THE GEOMETRY OF THE TENSOR PRODUCT OF C^*-ALGEBRAS

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ABSTRACT.

This work is concerned with the relationship between two concepts: the geometry of operator algebras, and their tensor products. First, Hermitian elements of a Banach algebra, and the special geometry of a C^*-algebra are discussed. The extremal Banach algebra generated by a Hermitian element is examined.

Some norms related to the matricial structure available in C^*-algebra are considered, and their relationships studied. The *symmetrized Haagerup norm* is defined, which corresponds to a variant of the notion of complete boundedness and a Christensen-Sinclair type representation theorem. A categorical definition of a tensor product of C^*-algebras is proposed, and an analysis of the geometry of such tensor products provides a complete description of the Hermitian elements and a characterization of the C^*-tensor norms.

Next the notion of a *tracially completely bounded* multilinear map is introduced, and the associated tensor norm is shown to be equivalent to the projective norm. Bounds are given for the relevant constants.

Finally non-self-adjoint operator algebras are considered. The projective and Haagerup tensor products of two C^*-algebras are shown not to be operator algebras. The problem of characterizing operator algebras up to complete isometry is considered. Examples are studied and necessary and sufficient conditions given.

As an appendix a criterion for the existence of invariant subspaces for an operator related to the Bishop operator is given.

To Michelle.

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INTRODUCTION.

The origins of the theory of tensor products of Banach spaces are to be found in work of von Neumann, Schatten [**Sc**] and Grothendieck [**Gr2**]. The tensor product is a fundamental construction in the category of Banach spaces and bounded maps; but like the direct sum or quotient constructions the tensor product is not merely a formal device: the geometry of Banach spaces and their tensor products are intimately related.

Naturally the relationship between two norms on the algebraic tensor product of two Banach spaces mirrors geometrical information about the spaces concerned. Classes of multilinear maps on Banach spaces are in duality with the tensor products of these spaces, thus to study a particular class of maps it is often useful and enlightening to consider the associated tensor product. Tensor products seem to be the correct framework to study factorization [**GL**], a concept central to the geometry of Banach spaces. There is also the interesting work of Varopoulos, Carne [**Va3**,**Ca3**] and others characterizing operator algebras in terms of tensor products.

The *injective norm* λ and *projective norm* γ, respectively the 'least' and 'greatest' tensor norms, have received the most attention. These norms have important applications in many fields, for example in harmonic analysis [**Va1**].

The theory of tensor products of C^*-algebras began in 1952 [**Tu**]. Since then it has been concerned with the case when the tensor product is again a C^*-algebra. Analysts were distressed to discover that there could exist more than one C^*-norm on the algebraic tensor product of two C^*-algebras. Naturally attention was drawn to the *nuclear* C^*-algebras: those C^*-algebras $\mathcal{A}$ for which there exists a unique C^*-norm on $\mathcal{A} \otimes \mathcal{B}$ for all C^*-algebras $\mathcal{B}$. Such C^*-algebras have been the subject of much research, and are now fairly well understood. The property of nuclearity plays a similar role to that of the approximation property in the metric theory of tensor products of Banach spaces, and has been found to be equivalent to a number of important spatial and geometric notions [**La3**,**To**].

One advantage C^*-algebras have over Banach spaces is the fact that a matrix of operators may be regarded as another operator in a canonical fashion. If $\mathcal{A}$ is a C^*-algebra, then the set $\mathcal{M}_n(\mathcal{A})$ of $n \times n$ matrices with elements in $\mathcal{A}$ may be identified with the C^*-algebraic tensor product of $\mathcal{A}$ with the C^*-algebra of complex $n \times n$ matrices. Recently the study of this attendant matricial superstructure of a C^*-algebra has proved to be most rewarding. The mappings respecting the natural order and metric in the matrix spaces over a C^*-algebra, the *completely positive* [**St**] and *completely bounded maps* [**Ar**] respectively, have deep applications in single operator and group representation theory as well as to operator algebras. The completely bounded multilinear maps were characterized by Christensen and Sinclair [**ChS1**]: this led to interesting results in the cohomology theory of C^*-algebras [**ChSE**,**ChS2**]. The sort of representation theorem that they obtained may be regarded as a factorization through a Hilbert space.

Under the usual algebraic correspondence between multilinear maps and tensor product spaces the space of completely bounded multilinear functionals on C^*-algebras is in duality with the *Haagerup tensor product* of these algebras [**EK**]. Certain questions arising naturally from the study of such maps relate to the geometry of this tensor product. For example, the commutative Grothendieck inequality may be regarded as the equivalence between the projective norm and the Haagerup norm on the tensor product of commutative C^*-algebras.

The work of Haagerup and Pisier [**Ha2**,**Ha3**,**Pr2**] on the Grothendieck-Pisier-Haagerup inequality and related geometrical topics (such as factorization of bilinear functionals through a Hilbert space) lead naturally to the consideration of other tensor norms which are not C^*-norms. However there has been no systematic theory of general norms on the tensor product of C^*-algebras, nor any attempt to make comparisons with the theory of Banach space tensor norms. Perhaps this is because until recently the *-representations have been assumed to be the only class of morphisms of C^*-algebras which behave well with respect to tensoring, and these correspond properly to the C^*-tensor norms. The serious study in the last decade or so of completely positive and completely bounded maps has provided a lot of machinery without which a general theory of tensor products is not possible.

In the late seventies and eighties the work of Choi, Effros, Paulsen, Smith, and Ruan appeared on the theory of matricial vector spaces and operator spaces [**Ru**,**ER1**]. Operator spaces are the natural setting for the study of completely bounded maps. With this theory came the notion of 'non-commutative' or 'quantized' functional analysis [**Ef2**]. The study of operator spaces and completely bounded maps is a strict generalization of classical functional analysis: there is a faithful functor ([**ER1**] Theorem 2.1) embedding the category of Banach spaces and bounded maps into the category of operator spaces and completely bounded maps. Thus the Hahn-Banach theorem becomes the Arveson-Wittstock-Hahn-Banach theorem on the existence of extensions for completely bounded maps.

The quotient construct for operator spaces and certain specific 'operator space tensor norms' were studied **[Ru]**. The operator space Haagerup tensor norm is the appropriate tensor norm corresponding to the class of completely bounded multilinear maps.

We now summarize the contents of this work. We shall be concise since most chapters have their own, more detailed, introduction.

In Chapter 1 we establish our notation and state some facts which will be of use later. Section 1.1 includes some approximate identity machinery which enables us in later chapters to extend results on unital C^*-algebras to the general case. A brief discussion of tensor products of Banach spaces is given in 1.2; the injective and projective C^*-tensor norms are defined at the end of this section.

Chapter 2 is concerned with the theory of numerical range and the geometry of Banach algebras. We establish in 2.2 some characterizations of C^*-norms which are interesting in their own right. For example it is shown that an algebra norm dominated by an (uncompleted) C^*-norm is itself a C^*-norm; and that if there exists a unital norm decreasing linear map from a C^*-algebra into a Banach algebra with dense range then there is an involution on the Banach algebra with respect to which it is a C^*-algebra. In 2.3 representations and the duality structure of the extremal algebra generated by a Hermitian element are studied. This section is self-contained and does not relate to the subsequent material.

In Chapter 3 we examine the matricial structure associated with a C^*-algebra. Section 3.1 is a quick review of the theory of completely positive linear maps and completely bounded multilinear maps; the Christensen-Sinclair representation theorem is stated. Operator spaces are introduced in 3.2 and a proof is given (due to E. G. Effros) of the Arveson-Wittstock-Hahn-Banach Theorem. In 3.3 we define and discuss operator space tensor norms. The *symmetrized Haagerup norm* is presented, which corresponds to multilinear maps having Christensen-Sinclair representations but with Jordan *-homomorphisms taking the place of the usual *-representations.

In Chapter 4 we investigate geometrical properties of general algebra norms on the tensor product of C^*-algebras, and also discuss some particular tensor norms and their geometrical relationships. A uniformity condition appropriate to tensor norms of C^*-algebras is introduced and some implications of this condition considered. It is shown that if $\mathcal{A}$ is a nuclear C^*-algebra then the canonical contraction $\mathcal{A} \otimes_\alpha \mathcal{B} \to \mathcal{A} \otimes_\lambda \mathcal{B}$ is injective for all C^*-algebras $\mathcal{B}$, and for any tensor norm α which is uniform in our new sense. In 4.3 we prove that for an algebra norm α which is uniform in this sense *either* $\mathcal{A} \otimes_\alpha \mathcal{B}$ is a C^*-algebra for all C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, *or* $\mathcal{A} \otimes_\alpha \mathcal{B}$ is never a C^*-algebra unless $\mathcal{A}$ or $\mathcal{B}$ is $\mathbb{C}$. In 4.4 it is found that for such α there is actually a dichotomy for Hermitian elements: if $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras then the set of Hermitian elements in $\mathcal{A} \otimes_\alpha \mathcal{B}$ is *either* a spanning set *or* is as small as it could possibly be.

In Chapter 5 we define the *tracially completely bounded* multilinear maps, and investigate some related geometrical questions. In the bilinear case these maps are essentially the same as the completely bounded maps of Itoh [It] from a C^*-algebra to its dual. In section 5.2 every bounded bilinear map of C^*-algebras is shown to be tracially completely bounded, and thus the tensor norm which corresponds to the class of tracially completely bounded bilinear functionals is equivalent to the projective norm. Some bounds for this equivalence are found. An example is given in 5.3 of a trilinear bounded map which is not tracially completely bounded; and some comments made on the possibility of a Christensen-Sinclair type representation theorem for tracially completely bounded maps.

In Chapter 6 we discuss characterizations of subalgebras of C^*-algebras. This subject is closely related to the study of certain tensor norms [**Va3**,**Ca3**]. In 6.1 we show that the projective and Haagerup tensor products of two C^*-algebras are not subalgebras of a C^*-algebra, but are often subalgebras of $B(B(\mathcal{H}))$ for some Hilbert space $\mathcal{H}$. In 6.2 we consider the problem of characterizing subalgebras of C^*-algebras up to complete isometry. Examples are studied and necessary and sufficient conditions given. A result of Cole, that the quotient of a subalgebra of a C^*-algebra by a closed two-sided ideal is again a subalgebra of a C^*-algebra [**Wr2**], is generalized. Hopefully these characterizations will also shed some light on the tensor product construct for subalgebras of C^*-algebras.

As an appendix we give a sufficient condition for the existence of invariant subspaces for an operator on the space $L^2(\mathbb{T})$ composed of a multiplication operator and a translation (here $\mathbb{T}$ is the unit circle in the complex plane regarded as a topological group).

This work was completed under the supervision of A. M. Sinclair with the exception of the appendix and some of the material of Chapter 5 which was done in the summer of 1986 under the supervision of A. M. Davie while A. M. Sinclair was on sabbatical.

CHAPTER 1. PRELIMINARIES.

1.1 DEFINITIONS AND NOTATION.

We make some conventions and recall some definitions and facts, most of which are very well known and are stated here for completeness.

All linear spaces are over the complex field $\mathbb{C}$ unless explicitly stated to the contrary. As usual if α and α' are norms on a linear space E, and if $\alpha'(e) \geq \alpha(e)$ for each $e \in E$, then we say α' *dominates* α, and write $\alpha' \geq \alpha$. This determines a partial ordering on the set of norms on E. If (E, α) is a normed linear space then $BALL(E)$ denotes the set of elements $e \in E$ with $\alpha(e) \leq 1$. The dual space of E is denoted by E^*, and the natural pairing $E^* \times E \to \mathbb{C}$ is often written $< \cdot , \cdot >$; thus if $\psi \in E^*$ and $e \in E$ then

$$< \psi , e > = \psi(e) \quad .$$

Write $B(E)$ for the normed linear space of all bounded linear operators on E. The identity on E is denoted by I_E. A linear map $T : E \to F$ between normed linear spaces is said to be *bicontinuous* if T is invertible and if T and T^{-1} are continuous. If $E_1,\dots,E_m$ and F are normed linear spaces then we write $B(E_1 \times \dots \times E_m;F)$ for the normed linear space of all bounded m-linear maps $E_1 \times \dots \times E_m \to F$. An element of $BALL(B(E_1 \times \dots \times E_m;F))$ is said to be a *contraction*, or *contractive.*

For $n \in \mathbb{N}$ we write $E^{(n)}$ for $\oplus_{i=1}^{n} E$, the direct sum of n copies of E . If $\mathcal{H}$ is a Hilbert space then $\mathcal{H}^{(n)}$ is taken to have the natural Hilbert space structure. We write ζ_i for the i'th entry of an element $\underset{\sim}{\zeta} \in \mathcal{H}^{(n)}$; conversely if $\zeta_1 , \dots , \zeta_n \in \mathcal{H}$ then we write $\underset{\sim}{\zeta}$ for the element of $\mathcal{H}^{(n)}$ whose i'th entry is ζ_i . A *projection* on a Banach space E is an operator $P \in B(E)$ which is idempotent : i. e. $P^2 = P$. An *orthogonal projection on a Hilbert space* $\mathcal{H}$ is a projection $P \in B(\mathcal{H})$ which is either self-adjoint or a contraction [**Conw**].

Let $\mathcal{A}$ be an algebra. An *algebra norm* α on $\mathcal{A}$ is a norm which is sub-multiplicative:

$$\alpha(a\ b) \le \alpha(a)\ \alpha(b) \qquad (a , b \in \mathcal{A}) .$$

In this case the pair $(\mathcal{A} , \alpha)$ is called a *normed algebra.* An algebra $\mathcal{A}$ is *unital* if it possesses an identity 1 and $\alpha(1) = 1$. A linear map between unital algebras is called unital if it preserves the identity. We shall call an algebra norm α on $\mathcal{A}$ a **-algebra norm* (*respectively* C^**-norm*) if there is an isometric involution on the α-completion of $\mathcal{A}$ making it into a Banach *-algebra (respectively C^*-algebra); if $\mathcal{A}$ was already a *-algebra it is usually assumed that the involutions coincide. We shall always assume that the involution is isometric in a Banach *-algebra.

If E and F are linear spaces and $\mathcal{B}$ is an algebra, and if $S : E \to \mathcal{B}$ and $T : F \to \mathcal{B}$ are maps such that

$$S(e)\ T(f) = T(f)\ S(e) \quad ,$$

for each $e \in E$ and $f \in F$, then we say that S and T have *commuting ranges* (not to be confused with commutative ranges).

The *unitization* $\mathcal{A}^1$ of an algebra $\mathcal{A}$ is defined as follows: put $\mathcal{A}^1 = \mathcal{A}$ if $\mathcal{A}$ has an identity, otherwise let $\mathcal{A}^1$ be the algebra obtained by adjoining an identity. In other words, if $\mathcal{A}$ does not have an identity then $\mathcal{A}^1$ is the direct sum $\mathcal{A} \oplus \mathbb{C}$ with the algebra structure

$$(a,\lambda)\ (b,\mu) = (\ ab + \lambda b + \mu a\ ,\ \lambda\mu\ )\quad ,$$

for $a,b \in \mathcal{A}$ and $\lambda,\mu \in \mathbb{C}$. We write $a + \lambda 1$ for $(a,\lambda) \in \mathcal{A}^1$. If $\mathcal{A}$ is a C^*-algebra there is (see [**Di**] for example) a unique C^*-norm on $\mathcal{A}^1$ extending the original norm on $\mathcal{A}$; we call $\mathcal{A}^1$ with this norm the *C^*-unitization of* $\mathcal{A}$.

A *two-sided contractive approximate identity* for a normed algebra $(\ \mathcal{A}\ ,\ \alpha\ )$ is a net of elements (e_ν) in $\mathcal{A}$ such that $\alpha(e_\nu) \leq 1$ for each ν , and such that if $a \in \mathcal{A}$ then $a\ e_\nu$ and $e_\nu\ a$ both converge to a .

The following result shall be needed several times so we choose to state it in this place:

1.1.1 **PROPOSITION.** *Let* $\mathcal{A}$ *be a normed algebra, and suppose* I *is a two-sided ideal of* $\mathcal{A}$ *. If there exists a two-sided contractive approximate identity* (e_ν) *for* I *then for* $a \in \mathcal{A}$ *the following identities hold*:

$$\begin{aligned}\sup\ \{\ \|a\ b\|\ :\ b \in \mathrm{BALL}(I)\ \} &= \sup\ \{\ \|b\ a\|\ :\ b \in \mathrm{BALL}(I)\ \}\\ &= \sup\ \{\ \|b\ a\ c\|\ :\ b,c \in \mathrm{BALL}(I)\ \}\\ &= \lim_\nu\ \|a\ e_\nu\| = \lim_\nu\ \|e_\nu\ a\| = \lim_\nu\ \|e_\nu\ a\ e_\nu\|\quad .\end{aligned}$$

Proof. Let α be the expression on the left hand side. If $\alpha = 0$ then certainly $\lim_\nu \|a\, e_\nu\| = 0$; thus for $b \in I$ we have

$$b\, a = \lim_\nu (b\, a)\, e_\nu = 0 \ ,$$

and hence all the equalities hold.

Now suppose $\alpha \neq 0$; for $b \in \mathrm{BALL}(I)$ we have

$$\|b\, a\| = \lim_\nu \|(b\, a)\, e_\nu\| \leq \overline{\lim}_\nu \|a\, e_\nu\| \leq \alpha \ .$$

Let $\epsilon > 0$ be given, and choose $b \in \mathrm{BALL}(I)$ with $\|a\, b\| > \alpha - \epsilon$. Thus

$$\begin{aligned} \alpha - \epsilon < \|a\, b\| &= \lim_\nu \|e_\nu\, (a\, b)\| \\ &\leq \underline{\lim}_\nu \|e_\nu\, a\| = \underline{\lim}_\nu \lim_\lambda \|e_\nu\, a\, e_\lambda\| \\ &\leq \underline{\lim}_\lambda \|a\, e_\lambda\| \\ &\leq \alpha \ , \end{aligned}$$

which shows that $\lim_\nu \|e_\nu\, a\|$ and $\lim_\nu \|a\, e_\nu\|$ exist and equal α . Hence all the equalities except the last one have been established. To see this last equality observe firstly that $\|e_\nu\, a\, e_\nu\| \leq \alpha$ for each ν . Notice that for ϵ and b as above

$$\begin{aligned} \|e_\nu\, a\, e_\nu\, b - a\, b\| &\leq \|e_\nu\, a\, e_\nu\, b - e_\nu\, a\, b\| + \|e_\nu\, a\, b - a\, b\| \\ &\leq \|e_\nu\, b - b\|\, \|a\| + \|e_\nu\, a\, b - a\, b\| \end{aligned}$$

and the right hand side converges to 0 . Now

$$\alpha \geq \overline{\lim}_\nu \|e_\nu\, a\, e_\nu\| \geq \lim_\nu \|e_\nu\, a\, e_\nu\, b\| = \|a\, b\| > \alpha - \epsilon \ ,$$

which gives the last identity. □

For the remainder of this section the reader is referred to [**Di,Ta**] for further details.

1.1.2 COROLLARY. *Let* $\mathcal{A}$ *be a* C^**-algebra and suppose* (e_ν) *is a two-sided contractive approximate identity for* $\mathcal{A}$ *. The unique* C^**-norm on* $\mathcal{A}^1$ *extending the original norm is given by*

$$\|a + \lambda 1\| = \sup \{ \|a b + \lambda b\| : b \in \mathrm{BALL}(\mathcal{A}) \} = \lim_\nu \|a e_\nu + \lambda e_\nu\| ,$$

whenever $a \in \mathcal{A}$ *and* $\lambda \in \mathbb{C}$.

The set of self adjoint elements in a *-algebra $\mathcal{A}$ shall be denoted by $\mathcal{A}_{s.a.}$. If $\mathcal{A}$ is a C^*-algebra, we may define a cone $\mathcal{A}_+$ in $\mathcal{A}$ consisting of the positive elements of $\mathcal{A}$; i. e. those elements $a \in \mathcal{A}$ for which one (and hence all) of the following conditions hold:

(i) $a = b^* b$ for some $b \in \mathcal{A}$,

(ii) $a = h^2$ for some self-adjoint element $h \in \mathcal{A}$,

(iii) a is self-adjoint and the spectrum $\sigma(a)$ of a in $\mathcal{A}$ is contained in $[0,\infty)$,

(iv) if $\mathcal{A}$ is represented faithfully on a Hilbert space $\mathcal{H}$ then a is positive-definite as an operator on $\mathcal{H}$, i. e.

$$\langle a \zeta , \zeta \rangle \geq 0 \qquad (\zeta \in \mathcal{H}) .$$

If $\mathcal{S}$ is a subset of a C^*-algebra $\mathcal{A}$ then we write $\mathcal{S}_+$ for the set of positive elements in $\mathcal{A}$ which also lie in $\mathcal{S}$. If $\mathcal{A}$ is a C^*-algebra then one can always find a two-sided contractive approximate identity for $\mathcal{A}$ consisting of positive elements of $\mathcal{A}$.

If f is a linear functional on a C^*-algebra $\mathcal{A}$ then we say f is *positive* if $f(\mathcal{A}_+) \subset [0,\infty)$.

1.1.3 PROPOSITION. *For a linear functional* f *on a* C^**-algebra* $\mathcal{A}$ *any two of the following three conditions implies the third:*

(i) f *is positive,*

(ii) f *is contractive,*

(iii) *there is a two-sided contractive approximate identity* (e_ν) *for* $\mathcal{A}$ *such that* $f(e_\nu) \to 1$.

Proof. The only part of this that does not follow from **[Di]** Proposition 2.1.5 is the fact that together (iii) and (ii) imply (i). To see this notice that any functional f on a C^*-algebra $\mathcal{A}$ satisfying (iii) and (ii) may be extended to a unital linear functional $f^\sim$ on the C^*-unitization $\mathcal{A}^1$ of $\mathcal{A}$, and

$$|f^\sim(a + \lambda 1)| = \lim_\nu |f(a e_\nu + \lambda e_\nu)| \leq \lim_\nu \|a e_\nu + \lambda e_\nu\| ,$$

for $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. Corollary 1.1.2 now shows that $f^\sim$ is contractive. By **[Di]** 2.1.9 $f^\sim$ is positive on $\mathcal{A}^1$, and consequently f is positive on $\mathcal{A}$. □

We call a linear functional satisfying the conditions of Proposition 1.1.3 a *state* of $\mathcal{A}$. The proposition would still be true if the last condition was replaced by

(iii)' for all two-sided contractive approximate identities (e_ν) for $\mathcal{A}$ we have $f(e_\nu) \to 1$.

If $\mathcal{A}$ is a C^*-algebra let $\mathcal{M}_n(\mathcal{A})$ be the algebra of $n \times n$ matrices with elements in $\mathcal{A}$. We shall usually use a capital letter (e.g. A) for an element of $\mathcal{M}_n(\mathcal{A})$, and the (i,j) coordinate of that matrix shall be denoted by the same letter in lower case with the usual i-j subscript (e.g. a_{ij}) . Sometimes we may have reason to write a(i,j) for a_{ij} . If $\mathcal{A}$ is the *trivial C^*-algebra* $\mathbb{C}$ then we write $\mathcal{M}_n$ for $\mathcal{M}_n(\mathcal{A})$. We write e_{ij} for the usual system of matrix units in $\mathcal{M}_n$, and I_n for the identity element of $\mathcal{M}_n$.

Now $\mathcal{M}_n(\mathcal{A})$ has an obvious involution given by

$$[a_{i,j}]^* = [a^*_{j,i}] \quad ,$$

for $A \in \mathcal{M}_n(\mathcal{A})$. There is a unique way to make $\mathcal{M}_n(\mathcal{A})$ into a C^*-algebra: if $\mathcal{A}$ is faithfully represented on a Hilbert space $\mathcal{H}$ then $\mathcal{M}_n(\mathcal{A})$ may be naturally identified with a closed *-subalgebra of $B(\mathcal{H}^{(n)})$.

We shall write ^{t} for the transpose map

$$\mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{A}) \ : \ [a_{ij}] \mapsto [a_{ji}] \quad .$$

This mapping has a norm bounded by n , and is a contraction (and positive) if and only if $\mathcal{A}$ is commutative **[Tm2]**, i. e. if and only if $\mathcal{A}$ is the C^*-algebra $C_0(\Omega)$ of continuous functions converging to zero at infinity on some locally compact Hausdorff space Ω .

1.2 TENSOR PRODUCTS.

If E and F are linear spaces then we write $E \otimes F$ for their algebraic tensor product. If X is another linear space and $\Psi : E \times F \to X$ is a bilinear map then we shall usually write ψ for the canonical linear mapping $E \otimes F \to X$ induced by Ψ , namely

$$\psi(e \otimes f) = \Psi(e,f) \qquad\qquad (e \in E \ , \ f \in F) \ .$$

Conversely if ψ is a linear mapping on $E \otimes F$ then we write Ψ for the associated bilinear map.

If E_1 , E_2 , F_1 and F_2 are normed linear spaces, and if $T_i : E_i \to F_i$ $(i = 1,2)$ are linear maps, then we write $T_1 \otimes T_2$ for the map

$$E_1 \otimes E_2 \to F_1 \otimes F_2 : e_1 \otimes e_2 \mapsto T_1 e_1 \otimes T_2 e_2 \ .$$

If α is a norm on $E \otimes F$ we will usually write $E \otimes_\alpha F$ for the α-completion of $E \otimes F$. As usual α is called a *cross norm* if

$$\alpha(e \otimes f) = \|e\| \ \|f\| \quad ,$$

for each $e \in E$ and $f \in F$. There is (in a sense which we do not specify here) a least and a greatest cross norm on $E \otimes F$, the *injective* and *projective* tensor norms λ and γ respectively. These are defined by

$$\lambda(\Sigma_{i=1}^n e_i \otimes f_i) = \sup \{ \ | \ \Sigma_{i=1}^n \varphi(e_i) \ \psi(f_i) \ | \ : \varphi \in \mathrm{BALL}(E^*) \ , \ \psi \in \mathrm{BALL}(F^*) \ \} \ ;$$

and

$$\gamma(u) = \inf \{ \ \Sigma_{i=1}^n \|e_i\| \ \|f_i\| \ : \ u = \Sigma_{i=1}^n e_i \otimes f_i \ \} \ .$$

The injective norm is so called because it has the following property (injectivity): if $E_1 \subset F_1$ and $E_2 \subset F_2$ then $E_1 \otimes_\lambda E_2$ is contained isometrically in $F_1 \otimes_\lambda F_2$.

Let E and F be normed linear spaces. Associated to each bounded linear map $T : E \to F^*$ is a linear functional $\psi : E \otimes_\gamma F \to \mathbb{C}$ given by

$$\psi(e \otimes f) = \langle T(e) \ , \ f \rangle \qquad (\ e \in E \ , \ f \in F \ ) \ .$$

This association gives an isometric isomorphism from $B(E;F^*)$ onto the dual space of $E \otimes_\gamma F$.

