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GEOMETRY OF THE N POINT P SPACE FUNCTION OF QUANTUM FIELD THEORY *)

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A B S T R A C T

The algebra generated by time-ordered product operators is investigated from a new point of view which is applied to the study of the generalized retarded products.

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INTRODUCTION

The geometry of the n point p space function of quantum field theory was first simultaneously developed by Araki [1] and Ruelle [2], following previous but less extensive work by Polkinghorne [3] and Steinman [4]. The construction of these authors which can be found in original papers and reviews is intuitively based on a preliminary study of the one-dimensional space-time situation - conventional non-relativistic quantum mechanics - which leads to the relativistic situation studied by these authors.

A more recent approach [5] is suggested by perturbation theory studies [6] as well as experience gained in the study of local analyticity properties of the scattering amplitudes in quantum field theory [7]. One natural way to supplement the Wightman axioms [8] - or, with minor modifications which do not affect analyticity statements, to exploit the Haag-Araki [9] axioms - is to assume the existence of operator valued distributions depending on n space-time points $(x_1 \dots x_n) = X$, conventionally called time-ordered products of n field operators, whose products leave stable a dense domain in Hilbert space, which contains the vacuum vector Ω. Time-ordered products are assumed to fulfil the causal factorization property [6] :

$$T(X) = T(I)\,T(I')$$

if $I \gtrsim I'$ where $I' = X \setminus I$ and $I \gtrsim I' = (x \mid x_i - x_j \in \bar{V}^- \ \forall i \in I,\ j \in I')$. It is the latter condition that undergoes a minor modification in the Haag-Araki formulation of Q.F.T. One can thus naturally introduce the partially retarded operator [7]

$$\mathcal{R}_I(X) = T(X) - T(I')\,T(I)$$

with support $(x \mid x_i - x_j \in \bar{V}^+$ some $i \in I,\ j \in I')$. Let now

$$(\Omega, \mathcal{R}_I(X)\Omega) = r_I(X) \,; \quad (\Omega, T(X)\Omega) = t(X)$$

Applying spectrum properties to Fourier transforms, one obtains

$$\tilde{r}_I(P) = \tilde{t}(P) \quad \text{if } p_I \notin S_I^-$$

where $S_I^- = \{0\} \cup \bar{V}_I^-$, $\bar{V}_I^-$ being the closed convex hull of the lower sheet of a hyperboloid of mass M_I depending on selection rules and spectrum.

These properties can be extended to more general expressions which we shall now construct.

1. - THE ALGEBRA GENERATED BY TIME-ORDERED PRODUCTS

One considers the vector space $\mathcal{T}$ spanned by ordered monomials

$$\prod_{k=1}^{k=\nu} T(I_k) \equiv T(I_1)\dots T(I_\nu)\,,\quad I_1 \cup \dots \cup I_\nu = X\,,\quad I_i \cap I_j = \phi\,,\ i,j = 1\dots\nu\,,$$

whose generic element will be denoted Θ. If $I \subseteq X$, one defines $\hat{I}$ by linear extension of

$$\hat{I} \prod_{k=1}^{k=\nu} T(I_k) = \prod_{k=1}^{k=\nu} T(I\cap I_k)\, T(I'\cap I_k)$$

with the convention $\hat{X} = \hat{\emptyset} = 1$, $T(\emptyset) = 1$. One has the following properties :

(i) $\hat{I}^2 = \hat{I}$

(ii) $\hat{I}_1 \hat{I}_2 = \hat{I}_2 \hat{I}_1 \iff I_1 \subseteq I_2$ or $I_2 \subseteq I_1$

$$\left(I_1' \cap I_2 = I_1 \cap I_2' \rightarrow I_1' \cap I_2 = I_1 \cap I_2' \cap I_2 = \phi \right)$$

(iii) $$\left(1 - \widehat{I_1 \cup I_2}\right) \prod_{r=1}^{r=\nu} \hat{K}_r \hat{I}_1 \hat{I}_2 = \prod_{r=1}^{r=\nu} \hat{K}_r \hat{I}_1 \hat{I}_2 \left(1 - \widehat{I_1 \cup I_2}\right)$$

$$= \prod_{r=1}^{r=\nu} \hat{K}_r \hat{I}_1 \left(1 - \widehat{I_1 \cup I_2}\right) \hat{I}_2 = 0$$

(iv) same as (iii) with $\cup \rightarrow \cap$

(v) $$\hat{I}'\hat{I} = \hat{I}$$
$$\hat{I}\,\hat{I}' = \hat{I}'$$

(vi) T(X) is cyclic :

$$T(K_1)\dots T(K_\nu) = \hat{I}_1 \dots \hat{I}_\nu \; T(X)$$
$$I_r = \bigcup_{s=1}^{s=r} K_s$$

This allows to define a partial order on $\mathcal{T}$:

$$T(K_1)\dots T(K_\nu) > T(L_1)\dots T(L_\mu)$$

if the partition $(L_1 \dots L_\mu)$ refines the partition $(K_1 \dots K_\nu)$.

