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Geometry of $SU(2)$ Gauge Fields

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Abstract. We study $SU(2)$ Yang-Mills theory on $S^3 \times \mathbb{R}$ from the canonical view-point. We use topological and differential geometric techniques, identifying the "true" configuration space as the base-space of a principal bundle with the gauge-group as structure group.

1. Introduction

We study in this paper the space of connections on the trivial $SU(2)$ bundle on S^3 and the action of the gauge-group on this space. Let $\mathscr{C} = \mathscr{C}^k$ denote the space of connections belonging to Sobolev class (k), $k \geqq 3$. We introduce the groups Aut, Aut^o (see Sect. 2) of gauge transformations belonging to the Sobolev class $(k+1)$. We then define the space $\mathscr{C}_o$ of generic connections, which are the connections whose holonomy coincides with the whole group $SU(2)$, and prove that the above groups act properly on $\mathscr{C}$ (Proposition 2.4) and that $\mathscr{C}_o$ in a principal Aut (or Aut^o) bundle (Propisition 4.3). The proof involves deriving estimates for certain elliptic operators whose coefficients belong to Sobolev spaces and are not necessarily C^∞. We define the groups Aut_e, Aut_e^o (Sect. 4b)) and show that the Aut (resp. Aut^o) bundle cannot be reduced to the subgroup Aut_e [resp. Aut_e^o (Theorem 5.1)]. In particular gauge-fixing is not possible. This result is proved by looking at left-invariant differential forms on $S^3 = SU(2)$ with values in the Lie algebra of $SU(2)$ and by showing essentially that the principal $SO(3)$ bundle obtained by the action on 3×3 real matrices of rank $\geqq 2$, by multiplication on the left, is nontrivial (Theorem 6.2).

In Sect. 7 we introduce the Coulomb connection. We show (Theorem 7.5) that, in case we use the biinvariant metric on $S^3 = SU(2)$, the values of the curvature form of this connection at the point $\omega/2 \in \mathscr{C}_o$, where ω is the Maurer-Cartan form, span a dense subspace in the gauge algebra.

The study was motivated by the following physical considerations, taking Dirac's theory [1] of singular Lagrangians as starting point. We may recall that the Faddeev-Popov procedure was derived [2] by an extension of Dirac's

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constraint analysis programme. With the realisation due to Gribov [3] that the Coulomb gauge has ambiguities in the case of non-abelian theories, it has become necessary to examine anew the quantisation of such theories.

The $SU(2)$ Yang-Mills theory without matter-fields is described by the action

$$-\tfrac{1}{4}\int (F_{\mu\nu}F^{\mu\nu})d^4x$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu, A_\nu]$. We assume that the fields $A_\mu(x)$ fall off fast enough at space-like infinity, that they can be mapped into fields on $S^3\times\mathbb{R}$, $\mathbb{R}$ representing the time-co-ordinate. Because of gauge-invariance, the Lagrangian is singular and the problem as is well-known, reduces to the following.

Consider the phase-space $\{A_i, \pi_i\}$ $(i, 1, 2, 3)$ of the space-components. There is a constraint on this space, usually expressed as $\partial_i\pi_i+[A_i, \pi_i]=0$. On this constrained space a "Hamiltonian" is defined.

$$\int (\pi_i\pi_i+1/2F_{ij}F_{ij})d^3x\,.$$

The constrained space, however is not a symplectic manifold. The "true" configuration space, and its phase space are obtained by factoring out by the "time-independent" gauge transformations. More precisely, time-independent gauge-transformations act on the space of fields $\mathscr{C}=\{A_i(x)\}$. The gauge-invariant configuration space $\boldsymbol{\mathscr{C}}$ is the quotient by this action, and the gauge-invariant phase-space, the corresponding phase-space. In terms of diagrams:

$$\begin{array}{ccccc} T^*(\mathscr{C}) & \hookleftarrow & \mathscr{I} & \longrightarrow & \mathscr{C} \\ & & \downarrow & & \downarrow \text{gauge-group} \\ & & T^*(\boldsymbol{\mathscr{C}}) & \xrightarrow[p]{} & \boldsymbol{\mathscr{C}} \end{array}$$

Here p is the projection from $T^*(\boldsymbol{\mathscr{C}})$, and $\mathscr{I}$, the fibre product over $\boldsymbol{\mathscr{C}}$ of $\mathscr{C}$ and $T^*(\boldsymbol{\mathscr{C}})$, is precisely the *constrained phase-space*.

The "Hamiltonian" given above goes down to $T^*(\boldsymbol{\mathscr{C}})$ and becomes a true Hamiltonian there. Correspondingly there is a well-defined, non-singular Lagrangian on $\boldsymbol{\mathscr{C}}$.

Faddeev [2] quantises by identifying $T^*(\boldsymbol{\mathscr{C}})$ with a section of the bundle $\mathscr{I}\to T^*(\boldsymbol{\mathscr{C}})$, this section representing the subsidiary constraint, which together with the first, forms a second-class system. Since $T^*(\boldsymbol{\mathscr{C}})\to\boldsymbol{\mathscr{C}}$ admits the zero section, it is clear that the existence of such a section is equivalent to the existence of a section for $\mathscr{C}\to\boldsymbol{\mathscr{C}}$.

The Lagrangian on $\boldsymbol{\mathscr{C}}$ can be obtained directly by the simple procedure of letting $A^0=0$ in the original Lagrangian, thus getting

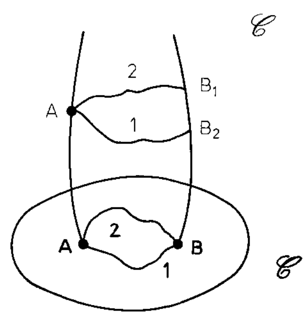
$$\int (\dot{A}_i\dot{A}_i-\tfrac{1}{2}F_{ij}F_{ij})d^3x\,.$$

This Lagrangian has "time-independent gauge-transformations" as a symmetry, and gives rise to a Lagrangian on $\boldsymbol{\mathscr{C}}$ in a natural way. This involves defining a "horizontal space" at each point A_i of $\mathscr{C}$: this is the space of tangent vectors $\dot{A}_i$ that satisfy

$$\partial_i\dot{A}_i+[A_i, \dot{A}_i]=0\,.$$

Note that in the abelian case the horizontal spaces form an integrable distribution, and the Coulomb gauge corresponds to taking a maximal integral manifold as the section $\mathbf{\mathscr{C}} \to \mathscr{C}$. Note also that in general this definition of horizontal spaces gives a connection in the bundle $\mathscr{C} \to \mathbf{\mathscr{C}}$. We call this the Coulomb connection.