1.2.1 ***Definition.*** Following Grothendieck **[Gr2]** we define a *reasonable tensor norm* α to be an assignment of a Banach space $E \otimes_\alpha F$ to each pair of Banach spaces (E, F) such that

(i) $E \otimes_\alpha F$ is the completion of $E \otimes F$ with respect to some cross norm which we write α or $\|\cdot\|_\alpha$, and

(ii) if E_1, E_2, F_1 and F_2 are Banach spaces, and if $T_i : E_i \to F_i$ $(i = 1,2)$ are bounded linear maps, then $T_1 \otimes T_2$ has a (unique) continuous extension $T_1 \otimes_\alpha T_2$ mapping $E_1 \otimes_\alpha E_2$ to $F_1 \otimes_\alpha F_2$ such that

$$\|T_1 \otimes_\alpha T_2\| \leq \|T_1\| \, \|T_2\| .$$

Thus a reasonable tensor norm may be regarded as a bifunctor from the category of Banach spaces and bounded linear maps to itself **[Ca4,Mi]**. Schatten **[Sc]** called a norm possessing property (ii) a *uniform* norm. This property allows us to 'tie' together the action of the tensor norm in some coherent fashion; to rule out arbitrary allocation of norms to different pairs of spaces.

Clearly λ and γ are reasonable tensor norms. Grothendieck in his influential paper on the metric theory of tensor products **[Gr1,Gr2]** produced a set of fourteen natural inequivalent reasonable tensor norms, including λ and γ. We shall say that a reasonable tensor norm α is an *algebra tensor norm* if whenever $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras then $\mathcal{A} \otimes_\alpha \mathcal{B}$ is again a Banach algebra. In **[Ca1,Ca4]** Carne gave a characterization of algebra tensor norms, and using this showed that of Grothendieck's natural norms only γ, H', $\gamma\backslash/$ and $\backslash/\gamma$ are algebra tensor norms. The norm H' is of some interest in the sequel; it may be defined by the statement

$\psi \in \mathrm{BALL}((E \otimes_{H'} F)^*)$ if and only if there exists a Hilbert space $\mathcal{H}$, and contractive linear maps $S : E \to \mathcal{H}^*$ and $T : F \to \mathcal{H}$, with

$$\psi(e \otimes f) = \langle S(e) , T(f) \rangle \qquad (e \in E , f \in F) .$$

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces then we write $\mathcal{H} \otimes \mathcal{K}$ for the (completed) Hilbert space tensor product **[Ta]** of $\mathcal{H}$ and $\mathcal{K}$. If $\zeta \in \mathcal{H}$, $\eta \in \mathcal{K}$ then we sometimes write $\zeta \otimes \eta$ for the operator $\mathcal{K} \to \mathcal{H} : \xi \mapsto \langle \xi, \eta \rangle \zeta$.

If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras then $\mathcal{A} \otimes \mathcal{B}$ is a *-algebra with the natural involution and multiplication

$$(a \otimes b)^* = a^* \otimes b^*$$

and

$$(a \otimes b)\ (c \otimes d) = (a\ c) \otimes (b\ d) ,$$

for $a,c \in \mathcal{A}$ and $b,d \in \mathcal{B}$. There is a least and a greatest C^*-norm on $\mathcal{A} \otimes \mathcal{B}$, namely the *injective* (or *spatial*) *C^*-norm* $\|\cdot\|_{min}$ and the *projective C^*-norm* $\|\cdot\|_{max}$ respectively **[Ta]**. Both of these norms are cross. If $\mathcal{A}$ and $\mathcal{B}$ are faithfully represented on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively then the norm $\|\cdot\|_{min}$ may be defined by identifying $\mathcal{A} \otimes \mathcal{B}$ with a *-subalgebra of $B(\mathcal{H} \otimes \mathcal{K})$ in the obvious way. This norm is independent of the specific Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ used to define it. The projective C^*-tensor norm is given on $\Sigma_{i=1}^n a_i \otimes b_i \in \mathcal{A} \otimes \mathcal{B}$ by

$$\| \Sigma_{i=1}^n a_i \otimes b_i \|_{max} = \sup \{ \| \Sigma_{i=1}^n \theta(a_i)\ \pi(b_i) \| \} ,$$

where the supremum is taken over all *-representations θ and π of $\mathcal{A}$ and $\mathcal{B}$ respectively on a Hilbert space $\mathcal{H}$ with commuting ranges. Note that $\|\cdot\|_{min}$ is injective: indeed if $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are C^*-algebras, with $\mathcal{A}_i \subset \mathcal{B}_i$ $(i = 1,2)$, then $\mathcal{A}_1 \otimes_{min} \mathcal{A}_2$ is a *-subalgebra of $\mathcal{B}_1 \otimes_{min} \mathcal{B}_2$.

A C^*-algebra $\mathcal{A}$ is said to be *nuclear* if $\|\cdot\|_{min} = \|\cdot\|_{max}$ on $\mathcal{A} \otimes \mathcal{B}$ for every C^*-algebra $\mathcal{B}$. Note that finite dimensional

C^*-algebras are nuclear; in particular $\mathcal{M}_n$ is nuclear. If $\mathcal{A}$ is a C^*-algebra then $\mathcal{A} \otimes \mathcal{M}_n$, endowed with its unique C^*-norm, is isometrically *-isomorphic to the space $\mathcal{M}_n(\mathcal{A})$ defined in 1.1.

We discuss C^*-tensor norms and nuclearity again in Section 4.1.

CHAPTER 2. GEOMETRY OF BANACH ALGEBRAS.

2.1 HERMITIAN ELEMENTS OF A BANACH ALGEBRA.

We refer the reader to [**BoD1**,**BoD2**,**BoD3**,**BD**] for details and a more thorough treatment of the ideas contained in this section.

Let $\mathcal{A}$ be a unital Banach algebra. A continuous linear functional f on $\mathcal{A}$ is said to be a *state* if $\|f\| = f(1) = 1$. If $\mathcal{A}$ is a unital C^*-algebra then this coincides with the former definition. We shall write $S(\mathcal{A})$ for the set of states on $\mathcal{A}$. For $a \in \mathcal{A}$ define the *numerical range* V(a) of a to be the compact convex sub-set of the plane given by

$$V(a) = \{ f(a) : f \in S(\mathcal{A}) \} .$$

It is well known that V(a) contains the spectrum $\sigma(a)$ of a . Define the *numerical radius* v(a) to be the number

$$v(a) = \sup \{ |\lambda| : \lambda \in V(a) \} .$$

It is clear that $r(a) \leq v(a) \leq \|a\|$, where r(a) is the spectral radius of a . In fact v is a norm on $\mathcal{A}$ equivalent to the original norm. Indeed is shown in [**BoD1**] Theorem 4.8 that

$$\|a^n\| \leq n!\ (e/n)^n\ v(a)^n \qquad (a \in \mathcal{A})$$

for $n = 1,2,\dots$, and this inequality is best possible: there is a Banach algebra (see section 2.3) where this bound is attained for each n .

2.1.1 THEOREM. *Let* h *be an element of a unital Banach algebra. The following three conditions are equivalent:*

(i) $V(h) \subset \mathbb{R}$,

(ii) $\|\exp(ith)\| = 1 \qquad (t \in \mathbb{R})$,

(iii) $\|1 + ith\| = 1 + o(t) \qquad (t \in \mathbb{R})$.

2.1.2 ***Definition.*** An element h of a unital Banach algebra is said to be *Hermitian* if one (and hence all) of the conditions of Theorem 2.1.1 is met.

It may also be shown that if h is a Hermitian element of a Banach algebra then $V(h)$ is the convex hull of $\sigma(h)$, and thus $v(h) = r(h)$. In fact more is true:

2.1.3 THEOREM [Si1]. *If* h *is a Hermitian element of a unital Banach algebra then* $r(h) = v(h) = \|h\|$.

We write $H(\mathcal{A})$ for the real Banach space of Hermitian elements in $\mathcal{A}$, and we put $J(\mathcal{A}) = H(\mathcal{A}) + i\,H(\mathcal{A})$. The following proposition is [**BoD1**] Lemma 5.8.

2.1.4 PROPOSITION. *Let* $\mathcal{A}$ *be a unital Banach algebra. Then* $J(\mathcal{A})$ *is a closed subspace of* $\mathcal{A}$ *, and the natural involution* $J(\mathcal{A}) \to J(\mathcal{A})$ *given by* $h + i\,k \mapsto h - i\,k$ *for* $h, k \in H(\mathcal{A})$ *is well defined and continuous.*

It follows directly from the definitions above that if

$T : \mathcal{A}_1 \to \mathcal{A}_2$ is a unital contraction between unital Banach algebras then $T(H(\mathcal{A}_1)) \subset H(\mathcal{A}_2)$. We shall use this fact extensively in the sequel.

2.2 GEOMETRICAL CHARACTERIZATIONS OF C^*-NORMS.

The following deep result is crucial in what follows:

2.2.1 THEOREM (Vidav - Palmer). *Let* $\mathcal{A}$ *be a unital Banach algebra such that* $J(\mathcal{A}) = \mathcal{A}$. *Then* $\mathcal{A}$ *is a* C^**-algebra with respect to the original norm and algebra structure, and the natural involution of* $J(\mathcal{A})$.

2.2.2 THEOREM. *Let* $\mathcal{A}$ *be a* C^**-algebra and let* $\mathcal{B}$ *be a Banach algebra. Suppose* $T : \mathcal{A} \to \mathcal{B}$ *is a linear contraction with dense range, mapping some two-sided contractive approximate identity for* $\mathcal{A}$ *to a two-sided contractive approximate identity for* $\mathcal{B}$. *Then there exists an involution on* $\mathcal{B}$ *such that* $\mathcal{B}$ *is a* C^**-algebra and* T *is involution preserving.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$ and T be as above, and suppose (e_ν) and (Te_ν) are two-sided contractive approximate identities in $\mathcal{A}$ and $\mathcal{B}$ respectively. Suppose firstly that $\mathcal{A}$ has an identity, then $e_\nu = e_\nu 1 \to 1$ and $T(e_\nu) \to T(1)$; consequently

$$T(1)\, b = \lim_\nu T(e_\nu)\, b = b = \lim_\nu b\, T(e_\nu) = b\, T(1) \quad ,$$

for $b \in \mathcal{B}$. Thus $\mathcal{B}$ has an identity and T is a unital contraction, whence

$$T(\mathcal{A}) = T(H(\mathcal{A})) + i\, T(H(\mathcal{A})) \subset H(\mathcal{B}) + i\, H(\mathcal{B}) \quad ,$$

and so the last set is dense in $\mathcal{B}$. Since $H(\mathcal{B}) + i\, H(\mathcal{B})$ is always closed (Proposition 2.1.4) it equals $\mathcal{B}$. An application of the Vidav-Palmer theorem (Theorem 2.2.1) now completes the proof.

Suppose now $\mathcal{A}$ has no identity; adjoin an identity in the usual way to obtain a C^*-algebra $\mathcal{A}^1$. Let $\mathcal{B}^1$ be the unitization of $\mathcal{B}$. Then $\mathcal{B}^1$ becomes a Banach algebra with the norm

$$\| b + \xi 1 \|_1 = \sup \{ \| b y + \xi y \| , \| y b + \xi y \| : y \in \mathrm{Ball}(\mathcal{B}) \}$$
$$= \max \{ \lim_\nu \|b\, Te_\nu + \xi\, Te_\nu\| , \lim_\nu \|Te_\nu\, b + \xi\, Te_\nu\| \}$$

where the equality holds by Proposition 1.1.1.

Define a unital linear mapping

$$T^\sim : \mathcal{A}^1 \to \mathcal{B}^1 : a + \xi 1 \mapsto Ta + \xi 1 .$$

Now $\lim_\nu \| Ta\, Te_\nu + \xi\, Te_\nu \| = \lim_\nu \| T(a\, e_\nu + \xi\, e_\nu) \|$

$$\leq \lim_\nu \| a\, e_\nu + \xi\, e_\nu \|$$
$$= \| a + \xi 1 \| \quad ;$$

similarly $\lim_\nu \| Te_\nu\, Ta + \xi\, Te_\nu \| \leq \| a + \xi 1 \|$; and so $T^\sim$ is a unital contraction. Clearly $T^\sim$ has dense range, and the result now follows from the first part. □

2.2.3 REMARK. The author is indebted to J. Feinstein for valuable discussions regarding the theorem above, and for the example below.

2.2.4 EXAMPLE. Given the hypotheses of Theorem 2.2.2 we cannot expect T to be surjective in general, even if T is injective. To see this consider the following example. Let c be the C^*-algebra of convergent complex sequences, and consider the linear mapping $T : c \to c$ given by

$$T\,\underset{\sim}{a} = (a_1 \,,\, a_2/2 \,,\, a_3/3 \,,\, \dots) + (\lim a_n)\,(0 \,,\, 1/2 \,,\, 2/3 \,,\, \dots)$$

for $\underset{\sim}{a} = (a_i)_{i=1}^{\infty} \in c$. It is clear that T is an injective unital contraction; and the range of T is certainly dense in c since it includes all sequences with only a finite number of non-zero terms. The mapping T is not surjective, because its range does not include the convergent sequence $(1 , 1/2 , 1/3 , \dots)$.

However if the mapping of Theorem 2.2.2 is a homomorphism then it is indeed surjective:

2.2.5 COROLLARY. *Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{B}$ be a Banach algebra, and suppose $\theta : \mathcal{A} \to \mathcal{B}$ is a contractive homomorphism. Then $\theta(\mathcal{A})$ possesses an involution which makes it into a C^*-algebra isometrically $*$-isomorphic to $\mathcal{A} \,/\, \ker\theta$, and θ is a $*$-homomorphism onto $\theta(\mathcal{A})$.*

Proof. Without loss of generality take $\mathcal{B}$ to be the closure of $\theta(\mathcal{A})$, and then θ satisfies the condition of Theorem 2.2.2. Thus $\mathcal{B}$ is a C^*-algebra and θ is a $*$-homomorphism; the corollary now follows from elementary C^*-algebra theory ([**Di**] Corollary 1.8.3). □

We note in passing that [**Di**] Corollary 1.8.3 can be proven directly from 2.2.2.

The following corollary shows that the C^*-norms are minimal amongst the algebra norms on an algebra.

2.2.6 COROLLARY. *Let* $\mathcal{A}$ *be an algebra. Any algebra norm on* $\mathcal{A}$ *dominated by a* C^**-norm is itself a* C^**-norm, and the canonical contraction between the two completions is surjective and involution preserving.*

2.3 THE EXTREMAL BANACH ALGEBRA GENERATED BY A HERMITIAN ELEMENT.

The results of 2.2 show that the C^*-algebras are extremal amongst the Banach algebras: they have the smallest norms, and are consequently the biggest algebras, in some sense. We consider in this section another extremal object in the category of Banach algebras.

We are concerned here with unital Banach algebras $\mathcal{A}$ which are generated by a Hermitian element h, with $\|h\| \le 1$; in other words the set of polynomials in h is dense in $\mathcal{A}$. We summarise this situation by writing $\mathcal{A} = \langle h \rangle$. Let $\mathcal{F}$ be the class of such Banach algebras. Via the Gelfand transform $C[-1,1]$ may be regarded in some sense as the largest algebra in $\mathcal{F}$, with the smallest norm. There is also in some sense a 'smallest' algebra $A[-1,1] = \langle u \rangle$ in $\mathcal{F}$, called the *extremal algebra generated by a Hermitian with numerical range* $[-1,1]$, and it may be identified algebraically with a dense subalgebra of $C[-1,1]$. It has the 'largest' norm in the following sense:

2.3.1 THEOREM. *Let* $\mathcal{B}$ *be a Banach algebra generated by a Hermitian element* h *, with* $\|h\| \le 1$ *. Then there exists a unique* *contractive homomorphism* $\theta : A[-1,1] \to \mathcal{B}$ *such that* $\theta(u^n) = h^n$ *for each* $n = 0,1,2,\ldots$.

We delay the proof of this theorem for a little while. The condition of Theorem 2.3.1 may be regarded as a universal property: there can only be one algebra in $\mathcal{F}$ which possesses this property. The mapping θ provided by Theorem 2.3.1 shall be called the *extremal homomorphism*, and may be regarded as a functional calculus for Hermitian elements of a Banach algebra. Note that the range of θ is dense in $\mathcal{B}$, and composing the extremal homomorphism θ with the Gelfand transform gives the canonical restriction map

$$A[-1,1] \to C(\sigma(h)) \ .$$

Interest has been shown **[Si2,Si3]** in using this functional calculus to understand Hermitian operators on Banach spaces; in particular inner derivations in $B(B(E))$ given by a Hermitian operator on E , where E is a Banach space (see example below). These objects are not very well understood, and if the functional calculus is bicontinuous then this would give much information about the structure of such operators.

2.3.2 EXAMPLE. Let $\mathcal{H}$ be a Hilbert space, and let T be a positive linear contraction on $\mathcal{H}$, with spectrum $\sigma(T)$. The *-derivation D on $B(\mathcal{H})$ given by

$$D(S) = T\,S - S\,T \qquad (\, S \in B(\mathcal{H})\,)$$

is Hermitian, since

$$1 = \|\exp(itD)\ (I)\|$$

$$\leq \|\exp(itD)\|$$

$$= \sup\{\|\exp(itT)\ S\ \exp(-itT)\| : S \in \mathrm{BALL}(B(\mathcal{X}))\}$$

$$\leq 1 \quad ,$$

using **[BoD3]** Proposition 18.8. It is easy to show that $\|D\| \leq 1$, and so by 2.3.1 there exists an extremal homomorphism

$$\theta : A[-1,1] \to \langle D\rangle .$$

It is shown in **[KaS]** that there is a bicontinuous homomorphism

$$\pi : \langle D\rangle \to C(\sigma(T)) \otimes_\gamma C(\sigma(T))$$

with $$\pi(D^n) = (z \otimes 1 - 1 \otimes z)^n \quad ,$$

for $n = 0,1,2\ldots$. Now $C(\sigma(T)) \otimes_\gamma C(\sigma(T))$ is semisimple (see **[Tm1]** or Chapter 4), and consequently so are $\langle z \otimes 1 - 1 \otimes z \rangle$ and $\langle D\rangle$. Thus in this case the extremal homomorphism θ is a monomorphism.

A particularly simple example is the situation where $\mathcal{X} = L^2[0,1]$, and T is the multiplication operator

$$(Tf)(t) = t\, f(t) \qquad (\ t \in [0,1]\ )$$

defined for $f \in L^2[0,1]$. Whether the extremal homomorphism θ is bicontinuous or not in this case is an open problem, posed in 1971 at the Aberdeen Conference on Numerical Range.

The extremal algebra $A[-1,1]$ can be constructed in many different ways (see **[Bo,Br,Si2]**) , but we choose to highlight one specific construction **[CrDM]** in terms of classical spaces of entire functions (see also **[Go]**) which displays its interesting duality structure and highlights a connection with derivations.

2.3.3 CONSTRUCTION. We merely sketch the construction, full details may be found in [**BoD2**] or [**CrDM**], whose notation we follow. We shall in fact construct a family of algebras $A(K)$, where K is a compact convex subset of the plane containing more than one point. We assume that K has been normalized so that $K \subset BALL(\mathbb{C})$; and either 0 is in the interior of K, or $K = [\alpha,1]$, where

$$-1 \le \alpha \le 0 .$$

For $\zeta \in \mathbb{C}$ put

$$\omega(\zeta) = \sup \{ |\exp(t\,\zeta)| : t \in K \} .$$

Let $\mathcal{M}(\mathbb{C})$ be the Banach algebra of regular Borel measures on the plane, with convolution product. Put

$$\mathcal{M}^{\omega}(\mathbb{C}) = \{ \mu \in \mathcal{M}(\mathbb{C}) : \int \omega \, d|\mu| < \infty \} ,$$

a Banach algebra with respect to convolution and the norm

$$\|\mu\|_{\omega} = \int \omega \, d|\mu| .$$

For $\mu \in \mathcal{M}^{\omega}(\mathbb{C})$ define a function $f_{\mu} \in C(K)$ by

$$f_{\mu}(t) = \int \exp(\zeta\, t) \, d\mu(\zeta) \qquad (t \in K) .$$

Put $A(K) = \{ f \in C(K) : f = f_{\mu}$ for some $\mu \in \mathcal{M}^{\omega}(\mathbb{C}) \}$, a Banach space with the norm

$$\|f\| = \inf \{ \|\mu\|_{\omega} : f = f_{\mu} \} .$$

Now $f_{\nu * \mu} = f_{\nu} f_{\mu}$ (pointwise) and so $A(K)$ is a subalgebra of $C(K)$.

The function $u(t) = t$ defined for $t \in K$ is in $A(K)$ since

$$(2\pi i)^{-1} \int_{\Gamma} \exp(\zeta\, t)\, \zeta^{-2} \, d\zeta = t ,$$

where Γ is the unit circle in $\mathbb{C}$. It is clear that the set of elements of $A(K)$ of form $\exp(\zeta u)$ for $\zeta \in \mathbb{C}$ spans a dense subspace of $A(K)$, and thus $A(K)$ is generated by u. The maximal ideal space of $A(K)$ is K, and consequently $A(K)$ is semisimple.

We now proceed to examine closely the duality structure of $A(K)$. Since the first part appears explicitly in **[CrDM]** we merely sketch the details, maintaining the notation of **[CrDM]** to avoid confusion.

Let $E(K)$ be the Banach space of entire functions ψ such that

$$\|\psi\| = \sup \{ |\psi(\zeta)|/\omega(\zeta) : \zeta \in \mathbb{C} \} < \infty ,$$

and let $E_0(K)$ be the closed subspace of $E(K)$ consisting of those functions $\psi \in E(K)$ with

$$|\psi(\zeta)|/\omega(\zeta) \to 0 \quad \text{as} \quad |\zeta| \to \infty .$$

When $K = [-1,1]$ then $E(K)$ is the Bernstein class **[Go]** of functions.

It is proved in **[CrDM]** that for $f_\mu \in A(K)$ and $\psi \in E(K)$ the pairing

$$< f_\mu , \psi > = \int \psi \, d\mu$$

is well defined and provides an isometric isomorphism

$$\Psi : E(K) \to A(K)^* : \psi \mapsto \Psi_\psi \ ;$$

where $$\Psi_\psi(f) = < f , \psi > \qquad (f \in A(K)) .$$

Proof of Theorem 2.3.1. Suppose $\mathcal{B} = \langle h \rangle$. Define a map

$$\theta : A[-1,1] \to \mathcal{B} : f_\mu \mapsto \int \exp(\zeta h) \, d\mu(\zeta) .$$

Now for any state g on $\mathcal{B}$ and any $\zeta \in \mathbb{C}$ we have

$$\|g(\exp(\zeta h))\| / \omega(\zeta) \leq \|\exp(\zeta h)\| / \omega(\zeta)$$

$$\leq \exp(|\mathrm{Re}\ \zeta|) \ / \ \omega(\zeta)$$
$$= 1 \quad .$$

Thus the function $\zeta \mapsto g(\exp(\zeta\ h))$ is in $E[-1,1]$. If $f_\mu = 0$ then

$$g(\int \exp(\zeta\ h)\ d\mu(\zeta)) = \int g(\exp(\zeta\ h))\ d\mu(\zeta)$$
$$= \langle f_\mu \ , \ g(\exp(\cdot\ h)) \rangle$$
$$= 0 \quad .$$

Since g was any state on $\mathcal{B}$ we see that $v(\theta(f_\mu)) = 0$ and consequently $\theta(f_\mu) = 0$. This shows that θ is a well defined function. It is easy to see that θ possesses the other properties that were promised. □

For $f \in A(K)$ define a functional F_f on $E_0(K)$ by

$$F_f(\psi) = \langle f \ , \ \psi \rangle \qquad\qquad (\ \psi \in E_0(K)\ ) \quad .$$

It is proved in **[CrDM]** that the mapping $f \mapsto F_f$ is an isometric isomorphism of $A(K)$ onto $E_0(K)^*$.

For $\zeta \in \mathbb{C}$ the element $\exp(\zeta\ u)$ of $A(K)$ may be represented by the discrete measure with unit mass at ζ , and so

$$\langle \exp\ (\zeta\ u)\ ,\ \psi \rangle = \psi(\zeta) \qquad\qquad (\ \psi \in E(K)\ ) \ .$$

Let $\lambda : \mathbb{C} \to B(E_0(K))$ be the group action of $\mathbb{C}$ on $E_0(K)$ by translation: if $\zeta \in \mathbb{C}$ and $\psi \in E_0(K)$ then

$$(\lambda(\zeta)\ \psi)\ (\eta) = \psi(\zeta + \eta) \qquad\qquad (\ \eta \in \mathbb{C}\ ) \ .$$

Let $\psi \in E(K)$, then for f_ν and f_μ in $A(K)$ we have

$$\langle f_\nu\ f_\mu \ , \ \psi \rangle = \iint \psi(\zeta + \eta)\ d\nu(\eta)\ d\mu(\zeta) = \langle f_\mu \ , \ \varphi \rangle \quad ,$$

where φ is the element of $E(K)$ given by

$$\varphi(\zeta) = \int \psi(\zeta + \eta)\ d\nu(\eta) \qquad (\ \zeta \in \mathbb{C}\ )\ .$$

We claim that if $\psi \in E_0(K)$ then $\varphi \in E_0(K)$. To see this notice firstly that

$$\varphi(\zeta) = \langle\ f_\nu\ ,\ \lambda(\zeta)\ \psi\ \rangle\ .$$

Now f_ν may be approximated arbitrarily closely in $A(K)$ by finite linear combinations of elements of the form $\exp(\xi\ u)$ for $\xi \in \mathbb{C}$, and certainly

$$|\langle\ \exp(\xi\ u)\ ,\ \lambda(\zeta)\ \psi\ \rangle|\ /\ \omega(\zeta) = |\psi(\zeta + \xi)|\ /\ \omega(\zeta) \to 0$$

as $|\zeta| \to \infty$.

