = p(b, d) = p(c, d = p$ $(0 < p < 1)$ and all remaining consistent with (A), (B), (C), may be embedded in complex Hilbertian space only for $p = \frac{1}{\sqrt{3}}$ [3].

For spaces composed of 10 elements (a, b, c, d, e and their complements) we have the following criteria for Hilbertian structure: (1)

[3] For $\frac{1}{\sqrt{3}} > p > 1$ this space may be embedded in the quaternionic Hilbertian space. For $0 > p > \frac{1}{\sqrt{3}}$ it may not be embedded in any Hilbertian spaces.

each 4 element subset of $\{a, b, c, d, e\}$ must fulfill previously formulated conditions; (2) the four-dimensional volume $V(a, b, c, d, e)$ of the 5-hedron with sides $\varrho(a, b)$, $\varrho(a, c)$, $\varrho(a, d)$, etc. must vanish:

$$V(a, b, c, d, e) = 0 . \tag{5.4}$$

Example. For the 10-element space $\{a, b, \ldots, e'\}$ with transition probabilities $p(a, b) = p(a, c) = p(a, d) = p(a, e) = p(b, c) = p(b, d) = p(b, c) = p(c, d) = p(c, e) = p(d, e) = \frac{1}{\sqrt{3}}$ we have $V(a, b, c, d, e) > 0$. Hence, this space cannot be embedded in the complex Hilbertian space, although every one of its 8-element subspaces can. For spaces containing more than 10 elements the question reduces to the consideration of 10-element subspaces since we have:

Theorem 3. *A 2-dimensional probability space may be embedded in* $S(2, 3)$ *if and only if each of its subspaces composed of no more than* 10 *elements may be embedded in* $S(2, 3)$.

Proof. We shall first consider finite spaces and proceed by induction. The theorem is a tautology for all spaces containing no more than 10 elements. Suppose now that the theorem is valid for spaces composed of $2K$ elements where $K \geqq 5$ and let $\{a_1, \ldots, a_K, a_{K+1}, a_1', \ldots, a_K', a_{K+1}'\}$ be a space composed of $2(K+1)$ elements. Because of our assumption $2K$ elements $a_1, \ldots, a_K, a_1', \ldots, a_K'$ may be represented as points $P_1, \ldots, P_K, P_1', \ldots, P_K'$ on the $S(2, 3)$ sphere in such a way that P_i' is the antipode of P_i and distances between P_i's are:

$$\varrho(P_i, P_j) = \sqrt{1 - p(a_i, a_j)}; \quad i, j = 1, \ldots, K .$$

We shall show that a_{K+1} may also be represented by a certain point of $S(2, 3)$, i.e., that there exists in $S(2, 3)$ at least one point P_{K+1} with the required distances from $P_1, \ldots, P_K$:

$$\varrho(P_{K+1}, P_i) = \sqrt{1 - p(a_{K+1}, a_i)} .$$

We shall introduce some notation. For each subset $X \subset \{P_1, \ldots, P_K\}$ we shall denote by X^* a subset of these points in $S(2, 3)$ which have "apropriate" distances from all points of X, i.e.,

$$X^* = \{x^* \in S(2, 3) : \varrho(x^*, P) = \sqrt{1 - p(a_{K+1}, P)} \quad \text{for} \quad P \in X\} .$$

If X is the empty set, $X^* = S(2, 3)$. If X is composed of one point, X^* is a circle on the surface $S(2, 3)$. For X containing more than one point, X^* contains no more than 2 points (see Fig. below):