In the absence of a section, a conceptually simple, although in practice difficult, path-integral procedure suggests itself. Suppose we consider transition amplitude between points **A**, **B**, in $\mathbf{\mathscr{C}}$. This involves integrating over all paths from **A** to **B**, using the Lagrangian in $\mathbf{\mathscr{C}}$. But for a given smooth path, the action is the same as the one given by lifting the path to a horizontal one in $\mathscr{C}$ (to a path satisfying $\partial_i \dot{A}_i + [A_i, \dot{A}_i] = 0$), between points A and B above. Thus the holonomy of the connection on $\mathscr{C}$ is clearly relevant. We calculate the holonomy for a special choice of a metric in S^3 and find that it is dense in the gauge-group. In other words, if we fix A on the fibre above **A**, a dense set of points above **B** can be joined to A by horizontal paths. (Thus the ambiguity is in some sense maximal.) Schematically:



Note that in the abelian case the holonomy is trivial, and a horizontal path in $\mathscr{C}$ starting from a point in the Coulomb gauge always stays within the Coulomb gauge. In particular all paths from **A** to **B** below, when lifted through A, end in the same point B above **B**.

Results on gauge-fixing, applicable when the base-space is S^3 or S^4, and the structure-group is a general compact semi-simple Lie group [in particular $SU(N)$], have been announced by Singer [4]. The present work was done independently and our approach is different. In the particular case that we consider, our first main result (Theorem 5.1) is stronger than the nonexistence of a section for the action of the group of gauge-transformations. The second main result (Theorem 7.5) of this paper, on the holonomy of the "Coulomb connection", is new.

2. The Space of Connections and the Action of the Gauge Group

We shall consider connections on the trivial $SU(2)$ bundle over S^3. We identify the set of connections with the set of 1-forms with coefficients in the Lie Algebra, $\mathfrak{SU}(2)$, of $SU(2)$ by means of the map $\alpha \mapsto \sigma^*(\alpha)$ where σ is the canonical section of the trivial $SU(2)$ bundle. We shall use connections which belong to the Sobolev class (k) with $k \geqq 3$. We denote the space of such connections by $\mathscr{C}^k$ or simply $\mathscr{C}$ when once we have fixed $k \geqq 3$. Let *Aut denote the gauge group consisting of maps from S^3 to $SU(2)$ which belong to the Sobolev class $(k+1)$. *Aut^o will refer to the subgroup of *Aut consisting of maps which are homotopic to the constant map $S^3 \to$ Identity.

In the rest of the paper, we will only occasionally need to distinguish between the groups *Aut and *Aut^o. We will let *G denote either one of them.

We will need

Lemma 2.1. *For* $i \geqq 2$,

i) *The Sobolev space* H^i *of functions from* S^3 *to* $\mathbb{C}$ *of class* (i) *forms a Banach algebra under pointwise multiplication.*

ii) *The multiplication* $H^i \times H^i \to H^i$ *is smooth.*

iii) *If we denote by* $\mathcal{M}$ *mappings of* S^3 *into* $M(2, \mathbb{C})$ *(complex* 2×2 *matrices) which are of Sobolov class* (i) *then the group* *G *is a closed* C^∞ *submanifold of* $\mathcal{M}$.

Proof. For a proof of (i) see [5], Theorem (5.23). Bilinearity and (i) imply (ii) and (iii) follows from [6, p. 78].

We have an action of *G on $\mathscr{C}$ given by

$$(\alpha, \varphi) \mapsto \varphi^{-1} \alpha \varphi + \varphi^{-1} d\varphi \equiv \alpha \circ \varphi \quad \text{for} \quad \alpha \in \mathscr{C}, \varphi \in {}^*\mathrm{G}.$$

We see from Lemma 1 that *G is a Lie group and that the above action is smooth.

The Lie algebra $\mathscr{G}$ of *G is identified with the Lie algebra of maps from S^3 to $\mathfrak{SU}(2)$ which are of Sobolev class $(k+1)$.

Lemma 2.2. *The isotropy of* *G *at any point of* $\mathscr{C}$ *is compact. In fact the isotropy group is isomorphic to the centraliser of the holonomy group in* $SU(2)$.

Proof. If φ belongs to the isotropy group at $\alpha \in \mathscr{C}$ then $\varphi^{-1}\alpha\varphi + \varphi^{-1} d\varphi = \alpha$ or $d\varphi + [\alpha, \varphi] = 0$. Thus φ is invariant under parallel translation, considered as a section of the bundle with $M(2, \mathbb{C})$ as fibre. Thus φ is determined by $\varphi(e)$ and $\varphi(e)$ commutes with the elements of the holonomy group.

Remark 2.3. The group of constant functions with values in the centre of $SU(2)$ acts trivially on $\mathscr{C}$. The isotropy group of *G at $\alpha \in \mathscr{C}$ coincides with this subgroup if and only if the holonomy group is $SU(2)$; this condition in turn is easily seen (e.g. by Schur's lemma) to be equivalent to the condition: if β is a 1-form with values in $\mathfrak{SU}(2)$, and $d\beta + [\alpha, \beta] = 0$ then $\beta = 0$. We call such connections, whose holonomy is the whole group $SU(2)$, *generic* and denote the set of generic connections by $\mathscr{C}_o$. Note that the gauge group *G acts on $\mathscr{C}_o$ and *G/($\mathbb{Z}$/(2)) acts freely. We will denote *G/($\mathbb{Z}$/(2)) by G.

Proposition 2.4. *The action of* *G *on* $\mathscr{C}$ *is proper.*

Proof. It is enough [7] to show that the map $\mu: \mathscr{C} \times {}^*\mathrm{G} \to \mathscr{C} \times \mathscr{C}$, $(\alpha, \varphi) \mapsto (\alpha \circ \varphi, \alpha)$ is closed and that the inverse image of each point by μ is compact. Lemma 2.2 shows that the inverse image of any point is compact. That μ is closed follows from

Lemma 2.5. *Let* $(\alpha_n, \varphi_n) \in \mathscr{C} \times {}^*\mathrm{G}$ *be a sequence such that* $\alpha_n \to \alpha$ *and* $\alpha_n \circ \varphi_n \equiv \beta_n \to \beta$ *in* $\mathscr{C}$. *Then there exists a subsequence* $\{\varphi_l\}$ *of* $\{\varphi_n\}$ *which tends to a limit* φ *(so that* $\alpha \circ \varphi = \beta$*).*

Lemma 2.5 will follow from Lemmas 2.6–2.8. In these lemmas we use the notation of Lemma 2.5.

Lemma 2.6. *Let U be an open co-ordinate cell in S^3 and p a point of U. If there exists a subsequence $\{\varphi_l\}$ of $\{\varphi_n\}$ so that $\varphi_l(p)$ tends to a limit g in $SU(2)$, then φ_l tends uniformly on compact sets to a limit $\varphi: U \to M(2,\mathbb{C})$.*

Proof. We have $d\varphi_l = \varphi_l\beta_l - \alpha_l\varphi_l$, $\varphi_l(p) \to g$. *Define* $\hat\varphi_l = \varphi_l\varphi_l^{-1}(p)$, $\hat\beta_l = \varphi_l(p)\beta_l\varphi_l^{-1}(p)$. Then $d\hat\varphi_l = \hat\varphi_l\hat\beta_l - \alpha_l\hat\varphi_l$ and $\hat\varphi_l(p) = \text{Identity}$.