We may now appeal to the following result:

2.3.4 PROPOSITION. *Let* $\mathcal{A}$ *be a unital Banach algebra and let* π *be the right regular representation of* $\mathcal{A}$ *on itself. Suppose that* $\mathcal{A}$ *satisfies the following two conditions:*

(i) *there is a Banach space* E *with* $E^* = \mathcal{A}$ *, and*

(ii) *the set of operators on* $\mathcal{A}^*$ *of the form* $\pi(a)^*$ *for* $a \in \mathcal{A}$ *leaves* $\hat{E}$ *, the canonical image of* E *in* $\mathcal{A}^*$ *, invariant (or, equivalently, that for each fixed* $a \in \mathcal{A}$ *and* $e \in E$ *the functional*

$$b \mapsto \langle\ ba\ ,\ e\ \rangle$$

on $\mathcal{A}$ *is in* E *).*

Then there exists a unique isometric homomorphism $\pi_* : \mathcal{A} \to B(E)$ *such that* $\pi_*(a)^* = \pi(a)$ *for each* $a \in \mathcal{A}$ *. Also if* ϵ *is the mapping* $B(E) \to \mathcal{A}$ *defined by*

$$\langle\ \epsilon(T)\ ,\ e\ \rangle = \langle\ 1\ ,\ Te\ \rangle \qquad (\ e \in E\ )\ ,$$

for $T \in B(E)$, *then* ϵ *is a unital projection (identifying* $\mathcal{A}$ *and* $\pi_*(\mathcal{A})$ *),* $\|\epsilon\| = 1$ *, and*

$$\epsilon(ST) = \epsilon(S) \circ T \quad ,$$

for $S , T \in B(E)$ *. If* $\mathcal{A}$ *is commutative then* $\pi_*(\mathcal{A})$ *is a maximal commutative subset of* $B(E)$ *.*

Proof. Define a mapping $\pi_* : \mathcal{A} \to B(E)$ by

$$\pi_*(a)\ e = (\ \pi(a)^* \ (\hat{e})\ )^{\check{}} = \pi(a)^*|_E \quad ,$$

for $a \in \mathcal{A}$ and $e \in E$. It is clear that π_* is a contractive unital homomorphism. By the Hahn-Banach theorem, for each $a \in \mathcal{A}$ there exists an element $e \in BALL(E)$ with $< a , e > = \|a\|$, and then

$$\|\pi_*(a)\| \geq \|\pi_*(a)\ e\| \geq |< \pi(a)^* \hat{e} , 1 >| = |< \hat{e} , a >| = \|a\| \ .$$

Thus π_* is an isometry. By definition $\pi_*(\cdot)^* = \pi(\cdot)$, and π_* is the only mapping with this property.

Defining $\epsilon : B(E) \to \mathcal{A}$ as in the statement of the proposition we see that

$$1 = \|\epsilon(I_E)\| \leq \|\epsilon\| \leq 1 \ ,$$

and so $\|\epsilon\| = 1$. For $a \in \mathcal{A}$ and $e \in E$ we have

$$< \epsilon(\pi_*(a)) \ , e > = < 1 \ , \pi_*(a)\ e > = < a , e > \quad ,$$

and so $\epsilon \circ \pi_* = I_{\mathcal{A}}$. Clearly

$$< \epsilon(ST) \ , e > = < 1 \ , ST\ e > = < \epsilon(S) \ , Te > = < \epsilon(S) \circ T \ , e > \quad ,$$

for $S , T \in B(E)$ and $e \in E$. If $\mathcal{A}$ is commutative, and if S is an operator on E in the commutant of $\pi_*(\mathcal{A})$, then for any $a \in \mathcal{A}$ and $e \in E$ we have

$$\begin{aligned} \langle a , T e \rangle &= \langle 1 , \pi_*(a) T e \rangle \\ &= \langle 1 , T \pi_*(a) e \rangle \\ &= \langle \epsilon(T) , \pi_*(a) e \rangle \\ &= \langle \epsilon(T) a , e \rangle \\ &= \langle a , \pi_*(\epsilon(T)) e \rangle , \end{aligned}$$

and so $T = \pi_*(\epsilon(T))$. □

There is a similar result for the left regular representation.

If $\mathcal{A}$ and E are as in 2.3.4 then we can deduce as a corollary of 2.3.4 that any mapping from $\mathcal{A}$ can be extended to a mapping on B(E) . Indeed if F is any Banach space, and α any reasonable tensor norm, then

$$\mathcal{A} \otimes_\alpha F \subset B(E) \otimes_\alpha F$$

isometrically.

2.3.5 EXAMPLE. Let (X , μ) be a measure space, let $\mathcal{A}$ be the space $L^\infty(X,\mu)$ of essentially bounded μ-measurable functions on X , and let $E = L^1(X,\mu)$. Then by 2.3.4 we may identify $\mathcal{A}$ with a subalgebra of B(E) , the commutant $\mathcal{A}'$ of $\mathcal{A}$ in B(E) equals $\mathcal{A}$, and there exists a contractive projection from B(E) onto $\mathcal{A}$.

Proposition 2.3.4 shows that the mapping $\pi_* : A(K) \to B(E_0(K))$ defined by

$$(\pi_*(f_\nu)\ \psi)\ (\zeta) = \int \psi(\zeta + \eta)\ d\nu$$

(for $f_\nu \in A(K)$, $\psi \in E_0(K)$ and $\zeta \in \mathbb{C}$) is actually an isometric homomorphism. Under this isometry it is clear that for $\zeta \in \mathbb{C}$ the element $\exp(\zeta u)$ of $A(K)$ corresponds to the translation operator $\lambda(\zeta)$. It is natural to ask which operator on $E_0(K)$ corresponds to the element $u \in A(K)$. Recall that $u = f_\mu$ where μ was the measure on the unit circle Γ given by

$$d\mu = (2\pi i)^{-1} \zeta^{-2} d\zeta \quad .$$

Thus

$$(\pi_*(u) \psi)(\zeta) = (2\pi i)^{-1} \int_\Gamma \psi(\zeta + \eta) / \eta^{-2} d\eta = \psi'(\zeta) ,$$

or in other words, $\pi_*(u)$ is the operation of differentiation on $E_0(K)$. Set $\pi_*(u) = D$.

If $K = \{ \zeta \in \mathbb{C} : |\zeta| \leq 1 \}$, and we define $\psi \in E_0(K)$ by

$$\psi(\zeta) = (e \zeta / n)^n \qquad (\zeta \in \mathbb{C}) ,$$

then we see that

$$\|u^n\| \geq |(D^n\psi)(0)| = n! (e/n)^n ,$$

and hence $A(K)$ is an algebra in which the extremal values mentioned in 2.1 are attained.

Putting these results together we have:

2.3.6 THEOREM. *The mapping* $\pi_* : A(K) \to B(E_0(K))$ *is a unital isometric monomorphism, and*

(i) $\pi_*(\cdot)^*$ *is the regular representation of* $A(K)$ *on itself,*

(ii) $\pi_*(u)$ *is the differentiation operator* D *on* $E_0(K)$,

(iii) *for each* $\zeta \in \mathbb{C}$ $\pi_*(\exp(\zeta u))$ *is the translation operator* $\lambda(\zeta)$,

(iv) *the map* $\epsilon : B(E_0(K)) \to A(K)$ *defined in Proposition 2.3.4.*

is (after identifying $A(K)$ *and* $\pi_*(A(K))$ *) a unital projection with* $\|\epsilon\| = 1$, *and*

$$\epsilon(T\,f) = \epsilon(T)\,f = f\,\epsilon(T) \quad ,$$

for $T \in B(E_0(K))$ *and* $f \in A(K)$,

(v) $\pi_*(A(K))$ *is a maximal commutative subset of* $B(E_0(K))$.

Thus $A(K)$ may be simultaneously regarded as

(a) the closed subalgebra of $B(E_0(K))$ generated by the translation operators $\lambda(\zeta)$ for $\zeta \in \mathbb{C}$; and

(b) the closed unital subalgebra of $B(E_0(K))$ generated by the differentiation operator D .

We now return to the case $K = [-1,1]$. Consider the derivation Δ on $B(E_0[-1,1])$ defined by

$$\Delta(T) = \tfrac{1}{2}\,(D\,T - T\,D) \qquad (\,T \in B(E_0[-1,1])\,)\ .$$

It is easy to see (as in Example 2.3.2) that Δ is Hermitian, and that $\|\Delta\| \leq 1$. As usual let $\langle\Delta\rangle$ be the unital Banach algebra generated by Δ .

2.3.7 THEOREM. *The extremal map* $\theta : A[-1,1] \to \langle\Delta\rangle$ *is an isometric isomorphism.*

Proof. By 2.3.6 it clearly suffices to show that

$$\|\zeta_0 + \zeta_1\,\Delta + \dots + \zeta_n\,\Delta^n\| \geq \|\zeta_0 + \zeta_1\,D + \dots + \zeta_n\,D^n\|$$

for $\zeta_0,\dots,\zeta_n \in \mathbb{C}$. Let R be the isometric reflection

$$(R\,\psi)(\zeta) = \psi(-\zeta) \qquad (\ \psi \in E_0(K)\ ,\ \zeta \in \mathbb{C}\ )\ .$$

Notice that R has the property

$$\Delta^m(R) = R\,D^m$$

for $m = 0,1,2,\dots$, whence

$$\begin{aligned}\|\zeta_0 + \zeta_1\,\Delta + \dots + \zeta_n\,\Delta^n\| &\geq \|(\zeta_0 + \zeta_1\,\Delta + \dots + \zeta_n\,\Delta^n)\ (R)\| \\ &= \|R\ (\zeta_0 + \zeta_1\,D + \dots + \zeta_n\,D^n)\| \\ &= \|\zeta_0 + \zeta_1\,D + \dots + \zeta_n\,D^n\|\ . \qquad \square\end{aligned}$$

We note that a similar calculation would show that $A[-1,1]$ is isometrically isomorphic to the Banach algebra generated by the inner derivation given by the differentiation operator on $E[-1,1]$.

CHAPTER 3. THE MATRICIAL SUPERSTRUCTURE OF A C^*-ALGEBRA.

In this chapter we explore the additional information about a C^*-algebra $\mathcal{A}$ that is obtained by considering the spaces of matrices $\mathcal{M}_n(\mathcal{A})$ over $\mathcal{A}$. For instance for an element a of a unital C^*-algebra $\mathcal{A}$ it is true [**Pn**] that

$$\|a\| \leq 1 \qquad \text{if and only if} \qquad \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \text{ is positive in } \mathcal{M}_2(\mathcal{A}) .$$

It is natural then to consider maps between C^*-algebras which respect the order and the norm of the associated spaces of matrices, respectively the *completely positive* [**St**] and *completely bounded* [**Ar**] maps. In 3.1 we discuss firstly the theory of completely positive maps, giving some of our own proofs; and then multilinear completely bounded maps and the Christensen-Sinclair representation theorems.

In 3.2 we review quickly the theory of operator spaces and completely bounded maps on operator spaces, and we discuss the sense in which this is a generalization of classical functional analysis. We also give a most illuminating proof (due to E. G. Effros) of the celebrated Arveson-Wittstock-Hahn-Banach theorem.

In 3.3 we define *operator space tensor norms* and discuss the operator space Haagerup norm and its relationship with completely bounded multilinear maps. We also introduce the *symmetrized Haagerup norm*. This corresponds to a variant of the notion of complete boundedness; and to maps which have representations of Christensen-Sinclair type, but with Jordan *-homomorphisms instead of the usual

*-representations.

3.1 COMPLETELY POSITIVE AND COMPLETELY BOUNDED MAPS.

A linear map $T : \mathcal{A} \to \mathcal{B}$ of C^*-algebras is said to be *positive* if $T(\mathcal{A}_+) \subset \mathcal{B}_+$ (this implies in particular that T is *-linear); and *n - positive* if the map

$$T_n : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B}) : [a_{ij}] \mapsto [Ta_{ij}]$$

is positive. If T is n - positive for each $n \in \mathbb{N}$ then T is *completely positive* [St]. If the maps T_n are uniformly bounded then T is said to be *completely bounded* and we put

$$\|T\|_{cb} = \sup \{ \|T_n\| : n \in \mathbb{N} \} .$$

We now collect together some facts which we shall need in the sequel. We refer the reader to **[Pn,Ta]** for details and a more thorough treatment. We do not dwell on the results on completely bounded maps since these shall be revisited in 3.2. Throughout this section $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, and $T : \mathcal{A} \to \mathcal{B}$ is a linear map.

3.1.1 THEOREM [St,BD]. *If $\mathcal{A}$ or $\mathcal{B}$ is commutative and $T : \mathcal{A} \to \mathcal{B}$ is a positive linear map, then T is completely positive.*

The following construction is fundamental. Let $\mathcal{A}$ be a C^*-algebra, let $\mathcal{H}$ be a Hilbert space and let $T : \mathcal{A} \to B(\mathcal{H})$ be a completely positive linear mapping. On the algebraic tensor product $\mathcal{A} \otimes \mathcal{H}$ we define a semi inner product by

$$\langle a \otimes \zeta , b \otimes \eta \rangle = \langle T(b^* a) \zeta , \eta \rangle ,$$

for $a , b \in \mathcal{A}$ and $\zeta , \eta \in \mathcal{H}$. The complete positivity of T ensures that $< \cdot , \cdot >$ is positive semi-definite. Let

$$\mathcal{N} = \{ \xi \in \mathcal{A} \otimes \mathcal{H} : < \xi , \xi > = 0 \}$$

then it is not hard to show (see **[KR]** Theorem 2.1.1) that $\mathcal{N}$ is a linear subspace of $\mathcal{A} \otimes \mathcal{H}$; write $\mathcal{A} \otimes_T \mathcal{H}$ for the Hilbert space completion of $\mathcal{A} \otimes \mathcal{H} / \mathcal{N}$ in the induced inner product. For $a \in \mathcal{A}$ and $\zeta \in \mathcal{H}$ we shall write $[a \otimes \zeta]$ for the coset of $a \otimes \zeta$ in $\mathcal{A} \otimes_T \mathcal{H}$.

3.1.2 THEOREM (STINESPRING). *Let* $\mathcal{A}$ *be a* C^**-algebra and let* $\mathcal{H}$ *be a Hilbert space. A linear map* $T : \mathcal{A} \to B(\mathcal{H})$ *is completely positive if and only if there exists a *-representation* π *of* $\mathcal{A}$ *on a Hilbert space* $\mathcal{K}$ *, and an operator* $V \in B(\mathcal{H},\mathcal{K})$ *, such that*

$$T(a) = V^* \pi(a) V \qquad (a \in \mathcal{A}) \quad .$$

In this case we can choose V *with* $\|V\| = \|T\|^{1/2}$ *. Further, if* $\mathcal{A}$ *is unital then* π *may be taken to be unital, and thus* $\|T\| = \|T(1)\|$ *. If* $\mathcal{A}$ *and* T *are unital then we may assume that* $\mathcal{K}$ *contains* $\mathcal{H}$ *as a subspace and* $T(\cdot) = P_{\mathcal{H}} \pi(\cdot)|_{\mathcal{H}}$.

Proof. The sufficiency is obvious. Suppose that $\mathcal{A}$ is unital. We merely sketch the proof of the necessity in this case, as it is standard **[Pn]**. Let $\mathcal{K}$ be the Hilbert space $\mathcal{A} \otimes_T \mathcal{H}$ defined immediately above the statement of this theorem. For $a \in \mathcal{A}$ define $\pi(a) : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$ by

$$\pi(a) \, (b \otimes \zeta) = (a \, b) \otimes \zeta \quad ,$$

for $b \in \mathcal{A} , \zeta \in \mathcal{H}$. The complete positivity of T ensures that

$\pi(a)$ extends to an operator on $\mathcal{K}$, and then it is immediate that π is a *-representation of $\mathcal{A}$ on $\mathcal{K}$. The operator $V : \mathcal{H} \to \mathcal{K}$ is defined by $V\zeta = [1 \otimes \zeta]$. If T is unital then V is an isometry.

Now suppose $\mathcal{A}$ is not unital and let $\mathcal{A}^1$ be the C^*-unitization of $\mathcal{A}$. If (e_ν) is a two-sided contractive approximate identity for $\mathcal{A}$ then $(T(e_\nu^* e_\nu))$ is a bounded net in $B(\mathcal{H})$; suppose E is a cluster point of this net in the weak operator topology. Define an extension $T^\sim$ of T to $\mathcal{A}^1$ by

$$T^\sim(a + \lambda 1) = Ta + \lambda E \quad ,$$

for $a + \lambda 1 \in \mathcal{A}^1$. It is easy to show that $T^\sim$ is completely positive, and now the result follows from the first part. □

3.1.3 COROLLARY. *If* $T : \mathcal{A} \to \mathcal{B}$ *is completely positive then it is completely bounded, and* $\|T\|_{cb} = \|T\|$.

3.1.4 COROLLARY (Generalized Schwarz inequality). *If* $T : \mathcal{A} \to \mathcal{B}$ *is completely positive then*

$$T(a)^* T(a) \le \|T\| \, T(a^* a) \quad ,$$

for each $a \in \mathcal{A}$.

3.1.5 COROLLARY. *Let* $\mathcal{B}$ *be a* C^**-algebra, and suppose* $\mathcal{A}$ *is a closed *-subalgebra of* $\mathcal{B}$. *If* $\mathcal{H}$ *is a Hilbert space, and if* $T : \mathcal{A} \to B(\mathcal{H})$ *is a completely positive linear map, then* T *has an extension to a completely positive map* $T^\sim : \mathcal{B} \to B(\mathcal{H})$.

Proof. This follows immediately from 3.1.2 and **[Di]** 2.10.2. □

3.1.6 THEOREM. *If $\mathcal{A}$ is a C^*-algebra and if $P : \mathcal{A} \to \mathcal{A}$ is a completely positive contractive projection, then there is a multiplication on the range of P with respect to which, with the usual norm and involution, it is a C^*-algebra.*

Proof. Suppose $\mathcal{A}$ is represented on a Hilbert space $\mathcal{H}$. Let $\mathcal{B} = P(\mathcal{A})$, then it is immediate that $\mathcal{B}$ is closed. If we can show that

$$P(P(a)\ P(b)) = P(P(a)\ b) = P(a\ P(b)) \quad ,$$

for all $a , b \in \mathcal{A}$, then the contractive bilinear map

$$\mathcal{B} \times \mathcal{B} \to \mathcal{B} : (\ b_1\ ,\ b_2\ ) \mapsto P(b_1\ b_2)$$

is an associative multiplication. The statement of the theorem shall then follow from Theorem 2.2.2, since P preserves two-sided approximate identities with respect to this multiplication on $\mathcal{B}$.

Construct the Hilbert space $\mathcal{K} = \mathcal{A} \otimes_P \mathcal{H}$ defined immediately before Theorem 3.1.2. Define a map $Q : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A} \otimes \mathcal{H}$ taking $a \otimes \zeta$ to $P(a) \otimes \zeta$. Now

$$\|[\Sigma_{i=1}^n P(a_i) \otimes \zeta_i]\|^2 = \Sigma_{i,j=1}^n < P(P(a_j)^* P(a_i))\ \zeta_i\ ,\ \zeta_j >$$
$$= < P_n(P_n(A)^*\ P_n(A))\ \underset{\sim}{\zeta}\ ,\ \underset{\sim}{\zeta} > \quad ,$$

where $A = \Sigma_{j=1}^n a_j \otimes e_{1j}$; thus by the generalized Schwarz inequality

$$\|[\Sigma_{i=1}^n P(a_i) \otimes \zeta_i]\|^2 \le < P_n(A^*\ A)\ \underset{\sim}{\zeta}\ ,\ \underset{\sim}{\zeta} > = \|[\Sigma_{i=1}^n a_i \otimes \zeta_i]\|^2 \ .$$

Thus Q extends to a contractive operator on $\mathcal{K}$. Since Q is also an idempotent operator it is an orthogonal projection, and so

$$< Q\ [a \otimes \zeta]\ ,\ Q\ [b \otimes \eta] > = < Q\ [a \otimes \zeta]\ ,\ [b \otimes \eta] >$$
$$= < [a \otimes \zeta]\ ,\ Q\ [b \otimes \eta] > \quad ,$$

for $a, b \in \mathcal{A}$ and $\zeta, \eta \in \mathcal{H}$. In other words

$$\langle P(P(b)\ P(a))\ \zeta, \eta \rangle = \langle P(b\ P(a))\ \zeta, \eta \rangle$$
$$= \langle P(P(b)\ a)\ \zeta, \eta \rangle ,$$

for $a, b \in \mathcal{A}$ and $\zeta, \eta \in \mathcal{H}$, which proves the result. □

I believe the result above first appeared, with a different proof, in [ChE1]. Our proof gives explicitly a Hilbert space on which the C^*-algebra may be represented. Some of the ideas above are in [Hm].

3.1.7 THEOREM [Pn]. *If* $T : \mathcal{A} \to \mathcal{B}$ *is completely bounded then there is a *-representation* π *of* $\mathcal{A}$ *on a Hilbert space* $\mathcal{K}$, *and operators* $U, V \in B(\mathcal{H},\mathcal{K})$ *with* $\|U\|\ \|V\| = \|T\|_{cb}$, *such that*

$$T(\cdot) = U^*\ \pi(\cdot)\ V .$$

If $\mathcal{A}$ *is unital then* π *can be chosen to be unital.*

3.1.8 ***Definition*** [ChS1]. Let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be C^*-algebras, $\mathcal{H}$ a Hilbert space and let $\Psi : \mathcal{A}_1 \times \ldots \times \mathcal{A}_m \to B(\mathcal{H})$ be an m-linear map. For each $n \in \mathbb{N}$ define an m-linear map

$$\Psi_n : \mathcal{M}_n(\mathcal{A}_1) \times \ldots \times \mathcal{M}_n(\mathcal{A}_m) \to \mathcal{M}_n(B(\mathcal{H})) ,$$

the n-fold amplification of Ψ, by

$$\Psi_n(A_1,\ldots,A_m) = [\Sigma^n_{i_1,\ldots,i_{m-1}=1}\ \Psi(A_1(i,i_1)\ , \ \ldots\ , \ A_m(i_{m-1},j))]_{i,j}$$

for $A_1 \in \mathcal{A}_1, \ldots, A_m \in \mathcal{A}_m$. We say Ψ is *completely bounded* if $\sup \{ \|\Psi_n\| : n \in \mathbb{N} \} < \infty$, and then we define $\|\Psi\|_{cb}$ to be this supremum. In the case $m = 1$ this coincides with the earlier definition. The space $CB(\mathcal{A}_1 \times \ldots \times \mathcal{A}_m;B(\mathcal{H}))$ of completely bounded maps $\mathcal{A}_1 \times \ldots \times \mathcal{A}_m \to B(\mathcal{H})$ is a Banach space with the norm $\|\cdot\|_{cb}$.

If $\mathcal{A}_1 = \ldots = \mathcal{A}_m = \mathcal{A}$ in the above, then the map Ψ is said to be *symmetric* if $\Psi = \Psi^*$, where Ψ^* is defined by

$$\Psi^*(a_1,\ldots,a_m) = \Psi(a_m^*,\ldots,a_1^*)^* \quad ,$$

for $a_1,\ldots,a_m \in \mathcal{A}$.

The next result is a generalization of Theorem 3.1.7 to the multilinear case.

3.1.9 THEOREM [ChS1]. *Let* $\mathcal{A}_1 , \ldots , \mathcal{A}_m$ *be* C^**-algebras, let* $\mathcal{H}$ *be a Hilbert space, and let* $\Psi : \mathcal{A}_1 \times \ldots \times \mathcal{A}_m \to B(\mathcal{H})$ *be an* m*-linear map. Then* Ψ *is completely bounded if and only if there are* **-representations* $\pi_1 , \ldots , \pi_m$ *of* $\mathcal{A}_1 , \ldots , \mathcal{A}_m$ *on Hilbert spaces* $\mathcal{H}_1 , \ldots , \mathcal{H}_m$ *respectively, and operators* $T_k \in B(\mathcal{H}_k,\mathcal{H}_{k-1})$ *for* $1 \leq k \leq m+1$ *, where* $\mathcal{H}_0 = \mathcal{H}_{m+1} = \mathcal{H}$ *, such that*

$$\Psi(a_1,\ldots,a_m) = T_1\ \pi_1(a_1)\ T_2 \ldots T_m\ \pi_m(a_m)\ T_{m+1} \quad ,$$

for $a_1 \in \mathcal{A}_1 , \ldots , a_m \in \mathcal{A}_m$. *In this case we can choose* $T_1 , \ldots , T_{m+1}$ *such that* $\|T_1\| \ldots \|T_m\| = \|\Psi\|_{cb}$.

If $\mathcal{A}_1 , \ldots , \mathcal{A}_m$ *are unital then we can choose* $\pi_1 , \ldots , \pi_m$ *unital.*

The expression given for Ψ in Theorem 3.1.9 is called by some a Christensen-Sinclair representation. The operators T_i occuring in the representation are sometimes called *bridging maps.*

Christensen and Sinclair also characterized the symmetric completely bounded maps. We shall need the following:

3.1.10 THEOREM [ChS1]. *Let* $\mathcal{A}$ *be a* C^**-algebra and let* $\mathcal{H}$ *be a Hilbert space. If* $\Psi : \mathcal{A} \times \mathcal{A} \to B(\mathcal{H})$ *is a symmetric bilinear completely bounded map then we can find a representation* π *of* $\mathcal{A}$ *on a Hilbert space* $\mathcal{K}$ *, a contractive operator* $U : \mathcal{H} \to \mathcal{K}$ *, and a self-adjoint operator* V *on* $\mathcal{K}$ *with* $\|V\| = \|\Psi\|_{cb}$ *, such that*

$$\Psi(a,b) = U^* \pi(a) V \pi(b) U \ ,$$

for $a , b \in \mathcal{A}$ *. If* $\mathcal{A}$ *is unital then* π *may be chosen to be unital.*

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Define a positive function $\|\cdot\|_h$ on $\mathcal{A} \otimes \mathcal{B}$ by

$$\|u\|_h = \inf \{ \| \Sigma_{i=1}^n a_i a_i^* \|^{1/2} \| \Sigma_{i=1}^n b_i^* b_i \|^{1/2} : u = \Sigma_{i=1}^n a_i \otimes b_i \} \ .$$

Let $u = \Sigma_{i=1}^n a_i \otimes b_i$ and $v = \Sigma_{i=1}^m c_i \otimes d_i$ be elements of $\mathcal{A} \otimes \mathcal{B}$, without loss of generality we may assume that

$$\|\Sigma_{i=1}^n a_i a_i^*\|^{1/2} = \|\Sigma_{i=1}^n b_i^* b_i\|^{1/2} \text{ and } \|\Sigma_{i=1}^m c_i c_i^*\|^{1/2} = \|\Sigma_{i=1}^m d_i^* d_i\|^{1/2} \ .$$

Then

$$\begin{aligned}\|u + v\|_h &\le \| \Sigma_{i=1}^n a_i a_i^* + \Sigma_{i=1}^m c_i c_i^* \|^{1/2} \| \Sigma_{i=1}^n b_i^* b_i + \Sigma_{i=1}^m d_i^* d_i \|^{1/2} \\ &\le \| \Sigma_{i=1}^n a_i a_i^* \| + \| \Sigma_{i=1}^m c_i c_i^* \| \\ &= \|\Sigma_{i=1}^n a_i a_i^*\|^{1/2} \|\Sigma_{i=1}^n b_i^* b_i\|^{1/2} + \|\Sigma_{i=1}^m c_i c_i^*\|^{1/2} \|\Sigma_{i=1}^m d_i^* d_i\|^{1/2} \ .\end{aligned}$$

Thus $\|\cdot\|_h$ is sub-additive.