(vii) If $I_1 \cap I_2 = \phi$, $\hat{I}_1\,\hat{I}_2 = \widehat{I_1 \cup I_2}\;\hat{I}_2 = \hat{I}_2\;\widehat{I_1 \cup I_2}$

(viii) If $I_1 \cup I_2 = X$, $\hat{I}_1\,\hat{I}_2 = \widehat{I_1 \cap I_2}\;\hat{I}_2 = \hat{I}_2\;\widehat{I_1 \cap I_2}$

(ix) Eigenstates of $\hat{I}$:

$$\hat{I}\,\Theta = \Theta \overset{Def.}{\Longleftrightarrow} \Theta \in \mathcal{N}(I).$$

vectors of the form

$$\Theta = \prod_{k=1}^{k=\nu} T(I_k)$$

with $I_k \subset I$ or $I_k \subset I'$ belong to $\mathcal{N}(I)$.

Conversely, all elements of $\mathcal{N}(I)$ are of this form :

If $$\hat{I}\prod_{k=1}^{k=\nu} T(I_k) \neq \prod_{k=1}^{k=\nu} T(I_k) \in \mathcal{D}(I),$$
$$(\mathbb{1}-\hat{I})\prod_{k=1}^{k=\nu} T(I_k) \neq 0, \quad \hat{I}(\mathbb{1}-\hat{I})\prod_{k=1}^{k=\nu} T(I_k) = 0.$$

In other words, either a monomial is non-decomposable [in $\mathcal{N}(I)$], and is an eigenstate of $\hat{I}$ with eigenvalue 1, or it is decomposable [in $\mathcal{D}(I)$] and there corresponds to it an eigenstate of $\hat{I}$ with eigenvalue 0 - such eigenstates of $\hat{I}$ are thus put in a one-to-one correspondence with all partitions of X and thus span the whole of $\mathcal{C}$.

2. - PRECELLS [5),10)]

We now look for generalizations of the r_I's, with support properties in x space, and with coincidence properties in p space with $\tilde{t}(P)$.

Def [5)]

A precell $\mathcal{S}$ is a set of proper parts of X ($\neq \emptyset$, X) such that

1) If $I \in \mathcal{S}$, $I' \notin \mathcal{S}$,
   Let $\mathcal{S}' = \{ I' \mid I \in \mathcal{S} \}$.
   If $I \notin \mathcal{S}$, then $I' \in \mathcal{S}$.

2) If $I_1 \in \mathcal{S}$, $I_2 \in \mathcal{S}$, $I_1 \cap I_2 = \emptyset$
   then $I_1 \cup I_2 \in \mathcal{S}$.

3) If $I_1 \in \mathcal{S}$, $I_2 \in \mathcal{S}$, $I_1 \cup I_2 = X$
   then $I_1 \cap I_2 \in \mathcal{S}$

   [a consequence of 1) and 2) applied to $\mathcal{S}'$]

Def

$$\Theta_{\mathcal{S}'} = \prod_{I \in \mathcal{S}} (1 - \hat{I}) \, \Theta$$

$$\theta_{\mathcal{S}'} = (\Omega, \Theta_{\mathcal{S}'} \Omega)$$

Precell Lemma [5] (elementary)

If $\mathcal{I}_1 \in \mathcal{S}$, $\mathcal{I}_2 \in \mathcal{S}$

Then $\mathcal{I}_1 \cap \mathcal{I}_2 \in \mathcal{S}$ or $\mathcal{I}_1 \cup \mathcal{I}_2 \in \mathcal{S}$

Def (paracell) [10]

A paracell is a set $\mathcal{S}$ of proper parts of X for which the precell lemma holds.

Th

If $\mathcal{S}$ is a paracell, the product

$$\prod_{I \in \mathcal{S}} (\mathbb{1} - \hat{I})$$

is independent of the order of the factors.

Sketch of proof

Let $I_1 \ldots I_N$ be an arbitrary ordering of $\mathcal{S}$, we have to show that for any permutation π

$$(\mathbb{1} - \hat{I}_1) \ldots (\mathbb{1} - \hat{I}_N) = (\mathbb{1} - \hat{I}_{\pi(1)}) \ldots (\mathbb{1} - \hat{I}_{\pi(N)})$$

Suppose π is the transposition $N \rightleftarrows N-1$. The property is true by (ii), (iii), (iv). Suppose the property has been proved for all π 's such that

$$\pi(1) = 1 \ , \ \pi(2) = 2, \ \ldots \ \pi(N-p) = N-p \, ,$$

The passage from p to p+1 follows easily from the induction hypothesis and application if (ii), (iii), (iv). In fact the proof shows the following.