Introduce co-ordinates (x_i) on U with $x_i(p)=0$, U being mapped onto $\mathbb{R}^3$ by (x_i). Denote by $\alpha_i(x)$ the components of a connection $\alpha \in \mathscr{C}$ in this co-ordinate system. When $x \in U$, α, $\beta \in \mathscr{C}$ consider the system [with $t \in \mathbb{R}$, $y \in M(2,\mathbb{C})$]

$$\frac{dy_{(\alpha,\beta,x)}(t)}{dt} = \Big(\sum_i x_i\beta_i(xt)\Big)y(t) - y(t)\Big(\sum_i x_i\alpha_i(xt)\Big)$$

$$y_{(\alpha,\beta,x)}(0) = \text{Identity}.$$

This is an ordinary linear differential equation in $y(t)$ with α, $\beta \in \mathscr{C}$ and $x \in U$ as parameters and a fixed initial condition. Then by (10.7.2.) of [8] the system has a unique solution $y_{(\alpha,\beta,x)}(t)$, defined for all t, x, α, β, which is continuous in all four variables and differentiable in the first.

Now, note that by uniqueness $y_{(\alpha_l,\hat\beta_l,x)}(t) = \hat\varphi_l(x,t)$ and $\hat\varphi_l(x) = y_{(\alpha_l,\hat\beta_l,x)}(1)$. The lemma follows easily by continuity in (α,β).

Lemma 2.7. *There exists a subsequence $\{\varphi_l\}$ of $\{\varphi_n\}$ which tends uniformly on S^3 to a limit φ which is a continuous map $\varphi: S^3 \to SU(2)$.*

Proof. Cover S^3 by two open cells U and U', choose a point p in their intersection. Since $SU(2)$ is compact there exists a subsequence $\{\varphi_l\}$ such that $\varphi_l(p)$ tends to a limit. Let V, V' be compact sets in U and U' respectively which also cover S^3. Then by Lemma 2.6, φ_l converges uniformly on both V and V' and hence on S^3 to a continuous function $\varphi: S^3 \to M(2,\mathbb{C})$. Since $SU(2)$ is closed in $M(2,\mathbb{C})$, φ has values in $SU(2)$.

Lemma 2.8. *If a subsequence $\{\varphi_l\}$ of $\{\varphi_n\}$ tends uniformly to a continuous function $\varphi: S^3 \to SU(2)$, then φ is of class $(k+1)$ and $\varphi_l \to \varphi$ in *G.*

Proof. We have $\alpha_l \to \alpha$ and $\beta_l \to \beta$ in H^k and $\varphi_l \to \varphi$ in C^o. But $d\varphi_l = \varphi_l\beta_l - \alpha_l\varphi_l \to \varphi\beta - \alpha\varphi$ in C^o by Sobolev lemma; this implies that φ is in C^1 and $\varphi_l \to \varphi$ in C^1. Similarly $\varphi_l \to \varphi$ in C^2 topology as α_l, $\beta_l \in C^1$ by Sobolev. In particular $\varphi_l \to \varphi$ in the Sobolev space H^2. Now, since d is an elliptic operator with injective symbol (on 0-forms) we see that $\varphi_l \to \varphi$ in H^3. We conclude by induction that $\varphi_l \to \varphi$ in H^{k+1}.

3. Some Estimates

In this section we shall derive some estimates connected with elliptic operators (whose coefficients are not necessarily C^∞) arising from connections belonging to Sobolev class (k). These will be needed in the rest of the paper and in particular to prove that the set $\mathscr{C}_o$ of connections whose holonomy is the whole group is a principal G-bundle.

Consider the tangent space to $\mathscr{C}$ at any point α. This can be identified with $\mathscr{C}$ itself. Given a metric on S^3, we can define an inner product on $\mathscr{C}$ by

$$(\gamma,\beta) = -\int \operatorname{Tr}(\gamma \wedge {}^*\beta).$$

This gives rise to a (weak) Riemannian metric on $\mathscr{C}$.

Let $\mathscr{G}^i$ denote sections of Sobolev class (i) of the adjoint bundle. For any $\alpha\in\mathscr{C}$ define $\partial_\alpha:\mathscr{C}^k\to\mathscr{G}^{k-1}$ by $(\beta, d_\alpha\Gamma)=(\partial_\alpha\beta,\Gamma)$ for $\beta\in\mathscr{C}^k$, $\Gamma\in\mathscr{G}^{k+1}$. Note that if $e(\alpha)$ denotes exterior multiplication by α with respect to the Lie algebra multiplication in $\mathfrak{SU}(2)$ and $i(\alpha)$ is the adjoint of $e(\alpha)$, then $d_\alpha=d+e(\alpha)$ and $\partial_\alpha=\partial+i(\alpha)$. [Note that $i(\alpha)$ is the interior multiplication by α defined using the metric and the Lie algebra multiplication in $\mathfrak{SU}(2)$]. Then $\Delta_\alpha=\partial_\alpha d_\alpha$ takes $\mathscr{G}^{k+1}$ to $\mathscr{G}^{k-1}$.

Note that for $\Gamma\in\mathscr{G}^{k+1}$, $\partial_\alpha d_\alpha\Gamma=0$ if and only if $d_\alpha\Gamma=0$ so that if $\alpha\in\mathscr{C}_o$, Δ_α is injective, by Remark 2.3.

We now prove two lemmas which we will need in the proof of the next proposition.

Lemma 3.1. *If $v\in H^k$, $k\geqq 3$, then v is a multiplier in H^m for $-k\leqq m\leqq k$.*

Proof. It is enough to show that for $u\in H^m$, $0\leqq m\leqq k$, $vu\in H^m$ and $u\to vu$ is continuous, for then we can define vT for $T\in H^{-m}$ by duality: $\langle vT,u\rangle=\langle T,vu\rangle$. For $m\geqq 2$ this follows from the fact that H^m, $m\geqq 2$ forms a Banach algebra. Since $k\geqq 3$, φ is C^1 (by Sobolev) and it is easy to show that φ is a multiplier in H^o and H^1 also.

Lemma 3.2. *Let $\alpha\in\mathscr{C}$, If $\Delta_\alpha u=0$ and $u\in\mathscr{G}^{-(k-1)}$, then $u\in\mathscr{G}^{k+1}$.*

Proof. Write $\Delta_\alpha=\partial_\alpha d_\alpha=\Delta+\mathrm{B}$ where $\Delta=\partial d$ and $B=\partial e(\alpha)+i(\alpha)d+i(\alpha)e(\alpha)$. Then if $\Delta_\alpha u=0$, $\Delta u=-Bu$. Since $u\in\mathscr{G}^{-(k-1)}$ we have by Lemma 3.1, $Bu\in\mathscr{G}^{-k}$. Since Δ has C^∞ coefficients, we have $u\in\mathscr{G}^{-k+2}$. We see by induction that $u\in\mathscr{G}^{k+1}$.

