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Geometry of Skyrmions

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Abstract. A Skyrmion may be regarded as a topologically non-trivial map from one Riemannian manifold to another, minimizing a particular energy functional. We discuss the geometrical interpretation of this energy functional and give examples of Skyrmions on various manifolds. We show how the existence of conformal transformations can cause a Skyrmion on a 3-sphere to become unstable, and how this may be related to chiral symmetry breaking.

1. Introduction

A Skyrmion is a classical static field configuration of minimal energy in a non-linear scalar field theory. The scalar field is the pion field, and the Skyrmion represents a baryon. The Skyrmion has a topological charge which prevents it being continuously deformed to the vacuum field configuration. This charge is identified with the conserved baryon number which prevents a proton from decaying into pions [1]. The Skyrmion picture is in fair quantitative agreement with experimentally determined properties of protons and neutrons and their excited states [2].

Mathematically the Skyrmion is a topologically non-trivial map from physical 3-dimensional space S to a target manifold Σ [3]. The metrics on both S and Σ are essential and the energy of the Skyrmion is a measure of the geometrical distortion induced by the map. This is real "rubber-sheet" geometry. Indeed, the Skyrmion's energy is very like the strain energy of a deformed material in one version of non-linear elasticity theory [4], generalized to curved space. The Skyrmion may also be regarded as a generalized harmonic map [5].

In Sect. 2 we review the geometry of the strain tensor, and show that the Skyrme model's natural setting is Riemannian geometry. There is no need for the target manifold to be a Lie group, as is often assumed. In Sect. 3 we study maps between a domain and target which differ only by a constant scale factor. The identity map is always a stationary point of the energy functional in this situation, but it is not always stable. We illustrate this in Sect. 4 by the example of Skyrmions on a 3-sphere and on a 3-torus. The standard Skyrmion in flat space emerges when the domain is a 3-sphere of infinite radius. In Sect. 5 we discuss how Skyrmion instabilities may be related to chiral symmetry restoration and deconfinement of quarks.

This work developed from a paper written in collaboration with Peter J. Ruback [6]. Some of the material here appears there in a less geometrical form. I would like to thank Sir Michael Atiyah, Alfred Goldhaber, and Peter Ruback for discussions.

2. A Mathematical Framework for Skyrmions

In a non-linear scalar field theory, a field configuration is a map π from physical space S to a target space Σ. Both these spaces are Riemannian manifolds, with metrics t and τ respectively. We shall assume immediately that S and Σ are 3-dimensional, orientable and connected. The energy functional will be a measure of the extent to which the map π is metric preserving. It is well-known in elasticity theory [4], and in the theory of harmonic maps [5], how to construct such functionals. The field equations are the associated Euler-Lagrange equations, and the Skyrmion is a minimal energy solution.

Consider a small neighbourhood of a point $p \in S$, and its image under π. Let us choose normal coordinates p^i centred at p, and normal coordinates π^α centred at $\pi(p)$, so the metrics t and τ are unit matrices at p and $\pi(p)$. The map π is represented by functions $\pi^\alpha(p^1, p^2, p^3)$, and the Jacobian matrix $J_{i\alpha} = \frac{\partial \pi^\alpha}{\partial p^i}$, evaluated at p, is the basic measure of the deformation induced by the map there.

Normal coordinate systems are not unique, as they may be independently rotated by rotation matrices O and Ω at p and $\pi(p)$. Under such a transformation

$$J \to O^{-1} J \Omega\,, \tag{2.1}$$

but the geometrical distortion is unaffected, and the energy density should be invariant. This motivates the introduction of the strain tensor $D = JJ^T$. D is symmetric and positive definite, but still not invariant. Under the transformation (2.1),

$$D \to O^{-1} D O\,. \tag{2.2}$$

However, the invariants of D are well-known. They are the permutation-symmetric functions of the eigenvalues of D. These eigenvalues are our main tool in what follows. Let us denote them by λ_1^2, λ_2^2, and λ_3^2. All invariants which can be constructed from D can be expressed as functions of the basic invariants,

$$\begin{aligned} \operatorname{Tr} D &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2\,, \\ \tfrac{1}{2}(\operatorname{Tr} D)^2 - \tfrac{1}{2}\operatorname{Tr} D^2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2\,, \\ \det D &= \lambda_1^2\lambda_2^2\lambda_3^2\,, \end{aligned} \tag{2.3}$$

and one such invariant will be chosen as the energy density at p. The distortion induced by the map is characterized by how far the eigenvalues of the strain tensor differ from unity. The map is locally an isometry, and there is no distortion, only if the strain tensor is the unit matrix.