Let π and θ be *-representations of $\mathcal{A}$ and $\mathcal{B}$ respectively on some Hilbert space $\mathcal{H}$, with commuting ranges. For $a_1,\dots,a_n \in \mathcal{A}$, $b_1,\dots,b_n \in \mathcal{B}$, and $\zeta , \eta \in \mathrm{BALL}(\mathcal{H})$

$$| < \Sigma_{i=1}^N \pi(a_i) \theta(b_i) \zeta , \eta > | \le \Sigma_{i=1}^N \| \theta(b_i) \zeta \| \| \pi(a_i^*) \eta \|$$

$$\leq \{ \Sigma_{i=1}^N \| \theta(b_i) \zeta \|^2 \}^{½} \{ \Sigma_{i=1}^N \| \pi(a_i^*) \eta \|^2 \}^{½}$$
$$= \{ \Sigma_{i=1}^N < \theta(b_i^* b_i) \zeta , \zeta > \}^{½} \{ \Sigma_{i=1}^N < \pi(a_i a_i^*) \eta , \eta > \}^{½}$$
$$\leq \| \Sigma_{i=1}^N a_i a_i^* \|^{½} \| \Sigma_{i=1}^N b_i^* b_i \|^{½} ,$$

and thus $\|\cdot\|_{max} \leq \|\cdot\|_h$. This shows that $\|\cdot\|_h$ is a norm.

We call $\|\cdot\|_h$ the *Haagerup tensor norm* **[EK]**. Following the custom we often write $\mathcal{A} \otimes_h \mathcal{B}$ for the *uncompleted* normed vector space ($\mathcal{A} \otimes \mathcal{B}$, $\|\cdot\|_h$) . This is an abuse of our earlier convention and we hope it does not confuse the reader. The Haagerup norm is the tensor norm corresponding to the notion of completely bounded bilinear maps.

3.1.11 THEOREM [EK]. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* C^**-algebras, and let* $\mathcal{H}$ *be a Hilbert space. If* $\Psi : \mathcal{A} \times \mathcal{B} \to B(\mathcal{H})$ *is a completely bounded bilinear map then the associated linear map* $\psi : \mathcal{A} \otimes_h \mathcal{B} \to B(\mathcal{H})$ *is bounded, and then* $\|\psi\| \leq \|\Psi\|_{cb}$. *A bilinear functional* $\Psi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ *is completely bounded if and only if the associated linear functional* ψ *on* $\mathcal{A} \otimes_h \mathcal{B}$ *is bounded, and then*

$$\|\Psi\|_{cb} = \|\psi\| .$$

3.2 OPERATOR SPACES.

The reader is referred in this section to **[Ru]** for further details, also to **[ER1,Ef2,Pn]**.

Let X be a linear space. Then for each $n \in \mathbb{N}$ the linear space $\mathcal{M}_n(X)$ of $n \times n$ matrices with entries in X is an $\mathcal{M}_n$ - bimodule in the obvious fashion. If $A \in \mathcal{M}_n(X)$, and $B \in \mathcal{M}_m(X)$ we may define the *direct sum* $A \oplus B$ in $\mathcal{M}_{n+m}(X)$ by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} .$$

We write $\mathcal{M}_{n,m}(X)$ for the linear space of $n \times m$ matrices with entries in X . This is a left $\mathcal{M}_n$ - module and a right $\mathcal{M}_m$ - module, and may be identified with a subspace of $\mathcal{M}_{\max\{n,m\}}(X)$ in the obvious way.

3.2.1 ***Definition.*** Let X be a linear space, and suppose that for each $n \in \mathbb{N}$ there is a norm $\|\cdot\|_n$ specified on $\mathcal{M}_n(X)$, such that for all A , $B \in \mathcal{M}_n(X)$ and Λ_1 , $\Lambda_2 \in \mathcal{M}_n$ the two conditions

(i) $\| \Lambda_1 A \Lambda_2 \|_n \leq \|\Lambda_1\| \, \|A\|_n \, \|\Lambda_2\|$

and (ii) $\| A \oplus B \|_{n+m} = \max \{ \|A\|_n , \|B\|_m \}$

hold. Then we say that $\{ \|\cdot\|_n \}$ is an *L^∞-matricial structure for* X , and that $(X , \|\cdot\|_n)$ is an *L^∞-matricial vector space.* Often we shall simply write X or $(X , \|\cdot\|)$ for $(X , \|\cdot\|_n)$ if there is no danger of confusion.

3.2.2 **EXAMPLE.** Let $\mathcal{H}$ be a Hilbert space, and suppose that X is a linear subspace of $B(\mathcal{H})$. Then X , together with the attendant norms $\|\cdot\|_n$ on $\mathcal{M}_n(X)$ inherited from $B(\mathcal{H}^{(n)})$, is an L^∞-matricial vector space. We call such an L^∞-matricial vector space $(X , \|\cdot\|_n)$ an *operator space.*

Let Y be a linear subspace of an L^∞-matricial vector space X . Clearly Y is again an L^∞-matricial vector space. One may also verify **[Ru]** that X / Y is an L^∞-matricial vector space with respect to the *quotient matricial norms* obtained from the identifications

$$\mathcal{M}_n(X / Y) = \mathcal{M}_n(X) / \mathcal{M}_n(Y) .$$

Let $(X , \|\cdot\|_n)$ and $(Y , \|\cdot\|_n)$ be two L^∞-matricial vector spaces and suppose $T : X \to Y$ is a linear map. If there exists a positive constant K such that

$$\| [T(a_{ij})] \|_n \leq K \|A\|_n ,$$

for $A \in \mathcal{M}_n(X)$, then T is said to be *completely bounded*, and we put $\|T\|_{cb}$ to be the least such K which will suffice. If $\|T\|_{cb} \leq 1$ then we say T is *completely contractive*. If T has an inverse defined on its range, and if T and T^{-1} are completely bounded, then we say T is *completely bicontinuous*. If in addition T and T^{-1} are completely contractive then T is said to be a *complete isometry*.

More generally we can define completely bounded multilinear maps of operator spaces by mimicking Definition 3.1.8.

The following theorem due to Z-J. Ruan shows that all L^∞-matricial vector spaces are operator spaces, and consequently provides an abstract characterization of operator spaces.

3.2.3 THEOREM [Ru]. *Let* $(X , \|\cdot\|_n)$ *be an* L^∞*-matricial vector space. Then there exists a Hilbert space* $\mathcal{H}$ *and a complete isometry of* X *into* $B(\mathcal{H})$.

3.2.4 COROLLARY. *Let* X *be an operator space, and* Y *a linear subspace of* X . *Then* X / Y *, with the quotient matricial norms, is an operator space.*

By the Bourbaki-Alaoglu Theorem ([**Conw**] Ex. 5.3.3) if E is a normed linear space then there is a compact Hausdorff space Ω such that $E \subset C(\Omega)$ isometrically. Now $C(\Omega) \otimes_\lambda \mathcal{M}_n$ is the C^*-algebra $\mathcal{M}_n(C(\Omega))$ [**Ta**], and consequently giving the spaces $\mathcal{M}_n(E) = E \otimes \mathcal{M}_n$ the injective tensor norm makes E into an operator space, completely isometrically contained in the operator space $C(\Omega)$. Conversely, operator spaces contained completely isometrically in a commutative C^*-algebra may be described by the construction above. In addition there is the following result:

3.2.5 PROPOSITION [Pn]. *Let* X *be an operator space, let* Ω *be a compact Hausdorff space, and let* $T : X \to C(\Omega)$ *be a bounded linear map. Then* T *is completely bounded and* $\|T\|_{cb} = \|T\|$.

Let INJ be the functor which takes a Banach space E to the operator space whose L^∞-matricial structure $\{ \|\cdot\|_n \}$ is specified by taking $\|\cdot\|_n$ to be the injective norm on $E \otimes \mathcal{M}_n$, and which takes a bounded linear map T between Banach spaces to the same map between the corresponding operator spaces. Then INJ is a full embedding of the category of Banach spaces and bounded linear maps onto a full subcategory of the category of operator spaces and completely bounded maps.

Thus we may regard normed linear spaces as the 'commutative operator spaces', or, conversely, regard the theory of operator spaces and completely bounded maps as 'non - commutative functional analysis' [**Ef2**]. Under this meta-transformation 'normed spaces' $\to$ 'operator spaces' the complex scalars become $B(\mathcal{H})$ in some sense. Often theorems from functional analysis carry over under this transformation to theorems about operator spaces.

The pièce de résistance of this analogy is the following theorem, the Arveson-Wittstock-Hahn-Banach theorem, so called by analogy with the ordinary Hahn-Banach theorem. Many proofs of this result have appeared in the literature, each succeeding proof more elementary [**Pn**]. The proof given here is due to E. G. Effros, and was presented at the Durham International Symposium on Operator Algebras in 1987. We give simpler proofs of his two lemmas (one of which is due originally to R.R. Smith).

3.2.6 THEOREM (Arveson-Wittstock-Hahn-Banach). *Let* X *be an operator space which is contained in an operator space* Y . *Suppose* $\mathcal{H}$ *is a Hilbert space and* $T : X \to B(\mathcal{H})$ *is a completely bounded linear map. Then* T *extends to a completely bounded map* $T^{\sim} : Y \to B(\mathcal{H})$ *with* $\|T^{\sim}\|_{cb} = \|T\|_{cb}$.

First we establish some notation. For the moment let $\mathcal{H} = \mathbb{C}^n$, and write $\mathcal{H}^*$ for the Hilbert space dual to $\mathcal{H}$. Let X be an operator space, and let $(CB(X;\mathcal{M}_n) , \|\cdot\|_{cb})$ be the Banach space of completely bounded maps from X into $\mathcal{M}_n$. Now the pairing

$$< T , \zeta^* \otimes x \otimes \eta > = < T(x) \eta , \zeta >$$

gives a duality between $CB(X;\mathcal{M}_n)$ and $\mathcal{H}^* \otimes X \otimes \mathcal{H}$, and thus defines a semi-norm $\|\cdot\|_{\sim}$ on $\mathcal{H}^* \otimes X \otimes \mathcal{H}$. In fact $\|\cdot\|_{\sim}$ is a norm, because if $\| \Sigma_{i=1}^N \zeta_i^* \otimes x_i \otimes \eta_i \|_{\sim} = 0$ then in particular

$$\Sigma_{i=1}^N f(x_i) < \zeta_i , \zeta > < \eta , \eta_i > = 0 \quad ,$$

for each $f \in X^*$ and $\zeta , \eta \in \mathcal{H}$, and consequently

$$\| \Sigma_{i=1}^N \zeta_i^* \otimes x_i \otimes \eta_i \|_{\lambda} = 0 \ .$$

If $\mathcal{H}^* \tilde{\otimes} X \tilde{\otimes} \mathcal{H}$ is the completion of $\mathcal{H}^* \otimes X \otimes \mathcal{H}$ in $\|\cdot\|_\sim$ then

$$(\mathcal{H}^* \tilde{\otimes} X \tilde{\otimes} \mathcal{H})^* = CB(X;\mathcal{M}_n) .$$

If p is a positive integer, if $A \in \mathcal{M}_p(X)$, and if $\underset{\sim}{\zeta}, \underset{\sim}{\eta} \in \mathcal{H}^{(P)}$, then define

$$\underset{\sim}{\zeta}^* \times A \times \underset{\sim}{\eta} = \Sigma_{i,j=1}^{p} \zeta_i^* \otimes a_{ij} \otimes \eta_j .$$

This is an element of $\mathcal{H}^* \otimes X \otimes \mathcal{H}$. If $V \in \mathcal{H}^* \otimes X \otimes \mathcal{H}$ then certainly $\|V\|_\sim$ is dominated by the expression

$$\inf \{ \Sigma_{j=1}^{N} \|\underset{\sim}{\zeta}_j\| \, \|A_j\| \, \|\underset{\sim}{\eta}_j\| : V = \Sigma_{j=1}^{N} \underset{\sim}{\zeta}_j^* \times A_j \times \underset{\sim}{\eta}_j ;$$
$$A_j \in \mathcal{M}_{p_j}(X) ; \underset{\sim}{\zeta}_j , \underset{\sim}{\eta}_j \in \mathcal{H}^{(p_j)} \}$$

and so this expression defines a norm on $\mathcal{H}^* \otimes X \otimes \mathcal{H}$. Now the Hahn-Banach theorem shows that this new norm is the same as $\|\cdot\|_\sim$.

Write $V \in \mathcal{H}^* \otimes X \otimes \mathcal{H}$ as

$$V = \Sigma_{j=1}^{N} \underset{\sim}{\zeta}_j^* \times A_j \times \underset{\sim}{\eta}_j ,$$

with $A_j \in \mathcal{M}_{p_j}(X)$ and $\underset{\sim}{\zeta}_j , \underset{\sim}{\eta}_j \in \mathcal{H}^{(p_j)}$, such that

$$\Sigma_{j=1}^{N} \|\underset{\sim}{\zeta}_j\| \, \|A_j\| \, \|\underset{\sim}{\eta}_j\| \le \|V\|_\sim + \epsilon .$$

By adding in zero entries if necessary we may assume that $p_1 = \ldots = p_N$ (= p, say). We may also assume without loss of generality that $\|A_j\| = 1$, and that $\|\underset{\sim}{\zeta}_j\| = \|\underset{\sim}{\eta}_j\|$ for each $j = 1,\ldots,N$. Define $\underset{\sim}{\zeta}$ and $\underset{\sim}{\eta} \in \mathcal{H}^{(N\times p)}$ to be the concatenations of the $\underset{\sim}{\zeta}_j$ and the $\underset{\sim}{\eta}_j$ respectively, and define a matrix $A = A_1 \oplus \ldots \oplus A_N$ in $\mathcal{M}_{(N\times p)}(X)$. Then $V = \underset{\sim}{\zeta}^* \times A \times \underset{\sim}{\eta}$ and $\|\underset{\sim}{\zeta}\| \, \|A\| \, \|\underset{\sim}{\eta}\| \le \|V\|_\sim + \epsilon$. Thus we may as well take $N = 1$ in the infimum above. Now notice that by the next lemma we can in fact take $p = n$ for the representations considered in the infimum.

3.2.7 LEMMA. *If* $p \geq n$ *, and if* $\eta_1, \ldots, \eta_p \in \mathbb{C}^n$ *, then there exists* $\xi_1, \ldots, \xi_n \in \mathbb{C}^n$ *, and a unitary matrix* $U \in \mathcal{M}_p$ *, with*

$$\eta_i = \Sigma_{j=1}^n u_{ij} \xi_j .$$

Proof of lemma. Let A be the $p \times p$ matrix with the η_i as rows (inserting zeroes in the last columns). Then by the polar decomposition in finite dimensions we may write $A = U (A^* A)^{1/2}$, where U is unitary. Clearly $(A^* A)^{1/2}$ consists of an $n \times n$ block in the top left hand corner and zeroes elsewhere. Take $\xi_1, \ldots, \xi_n$ to be the first n rows of $(A^* A)^{1/2}$ (ignoring the last $(p-n+1)$ zero columns). □

Thus we have established for $V \in \mathcal{H}^* \tilde{\otimes} X \tilde{\otimes} \mathcal{H}$ that

$$\|V\|_\sim = \inf\{\|\underset{\sim}{\zeta}\| \; \|A\| \; \|\underset{\sim}{\eta}\| : V = \underset{\sim}{\zeta}^* \times A \times \underset{\sim}{\eta} \; ; \; A \in \mathcal{M}_n(X) \; ; \; \underset{\sim}{\zeta} , \underset{\sim}{\eta} \in \mathcal{H}^{(n)}\} .$$

Proof of Theorem 3.2.6. As usual **[Pn]** it suffices to prove the theorem for all finite dimensional subspaces $\mathcal{F}$ of $\mathcal{H}$. For then letting $T_{\mathcal{F}} = P_{\mathcal{F}} T(\cdot)|_{\mathcal{F}}$, and extending to an operator $\tilde{T}_{\mathcal{F}}$ defined on Y, we obtain a bounded net of operators $\{ \tilde{T}_{\mathcal{F}} \}$ in $B(X;B(\mathcal{H}))$ directed by the finite dimensional subspaces $\mathcal{F}$ of $\mathcal{H}$. Now there exists a sub-net convergent in the bounded weak topology **[Pn]** to a limit which has the desired property.

Thus we may assume that $\mathcal{H} = \mathbb{C}^n$. If we can show that $\mathcal{H}^* \tilde{\otimes} X \tilde{\otimes} \mathcal{H} \subset \mathcal{H}^* \tilde{\otimes} Y \tilde{\otimes} \mathcal{H}$ isometrically then an application of the ordinary Hahn-Banach theorem completes the proof. To this purpose let $V \in \mathcal{H}^* \tilde{\otimes} X \tilde{\otimes} \mathcal{H}$ be given, with $\|V\|_{\mathcal{H}^* \tilde{\otimes} Y \tilde{\otimes} \mathcal{H}} < 1$. Thus we can write $V = \underset{\sim}{\zeta}^* \otimes B_0 \otimes \underset{\sim}{\eta}$, with $B_0 \in \mathcal{M}_n(Y)$ and $\underset{\sim}{\zeta}, \underset{\sim}{\eta} \in \mathcal{H}^{(n)}$, such that $\|\underset{\sim}{\zeta}\| = \|\underset{\sim}{\eta}\| = 1$, and $\|B_0\| < 1$. Decompose $\underset{\sim}{\zeta}$ as a direct sum of n vectors $\zeta_1,\ldots,\zeta_n$ each in $\mathcal{H}$, similarly write $\underset{\sim}{\eta} = \eta_1 \oplus \ldots \oplus \eta_n$. We can assume that $\zeta_1,\ldots,\zeta_n$ are linearly independent, for if they were not proceed as follows. Let A be the matrix with columns $\zeta_1,\ldots,\zeta_n$, and using the polar decomposition and spectral theorem in finite dimensions write

$$A = U \operatorname{diag}\{\lambda_1,\ldots,\lambda_k,0,\ldots,0\} V ,$$

where U and V are unitary and each $\lambda_i > 0$. Let ζ_i' be the i'th column of

$$(1+(n-k)\epsilon^2)^{-\frac{1}{2}} U \operatorname{diag}\{\lambda_1,\ldots,\lambda_k,\epsilon,\ldots,\epsilon\} V ,$$

where ϵ is small enough to ensure $(1+(n-k)\epsilon^2)^{\frac{1}{2}} \|B_0\| < 1$. It is not difficult to see that $\{ \Sigma_{i=1}^n \|\zeta_i'\|^2 \}^{\frac{1}{2}} = 1$, and that the ζ_i' are linearly independent. If $\zeta_i = \Sigma_{j=1}^n \alpha_{ij} \zeta_j'$ we have

$$\|[\alpha_{ij}]\| \leq (1+(n-k)\epsilon^2)^{\frac{1}{2}} ,$$

and $V = \underset{\sim}{\zeta}'^* \otimes [\overline{\alpha_{ji}}] B_0 \otimes \underset{\sim}{\eta}$. Similarly we can assume $\eta_1,\ldots,\eta_n$ are linearly independent.

Define $\theta_X : \mathcal{M}_n(X) \to \mathcal{H}^* \tilde{\otimes} X \tilde{\otimes} \mathcal{H} : A \mapsto \underset{\sim}{\zeta}^* \times A \times \underset{\sim}{\eta}$; and

$$\theta_Y : \mathcal{M}_n(Y) \to \mathcal{H}^* \tilde{\otimes} X \tilde{\otimes} \mathcal{H} : B \mapsto \underset{\sim}{\zeta}^* \times B \times \underset{\sim}{\eta} .$$

These are isomorphisms of vector spaces, and the diagram below commutes.

$$\begin{array}{ccc}
\mathcal{M}_n(X) & \xrightarrow{\text{inclusion}} & \mathcal{M}_n(Y) \\
\downarrow \theta_X & & \downarrow \theta_Y \\
\mathcal{K}^* \tilde{\otimes} X \tilde{\otimes} \mathcal{K} & \xrightarrow[\text{inclusion}]{} & \mathcal{K}^* \tilde{\otimes} Y \tilde{\otimes} \mathcal{K}
\end{array}$$

Now if $\theta_X(A_0) = V$ then we see that $A_0 = B_0$, and so $\|V\|_{\mathcal{K}^* \tilde{\otimes} X \tilde{\otimes} \mathcal{K}} < 1$. This completes the proof. □

3.3 OPERATOR SPACE TENSOR NORMS.

3.3.1 ***Definition.*** An *operator space tensor norm* α is an assignment of an L^∞-matricial structure $\{ \alpha_n \}$ to the algebraic tensor product $X \otimes Y$ of X and Y , for every pair of operator spaces X and Y , such that

(i) α_1 is a cross norm , and

(ii) if $T_1 : X_1 \to Y_1$ and $T_2 : X_2 \to Y_2$ are completely bounded linear maps then $T_1 \otimes T_2 : X_1 \otimes X_2 \to Y_1 \otimes Y_2$ is completely bounded with respect to the L^∞-matricial structures $\{ \alpha_n \}$ on $X_1 \otimes X_2$ and $Y_1 \otimes Y_2$, and

$$\| T_1 \otimes T_2 \|_{cb} \le \|T_1\|_{cb} \|T_2\|_{cb} .$$

We sometimes write $X \otimes_\alpha Y$ for the operator space $(X \otimes Y , \alpha_n)$; we shall not be too particular about whether the α_n are completed or not. The norm α_1 shall sometimes be called the *commutative* α *norm*.

This is a very general definition of an operator space tensor norm. One might possibly require, as an additional 'cross norm' condition, that

$$\|A \times B\|_n \leq \|A\|_m \|B\|_m$$

whenever $A \in \mathcal{M}_{n,p}(X)$ and $B \in \mathcal{M}_{p,n}(Y)$; where and $m = \max\{n,p\}$. $A \times B$ is defined to be the matrix $[\Sigma_{k=1}^{p} a_{ik} \otimes b_{kj}]$ in $\mathcal{M}_n(X \otimes Y)$. If we did insist on this condition there is a 'biggest' operator space tensor norm, namely the operator space Haagerup tensor norm defined below. One might also require that there be a least operator space tensor norm, the spatial operator space tensor norm $\|\cdot\|_{min}$, also defined below.

In the light of Proposition 3.2.5 and the remarks after Corollary 3.2.4 the notion of an operator space tensor norm generalizes the notion of a reasonable Banach space tensor norm (Definition 1.2.1).

If $X \subset B(\mathcal{H})$ and $Y \subset B(\mathcal{K})$ are operator spaces one can define the *spatial operator space tensor norm* $\|\cdot\|_{min}$ on $X \otimes Y$ by giving $X \otimes Y$ the L^∞-matricial structure it inherits as a subspace of $B(\mathcal{H} \otimes \mathcal{K})$. This structure is independent of the specific Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ that X and Y were realized upon. Condition (ii) of Definition 3.3.1 is verified in **[Pn]** Theorem 10.3.

We can also define the *Haagerup operator space tensor norm* **[PnS]**. Namely if $U \in \mathcal{M}_n(X \otimes Y)$ define $\|U\|_h$ to be the expression

$$\inf \{\Sigma_{k=1}^{m} \|A_k\| \|B_k\| : U = \Sigma_{k=1}^{m} A_k \times B_k , A_k \in \mathcal{M}_{n,p}(X) , B_k \in \mathcal{M}_{p,n}(Y)\}$$

$$= \inf \{ \|A\| \|B\| : U = A \times B , A \in \mathcal{M}_{n,p}(X) , B \in \mathcal{M}_{p,n}(Y) \} ,$$

where $\times$ is as defined above. It is not difficult to see that $\|\cdot\|_h \geq \|\cdot\|_{min}$, and thus $\|\cdot\|_h$ is a norm. It is easily checked **[Ru]** that $(X \otimes Y, \|\cdot\|_h)$ is an L^∞-matricial vector space and consequently an operator space. If X and Y are contained in C^*-algebras $\mathcal{A}$ and

$\mathcal{B}$ respectively then one can write down an explicit complete isometry of $X \otimes_h Y$ into the C^*-algebraic free product of $\mathcal{A}$ and $\mathcal{B}$ [ChSE]. The commutative Haagerup norm on C^*-algebras is what was called the Haagerup norm in Section 3.1.

One can easily check that the Haagerup tensor product is associative; i. e. if X_1, X_2, and X_3 are operator spaces then

$$(X_1 \otimes_h X_2) \otimes_h X_3 = X_1 \otimes_h (X_2 \otimes_h X_3)$$

as operator spaces; thus there is no confusion in writing

$$X_1 \otimes_h X_2 \otimes_h X_3 .$$

Just as in the C^*-algebra case, the operator space Haagerup norm is the 'correct' tensor norm when considering completely bounded multilinear maps:

3.3.2 PROPOSITION. *Suppose* $X_1, \ldots, X_m$ *are operator spaces. If* $\mathcal{H}$ *is a Hilbert space, and if* $\Psi : X_1 \times \ldots \times X_m \to B(\mathcal{H})$ *is an* m*-linear map, then* Ψ *is completely bounded if and only if the associated linear map* $\psi : X_1 \otimes_h \ldots \otimes_h X_m \to B(\mathcal{H})$ *is completely bounded, and then we have*

$$\|\Psi\|_{cb} = \|\psi\|_{cb} .$$

There is a Christensen-Sinclair representation theorem for completely bounded multilinear maps on operator spaces. For other formulations of the representation theorem see [ChSE].

3.3.3 THEOREM [PnS]. *Suppose* $X_1, \ldots, X_m$ *are operator spaces, contained in unital* C^**-algebras* $\mathcal{A}_1, \ldots, \mathcal{A}_m$ *respectively.*

If $\mathcal{H}$ *is a Hilbert space and* $\Psi : X_1 \times \ldots \times X_m \to B(\mathcal{H})$ *is a completely bounded* m*-linear map then there are unital* *-*representations* $\pi_1, \ldots, \pi_m$ *of* $\mathcal{A}_1, \ldots, \mathcal{A}_m$ *on Hilbert spaces* $\mathcal{H}_1, \ldots, \mathcal{H}_m$ *respectively, and operators* $T_k \in B(\mathcal{H}_k, \mathcal{H}_{k-1})$ *for* $1 \le k \le m+1$ *, where* $\mathcal{H}_0 = \mathcal{H}_{m+1} = \mathcal{H}$ *, such that*

$$\|T_1\| \cdots \|T_m\| = \|\Psi\|_{cb} ,$$

and such that

$$\Psi(x_1,\ldots,x_m) = T_1 \, \pi_1(x_1) \, T_2 \cdots T_m \, \pi_m(x_m) \, T_{m+1} ,$$

for $x_1 \in X_1 , \ldots , x_m \in X_m$.