Lemma

Let $A_1 \ldots A_N$ be a sequence of not necessarily distinct proper subsets of X, such that for every pair i, j, there exists k such that $A_k = A_i \cup A_j$ or $A_k = A_i \cap A_j$, then for every permutation π of $1 \ldots N$

$$(\mathbb{1} - \hat{A}_1) \ldots (\mathbb{1} - \hat{A}_N) = (\mathbb{1} - \hat{A}_{\pi(1)}) \ldots (\mathbb{1} - \hat{A}_{\pi(N)}) \, .$$

- Support properties

Lemma

$$(\mathbb{1}-\hat{I})\,\Theta_{\mathcal{S}'} = 0 \quad \text{if} \quad I' \gtrsim I \;,\; I \in \mathcal{S}$$

hence support $\Theta_{\mathcal{S}'} \subset C_I = \{x \,/\, x_i - x_j \in \overline{V}_+ \text{ , some } i \in I, j \in I'\}$

$$= \bigcup_{\substack{i\in I\\ j\in I'}} (x \,/\, x_i - x_j \in \overline{V}^+) = \bigcup_{\substack{i\in I\\ j\in I'}} C_{ij}$$

(use causal factorization property).

- Corollary

$$\Theta_{\mathcal{S}'} = \prod_{I\in\mathcal{S}} (\mathbb{1}-\hat{I})\,\Theta$$

has support contained in

$$\bigcap_{\substack{I\in\mathcal{S}\\ I'\in\mathcal{S}'}} \bigcup_{\substack{i\in I\\ j\in I'}} C_{ij}$$

This support can be analyzed as follows. A choice [5),7)] h is a map I → X such that $h(I) \in I$. Let

$$C_h^{\mathcal{S}} = \bigcap_{I\in\mathcal{S}} C_{h(I)\,h(I')}$$

With this notation

$$\bigcap_{\substack{I\in\mathcal{S}\\ I'\in\mathcal{S}'}} \bigcup_{\substack{i\in I\\ j\in I'}} C_{ij} = \bigcup_h C_h^{\mathcal{S}}$$

Choices h which yield the support can be restricted to choices compatible [5] with $\mathcal{S}$ as follows. Let

$$\mathcal{S}^+ = \{ j \in X , \{j\} \in \mathcal{S} \}$$
$$\mathcal{S}^- = X \setminus \mathcal{S}^+ = \mathcal{S}'^+$$

One can show [5] that for each choice h, there exists a "compatible choice" h' such that

$$h'(I) \in \mathcal{S}^+ \quad \text{if} \quad I \in \mathcal{S}$$

$$h'(I) \in \mathcal{S}^- \quad \text{if} \quad I \in \mathcal{S}'$$

and

$$C_h^{\mathcal{S}} \subset C_{h'}^{\mathcal{S}}$$

from which there follows

$$\text{supp.}\, \textcircled{H}_{\mathcal{S}'} \subset \bigcup_{h \text{ compatible}} C_h^{\mathcal{S}}$$

One can furthermore show [5] that each $C_h^{\mathcal{S}}$ is of the form

$$\bigcap_{\substack{i \in \mathcal{S}^+ \\ j \in \mathcal{S}^- \\ (ij) \in \mathcal{T}_h}} \{ x \mid x_i - x_j \in \bar{V}^+ \}$$

where $\mathcal{T}_h$ is a tree graph with vertices $i \in X$ and links (ij), $i \in \mathcal{S}^+$, $j \in \mathcal{S}^-$. Such trees were first discovered by Bros [11].

- <u>Analyticity properties</u>

It is easy to see that the Fourier transform of a translation invariant distribution with support in C_h can be extended into a holomorphic function defined on the hyperplane

$$p_X \equiv \sum_{i=1}^{n} p_i = 0$$

and analytic in the tube with imaginary basis $\tilde{C}_h$ dual to $C_{h'}$, given by the parametric form [1),9]

$$p = \sum_{(ij) \in \mathcal{T}_h} s_{(ij)} \, t_{(ij)}$$

where $\mu_{(ij)} \in V^+, \quad S_{(ij)} = (S^1_{(ij)}, \dots S^n_{(ij)}) ,$

$$S^k_{(ij)} = \delta^k_i - \delta^k_j$$

This parametric form is due to Araki [1),9)] and has the following nice property : if a translation invariant distribution has support $C^{\mathcal{S}}_h \cap C^{\mathcal{S}}_{h'}$, its Fourier transform can be extended into the tube with imaginary basis

$$p = \sum S_\lambda \mu_\lambda$$

where $\mu_\lambda \in V^+$ and S_λ are the co-ordinates of exposed one-dimensional facets of the convex hull

$$S = t \sum_{(ij)\in C_h} S_{(ij)} \rho_{(ij)} + (1-t) \sum_{(ij)\in C_{h'}} S'_{(ij)} \rho'_{(ij)}$$

$$0 \leq t \leq 1 , \quad 0 \leq \rho_{(ij)} , \quad 0 \leq \rho'_{(ij)} .$$

Alternatively, the tube corresponding to C_h can be described by the set of $n-1$ conditions of the type $\mathrm{Im}\, p_{I(ij)} \in V^+$, $I_{(ij)} \in \mathcal{S}$, obtained by writing

$$\sum_{i=1}^{i=n} p_i x_i = \sum_{(ij)\in \mathcal{C}_h} p_{I(ij)} (x_i - x_j) \quad \mathrm{mod} \sum_{i=1}^{i=n} p_i .$$

A description of the latter kind does not exist for convex hulls, in general.