Proposition 3.3. i) *Let $\alpha\in\mathscr{C}$. Then $\Delta_\alpha:\mathscr{G}^{k+1}\to\mathscr{G}^{k-1}$ is a quasi-monomorphism, i.e., its kernel is finite-dimensional and $\Delta_\alpha(\mathscr{G}^{k+1})$ is closed in $\mathscr{G}^{k-1}$. For $u\in\mathscr{G}^{k+1}$, we have*

$$|u|_{k+1}\leqq C\{|\Delta_\alpha u|_{k-1}+|u|_o\}$$

for some constant C. Here $|u|_i$ denotes sum of the L^2-norms of partial derivatives of order i.

ii) *Let $\alpha\in\mathscr{C}_o$, so that Δ_α is injective. Then Δ_α is actually an isomorphism.*

Proof. Write, as in the proof of Lemma 3.2, $\Delta_\alpha=\Delta+B$. Since Δ has smooth coefficients, we have as is well-known,

$$\begin{aligned}|u|_{k+1}&\leqq C\{|-Bu+\Delta_\alpha u|_{k-1}+|u|_o\}\\&\leqq C\{|Bu|_{k-1}+|\Delta_\alpha u|_{k-1}+|u|_o\}\end{aligned}$$

for some constant C. Also

$$\begin{aligned}|Bu|_{k-1}&\leqq|(i)\alpha)d+\partial e(\alpha))u|_{k-1}+|i(\alpha)e(\alpha)u|_{k-1}\\&\leqq C'\{|u|_k+|u|_{k-1}\}\end{aligned}$$

for some constant C'. On the other hand

$$|u|_l\leqq\varepsilon|u|_{k+1}+C(\varepsilon)|u|_o \quad\text{for}\quad 0<l<k+1 \quad\text{for}\quad \varepsilon>0 \text{ and some function } C(\varepsilon).$$

Thus we see that, with a suitable constant C, we have

$$|u|_{k+1} \leqq C\{|\Delta_\alpha u|_{k-1} + |u|_o\}. \tag{1}$$

By Rellich lemma it follows that the kernel of Δ_α is locally compact in the L^2-norm and hence is finite dimensional.

To see that Δ_α has closed image, let $\mathscr{H}$ be the kernel and W a topological supplement of $\mathscr{H}$. We see using the above estimate and Rellich lemma that there exists a constant C'' such that

$$|u|_o \leqq C''|\Delta_\alpha u|_{k-1} \quad \text{for} \quad u \in W \tag{2}$$

(see, for example [9, p. 456]). From (1) and (2) it is clear that $\operatorname{Im}\Delta_\alpha$ is closed.

ii) By i) it suffices to show that $\Delta_\alpha(\mathscr{G}^{k+1})$ is dense in $\mathscr{G}^{k-1}$, if $\alpha \in \mathscr{C}_o$. Let therefore $T \in \mathscr{G}^{-(k-1)}$ such that T is zero on $\Delta_\alpha(\mathscr{G}^{k+1})$. Then we have $\Delta_\alpha * T = 0$, which implies by Lemma 3.2 that $*T \in \mathscr{G}^{k+1}$ so that since $\alpha \in \mathscr{C}_o$, $T = 0$.

We can now prove

Proposition 3.4. *For any $\alpha \in \mathscr{C}_o$ we have*

$$\mathscr{C} = d_\alpha(\mathscr{G}^{k+1}) \oplus (\ker \partial_\alpha)$$

where $d_\alpha(\mathscr{G}^{k+1})$ and $\ker \partial_\alpha$ are closed subspaces.

Proof. Let $G_\alpha = (\Delta_\alpha)^{-1} : \mathscr{G}^{k-1} \to \mathscr{G}^{k+1}$. G_α is continuous by Proposition 3.3, ii). Then we have $d_\alpha(\mathscr{G}^{k+1}) = \ker(1 - d_\alpha G_\alpha \partial_\alpha)$ and both spaces, being kernels of continuous operators, are closed. Since $\partial_\alpha d_\alpha \Gamma = 0$ if and only if $d_\alpha \Gamma = 0$, the sum is direct. Finally, if $\beta \in \mathscr{C}$, we have

$$\beta = d_\alpha G_\alpha \partial_\alpha \beta + (\beta - d_\alpha G_\alpha \partial_\alpha \beta)$$

with $\partial_\alpha(\beta - d_\alpha G_\alpha \beta) = 0$.

Remark 3.5. The above direct sum decomposition of $\mathscr{C}$ holds even if $\alpha \notin \mathscr{C}_o$, as can be seen by suitably defining G_α.

4. The Space of Connections as a Principal Bundle

a) The Generic Connections

Lemma 4.1. *The space $\mathscr{C}_o$ of generic connections is open in $\mathscr{C}$.*

Proof. By Remark 2.3 an element α_o of $\mathscr{C}$ belongs to $\mathscr{C}_o$ if and only if $d_{\alpha_o}: \mathscr{G}^{k+1} \to \mathscr{C}$ is injective. By Proposition 3.3 the image of d_{α_o} is also closed. Moreover, for $\alpha \in \mathscr{C}$, $\alpha \mapsto d_\alpha$ is a continuous map when we put the strong topology on the space of continuous linear maps from $\mathscr{G}^{k+1}$ to $\mathscr{C}$. From this it follows that d_α is injective in a neighbourhood of α_o.

Lemma 4.2. *For every $\alpha \in \mathscr{C}_o$ the map $\mathrm{G} \to \mathscr{C}$ given by $\varphi \mapsto \alpha \circ \varphi$ is an injective immersion.*

Proof. The differential of the map at any point β in the orbit is $d_\beta : \mathscr{G}^{k+1} \to \mathscr{C}$. By Proposition 3.4 the image is closed and admits a topological supplement, so that the lemma follows.

Proposition 4.3. *The action of* G *makes* $\mathscr{C}_o$ *a principal* G*-bundle.*

Proof. This proposition follows from Proposition 2.4 and Lemmas 4.12 and 4.2, using (6.2.3) of [10].

b) The Groups Aut_e, Aut_e^o

We now define the groups Aut_e, Aut_e^o. Aut_e is the (normal) subgroup of $^*\mathrm{Aut}$ consisting of those elements $\varphi \in {}^*\mathrm{Aut}$ which take the value identity at a fixed point e on S^3. (Note that as $k \geqq 3$, by Sobolev lemma φ is of class C^1). Let $\mathrm{Aut}_e^o = \mathrm{Aut}_e \cap {}^*\mathrm{Aut}^o$. We will let G_e denote either Aut_e or Aut_e^o.

Note that the groups Aut_e, Aut_e^o act freely on $\mathscr{C}$. The Lie algebra $\mathscr{G}_e$ consists of elements of $\mathscr{G}$ which vanish at e. As in the case of G, G_e operates properly on $\mathscr{C}$. Also

Lemma 4.4. *For* $\alpha \in \mathscr{C}$ *the map* $\mathrm{G}_e \to \mathscr{C}$ *given by* $\alpha \mapsto \alpha \circ \varphi$ *is an injective immersion.*

Proof. Note that $\mathscr{G}_e$ is of finite co-dimension in $\mathscr{G}$ and we can write $\mathscr{G} = \mathscr{G}_e \oplus F$, where F is a finite-dimensional space. The differential of the map $\mathrm{G}_e \to \mathscr{C}$ at any point β in the orbit is $d_\beta : \mathscr{G}_e \to \mathscr{C}$. This is easily seen to be injective. By Remark (3.5), $\mathscr{C} = d_\beta(\mathscr{G}^{k+1}) \oplus \ker \partial_\beta = d_\beta(\mathscr{G}_e^{k+1}) \oplus d_\beta(F) \oplus \ker \partial_\beta$.

Finally, we have

Proposition 4.5. *The action of* G_e *makes* $\mathscr{C}$ *a principal* G_e *bundle.*

Proof. Same as Proposition 4.3.