The Haagerup norm is injective in the following sense:

3.3.4 THEOREM [PnS]. *Suppose* X_1 *and* X_2 *are operator spaces, contained in operator spaces* Y_1 *and* Y_2 *respectively. Then* $X_1 \otimes_h X_2$ *is contained as an operator space in* $Y_1 \otimes_h Y_2$.

We now introduce a new tensor norm related to the Haagerup norm; the class of associated multilinear maps having a Christensen-Sinclair type representation, but with Jordan *-homomorphisms taking the place of the usual *-representations.

3.3.5 ***Definition.*** Let X be an operator space, contained in a C^*-algebra $\mathcal{A}$. Let $\mathcal{A}^o$ be the opposite C^*-algebra of $\mathcal{A}$ (i. e. the C^*-algebra with the same Banach space structure and involution as $\mathcal{A}$ but with the reversed multiplication), and write $a \mapsto a^o$ for the identity map $\mathcal{A} \to \mathcal{A}^o$. It is easy to see that the transpose map

$$ {}^t : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{A}^\circ) $$

is an isometric *-anti-isomorphism.

Define the *symmetrized space of* X to be the operator space

$$ SYM(X) = \{ x \oplus x^\circ \in \mathcal{A} \oplus \mathcal{A}^\circ : x \in X \} . $$

As an operator space this is independent of the particular C^*-algebra $\mathcal{A}$ containing X .

3.3.6 *Definition.* Define the *operator space symmetrized Haagerup norm* $\|\cdot\|_{sh}$ to be the L^∞-matricial structure on the tensor product $X \otimes Y$ of two operator spaces X and Y whose value on $U \in \mathcal{M}_n(X \otimes Y)$ is

$$ \|U\|_{sh} = \inf \{ \max(\|A\|,\|A^t\|) \max(\|B\|,\|B^t\|) : U = A \times B , \quad A \in \mathcal{M}_{n,p}(X) , B \in \mathcal{M}_{p,n}(Y) \} . $$

The sub-additivity is proven as for the Haagerup norm, and it is clear that $\|\cdot\|_{sh}$ dominates the Haagerup norm, thus $\|\cdot\|_{sh}$ is indeed a norm. The commutative symmetrized Haagerup norm is given on $u \in X \otimes Y$ by:

$$ \inf \{\max (\|\Sigma_{i=1}^n x_i x_i^*\|,\|\Sigma_{i=1}^n x_i^* x_i\|)^{1/2} \max (\|\Sigma_{i=1}^n y_i y_i^*\|,\|\Sigma_{i=1}^n y_i^* y_i\|)^{1/2} : u = \Sigma_{i=1}^n x_i \otimes y_i\} . $$

In fact we shall see next that it is unnecessary to explicitly verify that $\|\cdot\|_{sh}$ is a norm. The crucial fact about the symmetrized Haagerup norm is the following observation:

3.3.7 PROPOSITION. *Let* X *and* Y *be operator spaces. The map* $X \otimes_{sh} Y \to SYM(X) \otimes_h SYM(Y)$ *defined by*

$$x \otimes y \mapsto (x \oplus x^o) \otimes (y \oplus y^o)$$

is a complete isometry.

Thus we could have used Proposition 3.3.7 to define $\|\cdot\|_{sh}$.

3.3.8 ***Definition.*** Let X and Y be operator spaces, let $\mathcal{L}$ be a Hilbert space and let $\Psi : X \times Y \to B(\mathcal{L})$ be a bilinear map. We shall say that Ψ is *Jordan completely bounded* if there is a constant $K > 0$ with

$$\|\Psi_n(A,B)\| \leq K \max\{ \|A\| , \|A^t\| \} \max \{ \|B\| , \|B^t\| \} ,$$

for all $A \in \mathcal{M}_n(X)$ and $B \in \mathcal{M}_n(Y)$. In this case we put $\|\Psi\|_{Jcb}$ equal to the least such K which suffices.

3.3.9 ***Definition.*** Let X and Y be operator spaces , let $\mathcal{L}$ be a Hilbert space and let $\Psi : X \times Y \to B(\mathcal{L})$ be a bilinear mapping. We shall say that Ψ is *Jordan representable* if there exists a *Jordan representation* of Ψ : i. e. if X and Y are subspaces of unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, then there exist Hilbert spaces $\mathcal{H}_+$, $\mathcal{H}_-$ and $\mathcal{K}_+$, $\mathcal{K}_-$, unital *-representations θ_+ and π_+ of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}_+$ and $\mathcal{K}_+$ respectively, unital *-anti-representations θ_- and π_- of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}_-$ and $\mathcal{K}_-$ respectively, an operator R from $\mathcal{H}_+ \oplus \mathcal{H}_-$ to $\mathcal{L}$, an operator S from $\mathcal{K}_+ \oplus \mathcal{K}_-$ to $\mathcal{H}_+ \oplus \mathcal{H}_-$, and an operator T from $\mathcal{L}$ to $\mathcal{K}_+ \oplus \mathcal{K}_-$, such that

$$\Psi(x,y) = R\ (\theta_+ \oplus \theta_-)(x)\ S\ (\pi_+ \oplus \pi_-)(y)\ T ,$$

for each $x \in X$, $y \in Y$.

In this case we put $\|\Psi\|_{Jrep}$ equal to the infimum of $\|R\| \|S\| \|T\|$ taken over all such Jordan representations of Ψ.

3.3.10 THEOREM. *Let* X *and* Y *be operator spaces, let* $\mathcal{L}$ *be a Hilbert space, and suppose* $\Psi : X \times Y \to B(\mathcal{L})$ *is a bilinear map. The following are equivalent*

(i) Ψ *is Jordan completely bounded with* $\|\Psi\|_{Jcb} \leq 1$,

(ii) the linear operator $X \otimes_{sh} Y \to B(\mathcal{L})$ *induced by* Ψ *is completely contractive,*

(iii) Ψ *is Jordan representable with* $\|\Psi\|_{Jrep} \leq 1$.

Proof. The equivalence of (i) and (ii) is easy, as is the fact that (iii) implies (ii) .

Now suppose $\psi : X \otimes_{sh} Y \to B(\mathcal{L})$ is completely contractive, and suppose X and Y are contained in unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. By Proposition 3.3.7 ψ induces a completely contractive map $SYM(X) \otimes_{sh} SYM(Y) \to B(\mathcal{L})$, and by Theorem 3.3.3 there exist unital *-representations θ and π of $\mathcal{A} \oplus \mathcal{A}^o$ and $\mathcal{B} \oplus \mathcal{B}^o$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, and bounded linear operators $R : \mathcal{H} \to \mathcal{L}$, $S : \mathcal{K} \to \mathcal{H}$, and $T : \mathcal{L} \to \mathcal{K}$, such that

$$\psi(x \otimes y) = R\ \theta(x \oplus x^o)\ S\ \pi(y \oplus y^o)\ T \quad ,$$

for each $x \in X$, $y \in Y$.

Let $\mathcal{H}_+$ be the closure in $\mathcal{H}$ of the subspace of $\mathcal{H}$ spanned by elements of the form $\theta(a \oplus 0)\ \zeta$ for $a \in \mathcal{A}$, $\zeta \in \mathcal{H}$. Similarly let $\mathcal{H}_-$ be the closure in $\mathcal{H}$ of the subspace of $\mathcal{H}$ spanned by elements of the form $\theta(0 \oplus a^o)\ \zeta$ for $a \in \mathcal{A}$, $\zeta \in \mathcal{H}$. Define a unital

*-representation θ_+ of $\mathcal{A}$ on $\mathcal{H}_+$ by

$$\theta_+(\cdot) = P_{\mathcal{H}_+} \theta(\cdot \oplus 0)|_{\mathcal{H}_+} ,$$

and define a unital *-anti-representation of $\mathcal{A}$ on $\mathcal{H}_-$ by

$$\theta_-(\cdot) = P_{\mathcal{H}_-} \theta(0 \oplus \cdot^{\circ})|_{\mathcal{H}_-} .$$

It is easy to see that $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\theta = \theta_+ \oplus \theta_-$. Define subspaces $\mathcal{K}_+$ and $\mathcal{K}_-$ of $\mathcal{K}$, and representations π_+ and π_- in an analagous fashion. We have for $x \in X$ and $y \in Y$ that

$$\psi(x \otimes y) = R\ (\theta_+ \oplus \theta_-)(x)\ S\ (\pi_+ \oplus \pi_-)(y)\ T \quad . \qquad \square$$

It is clear that the results of 3.3.2, 3.3.3 and 3.3.4 above carry over to the multilinear analogues of the symmetrized Haagerup norm and the corresponding class of completely contractive maps. Thus we get Hahn-Banach type extension theorems for a larger class of multilinear maps than we had before.

The commutative symmetrized Haagerup norm is discussed further in Section 4.2.

CHAPTER 4. GEOMETRY OF THE TENSOR PRODUCT OF C^*-ALGEBRAS.

Recall that when $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras their algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ is a $*$-algebra in a natural way. Until recently, work on tensor products of C^*-algebras has concentrated on C^*-tensor norms; i. e. norms α which make the completion $\mathcal{A} \otimes_\alpha \mathcal{B}$ into a C^*-algebra. The crucial role played by the Haagerup norm in the theory of operator spaces and completely bounded maps has produced some interest in more general norms (see for instance Chapter 3, [**KaS**] and [**It**]). In this chapter we investigate geometrical properties of algebra norms on $\mathcal{A} \otimes \mathcal{B}$, as well as discussing some particular tensor norms and their geometrical relationships.

The theory of tensor products of Banach spaces following on from A. Grothendieck's fundamental papers [**Gr1,Gr2**] studies so called 'reasonable' norms (see 1.2). These are norms α satisfying a certain uniformity condition

$$\alpha(S \otimes T (u)) \leq \|S\| \alpha(u) \|T\|$$

for all bounded linear operators S and T between Banach spaces. No C^*-tensor norm is reasonable in this sense - to see this consider the $*$-algebra $\mathcal{M}_n(\mathcal{M}_n) = \mathcal{M}_n \otimes \mathcal{M}_n$ on which all C^*-norms coincide; the transpose map $^t : \mathcal{M}_n \to \mathcal{M}_n$ is an isometry, however the map

$$^t \otimes I_{\mathcal{M}_n} : \mathcal{M}_n \otimes_{min} \mathcal{M}_n \to \mathcal{M}_n \otimes_{min} \mathcal{M}_n$$

can easily be shown [**Ok**] to have norm n. In 4.1 we introduce a uniformity condition appropriate to tensor norms of C^*-algebras, namely in the condition above we require the maps S and T to be completely positive linear operators between C^*-algebras; a norm α which satisfies this condition shall be called *completely positive uniform*. If $\mathcal{A}$ is a nuclear C^*-algebra the canonical map $\mathcal{A} \otimes_\alpha \mathcal{B} \to \mathcal{A} \otimes_\lambda \mathcal{B}$ is shown to be injective for all C^*-algebras $\mathcal{B}$ and tensor norms α which are completely positive uniform.

In 4.3 and 4.4 we consider completely positive uniform algebra tensor norms α. In Theorem 4.3.3 we prove that for such an α *either* $\mathcal{A} \otimes_\alpha \mathcal{B}$ is a C^*-algebra for all C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, *or* $\mathcal{A} \otimes_\alpha \mathcal{B}$ is never a C^*-algebra unless $\mathcal{A}$ or $\mathcal{B}$ is $\mathbb{C}$. To prove this we use the characterizations of C^*-norms that we established in Chapter 3. It is shown in Theorem 4.4.2 that for α as above there is actually a dichotomy for Hermitian elements: if $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras then the set of Hermitian elements in $\mathcal{A} \otimes_\alpha \mathcal{B}$ is *either* a spanning set *or* is as small as it could possibly be. Thus for α again as above, if we wish to calculate the Hermitian elements of $\mathcal{A} \otimes_\alpha \mathcal{B}$ for arbitrary C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ it suffices to consider the first non-trivial tensor product $\ell_2^\infty \otimes_\alpha \ell_2^\infty$; where ℓ_2^∞ is the two dimensional C^*-algebra.

4.1 NORMS ON THE TENSOR PRODUCT OF C^*-ALGEBRAS.

We begin with some results about C^*-tensor norms. Good surveys of the theory of C^*-tensor norms and aspects of nuclearity may be found in [**La3**,**To**].

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, and let $PS(\mathcal{A} \otimes_\gamma \mathcal{B})$ be the set of positive states of $\mathcal{A} \otimes_\gamma \mathcal{B}$, i. e. those states ψ for which $\psi(u^*u) \geq 0$ for each $u \in \mathcal{A} \otimes_\gamma \mathcal{B}$. The GNS construction assigns in a canonical fashion a cyclic *-representation π_ψ of $\mathcal{A} \otimes_\gamma \mathcal{B}$ on a Hilbert space to each element $\psi \in PS(\mathcal{A} \otimes_\gamma \mathcal{B})$. We shall say a subset Γ of $PS(\mathcal{A} \otimes_\gamma \mathcal{B})$ is *separating* if

$$p_\Gamma(\cdot) = \sup \{ \| \pi_\psi(\cdot) \| : \psi \in \Gamma \}$$

is a norm on $\mathcal{A} \otimes \mathcal{B}$. We call a subset Γ of $PS(\mathcal{A} \otimes_\gamma \mathcal{B})$ a C^**-set* if it is convex, weak*-closed and separating, and for all $\psi \in \Gamma$ and $u \in \mathcal{A} \otimes \mathcal{B}$ with $\psi(u^* u) \neq 0$ the state φ defined by

$$\varphi(v) = \psi(u^* v u) / \psi(u^* u) \quad ,$$

for $v \in \mathcal{A} \otimes \mathcal{B}$, is an element of Γ .

4.1.1 **THEOREM** [EL]. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be unital* C^**-algebras. There is a bijective correspondence between* C^**-norms* α *on* $\mathcal{A} \otimes \mathcal{B}$ *and* C^**-sets* Γ *of* $PS(\mathcal{A} \otimes_\gamma \mathcal{B})$ *, given by* $\alpha \to \Gamma_\alpha$ *, and* $\Gamma \to \alpha_\Gamma$ *, where*

$$\Gamma_\alpha = \{ \psi \in PS(\mathcal{A} \otimes_\gamma \mathcal{B}) : |\psi(u)| \leq \alpha(u) \ \textit{for all} \ u \in \mathcal{A} \otimes \mathcal{B} \} \quad ,$$

and

$$\alpha_\Gamma(u) = \sup \{ \psi(u^* u) : \psi \in \Gamma \} \qquad (u \in \mathcal{A} \otimes \mathcal{B}) .$$

If $\Gamma = PS(\mathcal{A} \otimes_\gamma \mathcal{B})$ then $\alpha_\Gamma = \|\cdot\|_{max}$, whereas if Γ is $PS(\mathcal{A} \otimes_\gamma \mathcal{B}) \cap (\mathcal{A}^* \otimes \mathcal{B}^*)$ then $\alpha_\Gamma = \|\cdot\|_{min}$.

Theorem 4.1.1 shows us that the set of C^*-norms on $\mathcal{A} \otimes \mathcal{B}$ has a natural lattice structure; if α_1 and α_2 are C^*-norms on $\mathcal{A} \otimes \mathcal{B}$ then for $u \in \mathcal{A} \otimes \mathcal{B}$

$$\alpha_1 \vee \alpha_2 \ (u) = \max \{ \alpha_1(u) \ , \ \alpha_2(u) \}$$

as one might expect.

The following three propositions are useful when attempting to extend results about tensor products of unital C^*-algebras to the non-unital case. We assume throughout this chapter that all approximate identities are contractive.

4.1.2 PROPOSITION. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* C^**-algebras and let* $\mathcal{A} \otimes \mathcal{B}$ *be their algebraic tensor product. If* (e_ν) *and* (f_μ) *are two-sided approximate identities for* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively, then* $(e_\nu \otimes f_\mu)$ *is a two-sided approximate identity in* $\mathcal{A} \otimes_\alpha \mathcal{B}$ *whenever* α *is an algebra cross norm on* $\mathcal{A} \otimes \mathcal{B}$.

Proof. Let $u \in \mathcal{A} \otimes_\alpha \mathcal{B}$ and $\epsilon > 0$ be given, and suppose $\alpha(u - \Sigma_{i=1}^n a_i \otimes b_i) < \epsilon/3$. Then

$$\begin{aligned} \alpha(u - (e_\nu \otimes f_\mu)\ u) &< 2\epsilon/3 + \alpha(\Sigma_{i=1}^n a_i \otimes b_i - (e_\nu \otimes f_\mu)\ \Sigma_{i=1}^n a_i \otimes b_i) \\ &\leq 2\epsilon/3 + \Sigma_{i=1}^n \alpha(a_i \otimes b_i - (e_\nu \otimes f_\mu)\ (a_i \otimes b_i)) \\ &< \epsilon \end{aligned}$$

for $\lambda \geq \lambda_0$, $\mu \geq \mu_0$ say. Thus $(e_\nu \otimes f_\mu)$ is a left approximate identity. A similar argument shows that it is also a right approximate identity. □

4.1.3. PROPOSITION. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* C^**-algebras with approximate identities* (e_ν) *and* (f_μ) *respectively, and let* $\mathcal{A}^1$ *and* $\mathcal{B}^1$ *be the* C^**-unitizations of* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively. If* $u \in \mathcal{A}^1 \otimes \mathcal{B}^1$ *satisfies* $u\ (e_\nu \otimes f_\mu) = 0$ *for all* λ , μ *then* $u = 0$.

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ are represented non-degenerately on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. It is clear that (e_ν) and (f_μ) converge in the strong operator topology to the identity maps on $\mathcal{H}$ and $\mathcal{K}$ respectively, and thus $(e_\nu \otimes f_\mu)$ converges in the strong operator topology on $B(\mathcal{H} \otimes \mathcal{K})$ to the identity map on $\mathcal{H} \otimes \mathcal{K}$. Now $\mathcal{A}^1 \otimes \mathcal{B}^1$ is represented naturally on $\mathcal{H} \otimes \mathcal{K}$; and for $\zeta \in \mathcal{H}$, $\eta \in \mathcal{K}$ we have

$$u\ (\zeta \otimes \eta) = \lim_{(\nu,\mu)} u\ (e_\nu \otimes f_\mu)\ (\zeta \otimes \eta) = 0\ .$$

Thus $u = 0$. □

4.1.4. PROPOSITION. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, and let $\mathcal{A}^1$ and $\mathcal{B}^1$ be their C^*-unitizations. If α is an algebra cross norm on $\mathcal{A} \otimes \mathcal{B}$ then there is an algebra cross norm $\alpha^\sim$ on $\mathcal{A}^1 \otimes \mathcal{B}^1$ extending α, given by*

$$\alpha^\sim(u) = \sup\ \{\ \alpha(uv)\ :\ v \in \mathcal{A} \otimes \mathcal{B}\ ,\ \alpha(v) \leq 1\ \}\ ,$$

*for $u \in \mathcal{A}^1 \otimes \mathcal{B}^1$. If α is an algebra *-norm then so is $\alpha^\sim$, and if α is a C^*-norm then $\alpha^\sim$ is the unique C^*-norm on $\mathcal{A}^1 \otimes \mathcal{B}^1$ extending α.*

Proof. Let (e_ν) and (f_μ) be positive two-sided approximate identities for $\mathcal{A}$ and $\mathcal{B}$ respectively. From 1.1.1 we have the identities

$$\begin{aligned}\sup\ \{\ \alpha(uv)\ :\ v \in \mathcal{A} \otimes \mathcal{B}\ ,\ \alpha(v) \leq 1\ \} &= \lim_{\nu,\mu} \alpha(u\ (e_\nu \otimes f_\mu)) \\ &= \lim_{\nu,\mu} \alpha((e_\nu \otimes f_\mu)\ u\ (e_\nu \otimes f_\mu))\ .\end{aligned}$$

That $\alpha\sim(u) = 0$ implies $u = 0$ follows from the first identity and Proposition 4.1.3. The second identity shows that $\alpha^\sim$ is a *-algebra (C^*-) norm if α is a *-algebra (C^*-) norm. The uniqueness of extension of C^*-norms is shown in [**La2**]. □

4.1.5. PROPOSITION. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* C^**-algebras. If* α *is a norm on* $\mathcal{A} \otimes \mathcal{B}$ *such that* $\mathcal{A} \otimes_\alpha \mathcal{B}$ *is a Banach *-algebra* (with isometric involution) *with respect to the usual multiplication and involution then* $\alpha \geq \|\cdot\|_{\min}$.

Proof. First suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras. Let $\psi \in PS(\mathcal{A} \otimes_\gamma \mathcal{B}) \cap (\mathcal{A}^* \otimes \mathcal{B}^*)$ be given, and write $\psi = \Sigma_{i=1}^n f_i \otimes g_i$, with $f_1,\dots,f_n \in \mathcal{A}^*$ and $g_1,\dots,g_n \in \mathcal{B}^*$. Then

$$|\psi(u)| \leq (\Sigma_{i=1}^n \|f_i\| \|g_i\|)\ \lambda(u) \leq (\Sigma_{i=1}^n \|f_i\| \|g_i\|)\ \alpha(u) \qquad (u \in \mathcal{A} \otimes \mathcal{B}),$$

and so ψ may be extended to a functional on $\mathcal{A} \otimes_\alpha \mathcal{B}$ such that $\psi(u^*u) \geq 0$ for all $u \in \mathcal{A} \otimes_\alpha \mathcal{B}$. Now since $\mathcal{A} \otimes_\alpha \mathcal{B}$ is a Banach *-algebra the remark after Corollary 37.9 in **[BoD3]** implies that ψ is a state on $\mathcal{A} \otimes_\alpha \mathcal{B}$. Thus for $u \in \mathcal{A} \otimes \mathcal{B}$ we have

$$\begin{aligned}\|\cdot\|_{\min} &= \sup \{\ \psi(u^*u)^{1/2} : \psi \in PS(\mathcal{A} \otimes_\gamma \mathcal{B}) \cap (\mathcal{A}^* \otimes \mathcal{B}^*)\ \} \\ &\leq \alpha(u^*u)^{1/2} \\ &\leq \alpha(u) \quad .\end{aligned}$$

Now suppose $\mathcal{A}$ and $\mathcal{B}$ are arbitary C^*-algebras. Let $\mathcal{A}^1$ and $\mathcal{B}^1$ be the C^*-unitizations of $\mathcal{A}$ and $\mathcal{B}$ respectively, and let $\alpha^\sim$ be the extension of α to a Banach *-algebra norm $\alpha^\sim$ on $\mathcal{A}^1 \otimes \mathcal{B}^1$ defined in Proposition 4.1.4. Now $\alpha^\sim \geq \|\cdot\|_{\min}$ on $\mathcal{A}^1 \otimes \mathcal{B}^1$ by the first part, and the injectivity of $\|\cdot\|_{\min}$ implies that $\alpha \geq \|\cdot\|_{\min}$ on $\mathcal{A} \otimes \mathcal{B}$. □

It would be interesting if $\|\cdot\|_{\min}$ was dominated by every algebra norm.

4.1.6 THEOREM [Pn,Ta]. *Let* $\mathcal{A}_1$, $\mathcal{A}_2$ *and* $\mathcal{B}_1$, $\mathcal{B}_2$ *be* C^**-algebras, and let* $T_i : \mathcal{A}_i \to \mathcal{B}_i$ *be completely positive linear maps* (i = 1,2). *If* α *is either* $\|\cdot\|_{\min}$ *or* $\|\cdot\|_{\max}$ *then* $T_1 \otimes T_2$ *extends to a completely positive map* $\mathcal{A}_1 \otimes_\alpha \mathcal{A}_2 \to \mathcal{B}_1 \otimes_\alpha \mathcal{B}_2$. *If* $S_i : \mathcal{A}_i \to \mathcal{B}_i$ *are completely bounded linear maps* (i = 1,2) *then* $S_1 \otimes S_2$ *extends to a completely bounded map*

$$S_1 \otimes_{\min} S_2 : \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \to \mathcal{B}_1 \otimes_{\min} \mathcal{B}_2 ,$$

with

$$\|S_1 \otimes_{\min} S_2\|_{cb} = \|S_1\|_{cb} \, \|S_2\|_{cb} \quad .$$

We shall want to regard a tensor norm as a bifunctor on the category of C^*-algebras, and we would like to tie together the way that the norm acts on different pairs of C^*-algebras, to rule out arbitrary allocation of norms to different pairs of C^*-algebras. Theorem 4.1.6 would seem to suggest a uniformity condition involving completely positive maps. Note that the norm $\|\cdot\|_{\max}$ does not behave well with respect to the tensor product of completely bounded maps [**Hu**].

4.1.7 ***Definition.*** A *tensor norm of* C^**-algebras* α is an assignment of a Banach space $\mathcal{A} \otimes_\alpha \mathcal{B}$ to each ordered pair ($\mathcal{A}$, $\mathcal{B}$) of C^*-algebras such that

(*i*) $\mathcal{A} \otimes_\alpha \mathcal{B}$ is the completion of $\mathcal{A} \otimes \mathcal{B}$ in some norm which we write as α or $\|\cdot\|_\alpha$; and

(*ii*) On $\mathcal{A} \otimes \mathcal{B}$ we have $\lambda \le \|\cdot\|_\alpha \le \gamma$.

The second condition forces α to be a cross norm. Henceforth in this chapter a 'tensor norm' shall mean a tensor norm of C^*-algebras.

A tensor norm α is called an *algebra* (respectively **-algebra*, *C^*-) tensor norm* if α is an algebra (respectively *-algebra, C^*-) norm on $\mathcal{A} \otimes \mathcal{B}$ for every pair of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. If α is a *-algebra norm or C^*-norm on $\mathcal{A} \otimes \mathcal{B}$ it is assumed that the involution on $\mathcal{A} \otimes_\alpha \mathcal{B}$ extends the natural involution.

4.1.8 ***Definition.*** A tensor norm α is said to be *completely positive uniform* (or *uniform* if there is no danger of confusion) if whenever $T_i : \mathcal{A}_i \to \mathcal{B}_i$ $(i = 1,2)$ are completely positive linear maps of C^*-algebras, then $T_1 \otimes T_2$ has an extension

$$T_1 \otimes_\alpha T_2 : \mathcal{A}_1 \otimes_\alpha \mathcal{A}_2 \to \mathcal{B}_1 \otimes_\alpha \mathcal{B}_2$$

satisfying $$\| T_1 \otimes_\alpha T_2 \| \leq \|T_1\| \, \|T_2\| .$$

In some sense in view of 3.1.3 the notion of an operator space tensor norm (Definition 3.3.1) generalizes the notion of a completely positive uniform tensor norm.