3. - SPECTRUM PROPERTIES OF $T_{\mathcal{S}}$

We look at Fourier transforms of vacuum expectation values, $\tilde{t}_{\mathcal{S}}(P)$ obtained for

$$\Theta = T(X) \ .$$

Expanding the product

$$T_{\mathcal{S}}(X) = \prod_{I \in \mathcal{S}'} (\mathbb{1} - \hat{I}) \, T(X)$$

and looking at the last factor of each term, we see that

$$\tilde{t}_{\mathcal{S}}(P) = \tilde{t}(P) \qquad \text{if } p_I \notin S_I^- \quad \forall\, I \in \mathcal{S}$$

4. - DISCONTINUITY FORMULAE

Def

If $\mathcal{S}$ is a precell, I is called a boundary of $\mathcal{S}$ ($I \in \partial\mathcal{S}$) if

$$(\mathcal{S} \setminus \{I\}) \cup \{I'\}$$

is a precell.

Boundary Lemma [5)]

I is a boundary of $\mathcal{S}$ if

(i) $I_1 \cup I_2 = I \ , \ I_1 \cap I_2 = \phi \quad I_1 \neq I, \ I_1 \in \mathcal{S}$

$\rightarrow I_2 \in \mathcal{S}'$

(ii) $I_1' \cup I_2' = I', \ I_1' \cap I_2' = \phi \ , \ I_1' \neq I' \ , \ I_1' \in \mathcal{S}'$

$\rightarrow I_2' \in \mathcal{S}$

(i) and (ii) are equivalent to (i) and :

(iii) $I_1 \cap I_2 = I$, $I_1 \cup I_2 = X$, $I_1 \neq I$, $I_1 \in \mathcal{S}$

$\rightarrow I_2 \in \mathcal{S}'$

If I is a boundary of $\mathcal{S}$ we shall define

$$\mathcal{S}_I^+ = \mathcal{S} \quad , \quad \mathcal{S}_I^- = (\mathcal{S} \setminus \{I\}) \cup \{I'\} .$$
$$\mathcal{S}_I = (\mathcal{S} \setminus \{I\})$$

Using the commutativity of $\prod_{I\in\mathcal{S}}$ $(1-\hat{I})$ and leaving I as the last factor, one easily gets

$$\prod_{J\in\mathcal{S}_I^+} (1 - \hat{J}) - \prod_{J\in\mathcal{S}_I^-} (1 - \hat{J}) = \prod_{J\in\mathcal{S}_I} (1 - \hat{J}) [\hat{I}' - \hat{I}]$$

Hence the commutator formula

$$T_{\mathcal{S}_I^{+\prime}} - T_{\mathcal{S}_I^{-\prime}} = \prod_{J\in\mathcal{S}_I} (1 - \hat{J}) [T(I'), T(I)]_-$$

Write now

$$\hat{J} = \hat{J}_{I'} . \hat{J}_I = \widehat{J \cap I'} . \widehat{J \cap I}$$

as a direct product which allows to write, if $K \subset I$, $K' \subset I'$

$$\hat{J}\, T(K)\, T(K') = \hat{J}_{I'} . \hat{J}_I\, T(K)\, T(K') = \hat{J}_I\, T(K) . \hat{J}_{I'}\, T(K')$$

where

$$\hat{J}_I\, T(K) = T(K \cap J \cap I)\, T(K \cap J' \cap I)$$

$$\hat{J}_{I'}\, T(K') = T(K' \cap J \cap I')\, T(K' \cap J' \cap I')$$

Let now

$$\Sigma_I = \{ K \mid K \subset I , \quad K \in \mathcal{S}_I \}$$

$$\Sigma_{I'} = \{ K \mid K \subset I' \quad K \in \mathcal{S}_I \}$$

These are precells from the precell property of $\mathcal{S}$ and the boundary property of I.

Now if $J \in \mathcal{S}_I$, either $J \cap I$ is in Σ_I or $J \cap I'$ is in $\Sigma_{I'}$ (or both) because if it were not so, both $J \cap I$ and $J \cap I'$ would be in $\mathcal{S}'$ and since they are disjoint, their union, J, would be in $\mathcal{S}'$ which is not true. Hence J is of one of the following types :

(i) $K \cup K'$, $K \in \Sigma_I$, $K' \in \Sigma_{I'}$.