5. Nonexistence of a Continuous Gauge

Theorem 5.1. *The* Aut^o *(resp.* Aut*) bundle* $\mathscr{C}_o$ *cannot be reduced to* Aut_e^o *(resp.* Aut_e*). In particular these bundles do not admit sections.*

The rest of this section, and the next will be devoted to a proof of this theorem. But first we make the following

Remark. The Aut_e^o bundle is not trivial. In fact $\mathscr{C}$ is contractible while π_i (third loop space of $SU(2)) = \pi_{i+3}(SU(2))$. But $\pi_4(S^3) = \mathbb{Z}_2$.

If the bundle were trivial, $\mathscr{C}$ would be homeomorphic to the product of Aut_e^o and some other topological space and π_1 $(\mathscr{C})$ would be different from zero. Nor is the Aut_e bundle trivial, for $\mathscr{C}$ is connected and $\mathrm{Aut}_e/\mathrm{Aut}_e^o$ discrete.

We identify S^3 with $SU(2)$, and the point e on S^3 (used in the definition of Aut_e, Aut_e^o) with the identity in $SU(2)$.

The argument uses in a critical way, the space of left invariant forms on $SU(2)$ with values in $\mathfrak{SU}(2)$, the Lie algebra. Fix a basis of left-invariant vector fields X_a such that $[X_a, X_b] = \varepsilon_{abc} X_c$ where ε_{abc} is defined by $\varepsilon_{123} = +1$ and complete antisymmetry in the indices, and the corresponding dual basis of one-forms given by

$$\omega^a(X_b) = \delta_{ab}.$$

We also define a metric on $\mathfrak{SU}(2)$ by

$$(X_a, X_b) = \delta_{ab}.$$

A left-invariant Lie-algebra valued from ϱ can be written as

$$\varrho = \sum_a L_\alpha \omega^a$$

where L_a are elements of the Lie-algebra of $SU(2)$ (linear combinations of X_a). Of particular interest is the Maurer-Cartan form

$$\omega = \sum_a X_a \omega^a$$

which satisfies

$$d\omega + \tfrac{1}{2}[\omega, \omega] = 0.$$

A left-invariant form $\varrho = \sum_a L_a \omega^a$ gives a mapping $X_a \mapsto L_a$ and there is a one-one correspondence with 3×3 matrices M_ϱ [which represent vector space endomorphism of the Lie algebra $\mathfrak{SU}(2)$]:

$$\varrho \leftrightarrow M_\varrho \quad \text{by} \quad L_a = M_\varrho X_a.$$

The Maurer-Cartan form corresponds to the identity homomorphism: $M_\omega = \text{Identity}$.

The curvature is a Lie-algebra valued two form:

$$F = d\varrho + \tfrac{1}{2}[\varrho, \varrho].$$

On left invariant vector fields X, Y, we have

$$F(X, Y) = [M_\varrho(X), M_\varrho(Y)] - M_\varrho([X, Y])$$

so that $F = 0$ if and only if M_ϱ represents a Lie algebra homomorphism. With respect to our earlier choice of basis of left invariant vector fields, this means that $F = 0$ if and only if either $M_\varrho = 0$ or $M_\varrho \in SO(3)$ (with respect to the Lie algebra metric given earlier).

We will need the following lemmas

Lemmas 5.2. *Let N denote the space of left-invariant forms ϱ such that rank of M_ϱ is $\geqq 2$, and $M_\varrho \notin SO(3)$. Then* G *acts freely at any point in N. There are no equivalences in N under* G_e.

Proof. Let $\varrho = \sum_a L_a \omega^a$. If a gauge-transformation g fixes ϱ

$$g^{-1} F g = F.$$

By hypothesis $F \neq 0$. Consider the image of F at any point $x \in S^3$. If $\operatorname{Im} F$ is of dimension $\geqq 2$, $g(x) = \text{Identity}$; and if $\operatorname{Im} F$ is in the one-dimensional subspace $\mathfrak{h}$ of $\mathfrak{SU}(2)$, $g(x)$ is in the corresponding one-parameter subgroup H. By left invariance of F we have thus two possibilities. Either $g(x) = \pm\text{Identity}\ \forall x \in S^3$ or, $\operatorname{Im} F \subset \mathfrak{h}$ and $g(x) \in H\ \forall x \in S^3$. In the second case.

$$g^{-1} L_a g + (g^{-1} dg)_a = L_a$$

which implies $g^{-1}L_a g - L_a \in \mathfrak{h}$. But $g^{-1}L_a g - L_a$ is orthogonal to $\mathfrak{h}$, and hence zero. Thus $L_a \in \mathfrak{h}$ for each a, and $\varrho \notin N$.

Now suppose that $g \in \mathrm{G}_e$ takes $\varrho = \sum_a L_a \omega^a$ to $\varrho' = \sum L'_a \omega^a$, $\varrho' \neq \varrho$. Then since $g(e) = \text{Identity}$ we have $F = F'$ and again g takes values in a one-parameter subgroup H of $SU(2)$. We also have

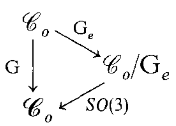
$$g^{-1}L_a g + (g^{-1}dg)_a = L_a$$

which implies $L_a - L_a = (g^{-1}dg)_a(e) \in \mathfrak{h}$

$$g(x)L'_a g^{-1}(x) - L'_a \in \mathfrak{h}$$

so that again ϱ, $\varrho' \notin N$.

Proof of Theorem 5.1. As in Lemma 5.2, let N denote the space of left invariant forms ϱ such that rank $M_\varrho \geqq 2$ and ϱ is not in the adjoint orbit of the Maurer-Cartan form. Consider the map $\eta : N \to \mathscr{C}_o/\mathrm{G}_e$ induced by the canonical map $\mathscr{C}_o \to \mathscr{C}_o/\mathrm{G}_e$. By Lemma 5.2 this map is injective. Let $N' = \eta(N)$



If the G bundle $\mathscr{C}_o$ could be reduced to the normal subgroup G_e, the $\mathrm{G}/\mathrm{G}_e = SO(3)$ fibration $\mathscr{C}_o/\mathrm{G}_e \to \mathscr{C}_o/\mathrm{G}$ would admit a (continuous) section. Note that the action of $SO(3)$ on w' and the action of $SU(2)/\mathbb{Z}_2 \approx SO(3)$ on N by $M_\varrho \to g M_\varrho g^{-1}$ commute. Hence the $SO(3)$ bundle N would be trivial. But this is not the case as will be proved in the next section (Theorem 6.2).

6. Nontriviality of the "Three-Body" Bundle

Let $M(3)$ denote the vector space of 3×3 real matrices. Consider the right action of $SO(3)$ on $M(3)$ by $(B, g) \to g^{-1}B$, $g \in SO(3)$, $B \in M(3)$.

Remark. If we identify $M(3)$ with $(\mathbb{R}^3)^3$ by means of the map $B \mapsto (Be_1, Be_2, Be_3)$ where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$, the above action goes over to the diagonal action $((f_1, f_2, f_3), g) = (g^{-1}f_1, g^{-1}f_2, g^{-1}f_3)$, $f_i \in \mathbb{R}^3$. Hence the term "Three-Body Bundle".