Examples of completely positive uniform norms include $\|\cdot\|_{\min}$ and $\|\cdot\|_{\max}$ (Theorem 4.1.6); the fourteen natural norms of Grothendieck including λ and the four algebra norms γ , H' , $\gamma\backslash/$, and $\backslash/\gamma$ (see section 1.2); and the commutative Haagerup and symmetrized Haagerup norms of 3.1 and 3.3 (the completely positive uniformity of these norms follows from Corollary 3.1.3).

The definition of a completely positive uniform tensor norm may be generalized to tensor products of operator systems [**Pn**] (i.e. self-adjoint subspaces of C^*-algebras containing the identity). Given a tensor norm defined for pairs of unital C^*-algebras one can use the uniformity property as the defining condition for a norm on tensor products of operator systems.

If α is a tensor norm, and if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, then we shall write ϵ_α for the canonical contraction $\mathcal{A} \otimes_\alpha \mathcal{B} \to \mathcal{A} \otimes_\lambda \mathcal{B}$. The situation when ϵ_α is injective is often of interest; for example it is not hard to show that $\mathcal{A} \otimes_\alpha \mathcal{B}$ is a semisimple A^*-algebra whenever α is a $*$-algebra tensor norm with ϵ_α injective. Indeed $\mathcal{A} \otimes_\alpha \mathcal{B}$ is a $*$-semisimple A^*-algebra if and only if the canonical map $\mathcal{A} \otimes_\alpha \mathcal{B} \to \mathcal{A} \otimes_m \mathcal{B}$ is injective; where m is the greatest C^*-norm on $\mathcal{A} \otimes \mathcal{B}$ dominated by α. This last statement follows from **[BoD3]** Chapter 40 Corollary 11, because if p is a C^*-norm on $\mathcal{A} \otimes_\alpha \mathcal{B}$ then $p \le \alpha$ by **[Di]** Proposition 1.3.7, thus α dominates the greatest C^*-norm on $\mathcal{A} \otimes_\alpha \mathcal{B}$. Note that by Proposition 4.1.5 there exists at least one C^*-norm dominated by α.

A reasonable tensor norm of Banach spaces α is called *nuclear* **[Ca4]** if the canonical map $E \otimes_\alpha F \to E \otimes_\lambda F$ is injective for all Banach spaces E and F. The next proposition relates this notion in some sense to the notion of nuclearity for C^*-algebras.

4.1.9 **PROPOSITION.** *A C^*-algebra $\mathcal{A}$ is nuclear if and only if for all C^*-algebras $\mathcal{B}$, and all completely positive uniform tensor norms α, the canonical map $\epsilon_\alpha : \mathcal{A} \otimes_\alpha \mathcal{B} \to \mathcal{A} \otimes_\lambda \mathcal{B}$ is injective.*

Proof. Suppose the second condition holds. Let $\mathcal{B}$ be a C^*-algebra and choose $\alpha = \|\cdot\|_{max}$. The condition implies that the canonical surjection $\mathcal{A} \otimes_{max} \mathcal{B} \to \mathcal{A} \otimes_{min} \mathcal{B}$ is one-to-one and consequently an isometry.

Now suppose that $\mathcal{A}$ is nuclear. It was shown in **[ChE2]** that this is equivalent to the existence of a net (T_ν) of completely positive contractive finite rank operators on $\mathcal{A}$ converging strongly to the identity mapping $I_{\mathcal{A}}$. Let $\mathcal{B}$ be a C^*-algebra and α be a completely positive uniform tensor norm. Suppose $u \in \ker \epsilon_\alpha$; choose $\psi \in (\mathcal{A} \otimes_\alpha \mathcal{B})^*$, and for each ν put $\psi_\nu = \psi \circ (T_\nu \otimes I_{\mathcal{B}})$. The uniformity implies that the net (ψ_ν) is uniformly bounded, and since $\psi_\nu(u) \to \psi(u)$ for u in a dense subset of $\mathcal{A} \otimes_\alpha \mathcal{B}$, we see that $\psi_\nu \to \psi$ in the weak*-topology on $(\mathcal{A} \otimes_\alpha \mathcal{B})^*$. Now each ψ_ν factors through $\mathcal{A} \otimes_\lambda \mathcal{B}$ since T_ν has finite rank, and so $\psi_\nu(u) = 0$ for each ν. Thus $\psi(u) = 0$; since ψ was chosen arbitrarily we see that $u = 0$. □

4.2 SOME SPECIAL TENSOR NORMS AND THEIR RELATIONSHIPS.

In this chapter we discuss the ordering of some of the norms we have met, and the resulting geometry of the dual spaces. By the end of the section we shall also have established exactly when any pair of the norms λ, $\|\cdot\|_{min}$, $\|\cdot\|_{max}$ and γ are mutually equivalent. If $\mathcal{A}$ is a C^*-algebra then by definition $\mathcal{A}$ is nuclear if and only $\|\cdot\|_{min}$ is equivalent (equal) to $\|\cdot\|_{max}$ on $\mathcal{A} \otimes \mathcal{B}$ for all C^*-algebras $\mathcal{B}$. The other characterizations we shall obtain will be of the following type: α is equivalent to β on $\mathcal{A} \otimes \mathcal{B}$ if and only if either $\mathcal{A}$ or $\mathcal{B}$ satisfy some condition C. Firstly however we establish some properties of the Haagerup norm.

4.2.1 PROPOSITION. *The Haagerup norm is a completely positive uniform algebra tensor norm dominated by* H' *, and the map* ϵ_h *defined above is always injective.*

Proof. Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and suppose $S_1,\dots,S_n$ are bounded operators on $\mathcal{H}$. Then we have

$$\begin{aligned}\| \Sigma_{i=1}^n S_i\, T\, S_i^* \| &= \sup \{ \mid \Sigma_{i=1}^n < T\, S_i^*(\xi) \, , \, S_i^*(\xi) > \mid \, : \, \|\xi\| \le 1 \} \\ &\le \|T\| \sup \{ \Sigma_{i=1}^n \mid < S_i^*(\xi) \, , \, S_i^*(\xi) > \mid \, : \, \|\xi\| \le 1 \} \\ &= \|T\| \; \| \Sigma_{i=1}^n S_i\, S_i^* \| \, .\end{aligned}$$

Thus if $u = \Sigma_{i=1}^n a_i \otimes b_i$ and $v = \Sigma_{j=1}^m x_j \otimes y_j$ are in the algebraic tensor product of two C^*-algebras, then

$$\begin{aligned}\| u\, v \|_h &\le \|\Sigma_{i=1}^n \Sigma_{j=1}^m a_i x_j x_j^* a_i^*\|^{1/2} \, \|\Sigma_{i=1}^n \Sigma_{j=1}^m y_j^* b_i^* b_i y_j\|^{1/2} \\ &\le \|\Sigma_{i=1}^n a_i a_i^*\|^{1/2} \, \|\Sigma_{i=1}^n b_i^* b_i\|^{1/2} \, \|\Sigma_{j=1}^m x_j x_j^*\|^{1/2} \, \|\Sigma_{j=1}^m y_j^* y_j\|^{1/2} \, ,\end{aligned}$$

and so $\| u\, v \|_h \le \|u\|_h \, \|v\|_h$.

As we remarked earlier the completely positive uniformity follows from Corollary 3.1.3. Propositions 3.1.9 and 3.1.11 show that $\|\cdot\|_h \le H'$.

Now suppose $\epsilon_h(u) = 0$ and choose $\psi \in (\mathcal{A} \otimes_h \mathcal{B})^*$. By 3.1.9 and 3.1.11 there exist Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, elements $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$, a bounded operator $T : \mathcal{K} \to \mathcal{H}$, and representations θ and π of $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{H}$ and $\mathcal{K}$ respectively, such that

$$\psi(a \otimes b) = < \theta(a)\; T\; \pi(b)\; \eta \; , \; \xi > \quad ,$$

for every $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Let (P_ν) be a net of finite dimensional orthogonal projections converging strongly to the identity mapping on $\mathcal{H}$, and define for each ν a functional

$$\psi_\nu : a \otimes b \mapsto < \theta(a)\; P_\nu\; T\; \pi(b)\; \eta \; , \; \xi >$$

on $\mathcal{A} \otimes_h \mathcal{B}$. By 3.1.11 the net (ψ_ν) is uniformly bounded by $\|T\| \; \|\xi\| \; \|\eta\|$ and the proof is now completed as in Proposition 4.1.9.□

The fact that the Haagerup norm is an algebra norm and its proof above is due to R. R. Smith. We note that $\|\cdot\|_h$ is not a *-algebra norm, to see this consider the following example [**Ha3**]: take $\mathcal{A}$ to be the bounded operators on a Hilbert space $\mathcal{H}$ and define a contractive functional $V(S \otimes T) = \psi(ST)$ on $\mathcal{A} \otimes_h \mathcal{A}$, where ψ is a fixed state on $B(\mathcal{H})$. For $n \in \mathbb{N}$ take isometries $u_1, \dots, u_n$ in $\mathcal{H}$ with $\Sigma_{i=1}^n u_i u_i^* = I_{\mathcal{H}}$. Then $\| \Sigma_{i=1}^n u_i \otimes u_i^* \|_h \le 1$, however

$$\|(\Sigma_{i=1}^n u_i \otimes u_i^*)^*\|_h \ge V((\Sigma_{i=1}^n u_i \otimes u_i^*)^*) = n .$$

This also shows that in general $\|\cdot\|_h$ is not equivalent to γ.

The next results focus on the intimate relationship between the three themes of equivalence of tensor norms, Grothendieck type inequalities, and the representability of bilinear functionals.

4.2.2 THEOREM (Grothendieck's inequality [Pr3]). *There exists a (smallest) universal constant* K_G *such that if* X *and* Y *are locally compact Hausdorff spaces, and* $\Psi : C_0(X) \times C_0(Y) \to \mathbb{C}$ *is a bounded bilinear functional, then there are probability measures* $\mathbb{P}_X$ *on* X *and* $\mathbb{P}_Y$ *on* Y *with*

$$| \Psi(f,g) | \le K_G \|\Psi\| \{\textstyle\int_X |f|^2 \mathbb{P}_X(dx)\}^{1/2} \{\textstyle\int_Y |g|^2 \mathbb{P}_Y(dy)\}^{1/2} ,$$

for $f \in C_0(X)$ *and* $g \in C_0(Y)$.

It has been shown that $1.33807\ldots < K_G < 1.4049\ldots$ (the lower bound is due to A. M. Davie (unpublished), the upper bound to U. Haagerup [**Ha4**]). We note that the theorem is usually stated for compact spaces, however by passing to the second dual (as in [**Ha3**]) we obtain the result as stated.

The next result appeared in [**KaS**], we provide a proof to illustrate the principles involved.

4.2.3 COROLLARY. *On the tensor product of commutative C^*-algebras we have*

$$\|\cdot\|_h \leq \gamma \leq K_G \|\cdot\|_h .$$

Indeed the last line is a restatement of the Grothendieck inequality.

Proof. Let $\Psi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ be a bounded bilinear functional on commutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. For $f_1,\ldots,f_n \in \mathcal{A}$ and $g_1,\ldots,g_n \in \mathcal{B}$ we have from 4.2.2 that

$$\begin{aligned}|\Sigma_{i=1}^N \Psi(f_i,g_i)| &\leq \Sigma_{i=1}^N |\Psi(f_i,g_i)| \\ &\leq K_G \|\Psi\| \Sigma_{i=1}^N \{\textstyle\int_X |f_i|^2 \mathbb{P}_X(dx)\}^{\frac12} \{\textstyle\int_Y |g_i|^2 \mathbb{P}_Y(dy)\}^{\frac12} \\ &\leq K_G \|\Psi\| \{\Sigma_{i=1}^N \textstyle\int_X |f_i|^2 \mathbb{P}_X(dx)\}^{\frac12} \{\Sigma_{i=1}^N \textstyle\int_Y |g_i|^2 \mathbb{P}_Y(dy)\}^{\frac12} \\ &\leq K_G \|\Psi\| \|\Sigma_{i=1}^N |f_i|^2 \|^{\frac12} \|\Sigma_{i=1}^N |g_i|^2 \|^{\frac12}\end{aligned}$$

which proves the first assertion.

Suppose K is a constant such that $\gamma \leq K \|\cdot\|_h$ on $\mathcal{A} \otimes \mathcal{B}$ for commutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Then for every bilinear functional $\Psi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ we have $\|\Psi\|_{cb} \leq K \|\Psi\|$ by 3.1.11. For such Ψ there exist $*$-representations π and θ of $\mathcal{A}$ and $\mathcal{B}$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, elements $\zeta \in \mathrm{BALL}(\mathcal{K})$ and $\eta \in \mathrm{BALL}(\mathcal{H})$, and an operator $T : \mathcal{K} \to \mathcal{H}$ with $\|T\| \leq K \|\Psi\|$, with

$$\Psi(f,g) = \langle \pi(f)\, T\, \theta(g)\, \zeta , \eta \rangle ,$$

for $f \in \mathcal{A}$ and $g \in \mathcal{B}$. Thus

$$\begin{aligned}|\Psi(f,g)| &\leq \|T\| \, \| \theta(g) \zeta \| \, \| \pi(f^*) \eta \| \\ &\leq K \|\Psi\| \langle \theta(|g|^2) \zeta , \zeta \rangle^{\frac12} \langle \pi(|f|^2) \eta , \eta \rangle^{\frac12} ,\end{aligned}$$

which is the Grothendieck inequality. □

4.2.4 THEOREM (Grothendieck-Pisier-Haagerup inequality [Ha3]). *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and suppose $\Psi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ is a bounded bilinear functional. Then there exist states φ_1 and φ_2 on $\mathcal{A}$, and states ψ_1 and ψ_2 on $\mathcal{B}$, such that for* $a \in \mathcal{A}$ *and* $b \in \mathcal{B}$

$$|\Psi(a,b)| \leq \|\Psi\| \{ \varphi_1(a^* a) + \varphi_2(a a^*) \}^{\frac{1}{2}} \{ \psi_1(b^* b) + \psi_2(b b^*) \}^{\frac{1}{2}} .$$

A calculation similar to the one after Proposition 4.2.1 shows that this inequality is best possible (in the sense that if one could replace $\|\Psi\|$ by $C \|\Psi\|$ in the inequality, for some universal constant C, then $C \geq 1$).

4.2.5 THEOREM. *On the tensor product of two C^*-algebras we have*

$$H' \leq \gamma \leq 2 H' .$$

Indeed this statement is equivalent to the Grothendieck-Pisier-Haagerup inequality, although the constants do not necessarily match.

Proof. The inequality is **[Ha3]** Proposition 2.1, again we give a proof to illustrate the technique. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let $\Psi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ be a bounded bilinear functional with $\|\Psi\| \leq 1$. Theorem 4.2.4 implies the existence of states φ_1 and φ_2 on $\mathcal{A}$, and states ψ_1 and ψ_2 on $\mathcal{B}$, such that for $a \in \mathcal{A}$ and $b \in \mathcal{B}$

$$|\Psi(a,b)| \leq \{ \varphi_1(a^* a) + \varphi_2(a a^*) \}^{\frac{1}{2}} \{ \psi_1(b^* b) + \psi_2(b b^*) \}^{\frac{1}{2}} .$$

Now define a semi inner product on $\mathcal{A}$ by

$$< a , b > = \varphi_1(b^* a) + \varphi_2(a b^*) ,$$

for $a , b \in \mathcal{A}$. Let $\mathcal{H}$ be the Hilbert space completion of the quotient

$$\mathcal{A} / \{ a : < a , a > = 0 \}$$

in the induced inner product. Let $\mathcal{K}$ be the Hilbert space derived similarly from $\mathcal{B}$ and ψ_1 , ψ_2 . The quotient mappings $p : \mathcal{A} \to \mathcal{H}$ and $q : \mathcal{B} \to \mathcal{K}$ each have norm $2^{1/2}$. The inequality above implies that the bilinear form

$$\mathcal{H} \times \mathcal{K} \to \mathbb{C} : (p(a) , q(b)) \to \Psi(a,b)$$

is well defined and contractive, thus there exists a contractive operator $T : \mathcal{H} \to \mathcal{K}^*$ with

$$< T(p(a)) , q(b) > = \Psi(a,b) .$$

This implies that the norm of Ψ as a functional on $\mathcal{A} \otimes_{H'} \mathcal{B}$ is not larger than 2 .

Now let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and suppose $\gamma \leq K H'$ on $\mathcal{A} \otimes \mathcal{B}$ for some positive constant K . Let $\Psi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ be a contractive bilinear mapping, then there exists a Hilbert space $\mathcal{H}$, and bounded linear maps $S : \mathcal{A} \to \mathcal{H}^*$ and $T : \mathcal{B} \to \mathcal{H}$ with $\|S\| \|T\| \leq K$, such that

$$\Psi(a,b) = < S(a) , T(b) > ,$$

for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. An application of **[Ha2]** Theorem 2.2 completes the proof. □

The norm whose equivalence with γ corresponds exactly to the Grothendieck-Pisier-Haagerup inequality, with the right constant, is given on $u \in \mathcal{A} \otimes \mathcal{B}$ by

$$\tfrac{1}{2} \inf\{(\|\Sigma_{i=1}^{N} a_i a_i^*\| + \|\Sigma_{i=1}^{N} a_i^* a_i\|)^{\frac{1}{2}} (\|\Sigma_{i=1}^{N} b_i b_i^*\| + \|\Sigma_{i=1}^{N} b_i^* b_i\|)^{\frac{1}{2}} : u = \Sigma_{i=1}^{N} a_i \otimes b_i\} .$$

We now consider the commutative symmetrized Haagerup norm of 3.3.6, which by a similar argument to that of 4.2.1 may be shown to be an algebra norm.

4.2.6 PROPOSITION. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* C^**-algebras, and suppose* $\Psi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ *is a bilinear map. The following are equivalent*

(i) Ψ *is symmetrically completely bounded with* $\|\Psi\|_{scb} \leq 1$ *;*

(ii) the linear functional on $\mathcal{A} \otimes_{sh} \mathcal{B}$ *corresponding to* Ψ *is contractive;*

(iii) we may write

$$\Psi(a,b) = \langle (\theta_+ \oplus \theta_-)(a)\ T\ (\pi_+ \oplus \pi_-)(b)\ \zeta\ ,\ \eta \rangle ,$$

for all $a \in \mathcal{A}$ *and* $b \in \mathcal{B}$ *; where* θ_+ *and* π_+ *are *-representations of* $\mathcal{A}$ *and* $\mathcal{B}$ *on Hilbert spaces* $\mathcal{H}_+$ *and* $\mathcal{K}_+$ *respectively,* θ_- *and* π_- *are *-anti-representations of* $\mathcal{A}$ *and* $\mathcal{B}$ *on Hilbert spaces* $\mathcal{H}_-$ *and* $\mathcal{K}_-$ *respectively,* $T : \mathcal{K}_+ \oplus \mathcal{K}_- \to \mathcal{H}_+ \oplus \mathcal{H}_-$ *is a contractive operator, and* $\zeta \in \mathrm{BALL}(\mathcal{K}_+ \oplus \mathcal{K}_-)$ *and* $\eta \in \mathrm{BALL}(\mathcal{H}_+ \oplus \mathcal{H}_-)$ *;*

(iv) there exist states φ_1 *and* φ_2 *on* $\mathcal{A}$ *, states* ψ_1 *and* ψ_2 *on* $\mathcal{B}$ *, and real numbers* s *and* t *in* $[0,1]$ *, such that*

$$|\Psi(a,b)| \leq \{s\varphi_1(a^*a)+(1-s)\varphi_2(aa^*)\}^{\frac{1}{2}} \{t\psi_1(b^*b)+(1-t)\ \psi_2(bb^*)\}^{\frac{1}{2}}$$

for all $a \in \mathcal{A}$ *and* $b \in \mathcal{B}$ *.*

Proof. The equivalence of (i) and (ii) and (iii) follow from Theorems 3.3.10 and 3.2.5. It is clear that (iii) implies (iv), and that (iv) implies (ii). □

4.2.7. PROPOSITION. *On the tensor product of any two* C^**-algebras we have*

$$\|\cdot\|_{sh} \le \gamma \le 2\,\|\cdot\|_{sh}\ , \textit{ and } \ \|\cdot\|_{sh} \le H' \le 2\,\|\cdot\|_{sh}\ ;$$

moreover the constant 2 *is best possible in each inequality.*

Proof. The Grothendieck-Pisier-Haagerup inequality and Proposition 4.2.6 shows that

$$\|\cdot\|_{sh} \le H' \le \gamma \le 2\,\|\cdot\|_{sh}\ .$$

Now suppose that $H' \le K\,\|\cdot\|_{sh}$ for some constant $K > 0$, to complete the proof it is sufficient to show that $K \ge 2$. Let $n \in \mathbb{N}$ be fixed, put $\mathcal{H} = \mathbb{C}^{2n+1}$ and let $\{e_k\}_{k=1}^{2n+1}$ be an orthonormal basis for $\mathcal{H}$. Form the exterior (or wedge, or alternating **[Lg]**, or antisymmetric) product spaces $\Lambda^n\,\mathcal{H}$ and $\Lambda^{n+1}\,\mathcal{H}$, which are complex vector spaces of the same dimension. Let N be the binomial coefficient $\binom{2n}{n}$.

Define for $k = 1,\ldots,2n+1$ linear maps $a_k : \Lambda^n\mathcal{H} \to \Lambda^{n+1}\mathcal{H}$ given by

$$a_k\,(\,\xi_1 \wedge \ldots \wedge \xi_n\,) = e_k \wedge \xi_1 \wedge \ldots \wedge \xi_n \qquad (\,\xi_1,\ldots,\xi_n \in \mathcal{H}\,)\ .$$

Now $\Lambda^n\mathcal{H}$ has a natural inner product which can be written as

$$<\xi_1 \wedge \ldots \wedge \xi_n\ ,\ \eta_1 \wedge \ldots \wedge \eta_n> = \det\,[<\xi_i\ ,\ \eta_j>]\quad ,$$

for $\xi_1,\ldots,\xi_n\ ,\ \eta_1,\ldots,\eta_n \in \mathcal{H}$. That this Hermitian sesquilinear form is positive-definite is most easily seen with reference to the basis

$$\{ e_{i_1} \wedge \dots \wedge e_{i_n} \}_{i_1 < \dots < i_n}$$

(which is orthonormal with respect to this form). Similarly we can define an inner product on $\Lambda^{n+1}\mathcal{X}$ such that

$$\{ e_{i_1} \wedge \dots \wedge e_{i_{n+1}} \}_{i_1 < \dots < i_{n+1}}$$

is an orthonormal basis for $\Lambda^{n+1}\mathcal{X}$, and such that the inner product is independent of the specific orthonormal basis $\{e_n\}$ that was chosen. With respect to these inner products one may verify that

$$\Sigma_{i=1}^{2n+1} a_k a_k^* = (n+1) I_{\Lambda^{n+1}\mathcal{X}} \quad \text{and} \quad \Sigma_{i=1}^{2n+1} a_k^* a_k = (n+1) I_{\Lambda^{n}\mathcal{X}} .$$

Let $\mathcal{A}$ be the C^*-algebra $B(\Lambda^n\mathcal{X},\Lambda^{n+1}\mathcal{X})$, considering $\Lambda^n\mathcal{X}$ as being identified with $\Lambda^{n+1}\mathcal{X}$ via some explicit isomorphism, and let E be the subspace of $\mathcal{A}$ spanned by $\{a_i\}$. Now $\mathcal{A}$ may also be regarded as a Hilbert space with the Schmidt class inner product [Ri]

$$\langle a , b \rangle_2 = \mathrm{Tr}(b^* a) \qquad (a , b \in \mathcal{A}) .$$

Write $\|\cdot\|_2$ for $\langle \cdot , \cdot \rangle_2^{1/2}$. It is not hard to see that $\{ N^{-1/2} a_k \}$ forms an orthonormal basis for E with respect to this inner product. Let $P : \mathcal{A} \to E$ be the orthogonal projection onto E with respect to $\langle \cdot , \cdot \rangle_2$. We make the following claims:

(i) $N^{1/2} \|e\| = \|e\|_2$ for $e \in E$,

(ii) if $e \neq 0$ then the rank of e equals N, and

(iii) $\|P(a)\| \leq \|a\|$ for $a \in \mathcal{A}$.

Identities (i) and (ii) may be seen by first verifying them in the case $e = a_1$, and then observing that the basis free nature of the inner product allows this assumption. Identity (iii) follows because if $P(a) = e$ then

$$N\,\|e\|^2 = \langle e, e\rangle_2 = \langle e, a\rangle_2 = \mathrm{tr}(a^* e)$$
$$\leq \|a^* e\|\ \mathrm{rank}(a^* e)$$
$$\leq \|e\|\ \|a\|\ \mathrm{rank}(e)$$
$$\leq N\,\|e\|\ \|a\|\ ,$$

and so $\|e\| \leq \|a\|$.

Define a linear map $T : \mathcal{A} \to \mathbb{C}^{2n+1}$ by

$$T(a) = N^{-1}\ (\ \mathrm{Tr}(a_i^*\ a)\ )_{i=1}^{2n+1}\ ,$$

for $a \in \mathcal{A}$. Identities (i) and (iii) above assert that $\|T\| \leq 1$. Write $\langle \cdot , \cdot \rangle$ for the bilinear form $\mathbb{C}^{2n+1} \times \mathbb{C}^{2n+1} \to \mathbb{C}$ giving the duality of $\mathbb{C}^{2n+1}$ with itself, and define a functional $\psi : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C}$ by

$$\psi(a \otimes b) = \langle T(a)\ , T(b) \rangle \qquad (\ a\ ,\ b \in \mathcal{A}\ )\ .$$

It is clear that $\psi(a_k \otimes a_k) = 1$, and that ψ is contractive with respect to the H' norm. Thus

$$2n+1 = |\Sigma_{i=1}^{2n+1}\ \psi(a_k \otimes a_k)| \leq K\ \|\psi\|\ \|\Sigma_{i=1}^{2n+1}\ a_k \otimes a_k\|_{sh} \leq K\ (n+1)\ ,$$

and since n was chosen arbitarily we see that $K \geq 2$. □

The construction above is due to U. Haagerup and was communicated to the author by A. M. Davie.

It is shown in **[KaS]** that one can represent every bounded bilinear map $\Psi : \mathcal{A} \times \mathcal{B} \to \mathbb{C}$ with $\|\Psi\| \leq \frac{1}{2}$ in the form quoted in (iii) of Proposition 4.2.6 - this is merely the equivalence of γ and $\|\cdot\|_{sh}$. From this it follows that if $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{A}_2$ and $\mathcal{B}_2$ are C^*-algebras, and if $\mathcal{A}_i \subset \mathcal{B}_i$ (i = 1,2) , then $\mathcal{A}_1 \otimes_\gamma \mathcal{A}_2 \subset \mathcal{B}_1 \otimes_\gamma \mathcal{B}_2$

as sets (but not isometrically), since bounded bilinear functionals on C^*-algebras extend to containing C^*-algebras (using [**Di**] 2.10.2).