(ii) $K \cup \emptyset$, $K \in \Sigma_I$.

(iii) $\emptyset \cup K'$, $K' \in \Sigma_{I'}$.

(iv) $K \cup I'$, $K \in \Sigma_I$.

(v) $I \cup K'$, $K' \in \Sigma_{I'}$.

(vi) $K \cup L'$, $K \in \Sigma_I$, $L' \subset I'$, $(L' \neq I', \emptyset)$.

(vii) $L \cup K'$, $L \subset I$, $(L \neq I, \emptyset)$, $K' \in \Sigma_{I'}$.

Using again the precell property of $\mathcal{S}$, the commutativity of $\prod_{J \in \mathcal{S}_I} (1-\hat{J})$ and the idempotency identities

$$(1 - \hat{K})^2 = (1 - \hat{K}) \quad , \quad (1 - \hat{K}')^2 = 1 - \hat{K}'$$

we can write

$$\prod_{J \in \mathcal{S}_I} (1 - \hat{J}) = \prod_{\substack{K \in \Sigma_I \\ K' \in \Sigma_{I'}}} (1 - \hat{K}' \hat{K})(1 - \hat{K})(1 - \hat{K}') \ldots$$

$$\ldots \quad \cdot \prod_{\substack{K' \in \Sigma_{I'} \\ L \subset I}} (\mathbb{1} - \hat{K}'.\hat{L})(\mathbb{1} - \hat{K}') \cdot \prod_{\substack{L' \subset I' \\ K \in \Sigma_I}} (\mathbb{1} - \hat{L}'.\hat{K})(\mathbb{1} - \hat{K})$$

$$= \prod_{\substack{K \in \Sigma_I \\ K' \in \Sigma_{I'}}} (\mathbb{1} - \hat{K}')(\mathbb{1} - \hat{K})$$

where we have used

$$(\mathbb{1} - \hat{K}'.\hat{K})(\mathbb{1} - \hat{K})(\mathbb{1} - \hat{K}') = (\mathbb{1} - \hat{K}').(\mathbb{1} - \hat{K})$$

$$(\mathbb{1} - \hat{K}'.\hat{L})(\mathbb{1} - \hat{K}') = (\mathbb{1} - \hat{K}')$$

$$(\mathbb{1} - \hat{L}'.\hat{K})(\mathbb{1} - \hat{K}) = (\mathbb{1} - \hat{K})$$

on account of the idempotency

$$\hat{K}^2 = \hat{K} \quad , \quad \hat{K}'^2 = \hat{K}'$$

The commutativity of the factors $(1-\hat{K})$, $(1-\hat{K}')$ allows to split these factors :

$$T_{\mathcal{S}_I^{+'}} - T_{\mathcal{S}_I^{-'}} = \prod_{\substack{K \in \Sigma_I \\ K' \in \Sigma_{I'}}} (\mathbb{1} - \hat{K})(\mathbb{1} - \hat{K}') \left[T(I'), T(I) \right]$$

$$= \left[\prod_{K' \in \Sigma_{I'}} (\mathbb{1} - \hat{K}') T(I') , \prod_{K \in \Sigma_I} (\mathbb{1} - \hat{K}) T(I) \right]$$

$$= \left[T_{\Sigma'_{I'}}(I') \quad , \quad T_{\Sigma'_I}(I) \right]$$

This is the so-called Ruelle discontinuity formula [2),5)].

5. - STEINMANN's IDENTITIES [4),1),2)]

Let I, J such that $I \not\subset J$, $I \not\subset J'$, $J \not\subset I$, $J \not\subset I$, hence $I' \not\subset J$, $I' \not\subset J'$, $J \not\subset I'$, $J' \not\subset I'$. Consider four cells admitting I, J, as boundaries :

$$\mathcal{S}_{IJ}^{++} = \mathcal{S}_{IJ} \cup I \cup J = \mathcal{S}_{I} \cup J = \mathcal{S}_{J} \cup I$$

$$\mathcal{S}_{IJ}^{+-} = \mathcal{S}_{IJ} \cup I \cup J' = \mathcal{S}_{I} \cup J' = \mathcal{S}_{J'} \cup I$$

$$\mathcal{S}_{IJ}^{-+} = \mathcal{S}_{IJ} \cup I' \cup J = \mathcal{S}_{I'} \cup J = \mathcal{S}_{J} \cup I'$$

$$\mathcal{S}_{IJ}^{--} = \mathcal{S}_{IJ} \cup I' \cup J' = \mathcal{S}_{I'} \cup J' = \mathcal{S}_{J'} \cup I'$$