Normal coordinates at p and at $\pi(p)$ are not normal everywhere, in general, so for completeness we give the formula for the strain tensor in an arbitrary coordinate system. Let e^i_m be an orthonormal frame field on S and ζ^α_μ an orthonormal frame field on Σ. The strain tensor is still JJ^T, but the deformation matrix is now $J_{m\mu}=e^i_m(\partial_i\pi^\alpha)\zeta_{\mu\alpha}$.

The simplest energy density is $e_2=\lambda_1^2+\lambda_2^2+\lambda_3^2$, with total energy

$$E_2=\int\lambda_1^2+\lambda_2^2+\lambda_3^2\,. \tag{2.4}$$

In this section and the next, all integrals are over S with integration measure $\sqrt{\det t}\,d^3p$. E_2 is the energy functional whose stationary points are harmonic maps [5]. The subscript 2 indicates that the energy density is quadratic in derivatives. e_2 has the following geometrical meaning. The frame vectors e^i_m, $m=1,2,3$ are mapped by π to the vectors $e^i_m\partial_i\pi^\alpha$ on Σ, whose squared lengths are

$$e^i_m\partial_i\pi^\alpha e^j_m\partial_j\pi^\beta\tau_{\alpha\beta} \tag{2.5}$$

(no sum over m). Summing over m gives e_2. So e_2 measures how the sum of the squared lengths of an orthonormal frame of vectors on S changes under the map, but doesn't directly measure angular changes.

The second energy density we consider is $e_4=\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2$, with total energy

$$E_4=\int\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2\,. \tag{2.6}$$

This has the following geometrical meaning.[1] Corresponding to the basis of vectors e^i_m, there is a basis of area elements constructed from pairs of these vectors: $\varepsilon_{qmn}e^i_me^j_n$, $q=1,2,3$. These are mapped by π into area elements on Σ

$$\varepsilon_{qmn}e^i_me^j_n\partial_i\pi^\alpha\partial_j\pi^\beta\,. \tag{2.7}$$

The natural squared norm of these is

$$\tfrac{1}{2}\varepsilon_{qmn}e^i_me^j_n\partial_i\pi^\alpha\partial_j\pi^\beta\varepsilon_{qrs}e^k_re^l_s\partial_k\pi^\gamma\partial_l\pi^\delta\tau_{\alpha\gamma}\tau_{\beta\delta}\,. \tag{2.8}$$

Summing over q gives e_4.

A very symmetric measure of the distortion produced by π, given that S is 3-dimensional, is the energy density $e=e_2+e_4$. This is the only energy density we shall consider from now on. It sums the squared norms of the image under π of a frame of 1-vectors and a frame of 2-vectors. The total energy is

$$E=\int\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2\,. \tag{2.9}$$

(This actually makes sense whether or not Σ is 3-dimensional. The only difference when Σ is not 3-dimensional is that the deformation matrix J is not square, although the strain tensor remains a 3×3 matrix. If S is N-dimensional, there are obvious generalizations of this energy functional involving n-vectors, where $0<n<N$.) E can be usefully rewritten as

$$E=\int(\lambda_1-\lambda_2\lambda_3)^2+(\lambda_2-\lambda_3\lambda_1)^2+(\lambda_3-\lambda_1\lambda_2)^2+6\int\lambda_1\lambda_2\lambda_3\,. \tag{2.10}$$

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[1] I thank Sir Michael Atiyah for pointing this out to me

Notice that the integrand in the last term is $\sqrt{\det D}$, which equals $\pm\det J$. Since S and Σ are orientable, the square root can be consistently defined to be $\det J$.

Expression (2.9) coincides with Skyrme's energy functional in the case that the target manifold Σ is $SU(2)$, with its standard metric. In the usual formulation of the Skyrme model there is a simple generalization for any target manifold which is a compact Lie group. Our formula doesn't generalize in the same way. In the usual formulation the structure constants appear in e_4, so e_4 does not depend on the metric of Σ alone. It depends on the curvature tensor, and the tangent space at a point on Σ is not treated isotropically [6]. For example, an area element on Σ is regarded as having zero norm if the generating vectors correspond to commuting elements of the Lie algebra. We believe that from a geometrical point of view our definition of the energy density is more interesting. Locally it knows nothing about the deviation of either S or Σ from flat space, but when the map is considered globally, or on any finite neighbourhood, the curvatures of the manifolds have an effect. Physics may dictate a non-isotropic form for e_4, but the evidence is not clear. The $SU(n)$ Skyrmion, for $n \geqq 3$, is just the $SU(2)$ Skyrmion embedded in the larger group, so the difference in energy functionals is not really explored.