Lemma 6.1. *The action of $SO(3)$ on $M(3)$ is free exactly at the set of matrices of rank $\geqq 2$. The isotropy group of a matrix of rank 1 is isomorphic to $SO(2)$.*

Proof. If $g^{-1}B = B$, every point of the image of B, considered as a linear map, is fixed by g^{-1}. Therefore if rank $B \geqq 2$, $g = \text{Identity}$ and if rank $B = 1$, the isotropy group is isomorphic to the special orthogonal group of the orthogonal complement of the image of B.

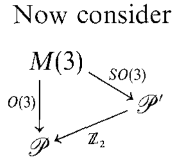
Theorem 6.2. *Let M_o denote the manifold of 3×3 real matrices of rank $k \geqq 2$. The principal $SO(3)$ bundle M_o (with the action $(B, g) \to g^{-1}B$) is not trivial on the complement of any point in $M_o/SO(3) = \boldsymbol{M}_o$.*

We first prove

Lemma 6.3. *The orbit space $M_0/SO(3)$ is homeomorphic to $\mathbb{R} \times (S^5 - P)$ where P is a submanifold of S^5 homeomorphic to the projective plane $\mathbb{P}^2(\mathbb{R})$.*

Proof. Consider first the action of $O(3)$ on $M(3)$ by multiplication on the left by g^{-1}, $g \in O(3)$. The quotient space is homeomorphic to the space of positive semidefinite matrices. This follows from remarking that if $B \in M(3)$ the non-negative square root $\sqrt{B'B}$ of $B'B$ is equivalent to B under this action. This fact is well-known for nonsingular B; if B is singular, let $B_n \to B$ with B_n nonsingular, so that $B_n = g_n \sqrt{B_n' B_n}$ with $g_n \in O(3)$. Choosing a subsequence of g_n tending to $g \in O(3)$ we see that $B = g\sqrt{B'B}$.

Now consider the diagram



where $\mathscr{P}'$ is a (ramified) two sheeted covering of $\mathscr{P}$. We claim that there is ramification precisely over the set of positive semidefinite matrices, which are not definite. This follows from the following fact: If B is a singular matrix there exists $g \in O(3)$ with $\det g = -1$ and $g^{-1}B = B$. (To see this it is sufficient to consider the case $B = P$ is singular and positive semidefinite. If $V \neq 0$ is the null space of P and $h^{-1}P = P$ ($h^{-1} \in O(3)$), h^{-1} leaves V and $V_\perp = \operatorname{Im} P$ invariant. We can multiply $h^{-1}|_V$ by a suitable constant, without changing $h^{-1}|_{V_\perp}$, to get $g \in O(3)$ with $\det g = -1$ and $g^{-1}P = P$).

Now let R denote the image of $\mathscr{P} - 0$ (0 denoting the zero matrix) in the five-dimensional projective space associated with the vector space of 3×3 symmetric matrics, R' will denote the subset of R corresponding to non-positive-definite matrices. The pair (R, R') is homeomorphic to (D^5, S^4) where D^5 denotes the closed 5-dimensional disc[1]. In fact, consider, in the space of symmetric matrices, the hyperplane S, consisting of elements B such that $\operatorname{Tr} B = 1$. Then $S \cap \mathscr{P}$ is mapped homeomorphically into R, mapping positive semidefinite matrices onto R'. It is clear that $S \cap \mathscr{P}$ is convex and compact and its interior is $S \cap \mathscr{P}_+$ where $\mathscr{P}_+$ denotes the set of positive definite matrices (Compactness is immediate since any element of $S \cap \mathscr{P}$ can be transformed by inner conjugation by $O(3)$ into a diagonal matrix $[\lambda_1, \lambda_2, \lambda_3]$ with $\lambda_i \geqq 0, \sum_i \lambda_i = 1$). It follows then, as is well known, that $(S \cap \mathscr{P}, \operatorname{bd} S \cap \mathscr{P})$ is homeomorphic to (D^5, S^4), (See, for example [11, p. 51]).

Now $(M(3) - 0)/SO(3)$ is homeomorphic to the product of $\mathbb{R}$ and the space obtained by doubling R along R'. This follows from the nature of the ramification locus of the map $\mathscr{P}' \to \mathscr{P}$. Since (R, R') is homeomorphic to (D^5, S^4) the corresponding double is homeomorphic to S^5. Hence $(M(3) - 0)/SO(3)$ is homeomorphic to $\mathbb{R} \times S^5$ (and $M(3)/SO(3)$ to $\mathbb{R}^6$).

The subspace of R' corresponding to quadratic forms of rank 1 is homeomorphic to $\mathbb{P}^2(\mathbb{R})$. In fact if Q is a (positive semidefinite) quadratic form of rank 1, Q defines a positive-definite quadratic form on the 1-dimensional space Q/(Nullity of

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[1] This fact pointed out to us by R. R. Simha, who also supplied the proof

Q) and in a one-dimensional space there is, upto a scalar multiple, a unique positive definite quadratic form Thus the above space is homeomorphic to the projective space of 1-dimensional quotient subspaces of $\mathbb{R}^3$. (A similar interpretation of quadratic forms of rank 2 gives a decomposition of $S^4 \simeq R'$ into $\mathbb{P}^2$ and a disc bundle over $\mathbb{P}^2$).

Thus $M_o = M_o/SO(3)$ is homeomorphic to $\mathbb{R} \times (S^5 - \mathbb{P}^2)$.

Proof of Theorem 6.2. We first compute some homology groups of $S^5 - \mathbb{P}^2$. We have the exact sequence

$$H^{i-1}(S^5) \to H^{i-1}(\mathbb{P}^2) \to H^i(S^5, \mathbb{P}^2) \to H^i(S^5)$$

and isomorphisms

$$H^i(S^5, \mathbb{P}^2) \simeq H^i_c(S^5 - \mathbb{P}^2) = H_{5-i}(S^5 - \mathbb{P}^2)$$

where H^i_c denotes cohomology with compact supports, the second isomorphism is given by Poincaré duality and the coefficient group is $\mathbb{Z}$. This gives, in particular.

$$H_2(S^5 - \mathbb{P}^2, \mathbb{Z}) \simeq H^2(\mathbb{P}^2, \mathbb{Z}) \simeq \mathbb{Z}/(2) \quad \text{(Alexander duality)}.$$

Now $M(3) - 0$ is homeomorphic to $\mathbb{R} \times S^8$, and the space of rank 1 matrices is homeomorphic to $\mathbb{R} \times E$ where E is a 4-dimensional subspace of S^8, so that M_o is homeomorphic to $\mathbb{R} \times (S^8 - E)$.

From the exact sequence

$$\begin{array}{c} H^{i-1}(S^8) \to H^{i-1}(E) \to H^i(S^8, E) \to H^i(S^8) \\ \wr\wr \\ H_{8-i}(S^8 - E) \end{array}$$

We see that $H_2(S^8 - E) \approx H^5(E) = 0$. Thus we have $H_2(M_o) = \mathbb{Z}/(2)$ and $H_2(M_o) = 0$. It now follows that the bundle M_o does not admit a section, for, if $\tilde{\sigma}$ were a section, the composite map $\pi_* \circ \tilde{\sigma}_*$ below would be the identity:

$$H_2(M_o) \xrightarrow[\tilde{\sigma}^*]{} H_2(M_o) \xrightarrow[\pi^*]{} H_2(M_o),$$

while $H_2(M_o) \neq 0$ and $H_2(M_o) = 0$.