The following result asserts that the class of C^*-algebras satisfies Grothendieck's conjecture [**Pr3**].

4.2.8 THEOREM. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. The projective norm γ is equivalent to the injective norm λ on $\mathcal{A} \otimes \mathcal{B}$ if and only if $\mathcal{A}$ or $\mathcal{B}$ is finite dimensional.*

Proof. If γ is equivalent to λ on $\mathcal{A} \otimes \mathcal{B}$, then $\|\cdot\|_h$ is certainly equivalent to λ on $\mathcal{A} \otimes \mathcal{B}$. Let $\mathcal{A}_1$ and $\mathcal{B}_1$ be maximal abelian *-subalgebras of $\mathcal{A}$ and $\mathcal{B}$ respectively; since λ and $\|\cdot\|_h$ are both injective (1.2 and Proposition 3.3.4) we have that $\|\cdot\|_h$ is equivalent to λ on $\mathcal{A}_1 \otimes \mathcal{B}_1$. This implies by 4.2.3 that γ and λ are equivalent on $\mathcal{A}_1 \otimes \mathcal{B}_1$, and so $\mathcal{A}_1$ or $\mathcal{B}_1$ is finite dimensional [**Pr1**]. This implies by [**KR**] Exercise 4.6.12 that $\mathcal{A}$ or $\mathcal{B}$ is finite dimensional. □

A more direct proof of 4.2.8 is given in Section 6.1. We note in passing here that if n and m are positive integers with $n \leq m$ then on $\ell_n^\infty \otimes \ell_m^\infty$ we have

$$\gamma \leq (2\,n)^{1/2}\,\lambda\ ,$$

and the best constant in this inequality is not smaller than $n^{1/2}$. The first statement follows directly from an inequality of Littlewood (sometimes called Khintchine's inequality) [**Ka**], the second we give a proof of in Section 6.1.

4.2.9 THEOREM [Wn]. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. The norms λ and $\|\cdot\|_{max}$ are equivalent* on $\mathcal{A} \otimes \mathcal{B}$ *if and only if the norms λ and $\|\cdot\|_{min}$ are equivalent on $\mathcal{A} \otimes \mathcal{B}$ if and only if either $\mathcal{A}$ or $\mathcal{B}$ satisfies the following condition: there exists a positive integer* n *such that all irreducible representations have range with dimension not greater than* n .

4.2.10 THEOREM. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. The following are equivalent:*

(*i*) *$\mathcal{A}$ or $\mathcal{B}$ is commutative,*

(*ii*) *the norms λ and $\|\cdot\|_{min}$ agree on $\mathcal{A} \otimes \mathcal{B}$* , and

(*iii*) *λ is an algebra norm on $\mathcal{A} \otimes \mathcal{B}$* .

Proof. The equivalence of (i) and (ii) is shown in **[Ta]** Theorem IV.4.14 for unital C^*-algebras and follows in general by the unitization technique of 4.1.4. That (ii) implies (iii) is trivial. That (iii) implies (ii) follows from Proposition 4.1.5. □

4.3 COMPLETELY POSITIVE UNIFORM ALGEBRA NORMS.

We now apply some of our results from Chapter 2 to investigate the geometry of the tensor product of C^*-algebras. The next result states that C^*-tensor norms are minimal amongst the algebra tensor norms.

4.3.1 PROPOSITION. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Then any algebra norm α on $\mathcal{A} \otimes \mathcal{B}$ which is dominated by $\|\cdot\|_{max}$ is a*

C^*-norm, and then the canonical contraction $\mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{A} \otimes_\alpha \mathcal{B}$ *is surjective.*

Proof. Apply Corollary 2.2.6. □

4.3.2 REMARK. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras then whenever $\mathcal{A} \otimes_\lambda \mathcal{B}$ is a Banach algebra it is a C^*-algebra by Proposition 4.3.1, and so λ coincides with $\|\cdot\|_{\min}$. Thus we obtain another proof that (iii) implies (ii) in 4.2.10.

4.3.3 THEOREM. *Let* α *be a completely positive uniform algebra tensor norm, which is a* C^**-norm on* $\mathcal{A}_0 \otimes \mathcal{B}_0$ *for some pair of non-trivial* C^**-algebras* $\mathcal{A}_0$ *and* $\mathcal{B}_0$ *. Then* α *is a* C^**-tensor norm.*

Proof. Let $\mathcal{A}_0$ and $\mathcal{B}_0$ be as above; let φ and φ' be two different states on $\mathcal{A}_0$, and let ψ and ψ' be two different states on $\mathcal{B}_0$. Define two positive contractions

$$J_1 : \mathcal{A}_0 \to \ell_2^\infty : a \mapsto (\varphi(a), \varphi'(a))$$

and

$$J_2 : \mathcal{B}_0 \to \ell_2^\infty : b \mapsto (\psi(b), \psi'(b)) .$$

Since J_1 and J_2 have commutative ranges they are completely positive by 3.1.1, and it is easily checked that they are surjective. By Proposition 1.1.3 they each preserve two-sided approximate identities. The uniformity implies that the map

$$J_1 \otimes_\alpha J_2 : \mathcal{A}_0 \otimes_\alpha \mathcal{B}_0 \to \ell_2^\infty \otimes_\alpha \ell_2^\infty$$

is contractive and it certainly is surjective. Moreover it clearly preserves a two-sided approximate identity, and so Theorem 2.2.2

implies that $\ell_2^\infty \otimes_\alpha \ell_2^\infty$ is a C*-algebra. Note that the induced involution on $\ell_2^\infty \otimes_\alpha \ell_2^\infty$ is the usual one.

Now let $\mathcal{A}$ and $\mathcal{B}$ be two unital C*-algebras, and choose $a \in (\text{Ball } \mathcal{A})_+$ and $b \in (\text{Ball } \mathcal{B})_+$. Consider the positive unital contractions

$$\chi_a : \ell_2^\infty \to \mathcal{A} : (\xi_1, \xi_2) \mapsto \xi_1 \; a + \xi_2 \; (1-a)$$

and $$\chi_b : \ell_2^\infty \to \mathcal{B} : (\xi_1, \xi_2) \mapsto \xi_1 \; b + \xi_2 \; (1-b) \; .$$

Since ℓ_2^∞ is commutative these maps are completely positive by 3.1.1, and so $\chi_a \otimes_\alpha \chi_b$ is a unital contraction by the uniformity. Thus $\chi_a \otimes_\alpha \chi_b \, (\, (1,0) \otimes (1,0) \,) = a \otimes b$ is Hermitian and we conclude that $\mathcal{A}_{s.a.} \underset{\mathbb{R}}{\otimes} \mathcal{B}_{s.a.} \subset H(\mathcal{A} \otimes_\alpha \mathcal{B})$.

Now $(\mathcal{A}_{s.a.} \underset{\mathbb{R}}{\otimes} \mathcal{B}_{s.a.}) = (\mathcal{A} \otimes \mathcal{B})_{s.a.}$ as real spaces, thus the set

$$(\mathcal{A}_{s.a.} \underset{\mathbb{R}}{\otimes} \mathcal{B}_{s.a.}) + i \, (\mathcal{A}_{s.a.} \underset{\mathbb{R}}{\otimes} \mathcal{B}_{s.a.})$$

is dense in $\mathcal{A} \otimes_\alpha \mathcal{B}$, and consequently

$$H(\mathcal{A} \otimes_\alpha \mathcal{B}) + i \, H(\mathcal{A} \otimes_\alpha \mathcal{B}) = \mathcal{A} \otimes_\alpha \mathcal{B}$$

(using Proposition 2.1.4). Applying the Vidav - Palmer theorem (Theorem 2.2.1) we find that $\mathcal{A} \otimes_\alpha \mathcal{B}$ is a C*-algebra.

Suppose now that $\mathcal{A}$ and $\mathcal{B}$ are arbitrary C*-algebras and let (e_ν) and (f_μ) be positive two-sided approximate identities for $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $\mathcal{A}^1$ and $\mathcal{B}^1$ be the unitizations of $\mathcal{A}$ and $\mathcal{B}$ respectively, and let $\alpha^\sim$ be the extension of α to $\mathcal{A}^1 \otimes \mathcal{B}^1$ defined in Proposition 4.1.4. The maps

$$\mathcal{A} \to \mathcal{A} : a \mapsto e_\lambda \, a \, e_\lambda \quad \text{and} \quad \mathcal{B} \to \mathcal{B} : b \mapsto f_\mu \, b \, f_\mu$$

are completely positive contractions, and so by the uniformity

$\alpha_{\sim} \le \alpha$ on $\mathcal{A}^1 \otimes \mathcal{B}^1$. Proposition 4.3.1 now shows that $\alpha_{\sim}$ is a C^*-norm. Since $\mathcal{A} \otimes_\alpha \mathcal{B}$ is embedded isometrically in $\mathcal{A}^1 \otimes_{\alpha_{\sim}} \mathcal{B}^1$ we find that α is a C^*-norm on $\mathcal{A} \otimes \mathcal{B}$. □

The proof above leads to the following characterization of C^*-norms in terms of values on linear combinations of four elementary tensors:

4.3.4 PROPOSITION. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. An algebra norm α on $\mathcal{A} \otimes \mathcal{B}$ is a C^*-norm if and only if for each* $a \in (\mathrm{BALL}\ \mathcal{A})_+$, $b \in (\mathrm{BALL}\ \mathcal{B})_+$ *and* $\xi_1, .., \xi_4 \in \mathbb{C}$, *we have*

$$\| \xi_1 a \otimes b + \xi_2 a \otimes 1 + \xi_3 1 \otimes b + \xi_4 1 \otimes 1 \|$$
$$\le \max\{|\xi_1+\xi_2+\xi_3+\xi_4|;|\xi_2+\xi_4|;|\xi_3+\xi_4|;|\xi_4|\} .$$

Note that we do not require α to be uniform here.

Proof. Suppose that the condition is satisfied, and let $a \in \mathrm{BALL}(\mathcal{A})_+$ and $b \in \mathrm{BALL}(\mathcal{B})_+$ be fixed. Define a map $\psi_{a,b} : \ell^\infty_{2\times 2} \to \mathcal{A} \otimes_\alpha \mathcal{B}$ taking an element $(\mu_{ij})^2_{i,j=1}$ to

$$\mu_{11}\ a \otimes b + \mu_{12}\ a \otimes (1-b) + \mu_{21}\ (1-a) \otimes b + \mu_{22}\ (1-a) \otimes (1-b) \quad .$$

The condition of the proposition says precisely that $\psi_{a,b}$ is contractive, and since it is certainly unital we see that $\psi_{a,b}(e_{11}) = a \otimes b$ is Hermitian. The argument used in Theorem 4.3.3 shows that α is a C^*-norm.

Now suppose that α is a C^*-norm, and choose $a \in \mathrm{BALL}(\mathcal{A})_+$, $b \in \mathrm{BALL}(\mathcal{B})_+$. Then certainly $\alpha \le \|\cdot\|_{\max}$. The map

$$\chi_a \otimes_{\max} \chi_b : \ell^\infty_2 \otimes_{\max} \ell^\infty_2 \to \mathcal{A} \otimes_{\max} \mathcal{B}$$

defined in Theorem 4.3.3 is contractive, and composing it with the natural contraction $\mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{A} \otimes_{\alpha} \mathcal{B}$ we obtain $\psi_{a,b}$. □

4.4 THE HERMITIAN DICHOTOMY.

Let α be an algebra tensor norm and suppose $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras. Since the maps

$$\mathcal{A} \to \mathcal{A} \otimes_{\alpha} \mathcal{B} : a \mapsto a \otimes 1 \quad \text{and} \quad \mathcal{B} \to \mathcal{A} \otimes_{\alpha} \mathcal{B} : b \mapsto 1 \otimes b$$

are unital contractions we see that

$$\mathcal{A}_{s.a.} \otimes 1 + 1 \otimes \mathcal{B}_{s.a.} \subset H(\mathcal{A} \otimes_{\alpha} \mathcal{B}) .$$

We call the set on the left hand side the *trivial Hermitians* of $\mathcal{A} \otimes_{\alpha} \mathcal{B}$. It is not hard to prove that if $a \in \mathcal{A}_{s.a.}$ and $b \in \mathcal{B}_{s.a.}$ then

$$\alpha(a \otimes 1 + 1 \otimes b) = \inf \{ \|a - t\| + \|b + t\| : t \in \mathbb{R} \}$$

for any cross norm α ; however we shall not use this fact.

It is clear from the above that for an algebra tensor norm α the real dimension of the Hermitians in $\ell_2^\infty \otimes_\alpha \ell_2^\infty$ is either 3 or 4 . The dimension is 3 if and only if every Hermitian element h , considered as a real valued function on $\{0;1\}^2$, satisfies

$$h(0,0) + h(1,1) = h(1,0) + h(0,1) .$$

In the previous section we saw that a uniform algebra tensor norm was a C^*-tensor norm (and consequently always gives rise to a spanning set of Hermitians) if and only if $\dim H(\ell_2^\infty \otimes_\alpha \ell_2^\infty) = 4$. We shall show that if $\dim H(\ell_2^\infty \otimes_\alpha \ell_2^\infty) = 3$ then $\mathcal{A} \otimes_\alpha \mathcal{B}$ commonly has only the trivial Hermitians. It is not true that exactly one of these two alternatives hold, as the following example shows:

4.4.1 **EXAMPLE.** Consider $\ell_3^\infty \otimes \ell_3^\infty$, which may be identified algebraically with $\ell_{3\times 3}^\infty$, with the cross norm

$$\left\|\begin{bmatrix}\mu_{11} & \mu_{12} & \mu_{13}\\ \mu_{21} & \mu_{22} & \mu_{23}\\ \mu_{31} & \mu_{32} & \mu_{33}\end{bmatrix}\right\| = \max\left\{ \left\|\begin{bmatrix}\mu_{11} & \mu_{12}\\ \mu_{21} & \mu_{22}\end{bmatrix}\right\|_\gamma ; |\mu_{ij}| : i,j = 1,2,3 \right\},$$

where we consider the top left square as an element of $\ell_2^\infty \otimes_\gamma \ell_2^\infty$. This is a Banach algebra isometrically isomorphic to $(\ell_2^\infty \otimes_\gamma \ell_2^\infty) \oplus_\infty \ell_5^\infty$. Thus its dual space is $(\ell_2^\infty \otimes_\gamma \ell_2^\infty) \oplus_1 \ell_5^1$, and it is not hard to show that the set of Hermitian elements is $H((\ell_2^\infty \otimes_\gamma \ell_2^\infty)) \oplus H(\ell_5^\infty)$, an 8 dimensional space.

Let α be an algebra tensor norm; for C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ let ϵ_α be the canonical contraction $\mathcal{A} \otimes_\alpha \mathcal{B} \to \mathcal{A} \otimes_\lambda \mathcal{B}$ as before. Put

$$Z_\alpha(\mathcal{A},\mathcal{B}) = H(\mathcal{A} \otimes_\alpha \mathcal{B}) \cap \ker \epsilon_\alpha ,$$

a closed subspace of $H(\mathcal{A} \otimes_\alpha \mathcal{B})$. Often $Z_\alpha(\mathcal{A},\mathcal{B}) = \{0\}$ as is the case when $\mathcal{A}$ or $\mathcal{B}$ is finite dimensional, or (by Proposition 4.1.9) when $\mathcal{A}$ or $\mathcal{B}$ is nuclear and α is completely positive uniform.

Also if $\mathcal{A}$ and $\mathcal{B}$ are commutative then ϵ_α is just the Gelfand transform, and since norm equals spectral radius on Hermitian elements (Theorem 2.1.3) we have $Z(\mathcal{A},\mathcal{B}) = \{0\}$.

4.4.2 **THEOREM.** *Let α be a completely positive uniform algebra tensor norm which is not a C^*-tensor norm. If $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras then $H(\mathcal{A} \otimes_\alpha \mathcal{B})$ is the real direct sum of the trivial Hermitians and $Z_\alpha(\mathcal{A},\mathcal{B})$. If in addition $\mathcal{A}$ or $\mathcal{B}$ is nuclear we obtain only the trivial Hermitians.*

Proof. By the observation at the beginning of this section

$H(\ell_2^\infty \otimes_\alpha \ell_2^\infty)$ is trivial. Suppose φ , $\varphi' \in S(\mathcal{A})$ and ψ , $\psi' \in S(\mathcal{B})$, and define J_1 and J_2 as in Theorem 4.3.3. Choose $u \in H(\mathcal{A} \otimes_\alpha \mathcal{B})$, then $J_1 \otimes_\alpha J_2$ (u) is Hermitian in $\ell_2^\infty \otimes_\alpha \ell_2^\infty$ and by the remark at the beginning of this section

$$< \varphi \otimes \psi + \varphi' \otimes \psi' , \epsilon_\alpha u > = < \varphi \otimes \psi' + \varphi' \otimes \psi , \epsilon_\alpha u > \qquad (*)$$

The following argument is given in [**KaS**]:

Now let $\varphi_0 \in S(\mathcal{A})$, $\psi_0 \in S(\mathcal{B})$ be fixed and define unital contractions

$$P_{\psi_0} : \mathcal{A} \otimes_\alpha \mathcal{B} \to \mathcal{A} : a \otimes b \mapsto \psi_0(b)\ a$$

and $$Q_{\varphi_0} : \mathcal{A} \otimes_\alpha \mathcal{B} \to \mathcal{B} : a \otimes b \mapsto \varphi_0(a)\ b \ .$$

Put $h = P_{\psi_0}(u)$ and $k = Q_{\varphi_0}(u) - < \varphi_0 \otimes \psi_0 , \epsilon_\alpha u > 1$. These are Hermitian and for $\varphi \in S(\mathcal{A})$ and $\psi \in S(\mathcal{B})$ we have

$$< \varphi \otimes \psi , \epsilon_\alpha(u-h\otimes 1-1\otimes k) > = < \varphi \otimes \psi , \epsilon_\alpha u > - \varphi(h) - \psi(k)$$

$$= < \varphi \otimes \psi , \epsilon_\alpha u > - < \varphi \otimes \psi_0 , \epsilon_\alpha u > - < \varphi_0 \otimes \psi - \varphi_0 \otimes \psi_0 , \epsilon_\alpha u >$$

$$= 0$$

by (*) . Recall that a continuous linear functional on a C^*-algebra may be written as a linear combination of four states; this shows that $\epsilon_\alpha(u-h\otimes 1-1\otimes k) = 0$ and so $u-h\otimes 1-1\otimes k \in Z_\alpha$.

The last statement of the theorem follows from Proposition 4.1.9.□

4.4.3 REMARK. The conclusion of the theorem remains true if $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras provided that $J_1 \otimes_\alpha J_2$ is a contraction and the dimension of $H(\ell_2^\infty \otimes_\alpha \ell_2^\infty)$ is 3 ; indeed $\mathcal{A} \otimes_\alpha \mathcal{B}$ does not even have to be an algebra so long as $H(\mathcal{A} \otimes_\alpha \mathcal{B})$ is taken to be the obvious set.

Finally, as examples we compute the Hermitian elements for some tensor norms we have encountered:

4.4.4 EXAMPLE. Grothendieck's natural algebra preserving norms γ, $\gamma\backslash/$, $\backslash/\gamma$ and H' : These norms are all equivalent by Theorem 4.2.5 and [**Gr2**] Théorème 7 Corollaire 2. Haagerup [**Ha3**] showed that ϵ_γ is always injective and consequently $\epsilon_{\gamma\backslash/}$, $\epsilon_{\backslash/\gamma}$ and H' are injective too (indeed $\epsilon_{H'}$ is always injective for Banach spaces [**Ca4**]). Since these are reasonable algebra norms and no C^*-tensor norm can be reasonable in Grothendieck's sense, Theorem 4.4.2 implies that these four norms always give only the trivial Hermitians.

4.4.5 EXAMPLE. The Haagerup and symmetrized Haagerup norms: it was shown earlier that $\|\cdot\|_h$ is not a *-algebra tensor norm so by Theorem 4.4.2 and Proposition 4.2.1 we obtain just the trivial Hermitians. Since $\|\cdot\|_{sh}$ is equivalent to γ Theorem 4.4.2 implies that $\|\cdot\|_{sh}$ gives only the trivial Hermitians.

CHAPTER 5.

TRACIALLY COMPLETELY BOUNDED MULTILINEAR MAPS ON C^*-ALGEBRAS.

In this chapter we define the class of *tracially completely bounded* multilinear maps, and investigate some related geometrical questions. This class includes all completely bounded multilinear maps on C^*-algebras. The author was led to this definition in an attempt to build invariance under cyclic permutation of variables into the assignment $\Psi \to \Psi_n$, the standard n-fold amplification of a multilinear map (Definition 3.1.8). This was in order to create a class of maps which would be suitable for a 'completely bounded' cyclic cohomology theory **[ChS2,Con]**. We explain our motivation in more detail in 5.1.

In section 5.2 we show that every bounded bilinear map of C^*-algebras $\Psi : \mathcal{A} \times \mathcal{B} \to B(\mathcal{H})$ is tracially completely bounded, and indeed

$$\|\Psi\| \leq \|\Psi\|_{tcb} \leq 2\,\|\Psi\| ,$$

where $\|\cdot\|_{tcb}$ is the norm appropriate to the space of tracially completely bounded maps. The norm on $\mathcal{A} \otimes \mathcal{B}$ which corresponds to the class of tracially completely bounded functionals is a completely positive uniform *-algebra tensor norm equivalent to the projective norm, and in general not equivalent to the Haagerup norm. We also show that the least constant that suffices in the inequality

$$\|\Psi\|_{tcb} \leq K\,\|\Psi\|$$

is not smaller than $4/\pi$. The inequality of the previous line is somewhat akin to the Grothendieck-Pisier-Haagerup inequality (Theorem

4.2.4).

Finally, we give an example of a trilinear bounded map which is not tracially completely bounded, and make some comments on the possibility of a Christensen-Sinclair type representation theorem for tracially completely bounded maps.

The work described above had been completed when the author obtained a copy of [It]. In his paper Itoh introduces a new definition of complete boundedness for linear maps $\mathcal{A} \to \mathcal{B}^*$, where $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. If $T : \mathcal{A} \to \mathcal{B}^*$ we may consider the associated bilinear functional $T^\sim$ defined by

$$T^\sim(a,b) = \langle T(a) , b \rangle \qquad (a \in \mathcal{A} , b \in \mathcal{B}) .$$

Itoh's completely bounded maps are the same as the bilinear tracially completely bounded maps via the correspondence $T \to T^\sim$, except for a slight twist which arises from the ambiguity in the definition of the duality of $\mathcal{M}_n^*$ and $\mathcal{M}_n$. Our results give alternative proofs of the theorems in [It]. In addition, the proof of Theorem 5.2.2 shows that every bounded operator $T : \mathcal{A} \to \mathcal{B}^*$ is completely bounded in Itoh's sense; and indeed

$$\|T\| \leq \|T\|_{cbd} \leq 2 \|T\| ,$$

where we write $\|\cdot\|_{cbd}$ for the completely bounded norm defined in [It]. Thus the tensor norm $\|\cdot\|_\nu$ of [It] is in fact equivalent to the projective norm. The proof of Theorem 5.2.6 can be also be modified fractionally to show that the least constant that suffices in the inequality $\|T\|_{cbd} \leq K \|T\|$ is not smaller than $4/\pi$.

5.1 DEFINITIONS AND MOTIVATION.

Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{M}_n(\mathcal{A})$ be the C^*-algebra of $n \times n$ matrices with entries in $\mathcal{A}$. Define the 'normalised trace'

$$\tau_n : \mathcal{M}_n(\mathcal{A}) \to \mathcal{A} : A \mapsto n^{-1} \Sigma_{i=1}^n a_{ii} \quad .$$

This a contractive mapping. We reserve the symbol Tr for the trace map defined on the trace class operators on a Hilbert space [**Ri**].

Now let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be C^*-algebras, and let $\mathcal{K}$ be a Hilbert space. Suppose Ψ is an m-linear map $\mathcal{A}_1 \times \ldots \times \mathcal{A}_m \to B(\mathcal{K})$. Define for each $n \in \mathbb{N}$ the m-linear map

$$\Psi^n : \mathcal{M}_n(\mathcal{A}_1) \times \ldots \times \mathcal{M}_n(\mathcal{A}_m) \to B(\mathcal{K})$$

by

$$\Psi^n = \tau_n \circ \Psi_n \quad ,$$

where Ψ_n is the n-fold amplification of Ψ defined in 3.1.8 . Written explicitly this is :

$$\Psi^n(X^1, \ldots, X^m) = n^{-1} \Sigma_{i_1=1}^n \ldots \Sigma_{i_m=1}^n \Psi(x^1_{i_1,i_2}, x^2_{i_2,i_3}, \ldots, x^m_{i_m,i_1}) \ ,$$

where $X^i \in \mathcal{M}_n(\mathcal{A}_i)$ for $i = 1,\ldots,m$.

We say Ψ is *tracially completely bounded* if

$$\sup \{ \|\Psi^n\| : n \in \mathbb{N} \} < \infty \quad ,$$

and then we write $\|\Psi\|_{tcb}$ for this supremum. Write $TCB(\mathcal{A}_1 \times \ldots \times \mathcal{A}_m;B(\mathcal{K}))$ for the Banach space of tracially completely bounded m-linear maps $\mathcal{A}_1 \times \ldots \times \mathcal{A}_m \to B(\mathcal{K})$, with the norm $\|\cdot\|_{tcb}$.

It is clear that if Ψ is completely bounded then it is tracially completely bounded and

$$\|\Psi\| \le \|\Psi\|_{tcb} \le \|\Psi\|_{cb} \quad .$$

Thus

$$CB(\mathcal{A}_1 \times \ldots \times \mathcal{A}_m ; B(\mathcal{H})) \subset TCB(\mathcal{A}_1 \times \ldots \times \mathcal{A}_m ; B(\mathcal{H})) \subset B(\mathcal{A}_1 \times \ldots \times \mathcal{A}_m ; B(\mathcal{H})) .$$

Also if Ψ is a linear map, then Ψ is tracially completely bounded if and only if it is bounded, indeed $\|\Psi\|_{tcb} = \|\Psi\|$ in this case.

Notice that the explicit expression for Ψ^n is invariant under cyclic permutation of the indices $i_1, \ldots, i_m$. Thus if ρ is the 'cyclic permutation of variables' map

$$(\rho \circ \Psi)(a_m, a_1, \ldots, a_{m-1}) = \Psi(a_1, \ldots, a_m)$$

then $\rho \circ \Psi^n = (\rho \circ \Psi)^n$. This is not the case for the map Ψ_n ; indeed as remarked in the introduction, the original motivation for the definition of Ψ^n was that it had this property. We give some further motivation below.