The integral of $\det J$, the last term in (2.10), is a topological invariant. Locally, $\det J$ just turns the integration measure on S into the integration measure on Σ, and the integral is the volume of Σ times the degree of the map π. The degree is an integer and a topological invariant. Since the other, non-topological integral in (2.10) is manifestly non-negative, the energy satisfies the topological bound given by Fadeev [7],

$$E \geqq 6(\deg \pi)(\mathrm{Vol}\,\Sigma). \tag{2.11}$$

The topological bound is attained if and only if

$$\lambda_1 = \lambda_2\lambda_3, \qquad \lambda_2 = \lambda_3\lambda_1, \qquad \lambda_3 = \lambda_1\lambda_2 \tag{2.12}$$

everywhere on S. This implies that $\lambda_1 = \lambda_2 = \lambda_3 = 1$, and that the strain tensor is the unit matrix, if the map is non-singular. The map therefore produces no distortion at all and is an isometry. The amount by which the energy exceeds this bound, i.e., the non-topological integral in (2.10), is a good measure of the geometrical distortion induced by the map. If Σ is not isometric to S, then the topological bound cannot be attained [6]. It would be valuable to have a better lower bound than (2.11) which takes into account any difference in shape between S and Σ.

The standard Skyrmion is a map of degree 1 from flat 3-dimensional space, topologically compactified at infinity, to a 3-sphere of unit radius. In this case, the topological bound is $E \geqq 12\pi^2$ because $\mathrm{Vol}\,\Sigma = 2\pi^2$. However, there is no isometry between these spaces, so the Skyrmion must have energy greater than $12\pi^2$. Numerically it has been found that the standard Skyrmion has energy $1.23\ldots \times 12\pi^2$ [2]. We would like to understand the constant 1.23... better. Clearly it is a measure of the difference in curvature between $\mathbb{R}^3$ and S^3. We have been using dimensionless energy and length units until now. To make contact with physics one must take the energy unit to be approximately 6 MeV, corresponding to $\frac{F_\pi}{4e}$ in the notation of [2], and the length unit to be approximately 0.6 fermi, corresponding to $\frac{2}{eF_\pi}$. The Skyrmion then has an energy, or mass, of about

870 MeV. The proton, with mass 938 MeV, is interpreted as the Skyrmion's lowest energy rotational excitation with spin $\frac{1}{2}$.

There is an energy bound which depends on the volumes of S and Σ and which is stronger than (2.11) when $\mathrm{Vol}\,\Sigma > \mathrm{Vol}\,S$. One minimizes the expression (2.9) for the energy subject only to the constraint

$$\int \lambda_1\lambda_2\lambda_3 = \mathrm{Vol}\,\Sigma\,. \tag{2.13}$$

Let $\mathrm{Vol}\,\Sigma = \sigma^3(\mathrm{Vol}\,S)$, with $\sigma > 1$. Using simple algebra and a Lagrange multiplier, it is easy to show that for a map of degree 1, E can be no less than what it would be if λ_1, λ_2, and λ_3 were everywhere constant, and equal. Because of the constraint, this constant is σ, so

$$E \geqq 3\left(\sigma + \frac{1}{\sigma}\right)\mathrm{Vol}\,\Sigma\,. \tag{2.14}$$

If $\mathrm{Vol}\,\Sigma < \mathrm{Vol}\,S$ there is no useful energy bound coming from the constraint (2.13) alone. For (2.13) would be satisfied if $\lambda_1 = \lambda_2 = \lambda_3 = 1$ on a part of S equal in volume to Σ and $\lambda_1 = \lambda_2 = \lambda_3 = 0$ on the remainder of S. Then the energy would be $6\,\mathrm{Vol}\,S$, which is just the topological bound.