Then

$$T_{\mathcal{S}_{IJ}^{++\prime}} - T_{\mathcal{S}_{IJ}^{-+\prime}} = \left[T_{\Sigma_I^{+\prime}}(I),\ T_{\Sigma_{I'}^{+\prime}}(I') \right]$$

where

$$\Sigma_I^{+} = \{ K \mid K \subset I \ , \ K \in \mathcal{S}_{IJ} \}$$
$$\text{since } J \not\subset I$$

$$\Sigma_{I'}^{+} = \{ K \mid K \subset I' \ , \ K \in \mathcal{S}_{IJ} \}$$
$$\text{since } J \not\subset I'$$

Similarly

$$T_{\mathcal{S}_{IJ}^{+-\prime}} - T_{\mathcal{S}_{IJ}^{--\prime}} = \left[T_{\Sigma_I^{-\prime}}(I),\ T_{\Sigma_{I'}^{-\prime}}(I') \right]$$

where

$$\Sigma_I^{-} = \{ K \mid K \subset I \ , \ K \in \mathcal{S}_{IJ} \}$$
$$\text{since } J' \not\subset I$$

$$\Sigma_{I'}^{-} = \{ K \mid K \subset I' , \ K \in \mathcal{S}_{IJ} \}$$
$$\text{since } J' \not\subset I'$$

Hence

$$\Sigma_I^+ = \Sigma_I^-$$
$$\Sigma_{I'}^+ = \Sigma_{I'}^-$$

and

$$T_{\mathcal{S}_{IJ}^{++'}} - T_{\mathcal{S}_{IJ}^{+-'}} - T_{\mathcal{S}_{IJ}^{-+'}} + T_{\mathcal{S}_{IJ}^{--'}} = 0$$

The so-called Steinmann-Ruelle relations [1),2)].

6. - CELLS [1),2)]

The interpretation of $\tilde{t}(P)$ as a piecewise boundary value of analytic functions goes via the consideration of cells.

Def [5)]

A precell $\mathcal{S}$ is a cell if, in R^n, on the hyperplane $S_X = \sum_i^n S_i = 0$, the set of conditions

$$S_I = \sum_{i \in I} S_i > 0 \qquad \forall\, I \in \mathcal{S}$$

is non empty. This set will be called $\Gamma_{\mathcal{S}}$. Clearly, if $S \in \Gamma_{\mathcal{S}}$ $S_I < 0$ if $I \in \mathcal{S}'$.

One can easily see that Steinmann-Ruelle relations can only connect together cells whose Γ's are within a hypercell Γ_I : $S_i > 0$ $i \in I$, $S_i < 0$ $i \in I'$, i.e., which all contain i, $i \in I$ and $[i]'$, $i \in I'$. The one dimensional boundaries of Γ_I are the vectors S_{ij}, $i \in I$, $j \in I'$, $S_{ij}^k = \delta_i^k - \delta_j^k$, we already met. One can prove that for a cell $\mathcal{S}$, the admissible $\tilde{C}_h$ are such that Γ_h :

$$S = \sum_{(ij) \in \mathcal{C}_h} S_{(ij)}\, \rho_{(ij)} \quad , \quad \rho_{(ij)} > 0$$

contain $\Gamma_{\mathcal{S}}$.

A more detailed result (Bros [11]) shows that for all $\mathcal{J}$'s such that $\Gamma_{\mathcal{J}} \subset \Gamma_I$ there is a decomposition of the form

$$\tilde{t}_{\mathcal{J}} = \sum_h \Theta_{\mathcal{J},h} \, \tilde{f}_h$$

where the $\tilde{f}_h$ are holomorphic in the tube with imaginary basis $\tilde{C}_h$ (the dual of C_h), where

$$\Theta_{\mathcal{J},h} = 1 \quad \text{if} \quad \Gamma_h \supset \Gamma_{\mathcal{J}}$$

$$\Theta_{\mathcal{J},h} = 0 \quad \text{if} \quad \Gamma_h \not\supset \Gamma_{\mathcal{J}}$$

The foregoing results are best summarized within the framework of the following geometrical construction.

Consider an n dimensional space $\{ S = (S_1, \ldots, S_n) \}$ and the hyperplane

$$\sum_{i=1}^{i=n} S_i = 0$$

One draws all the hyperplanes

$$S_I \equiv \sum_{i \in I} S_i = 0 \quad \left(= S_{I'} \right)$$

which bound geometrical cells $\Gamma_{\mathcal{J}}$ within which all S_I's have a prescribed sign. One can easily read off this diagram the various relevant S_{ij}'s in the preceding formulae.

Examples are shown on Figs. 1, 2, 3.