Now if $p \in M_o/SO(3)$, $\pi^{-1}(p)$ is a submanifold of M_o of co-dimension 6 and p is a point in a 6-dimensional manifold. It follows, as the co-dimension is $\geqq 4$, that $H_2(M_o - p) \simeq H_2(M_o) = \mathbb{Z}_2$ and $H_2(M_o - \pi^{-1}(p)) \simeq H_2(M_o) = 0$. (See [12, p. 41]). The theorem then follows, as above.

Remarks. 1) The $SO(3)$ bundle M_o cannot be reduced to any Lie subgroup of $SO(3)$. Any (connected) Lie subgroup of $SO(3) \neq \{e\}$, is isomorphic to $SO(2)$ and if there were a reduction, the corresponding complex line bundle would have Chern class zero as $H^2(M_o) = 0$. Hence the line-bundle would be trivial – but this would imply that M_o itself is trivial.

2) A similar proof shows the following: The $SO(n)$ bundle of $n \times n$ matrices $(n \geqq 2)$ B with rank $B \geqq n-1$ is nontrivial.

3) A simpler proof of Theorem 6.2, which however does not give information about the structure of M_o, can be given as follows. For $p \in M_o$, the codimension of $M(3) - M_o - \pi^{-1}(p)$ in $M(3)$ is greater than or equal to 3. Hence

$\pi_1(M_o-\pi^{-1}(p))=\pi_1(M(3))=0$. If the bundles were trivial, $M_o-\pi^{-1}(p)$ would be isomorphic to $SO(3)\times(M_o-p)$ and since $\pi_1(SO(3)\approx\mathbb{Z}/2$, $M_o-\pi^{-1}(p)$ could not be simply connected. The proof works for all n.

7. The Coulomb Connection, Its Curvature, and Holonomy

We now define a connection on the bundle $\mathscr{C}_o$: we take the horizontal space at $\alpha\in\mathscr{C}_o$ to be the space $H_\alpha=\{\beta\in\mathscr{C}\,|\,\partial_\alpha\beta=0\}$. The horizontal space is easily seen to be invariant under the group action, as the metric is invariant.

Lemma 7.1. *The above definition of horizontal space gives a connection on $\mathscr{C}_o$. The connection form at $\alpha\in\mathscr{C}_o$ is given by $G_\alpha\partial_\alpha$ where G_α is the inverse of $\Delta_\alpha=\partial_\alpha d_\alpha:\mathscr{G}^{k+1}\to\mathscr{G}^{k-1}$.*

Proof. By Proposition 3.3 (ii), G_α is well-defined. On a vertical vector $\beta=d_\alpha\Gamma$, $\Gamma\in\mathscr{G}^{k+1}$, we have $G_\alpha\partial_\alpha d_\alpha\Gamma=\Gamma$. On horizontal vectors $G_\alpha\partial_\alpha$ is zero by definition. Since $\Delta_\alpha:\mathscr{G}^{k+1}\to\mathscr{G}^{k-1}$ is a family of isomorphisms depending smoothly on α it follows that the inverse G_α also depends smoothly on α.

We will denote by $\hat{\omega}$ the above connection form, and call this the *Coulomb connection* on $\mathscr{C}_o$.

Lemma 7.2. *Let β_1,β_2 be horizontal vectors at $\alpha\in\mathscr{C}_o$. If Ω is the curvature form corresponding to $\hat{\omega}$, we have*

$$\Omega(\beta_1,\beta_2)=G_\alpha(i(\beta_1)\beta_2-i(\beta_2)\beta_1)$$

where $i(\beta)$ denotes interior product with respect to β.

Proof. Consider $\beta_i(i=1,2)$ to be the infinitesimal generator of the one-parameter group of transformations $\alpha\mapsto t_i\beta_i+\alpha$. Then the vector fields β_i satisfy $[\beta_1,\beta_2]=0$. Then

$$\begin{aligned}\Omega(\beta_1,\beta_2)&=d\hat{\omega}(\beta_1,\beta_2)\\&=\frac{\partial}{\partial t_1}\hat{\omega}_{(t_1,0)}(\beta_2)\Big|_{t_1=0}-\frac{\partial}{\partial t_2}\hat{\omega}_{(0,t_2)}(\beta_1)\Big|_{t_2=0}\\&=\frac{\partial}{\partial t_1}(G_{\alpha+t_1\beta_1}\partial_{\alpha+t_1\beta_1}(\beta_2))\Big|_{t_1=0}-(t_1\leftrightarrow t_2)\\&=G_\alpha(i(\beta_1)\beta_2-i(\beta_2)\beta_1)\quad\text{since}\quad\frac{\partial}{\partial t_1}\partial_{\alpha+t_1\beta_1}=i(\beta_1).\end{aligned}$$

We now calculate the 'holonomy group' of $\hat{\omega}$. *From now on we use as the metric on $S^3=SU(2)$. a biinvariant metric on $SU(2)$.*

Let ω be the Maurer-Cartan form and $\omega'=\omega/2$. Note that for left-invariant vector fields X and Y, $F_{\omega'}(X,Y)=-\frac{1}{2}[X,Y]$ so that $\omega'\in\mathscr{C}_o$. Then we have

Proposition 7.3. *The linear subspace generated by elements of the form $i(\beta_1)\beta_2-i(\beta_2)\beta_1$ where β_1,β_2 are smooth horizontal vectors at ω' coincides with the space of smooth $\mathfrak{SU}(2)$-valued functions (which we will denote by $\mathscr{G}^\infty$).*

Proof. Note that $\mathcal{G}^\infty$ can be identified with the space of smooth 1-forms by $X_a \leftrightarrow \omega^a$. We shall prove that under the above identification, we can cover all smooth 1-forms.

We first construct 'enough' horizontal vectors. Note that with respect to a biinvariant metric any left-invariant form γ satisfies $\partial\gamma = 0$ [By left-invariance of metric, γ is a constant function, so that we have a linear map from the space of left-invariant forms into $\mathbb{R}$. Also, by right-invariance of metric $\partial R_g^*\gamma = R_g^*\partial\gamma = \partial\gamma$, so that this is a homomorphism of the adjoint representation of $SU(2)$ into the trivial representation. By Schur's. Lemma, $\partial\gamma = 0$]. Therefore, since $i(\omega)\omega = 0$,

$$\partial_{\omega'}\omega = 0 .$$

Also, if ζ is a closed one-form (with values in $\mathbb{R}$) then $d_{\omega'}(\zeta \wedge \omega) = d\zeta \wedge \omega + \zeta \wedge d_{\omega'}\omega = 0$ since $d_{\omega'}\omega = d\omega + \frac{1}{2}[\omega, \omega] = 0$. Therefore