Generally, the energy does not saturate the bound (2.14), because the eigenvalues of the strain tensor, or more precisely their derivatives, are subject to further constraints than just (2.13). However, if Σ is geometrically similar to S, with lengths simply rescaled by σ, and with $\sigma > 1$, then the identity map attains the bound. It is therefore the map of lowest energy, and is automatically stable. We shall show in Sect. 3 that the identity map is not necessarily the map of lowest energy when $\sigma < 1$.

3. Eigenvalues and Stability

Throughout this section we assume that S and Σ are manifolds of similar shape, with Σ having linear dimension σ times that of S. We shall show that the identity map is always a stationary point of the energy functional, and shall investigate the stability of this map.

The identity map's strain tensor is σ^2 times the unit tensor everywhere, so $\lambda_1 = \lambda_2 = \lambda_3 = \sigma$. Its energy is

$$E = (3\sigma^2 + 3\sigma^4)(\mathrm{Vol}\,S) = 3\left(\frac{1}{\sigma} + \sigma\right)(\mathrm{Vol}\,\Sigma)\,. \tag{3.1}$$

A small deformation changes the strain tensor and its eigenvalues. So now

$$\lambda_1 = \sigma + \delta_1\,, \qquad \lambda_2 = \sigma + \delta_2\,, \qquad \lambda_3 = \sigma + \delta_3\,, \tag{3.2}$$

where δ_1, δ_2, and δ_3 vary over S, and are small. It is useful to define the quantities

$$\begin{aligned} I_1 &= \delta_1 + \delta_2 + \delta_3\,, \\ I_2 &= \delta_1^2 + \delta_2^2 + \delta_3^2\,, \\ I_3 &= \delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1\,. \end{aligned} \tag{3.3}$$

I_1 is of first order in δ, and I_2 and I_3 are second order. To second order in δ, the deformation results in a change of energy

$$\Delta E=(2\sigma+4\sigma^3)\int I_1+(1+2\sigma^2)\int I_2+4\sigma^2\int I_3+O(\delta^3). \tag{3.4}$$

The deformation of the eigenvalues is subject to the topological constraint (2.13), which implies

$$\sigma^2\int I_1+\sigma\int I_3=O(\delta^3). \tag{3.5}$$

This means that to first order in δ, the integral of I_1 vanishes, but more accurately, it is a second order small quantity proportional to the integral of I_3. Substituting (3.5) into ΔE, we obtain an expression which is of second order in δ,

$$\Delta E=\int[(1+2\sigma^2)I_2-2I_3]+O(\delta^3), \tag{3.6}$$

which shows that the identity map is a stationary point of the energy functional.

The quadratic form appearing in the integrand of (3.6)

$$(1+2\sigma^2)(\delta_1^2+\delta_2^2+\delta_3^2)-2(\delta_1\delta_2+\delta_2\delta_3+\delta_3\delta_1) \tag{3.7}$$

is positive definite if $\sigma>\frac{1}{\sqrt{2}}$. It is semi-definite if $\sigma=\frac{1}{\sqrt{2}}$ and can take negative values if $\sigma<\frac{1}{\sqrt{2}}$. In the last case, the decrease is most rapid when $\delta_1=\delta_2=\delta_3$. We conclude that for $\sigma>\frac{1}{\sqrt{2}}$ the identity map is stable, but for $\sigma<\frac{1}{\sqrt{2}}$ there is potentially an instability, where the deformation is predominantly a local scale change. A pure scale change, with $\delta_1=\delta_2=\delta_3$ everywhere, exists only if the manifold Σ, or equivalently S, admits an infinitesimal conformal transformation which is not an isometry; so only in this case is the identity map automatically unstable for all $\sigma<\frac{1}{\sqrt{2}}$. The 3-sphere with its standard metric admits a conformal transformation. On the other hand, a 3-torus does not, and the identity map is in fact stable for all σ. We study these examples in the next section. In general one may expect the identity map from S to Σ to be unstable for all σ less than some σ_0, where $\sigma_0<\frac{1}{\sqrt{2}}$.

4. Skyrmions on a 3-Sphere and on a 3-Torus

Let us see how the existence of conformal transformations on the 3-sphere affects the energy of maps from a 3-sphere S of radius L to a unit 3-sphere Σ.