7. - TRUNCATION

One has still to make a slight modification in order to make the foregoing construction useful. The regions in p space where the $\tilde{t}$ (P) coincide with $\tilde{t}(P)$ do not cover the whole of P space because of the occurrence of the vacuum contribution $\{0\}$ to the spectrum S_I^+. One can however, define connected distributions [6] $\tilde{t}_c(P)$, $\tilde{t}_{\mathcal{S},c}(P)$ [which turn out to be identical to $\tilde{t}_{\mathcal{S}}(P)$] such that all previously established properties hold. The regions where $\tilde{t}_c(P)$ coincide with the $\tilde{t}_{\mathcal{S}}(P)$'s cover all of P space when $\mathcal{S}$ goes over all possible cells, S_I^+ being then reduced to $\bar{V}_I^+$. Although this is quite important, we shall not give any more detail here.

The bridge can be made with the construction of Haraki and Ruelle by establishing the following identity [5),12)]

$$T_{\mathcal{S}} = \sum_{\nu} (-)^{\nu+1} \sum_{\substack{I_1 \cup \ldots \cup I_\nu = X \\ I_j \cap I_k = \phi \\ K_r = I_1 \cup \ldots \cup I_r \in \mathcal{S} \\ r = 1, \ldots, \nu-1}} T(I_1) \ldots T(I_\nu)$$

from which one easily recovers the spectrum properties, the discontinuity formula and the Steinmann identities, but from which support properties are hard to get.

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12) The first step in deriving this formula from the definition of Refs. 1), 2), was taken by C. Itzykson (1963), unpublished.