$$\partial_{\omega'} * (\zeta \wedge \omega) = 0 .$$

If $\beta_i = \sum_a \beta_{ia}\omega^a (i = 1, 2)$ we have $i(\beta_1)\beta_2 - i(\beta_2)\beta_1 = \sum_a [\beta_{1a}, \beta_{2a}]$. Now

i) Let ζ be a closed 1-form. Take $\beta_1 = *(\zeta \wedge \omega)$, $\beta_2 = \omega'$. Then

$$\beta_1 = \sum_{a<b} [\zeta_a X_b - \zeta_b X_a] * (\omega^a \wedge \omega^b)$$

$$= \sum_{a<b} [\zeta_a X_b - \zeta_b X_a]\, \varepsilon_{abc}\omega^c = \sum \varepsilon_{abc}\zeta_a X_b \omega^c$$

$$i(\beta_1)\beta_2 - i(\beta_2)\beta_1 = \tfrac{1}{2}\sum \varepsilon_{abc}\zeta_a [X_b, X_c]$$

$$= \tfrac{1}{2}\sum_a \varepsilon_{abc}\varepsilon_{dbc}\zeta_a X_d$$

$$= \sum \zeta_a X_a \leftrightarrow \sum_a \zeta_a \omega^a = \zeta .$$

where $\leftrightarrow$ denotes the above-mentioned identification.

ii) Take $\beta_1 = *(\zeta_1 \wedge \omega)$, $\beta_2 = *(\zeta_2 \wedge \omega)$ with ζ_1, ζ_2 closed 1-forms. Then

$$i(\beta_1)\beta_2 - i(\beta_2)\beta_1 = \sum [\varepsilon_{abc}\zeta_{1b}X_c, \varepsilon_{ade}\zeta_{2d}X_e]$$

$$= \sum (\zeta_{1b}\zeta_{2b}[X_c, X_c] - \zeta_{1b}\zeta_{2c}[X_c, X_b]$$

$$= \sum \varepsilon_{abc}\zeta_{1b}\zeta_{2c}X_a \leftrightarrow *(\zeta_1 \wedge \zeta_2) .$$

Thus closed 1-forms are clearly covered, and also co-closed 1-forms of the type $*(\beta_1 \wedge \beta_2)$ where β_1 and β_2 are closed. The next lemma completes the proof of the proposition.

Lemma 7.4. *Any smooth co-closed 1-form β can be written as a finite sum $*\sum_p \beta_{1p} \wedge \beta_{2p}$ with β_{1p}, β_{2p} closed, smooth 1-forms.*

Proof. It is enough to show that any smooth closed 2-form η can be written as $\sum_p \beta_{1p} \wedge \beta_{2p}$ with β_{1p}, β_{2p} smooth and closed. To see this, write $\eta = d\psi$ where ψ is some smooth 1-form. By embedding S^3 in $\mathbb{R}^4$ and using, for instance, the retraction $\mathbb{R}^3 - 0 \to S^3$ it is clear that ψ can be written as

$$\psi = \sum_{p=1}^{4} \varphi_p dx_p$$

where φ_p are smooth functions on S^3. Then $\eta = \sum_{p=1}^{4} d\varphi_p \wedge dx_p$ and the lemma is proved.

Now we can prove

Theorem 7.5. *Let $\omega' = \frac{\omega}{2}$ where ω is the Maurer-Cartan form. Then $\omega' \in \mathscr{C}_o$ and the set of values of the curvature form Ω of the Coulomb connection (defined using a biinvariant metric on $SU(2)$) at ω' is dense in the gauge algebra.*

Proof. Note that $\mathscr{G}^\infty$ is dense in $\mathscr{G}^{k-1}$. Then the theorem follows from Proposition 7.3 and the fact that $G_{\omega'}$ is an isomorphism.

Note. For the purposes of the present paper, the restricted holonomy group at a point of $\mathscr{C}_o$ is defined as in the finite-dimensional case. It is a differentiably arcwise connected subgroup of G.

Lemma 7.6. *Let $\beta_1, \beta_2 \in H_\alpha$, $\alpha \in \mathscr{C}_o$. Then $\Omega(\beta_1, \beta_2)$ is the tangent vector to a curve in the restricted holonomy group at α.*

Proof. This follows from the well-known geometric interpretation of curvature (see, e.g. [13, p. 75]).

Proposition 7.7. *Let* L *be a connected Banach Lie group,* L_o *a differentiably arcwise connected subgroup of* L. *Let* $\mathscr{L}_o$ *denote the subset of* $\mathscr{L}$, *the Lie algebra of* L, *consisting of tangent vectors to (piecewise smooth) curves in* L_o *through the Identity. If* $\mathscr{L}_o$ *is dense in* $\mathscr{L}$, *then* L_o *is dense in* L.

Proof. It is easily checked that $\mathscr{L}_o$ is a subalgebra of $\mathscr{L}$. Let then $X \in \mathscr{L}_o$. We shall show that $\exp X \in \bar{\mathrm{L}}_o$. Let $\gamma(t)$ be a curve in L_o with $\gamma(0) = e$ and $\dot{\gamma}(0) = X$. For small t, $\gamma(t) = \exp Z(t)$, $Z(t) \in \mathscr{L}$. Then $\frac{Z(t)}{t} \to \dot{Z}(0) = X$. Thus

$$\begin{aligned}\exp X &= \lim_{n\to\infty} \exp\left(nZ\left(\frac{1}{n}\right)\right)\\ &= \lim_{n\to\infty}\left[\exp\left(Z\left(\frac{1}{n}\right)\right)\right]^n\\ &= \lim_{n\to\infty}\left[\gamma\left(\frac{1}{n}\right)\right]^n.\end{aligned}$$

As $\left[\gamma\left(\frac{1}{n}\right)\right]^n \in \mathrm{L}_o$, it follows that $\exp X \in \bar{\mathrm{L}}_o$. Let $\mathscr{U}$(resp. U) be a neighbourhood of 0 (resp. e) in $\mathscr{L}$ (resp. L) such that $\exp : \mathscr{U} \to \mathrm{U}$ is a diffeomorphism. Now $\mathscr{L}_o \cap \mathscr{U}$ is dense in $\mathscr{L} \cap \mathscr{U}$ and $\exp(\mathscr{L}_o \cap \mathscr{U}) = \bar{\mathrm{L}}_o \cap L$; hence $\bar{\mathrm{L}}_o \cap U = U$ so that $\mathrm{L}_o \cap U$ is dense in U. Since U generates L, the lemma follows.

Lemma 7.8. *Let P be a principal bundle with structure group* L, *with both* P, L *connected. Let there be a connection on P with holonomy group* L_o, *such that* $\bar{\mathrm{L}}_o = \mathrm{L}$. *Then if $x \in P$, the set of points in P which can be joined to x by horizontal paths is dense in P.*

Proof. Note that P/L is connected. Then the lemma follows from the fact that a dense set of points in the fibre through x can be joined to x by horizontal paths.

If we let $^*\mathscr{C}_o$ denote the connected component of $\mathscr{C}_o$ containing ω', it is clear that $^*\mathscr{C}_o$ is a principal Aut^o bundle. From Theorem 7.5, Lemma 7.6 and Proposition 7.7 it follows that a dense set of points in $^*\mathscr{C}_o$ can be connected to ω' by horizontal paths.

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