Introduce coordinates (μ, θ, ϕ) on S, with μ the polar angle and (θ, ϕ) standard coordinates on the 2-sphere at polar angle μ. Let (μ', θ', ϕ') be similar coordinates on Σ. The map we are interested in is defined by

$$\tan\tfrac{1}{2}\mu'=\alpha\tan\tfrac{1}{2}\mu, \qquad \theta'=\theta, \qquad \phi'=\phi \tag{4.1}$$

with α a real positive constant. This is conformal. It can be thought of as a stereographic projection from S to $\mathbb{R}^3$, followed by a rescaling by α, followed by an inverse stereographic projection from $\mathbb{R}^3$ to Σ. A map of the $SO(3)$-symmetric form $\mu'=f(\mu)$, $\theta'=\theta$, $\phi'=\phi$ has energy [6]

$$E=4\pi\int_0^\pi\left[L\sin^2\mu\left[\left(\frac{df}{d\mu}\right)^2+2\frac{\sin^2 f}{\sin^2\mu}\right]\right.$$
$$\left.+\frac{1}{L}\sin^2 f\left[\frac{\sin^2 f}{\sin^2\mu}+2\left(\frac{df}{d\mu}\right)^2\right]\right]d\mu\,. \tag{4.2}$$

When $f(\mu)=2\tan^{-1}(\alpha\tan\frac{1}{2}\mu)$, this integral is elementary and the energy is

$$E=12\pi^2\left[\left(\frac{2L}{\alpha+\frac{1}{\alpha}+2}\right)+\frac{1}{4L}\left(\alpha+\frac{1}{\alpha}\right)\right]. \tag{4.3}$$

Note that for $\alpha=1$, (4.1) is the identity map, and the formula above gives as its energy

$$E=6\pi^2\left(L+\frac{1}{L}\right), \tag{4.4}$$

in agreement with (3.1), since $\sigma=L^{-1}$. Let us now seek the minimum of E. First set $\beta=\alpha+\frac{1}{\alpha}$. E has a single minimum with respect to β when $\beta=\sqrt{8}L-2$. For $L<\sqrt{2}$, this value of β is less than 2, and unattainable for real α, so the minimum energy actually occurs at $\alpha=1$. The identity map is therefore stable with respect to conformal transformations. This agrees with the general result of Sect. 3 which predicts stability against any deformation when $L\leqq\sqrt{2}$. On the other hand, for $L>\sqrt{2}$, the minimum occurs when α is either root of

$$\alpha+\frac{1}{\alpha}=\sqrt{8}L-2\,. \tag{4.5}$$

These two roots, whose product is 1, give geometrically equivalent maps, since one is transformed into the other by exchanging poles on the 3-sphere, and sending μ to $\pi-\mu$. Neither is the identity map. Their energy is

$$E=12\pi^2\left[\sqrt{2}-\frac{1}{2L}\right], \tag{4.6}$$

which is less than (4.4). The true Skyrmion is probably also $SO(3)$-symmetric, and qualitatively similar to these conformal maps of lowest energy, but the exact form of $f(\mu)$ is unknown.

In the limit $L\to\infty$, the map is from $\mathbb{R}^3$ to the unit 3-sphere. In this limit, the large value of α is $\sim\sqrt{8}L$, and the map is $\tan\frac{1}{2}\mu'\sim\sqrt{2}L\mu$. r, the distance from the origin in $\mathbb{R}^3$, should be identified with $L\mu$. The conformal map of lowest energy from $\mathbb{R}^3$ to the unit 3-sphere is therefore $\mu'=2\tan^{-1}(\sqrt{2}r)$ and its energy is $12\pi^2\sqrt{2}$. The true Skyrmion on $\mathbb{R}^3$ is again a qualitatively similar map, but is certainly different and has lower energy.

Figure 1 shows the energy of the identity map, and for $L>\sqrt{2}$ the energy (4.6), representing the map of lowest energy which is conformal to the identity map. Also