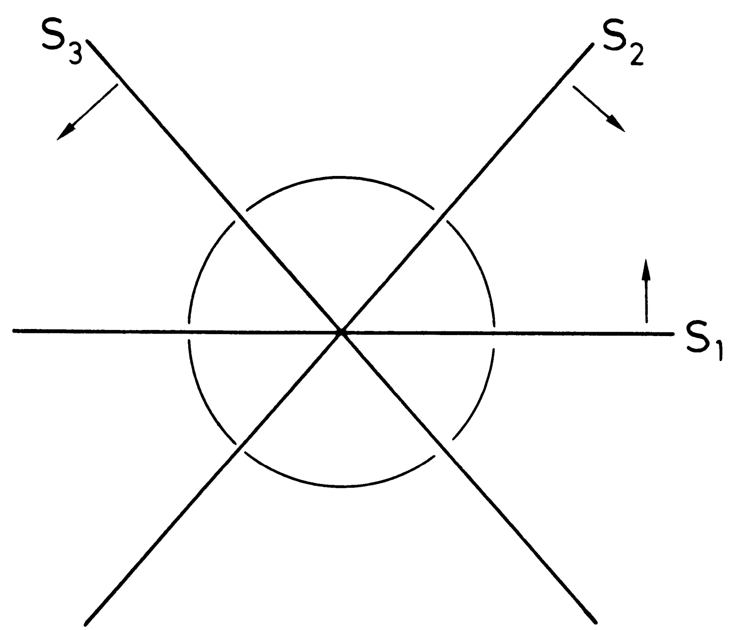


FIG. 1
$n = 3$

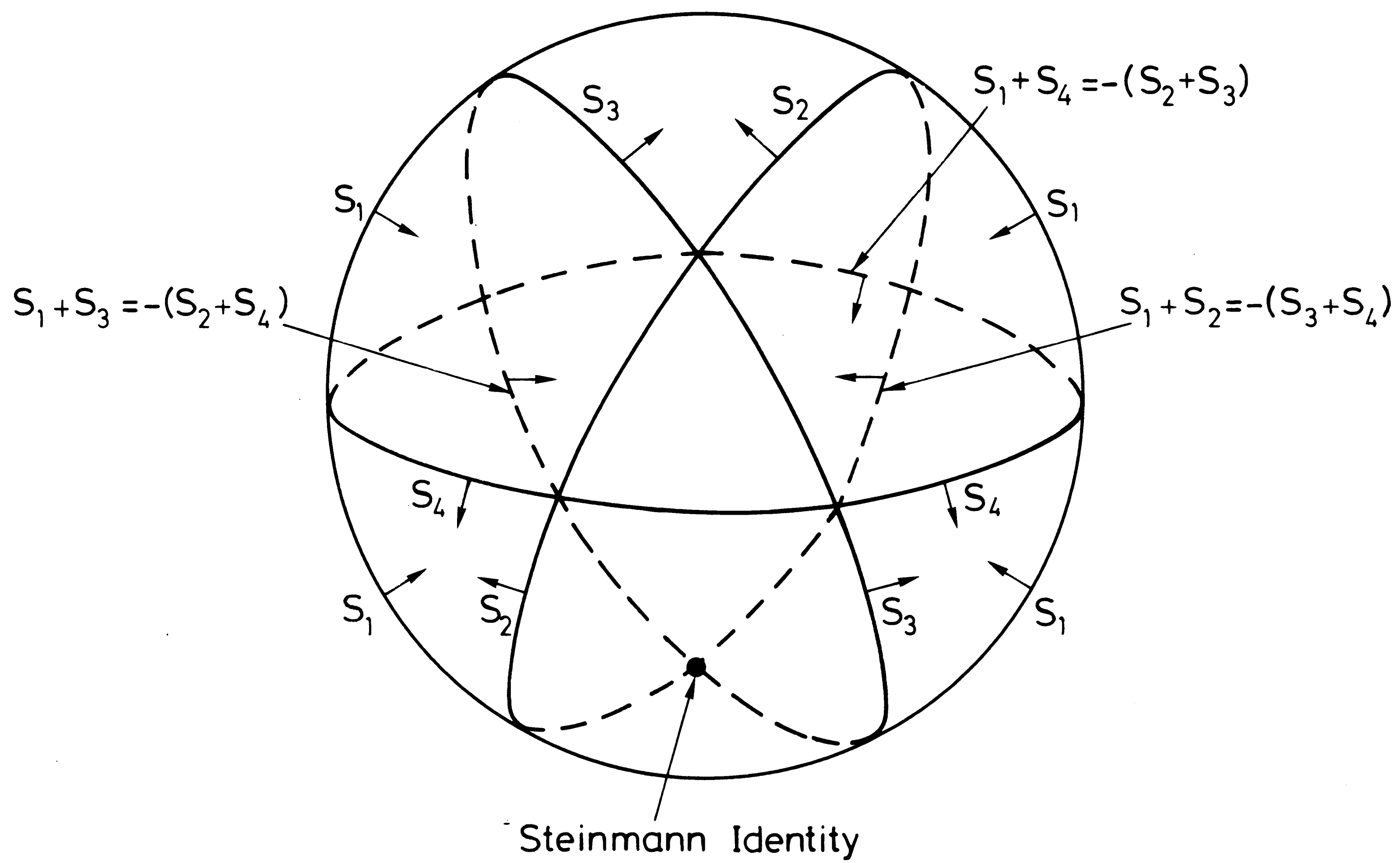
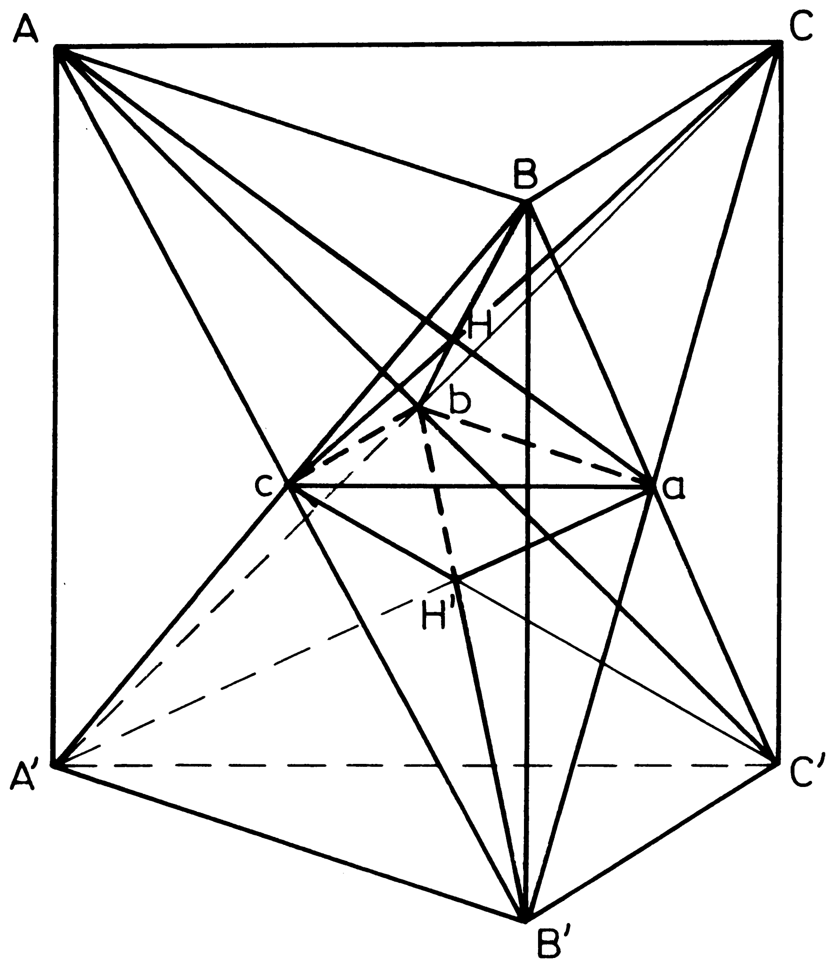


FIG. 2

$n = 4$



A hypercell ABCA'B'C'

Typical cells : AA'bc (3)

AbcH (6)

A'B'cH'(6)

abcHH'(1)

HABC (2)

Typical Bros Trees : AA'B'C'(6)

A'BB'C (6)

FIG. 3

n=5