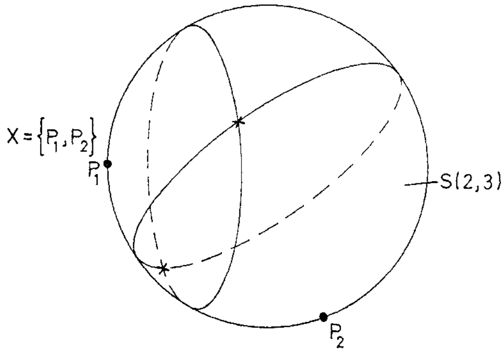

Fig. 16

The following simple rules hold:

$$X_1 \subset X_2 \Rightarrow X_1^* \supset X_2^* \tag{5.5}$$

$$(X_1 \cup X_2)^* = X_1^* \cap X_2^* \tag{5.6}$$

$$(X_1 \cap X_2)^* \supset X_1^* \cup X_2^* . \tag{5.7}$$

Our assumption implies that for every subset $X \subset \{P_1, \ldots, P_K\}$ composed of no more than $K - 1$ points the corresponding X^* is not empty. We have to show that $\{P_1, \ldots, P_K\}^*$ is not empty. This is a simple combinatorial fact, since only the following possibilities exist:

(a) A subset $X \subset \{P_1, \ldots, P_K\}$ composed of $K - 1$ points such that X^* contains 2 points exists. Consider now any subset Y composed also of $K - 1$ points and such that $X \cup Y = \{P_1, \ldots, P_K\}$. The common part $X \cap Y$ must contain at least 2 points. Hence $(X \cap Y)^*$ contains no more than 2 points. Now, $X^* \subset (X \cap Y)^*$ and $Y^* \subset (X \cap Y)^*$ because of (5.5). Since X^* contains 2 points, X^* and Y^* may not be disjoint. Hence, $X^* \cap Y^* = (X \cup Y)^* = \{P_1, \ldots, P_K\}^*$ is not empty.

(b) For each subset $X \subset \{P_1, \ldots, P_K\}$ composed of $K - 1$ points, X^* contains only one point. Then two subcases are possible:

(b 1) A subset Z composed of $K - 2$ points such that Z^* contains one point exists. Let now X and Y be two $(K - 1)$-element subsets such that $X \cup Y = \{P_1, \ldots, P_K\}$, $X \cap Y = Z$. Since X^* and Y^* are contained in Z^*, and Z^* contains only one point, then $X^* = Y^*$ and the set $X^* \cap Y^* = (X \cup Y)^* = \{P_1, \ldots, P_K\}^*$ is not empty.

(b 2) For each Z composed of $K - 2$ points Z^* contains 2 points. Represent now $\{P_1, \ldots, P_K\}$ as a sum of three $(K - 2)$-element subsets Z_1, Z_2, Z_3, such that each common part $Z_1 \cap Z_2$ and $Z_2 \cap Z_3$ contains at least 2 points. Now, 2-element subsets Z_1^*, Z_2^* are both contained in

$(Z_1 \cap Z_2)^*$ which also contains two elements; hence $Z_2^* = Z_1^*$. For similar reason $Z_2^* = Z_3^*$. Thus the set $\{P_1, \ldots, P_K\}^* = (Z_1 \cup Z_2 \cup Z_3)^* = Z_1^* \cap Z_2^* \cap Z_3^*$ is not empty.

In this manner the theorem is proved by induction for all finite 2-dimensional spaces. By applying the transfinite induction this result may be extended in all 2-dimensional spaces which concludes the proof of Theorem 3. ▮

We see that quantities $R(a, b, c, d)$ and $V(a, b, c, d, e)$ in (5.3) and (5.4) play a decisive role in the geometry of 2-dimensional probability space. When $R \equiv 1/2$ and $V \equiv 0$ the space is isomorphic with certain subspace of the complex Hilbertian space. This resolves the problem of how to determine the type of 2-dimensional "spaces of superpositions" considered in § 4.

We must return to the fundamental problem: should we believe that all physical phenomena can be described in terms of Hilbertian spaces? This is a general conviction of present day physics. There are strong arguments in its favour: among them are all successes of modern quantum theory. On the other hand, we cannot predict the future. It may easily happen that the present quantum theory which is based on a belief that each 2-dimensional space of superpositions is of type $S(2, 3)$ will be considered as naive as the ancients opinion that planets must move along circular orbits since the circle has the most perfect shape.

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