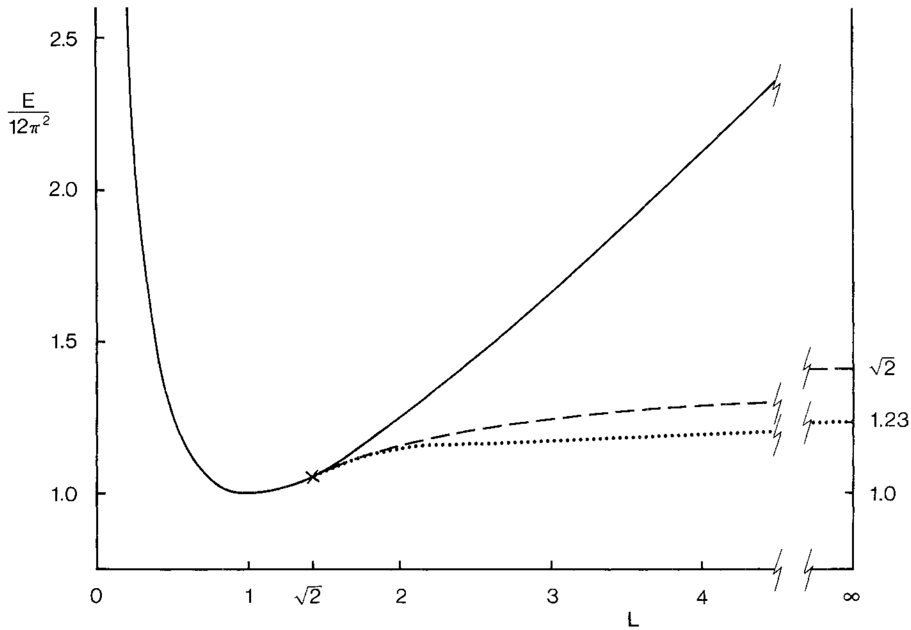

Fig. 1. The energy of maps of degree 1 from a 3-sphere of radius L to a unit 3-sphere: the identity map (solid), the conformal map of lowest energy (dashed), the likely form of the energy of the Skyrmion (dotted). The energy E is expressed as a multiple of $12\pi^2$

shown is the likely form of the energy of the map of degree 1 with lowest energy, which is the true Skyrmion.

As a second illustration of our formalism, let us consider maps of degree 1 from a 3-torus with linear dimension (or period) L to a unit 3-torus. Since a torus is flat, we are very close to elasticity theory. We are again interested in the stability of the identity map. Let us introduce Cartesian coordinates p_i on the domain and π_α on the target – all lying in the interval $[0, 1]$. A map close to the identity map is $\pi_\alpha = p_\alpha + \varepsilon_\alpha(p)$, where ε_α is small. The strain tensor is

$$D_{\alpha\beta} = \frac{1}{L^2}\left(\delta_{\alpha\beta} + \partial_\beta\varepsilon_\alpha + \partial_\alpha\varepsilon_\beta + \partial_\gamma\varepsilon_\alpha\partial_\gamma\varepsilon_\beta\right), \qquad (4.7)$$

and the energy, to second order in ε, is

$$E = \frac{1}{L^2}\int(3 + 2\partial_\alpha\varepsilon_\alpha + \partial_\beta\varepsilon_\alpha\partial_\beta\varepsilon_\alpha)d^3p + \frac{1}{L^4}\int(3 + 4\partial_\alpha\varepsilon_\alpha + 2\partial_\alpha\varepsilon_\alpha\partial_\beta\varepsilon_\beta + \partial_\alpha\varepsilon_\beta\partial_\alpha\varepsilon_\beta - \partial_\alpha\varepsilon_\beta\partial_\beta\varepsilon_\alpha)d^3p\,. \qquad (4.8)$$

The energy density is $\mathrm{Tr}\,D - \frac{1}{2}\mathrm{Tr}\,D^2 + \frac{1}{2}(\mathrm{Tr}\,D)^2$, which is here given directly, rather than in terms of the eigenvalues of D. ε_α and its derivatives are periodic, which implies that the following integrals of total divergences vanish,

$$\int\partial_\alpha\varepsilon_\alpha d^3p = 0\,, \qquad \int\partial_\alpha(\varepsilon_\beta\partial_\beta\varepsilon_\alpha - \varepsilon_\alpha\partial_\beta\varepsilon_\beta)d^3p = 0\,. \qquad (4.9)$$

It follows that the energy can be reexpressed as

$$E=3\left(\frac{1}{L^2}+\frac{1}{L^4}\right)+\frac{1}{L^2}\int\partial_\beta\varepsilon_\alpha\partial_\beta\varepsilon_\alpha d^3p+\frac{1}{L^4}\int(\partial_\alpha\varepsilon_\alpha\partial_\beta\varepsilon_\beta+\partial_\alpha\varepsilon_\beta\partial_\alpha\varepsilon_\beta)d^3p\,. \quad (4.10)$$

E has no linear dependence on ε now, so the identity map is a stationary point, as expected. The quadratic terms in ε are obviously non-negative. They are zero only if ε_α is constant, which corresponds to a translation of the identity map. The identity map is therefore stable for all L. Unlike the 3-sphere the 3-torus has no non-trivial conformal transformations producing an instability.

5. Physical Interpretation

It is believed that in certain extreme conditions of temperature or density, or in a highly curved universe, baryonic matter turns into a quark plasma. This is supposed to be a quantum effect. In this phase the quarks are unconfined and chiral symmetry is restored. However, there is considerable debate about whether the deconfinement phase transition and the chiral symmetry restoring transition occur at exactly the same time or not. Numerical investigations suggest that they probably do occur at the same time [8]. If one can trust that a Skyrmion remains a good description of a baryon and its quark content when the baryon is confined to a small universe, then our results concerning classical Skyrmion instabilities are relevant to the arguments about these phase transitions.

The full symmetry group of the Skyrme model, as described here, is the product of the isometry groups of S and Σ. This is the symmetry group of the Hamiltonian when the model is quantized. However, quantum states in the vacuum sector occur in multiplets not of this group, but of the unbroken subgroup which leaves invariant the classical vacuum configuration, which is a constant map. In the standard Skyrme model in flat space, with $S=\mathbb{R}^3$ and $\Sigma=S^3$, the full symmetry group is $E_3\times SO(4)$, with E_3 the Euclidean group of rotations and translations and $SO(4)$ the chiral symmetry group. The classical vacuum breaks this to $E_3\times SO(3)_{\text{isospin}}$, and this last group classifies pion states (by momentum, spin, and isospin) in the absence of a Skyrmion.

Consider now the quantum states in the Skyrmion sector suggested by the semi-classical approach to quantization. Here, states are classified by the subgroup of the full symmetry group leaving the Skyrmion invariant. The classical Skyrmion in flat space breaks translation invariance and is invariant only under a combined rotation and isospin rotation. The unbroken group is $SO(3)$. States representing pions scattering off a Skyrmion lie in multiplets of this last group. The incoming pions, far from the Skyrmion, have well-defined momenta and isospin, but this is not conserved during the scattering. (The total momentum, angular momentum, and isospin of pions and Skyrmion together are, however, conserved.) When space is a 3-sphere, the symmetry group of the Hamiltonian is $SO(4)_{\text{spatial}}\times SO(4)_{\text{chiral}}$. If this 3-sphere has a radius greater than $\sqrt{2}$, which is approximately 0.8 fermi in physical units, the Skyrmion is similar to the flat space Skyrmion. There is a preferred point, corresponding to the point at infinity in flat space, where the eigenvalues λ_1, λ_2, and λ_3 are smallest, and the image of this point is a preferred point in Σ. The unbroken group is therefore the diagonal $SO(3)$ subgroup of the full symmetry group which fixes these points, as in flat space. Chiral symmetry is certainly broken.

On a smaller sphere, things are different. The Skyrmion is now the identity map so no point in space is special and no point in Σ is special. The unbroken group is the diagonal $SO(4)$ subgroup of the full symmetry group. One may think of this as either chiral $SO(4)$ or spatial $SO(4)$ – either interpretation is partly right. The essential point is that quantum states of pions interacting with the Skyrmion are classified by a larger group then in flat space, and one may say that chiral symmetry is restored. Notice that this symmetry restoration does not occur in the absence of the Skyrmion, i.e., in the vacuum sector, because a constant map always singles out a particular point in Σ.

Not only is chiral symmetry restored at the critical radius, but one may say that quarks become deconfined simultaneously. In flat space or on a large 3-sphere, the energy density of a Skyrmion is concentrated in a finite region. This can be interpreted as a dynamical quark confinement. At the critical radius the energy becomes uniformly distributed, so the quarks are no longer confined but move freely in the small available space. In the Skyrme model, therefore, chiral symmetry restoration and deconfinement seem to occur together. However, more work is needed to see if semi-classical quantization is valid. It could be that large quantum fluctuations restore chiral symmetry even in the absence of a Skyrmion.

Note added in proof. A. D. Jackson has recently computed the energy of the Skyrmion on a 3-sphere of radius L, for $L>\sqrt{2}$. For $L=2$, 3, 4, 5, and 10, and 10, the energy, expressed as a multiple of $12\pi^2$, is 1.145, 1.192, 1.208, 1.216, and 1.227 respectively (cf. Fig. 1). I have learned that E. Lieb and M. Loss have independently obtained some of the results of Sects. 2 and 3 [Loss, M.: Lett. Math. Phys. (to appear)].

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