Let $n \in \mathbb{N}$ be fixed. We wish to consider functors a from the class of multilinear maps

$$\mathcal{A}_1 \times \ldots \times \mathcal{A}_m \to B(\mathcal{H})$$

(for all $m \in \mathbb{N}$, $\mathcal{A}_i$ C^*-algebras, $\mathcal{H}$ a Hilbert space) to the class of multilinear maps

$$M_n(\mathcal{A}_1) \times \ldots \times M_n(\mathcal{A}_m) \to B(\mathcal{H})$$

(for all $m \in \mathbb{N}$, $\mathcal{A}_i$ C^*-algebras, $\mathcal{H}$ a Hilbert space). The only sensible such functors a would seem to be those satisfying the following condition (*) : For each $m \in \mathbb{N}$, for all C^*-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$, for all Hilbert spaces $\mathcal{H}$, and for all m-linear maps $\Psi : \mathcal{A}_1 \times \ldots \times \mathcal{A}_m \to B(\mathcal{H})$, we have

$$(a\, \Psi)(a_1 \otimes e_{i_1,j_1}, \ldots, a_m \otimes e_{i_m,j_m}) = \alpha_{i_1 j_1, \ldots, i_m j_m} \Psi(a_1, \ldots, a_m)$$

for $a_1 \in \mathcal{A}_1$, ... , $a_m \in \mathcal{A}_m$. Here $\alpha_{i_1 j_1, \ldots, i_m j_m}$ is a complex number which is independent of the particular C^*-algebras

$\mathcal{A}_1$, ... , $\mathcal{A}_m$, Hilbert space $\mathcal{H}$, or mapping Ψ .

We shall want to consider functors a as above which also satisfy the normalizing condition (**) :

$$(a\ f)\ (x \otimes I_n) = f(x) \qquad\qquad (x \in \mathcal{A})$$

for linear functionals f on a C^*-algebra $\mathcal{A}$.

Suppose $\mathcal{A}$ is a C^*-algebra, and let ($C^m_\lambda(\mathcal{A})$, b) be the cyclic cohomology cochain complex [**Con**]. It is easy to show by mathematical induction that, subject to the conditions (*) and (**) above, there is only one functor a such that the following diagram commutes

$$\begin{array}{ccccccc} C^0_\lambda(\mathcal{A}) & \xrightarrow{b} & C^1_\lambda(\mathcal{A}) & \xrightarrow{b} & C^2_\lambda(\mathcal{A}) & \xrightarrow{b} & \cdots \\ a\downarrow & & a\downarrow & & a\downarrow & & \\ C^0_\lambda(\mathcal{M}_n(\mathcal{A})) & \xrightarrow{b} & C^1_\lambda(\mathcal{M}_n(\mathcal{A})) & \xrightarrow{b} & C^2_\lambda(\mathcal{M}_n(\mathcal{A})) & \xrightarrow{b} & \cdots \end{array}$$

for all C^*-algebras $\mathcal{A}$, and that is the functor $\Psi \to \Psi^n$ defined in the beginning of this section. It is not surprising then that this functor appears in various guises in Hochschild and cyclic cohomology theory (e. g. the Dennis map [**Ig**]; the cup product # Tr [**Con**]).

Thus if there was a useful representation theory for tracially completely bounded maps, perhaps similar to the representation of Theorem 3.1.9 for completely bounded maps, then the tracially completely bounded maps would be an appropriate setting for a 'completely bounded' cyclic cohomology theory (see [**ChS2**,**Con**]). We make some comments on representations in 5.3.

5.2 THE BILINEAR CASE.

Throughout this section $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras and $\mathcal{H}$ is a Hilbert space. Define a semi-norm $\|\cdot\|_{tcb}$ on $\mathcal{A} \otimes \mathcal{B}$ by

$$\|u\|_{tcb} = \inf \{ \Sigma_{k=1}^{N} \|A^k\| \ \|B^k\| : u = \Sigma_{k=1}^{N} n(k)^{-1} \Sigma_{i,j=1}^{n(k)} a_{ij}^k \otimes b_{ji}^k \}$$

$$= \inf \{ \Sigma_{k=1}^{N} \|A^k\| \ \|B^k\| : u = \Sigma_{k=1}^{N} n^{-1} \Sigma_{i,j=1}^{n} a_{ij}^k \otimes b_{ji}^k \}$$

as can be seen by building larger matrices of fixed size $\Pi_{k=1}^{N} n(k)$ made up of the smaller matrices repeated sufficiently many times along the main diagonal.

5.2.1 PROPOSITION. *The seminorm* $\|\cdot\|_{tcb}$ *defined above on* $\mathcal{A} \otimes \mathcal{B}$ *is actually a completely positive uniform *-algebra tensor norm satisfying*

$$\|\cdot\|_h \le \|\cdot\|_{tcb} \le \|\cdot\|_\gamma \quad .$$

If $\mathcal{A} \otimes_{tcb} \mathcal{B}$ *is the completion of* $\mathcal{A} \otimes \mathcal{B}$ *with respect to* $\|\cdot\|_{tcb}$ *then* $B(\mathcal{A} \otimes_{tcb} \mathcal{B}; B(\mathcal{H}))$ *is isometrically isomorphic to* $TCB(\mathcal{A} \times \mathcal{B}; B(\mathcal{H}))$.

Proof. To show that $\|\cdot\|_{tcb}$ is a norm it suffices to show that $\|\cdot\|_h \le \|\cdot\|_{tcb}$. Write $u = \Sigma_{k=1}^{N} n^{-1} \Sigma_{i,j=1}^{n} a_{ij}^k \otimes b_{ji}^k$, then

$$\|u\|_h \le \Sigma_{k=1}^{N} n^{-1} \| \Sigma_{i,j=1}^{n} a_{ij}^k (a_{ij}^k)^* \|^{½} \| \Sigma_{i,j=1}^{n} (b_{ji}^k)^* b_{ji}^k \|^{½}$$

$$\le \Sigma_{k=1}^{N} \| \tau_n(A^k (A^k)^*) \|^{½} \| \tau_n((B^k)^* B^k) \|^{½}$$

$$\le \Sigma_{k=1}^{N} \|A^k\| \ \|B^k\| \quad ,$$

and thus $\|\cdot\|_h \le \|\cdot\|_{tcb}$.

Now if $u , v \in \mathcal{A} \otimes \mathcal{B}$, with $u = \Sigma_{k=1}^{N} n^{-1} \Sigma_{i,j=1}^{n} a_{ij}^k \otimes b_{ji}^k$ and $v = \Sigma_{k=1}^{N} n^{-1} \Sigma_{i,j=1}^{n} c_{ij}^k \otimes d_{ji}^k$, then we have

$$u\ v = \Sigma_{k,l=1}^{N} n^{-2} \Sigma_{i,j,p,q=1}^{n} a_{ij}^{k} c_{pq}^{l} \otimes b_{ji}^{k} d_{qp}^{l} .$$

Consequently letting $A^k \otimes C^l$ and $B^k \otimes D^l$ be the matrices in $\mathcal{M}_{n^2}(\mathcal{A})$ given by $[\ a_{ij}^{k}\ c_{pq}^{l}\ ]_{(i,p),(j,q)}$ and $[\ b_{ij}^{k}\ d_{pq}^{l}\ ]_{(i,p),(j,q)}$ respectively we have

$$\|\ u\ v\ \|_{tcb} \le \Sigma_{k,l=1}^{N} \|\ A^k \otimes C^l\ \|\ \|\ B^k \otimes D^l\ \|$$
$$\le (\ \Sigma_{k=1}^{N} \|A^k\|\ \|B^k\|\ )\ (\ \Sigma_{l=1}^{N} \|C^l\|\ \|D^l\|\ )\ ,$$

and so $\|\ u\ v\ \|_{tcb} \le \|u\|_{tcb}\ \|v\|_{tcb}$. Thus $\|\cdot\|_{tcb}$ is an algebra norm. The complete positive uniformity follows from 3.1.3, and the other statements of the proposition are obvious. □

In fact the proof above shows that $\|\cdot\|_{sh} \le \|\cdot\|_{tcb}$, where $\|\cdot\|_{sh}$ is the commutative symmetrized Haagerup norm of 3.3.6. Notice that by the last statement of Proposition 5.2.1 there is no need to attempt to consider $\|\cdot\|_{tcb}$ as an operator space tensor norm.

5.2.2 THEOREM. *Let* $\Psi : \mathcal{A} \times \mathcal{B} \to B(\mathcal{H})$ *be a bilinear map. Then* Ψ *is tracially completely bounded if and only if it is bounded and then*

$$\|\Psi\| \le \|\Psi\|_{tcb} \le 2\ \|\Psi\|\ .$$

Thus the norm $\|\cdot\|_{tcb}$ *is equivalent to the projective norm.*

Proof. The necessity is clear. Suppose Ψ is bounded. If f is a linear functional on $B(\mathcal{H})$ then $f \circ \Psi^n = (f\circ\Psi)^n$, and the Hahn-Banach theorem allows us to assume without loss of generality that Ψ is a bilinear functional. By the Grothendieck-Pisier-Haagerup inequality (Theorem 4.2.4) there exist states φ_1 and φ_2

on $\mathcal{A}$, and states ψ_1 and ψ_2 on $\mathcal{B}$, such that

$$|\Psi(a,b)| \leq \|\Psi\| \{ \varphi_1(a^* a) + \varphi_2(a a^*) \}^{1/2} \{ \psi_1(b^* b) + \psi_2(b b^*) \}^{1/2} ,$$

for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Using this and the Cauchy-Schwarz inequality we obtain for each $A \in \mathcal{M}_n(\mathcal{A})$ and $B \in \mathcal{M}_n(\mathcal{B})$ that

$$\begin{aligned}
|\Psi^n(A,B)| &\leq n^{-1} \Sigma^n_{i,j=1} |\Psi(a_{ij},b_{ji})| \\
&\leq n^{-1} \|\Psi\| \Sigma^n_{i,j=1} \{ \varphi_1(a^*_{ij} a_{ij}) + \varphi_2(a_{ij} a^*_{ij}) \}^{1/2} \\
&\qquad\qquad \{ \psi_1(b^*_{ji} b_{ji}) + \psi_2(b_{ji} b^*_{ji}) \}^{1/2} \\
&\leq n^{-1} \|\Psi\| \{ \varphi_1(\Sigma^n_{i,j=1} a^*_{ij} a_{ij}) + \varphi_2(\Sigma^n_{i,j=1} a_{ij} a^*_{ij}) \}^{1/2} \\
&\qquad \{ \psi_1(\Sigma^n_{i,j=1} b^*_{ji} b_{ji}) + \psi_2(\Sigma^n_{i,j=1} b_{ji} b^*_{ji}) \}^{1/2} \\
&\leq n^{-1} \|\Psi\| \{ \| \Sigma^n_{i,j=1} a^*_{ij} a_{ij} \| + \| \Sigma^n_{i,j=1} a_{ij} a^*_{ij} \| \}^{1/2} \\
&\qquad \{ \| \Sigma^n_{i,j=1} b^*_{ji} b_{ji} \| + \| \Sigma^n_{i,j=1} b_{ji} b^*_{ji} \| \}^{1/2} \\
&\leq \|\Psi\| \{\|\tau_n(A^* A)\| + \|\tau_n(A A^*)\|\}^{1/2} \{\|\tau_n(B^* B)\| + \|\tau_n(B B^*)\|\}^{1/2} \\
&\leq \|\Psi\| \{ \| A^* A \| + \| A A^* \| \}^{1/2} \{ \| B^* B \| + \| B B^* \| \}^{1/2} \\
&\leq 2 \|\Psi\| \|A\| \|B\| .
\end{aligned}$$

□

5.2.3 REMARK. If $\mathcal{A}$ and $\mathcal{B}$ are commutative we can use Corollary 4.2.3 to improve the inequality to

$$\|\Psi\| \leq \|\Psi\|_{tcb} \leq \|\Psi\|_{cb} \leq K^{\mathbb{C}}_G \|\Psi\| ;$$

where $K^{\mathbb{C}}_G$ is the complex Grothendieck constant.

5.2.4 REMARK. The complete positive uniformity of $\|\cdot\|_{tcb}$ together with the results of Chapter 4 now inform us that if $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras then

$$H(\mathcal{A} \otimes_{tcb} \mathcal{B}) = (\mathcal{A}_{s.a.} \otimes 1) \oplus (1 \otimes \mathcal{B}_{s.a.}) .$$

If $\mathcal{A}$ and $\mathcal{B}$ are any two C^*-algebras then $\mathcal{A} \otimes_{tcb} \mathcal{B}$ is a semisimple A^*-algebra and never a C^*-algebra unless $\mathcal{A}$ or $\mathcal{B}$ is $\mathbb{C}$.

5.2.5 COROLLARY. *Every bounded linear map* $T : \mathcal{A} \to \mathcal{B}^*$ *is completely bounded in the sense of* [It], *and* $\|T\|_{cbd} \leq 2 \|T\|$.

Let K_{tcb} be the least constant such that

$$\|\Psi\|_{tcb} \leq K \|\Psi\| \quad ,$$

for all C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, all Hilbert spaces $\mathcal{H}$, and every bounded bilinear map $\Psi : \mathcal{A} \times \mathcal{B} \to B(\mathcal{H})$. The theorem above asserts that $1 \leq K_{tcb} \leq 2$. The next theorem gives a better lower bound for K_{tcb} .

5.2.6 THEOREM. *The constant* K_{tcb} *is not smaller than* $4/\pi$.

We shall need two lemmas and some notation. Both lemmas are well known, but we include a sketch proof of the first for completeness.

5.2.7 LEMMA. *For each* $\epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\left| (2\pi)^{-n} \int_{[0,2\pi)^n} \left| \Sigma_{j=1}^n r_j e^{i\theta_j} \right| d\theta_1 \ldots d\theta_n - \pi^{1/2} / 2 \right| < \epsilon$$

whenever $r_1,\ldots,r_n$ *are positive numbers, each smaller than* δ , *such that* $\Sigma_{j=1}^n r_j^2 = 1$.

Proof. For $n = 1,2,..$ let $\theta_{n1}, \ldots, \theta_{nn}$ be independent random variables, each uniformly distributed on $[0,2\pi)$; and let $r_{n1}, \ldots, r_{nn}$ be fixed positive scalars with

$$\Sigma_{j=1}^{n} r_{nj}^2 = 1 ,$$

and $$\sup \{ r_{nj} : 1 \le j \le n \} \to 0 \quad \text{as} \quad n \to \infty .$$

For $j = 1,\ldots,n$ put $X_{nj} = r_{nj} e^{i\theta_{nj}}$ and $S_n = \Sigma_{j=1}^{n} X_{nj}$; then $2^{1/2} S_n$ converges, by the Lindeberg form of the central limit theorem (see [**Bi**] for example), to the standard complex normal distribution with density ϕ . Thus given $f : \mathbb{C} \to \mathbb{C}$ continuous and bounded, and $\epsilon > 0$, there exists a positive number δ such that if $r_1,\ldots,r_n$ are positive numbers each smaller than δ , and $\Sigma_{j=1}^{n} r_j^2 = 1$, then

$$|(2\pi)^{-n} \int_{[0,2\pi)^n} f(2^{1/2} \Sigma_{j=1}^{n} r_j e^{i\theta_j}) \, d\theta_1 \ldots d\theta_n - \int_{\mathbb{C}} f\phi | < \epsilon .$$

To see this notice that the converse leads to a contradiction. The lemma will now follow after an appropriate choice of f . □

We endow $\mathcal{M}_n$ with the inner product

$$\langle A , B \rangle = \Sigma_{i,j=1}^{n} a_{ij} \overline{b_{ij}} = \operatorname{Tr}(A B^*) ,$$

and write $\|\cdot\|_2$ for the associated norm. The group $\mathcal{U}(n)$ of $n \times n$ unitary matrices inherits a topology from $(\mathcal{M}_n , \|\cdot\|_2)$ with respect to which it is a compact, and thus unimodular, topological group. Let $\mathbb{P}$ denote the normalized Haar measure on $\mathcal{U}(n)$, writing $d\mathbb{P}(U)$ as dU as usual. Let π_{ij} denote the (i,j) coordinate function on $\mathcal{U}(n)$, namely

$$\pi_{ij}(U) = u_{ij} \qquad (U \in \mathcal{U}(n)) .$$

We need the following facts, proofs of which may be found in [**HR**].

5.2.8 LEMMA. *For* i , j , k *and* l *in* $\{1,\ldots,n\}$ *we have the following orthogonality relations:*

(i) $\int_{\mathcal{U}(n)} \pi_{ij}\ \overline{\pi_{kl}}\ dU = n^{-1}\ \delta_{ik}\ \delta_{jl}$.

(ii) $\int_{\mathcal{U}(n)} |\pi_{ij}|^2\ |\pi_{kl}|^2\ dU = 1/(n^2-1)$ *if* $i\neq k$ *and* $j\neq l$

$= 2/(n^2+n)$ *if* $i=k$ *and* $j=l$

$= 1/(n^2+n)$ *otherwise.*

For any $\Lambda \in \mathcal{M}_n$ *we have*

(iii) $\int_{\mathcal{U}(n)} |\ \langle \Lambda\ ,\ U \rangle\ |^2\ dU = n^{-1}\ \|\Lambda\|_2^2$.

Proof of Theorem 5.2.6. Let $\mathcal{A}$ be the C^*-algebra of continuous complex valued functions on $\mathcal{U}(n)$. Define a bilinear functional $\Psi : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ by

$$\Psi(f,g) = \int_{\mathcal{U}(n)\ \times\ \mathcal{U}(n)} f(U)\ g(V)\ \langle U\ ,\ V \rangle\ dU\ dV \quad .$$

Lemma 5.2.8 (iii) gives

$$|\ \Psi^n([\pi_{ji}]^*,[\pi_{ji}])\ | = n^{-1} \iint |\ \langle U\ ,\ V \rangle\ |^2\ dU\ dV$$
$$= n^{-1} \quad ,$$

whereas

$$\|[\pi_{ji}]\| = \|\ [\ \Sigma_{k=1}^n\ \pi_{ki}\ \overline{\pi_{kj}}\ ]\ \|^{1/2} = 1 \quad .$$

Thus we see immediately that

$$K_{tcb} \geq 1\ /\ (\ n\ \|\Psi\|\ ) \quad .$$

The remainder of the proof of the theorem is the calculation of an asymptotic lower bound for the right hand side of this inequality.

Let f , $g \in \mathrm{BALL}(\mathcal{A})$ be fixed. By the Riesz representation theorem there exists $W \in \mathcal{M}_n$ such that

$$< U , W > = \int_{\mathcal{U}(n)} < U , V > g(V)\, dV \qquad (U \in \mathcal{M}_n) .$$

Now $\|W\|_2^2 = < W , W > \leq \int_{\mathcal{U}(n)} | < W , V > | \, dV \leq \|W\|_2 \; c_n$,

where c_n is defined by

$$c_n = \sup \{ \int_{\mathcal{U}(n)} | < A , U > | \, dU : A \in \mathcal{M}_n , \|A\|_2 \leq 1 \} .$$

Thus we see that $\|W\|_2 \leq c_n$. Using the definition of W we have

$$\begin{aligned} | \Psi(f,g) | &= | \int_{\mathcal{U}(n)} < U , W > f(U)\, dU | \\ &\leq \int_{\mathcal{U}(n)} | < U , W > | \, dU \\ &\leq \|W\|_2 \; c_n \\ &\leq c_n^2 . \end{aligned}$$

Since f and g were arbitrary elements in $\mathrm{BALL}(\mathcal{A})$ we conclude that $\|\Psi\| \leq c_n^2$.

To complete the proof it suffices to show that given $\epsilon > 0$, we have

$$n^{1/2} c_n \leq \pi^{1/2} / 2 + \epsilon$$

for n large enough. To that purpose let $\epsilon > 0$ be given, and choose $\delta > 0$ as in Lemma 5.2.7.

Now suppose A is an arbitrary element of $\mathcal{M}_n$, with $\|A\|_2 = 1$. Let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues, in increasing order, of the positive definite square root $|A|$ of $A^* A$. Thus

$$0 \leq \lambda_1 \leq \ldots \leq \lambda_n$$

and

$$\Sigma_{i=1}^n \lambda_i^2 = 1 . \qquad (1)$$

Put $\Lambda = \mathrm{diag}\{\lambda_1,\ldots,\lambda_n\}$. Using the polar decomposition and spectral theorem in finite dimensions we can find unitary matrices V_1 and V_2 such that

$$A = V_1 \; |A| \quad \text{and} \quad V_2^* \; |A| \; V_2 = \Lambda \; .$$

By the invariance of Haar measure it follows that

$$\int_{\mathcal{U}(n)} | < A \; , \; U > | \; dU = \int_{\mathcal{U}(n)} | < \Lambda \; , \; U > | \; dU$$

$$= \int_{\mathcal{U}(n)} | \; \Sigma_{i=1}^{n} \lambda_i \; u_{ii} \; | \; dU \; . \qquad (2)$$

A direct application of Lemma 5.2.7 will fail here if some of the λ_i are too large. To avoid this possibility we spread each large λ_i over its own column in Λ ; to retain independence of the columns we eliminate some of the smaller λ_i . More specifically, let C be some large positive number, to be chosen later, and suppose s is the smallest positive integer with

$$\lambda_s \geq C \; / \; n^{1/2} \; . \qquad (3)$$

If $\lambda_n < C \; / \; n^{1/2}$ the proof will be substantially easier, we leave it to the reader to prune the argument below in this case.

From (1) we have immediately that

$$n - s + 1 = \# \{ \; \lambda_i \geq C \; / \; n^{1/2} \; \} \; \leq n \; / \; C^2 \; . \qquad (4)$$

Write $[\cdot]$ for the 'integer part of' function :

$$[x] = \max \{ \; n = 0,1,\ldots : n \leq x \; \} \qquad (\; x \geq 0 \;)$$

(not to be confused with the square bracket matrix notation).

We now define some integers: $m_{s-1} = 0$,

$$m_i = [n^{1/2} \lambda_s] + \ldots + [n^{1/2} \lambda_i] \qquad (\; i = s,\ldots,n \;) \; ,$$

and put $m = m_n$. By (1),(4) and the Cauchy-Schwarz inequality it follows that

$$m \leq n^{1/2} \; \Sigma_{i=s}^{n} \; \lambda_i \leq n^{1/2} \; (n-s+1)^{1/2} \leq n \; / \; C \; . \qquad (5)$$

Thus if $C \geq 3$ then (4) and (5) give

$$s - m \geq n / 2 \quad . \tag{6}$$

Now from (1) we see that

$$\lambda_m^2 \leq 1 - \Sigma_{i=m+1}^{s-1} \lambda_i^2 \leq 1 - (s-m-1)\, \lambda_m^2 \quad ;$$

and so by (6)

$$\lambda_m^2 \leq (s-m)^{-1} \leq 2 / n \quad . \tag{7}$$

By Lemma 5.2.8 (iii) and the Cauchy-Schwarz inequality

$$\begin{aligned} n^{\frac{1}{2}} \int_{\mathcal{U}(n)} \mid \Sigma_{i=1}^{m} \lambda_i u_{ii} \mid dU &\leq n^{\frac{1}{2}} \{ \int_{\mathcal{U}(n)} \mid \Sigma_{i=1}^{m} \lambda_i u_{ii} \mid^2 dU \}^{\frac{1}{2}} \\ &= \{ \Sigma_{i=1}^{m} \lambda_i^2 \}^{\frac{1}{2}} \\ &\leq m^{\frac{1}{2}} \lambda_m \\ &\leq (2 / C)^{\frac{1}{2}} \end{aligned} \tag{8}$$

using (5) and (7).

Let $(e_k)_{k=1}^{n}$ denote the usual orthonormal basis of $\mathbb{C}^n$. Let V be an operator on $\mathbb{C}^n$ such that

(i) $V e_k = e_k$ for $k = (m+1),\ldots,(s-1)$,

(ii) $V e_k = [n^{\frac{1}{2}} \lambda_k]^{-\frac{1}{2}} (e_{m_{k-1}+1} + \ldots + e_{m_k})$ for $k = s,\ldots,n$,

(iii) $(Ve_k)_{k=1}^{n}$ is an orthonormal basis for $\mathbb{C}^n$;

this forces V to be unitary.

Let $\tilde{\lambda}_i = \lambda_i / [n^{\frac{1}{2}} \lambda_i]^{\frac{1}{2}}$ for $i = s,\ldots,n$. By the invariance of Haar measure we have

$$\begin{aligned} & n^{\frac{1}{2}} \int_{\mathcal{U}(n)} \mid \Sigma_{i=m+1}^{n} \lambda_i u_{ii} \mid dU \\ & = n^{\frac{1}{2}} \int_{\mathcal{U}(n)} \mid \mathrm{Tr} (\mathrm{diag}\{0,\ldots,0,\lambda_{m+1},\ldots,\lambda_n\}\, U) \mid dU \\ & = n^{\frac{1}{2}} \int_{\mathcal{U}(n)} \mid \mathrm{Tr} (\mathrm{diag}\{0,\ldots,0,\lambda_{m+1},\ldots,\lambda_n\}\, U V) \mid dU \end{aligned}$$

$$= \int_{\mathcal{U}(n)} | \Sigma_{i=m+1}^{s-1} n^{1/2} \lambda_i u_{ii} + \Sigma_{i=s}^{n} \Sigma_{j=m_{i-1}+1}^{m_i} n^{1/2} \tilde{\lambda}_i u_{ij} | dU \quad . \quad (9)$$

In what follows if an integrand is not specified it shall be the integrand on the right hand side of (9).

Define the following random variables on $\mathcal{U}(n)$:

$$r_{ii}(U) = n^{1/2} \lambda_i |u_{ii}| \qquad (i = m+1,\ldots,s-1) \quad ;$$

$$r_{ij}(U) = n^{1/2} \tilde{\lambda}_i |u_{ij}| \qquad (i = s,\ldots,n \ , \ j = m_{i-1}+1,\ldots,m_i) \quad ;$$

and $\quad \Delta(U) = \sigma^{-1} \{ \Sigma_{i=m+1}^{s-1} r_{ii}(U)^2 + \Sigma_{i=s}^{n} \Sigma_{j=m_{i-1}+1}^{m_i} r_{ij}(U)^2 \}^{1/2}$,

where $\quad \sigma^2 = \Sigma_{i=m+1}^{n} \lambda_i^2$.

From (1) and (8) it follows that if $C \geq 4$ then

$$1/2 \leq \sigma^2 \leq 1 \quad . \qquad (10)$$

Now let Ω be the set of matrices U in $\mathcal{U}(n)$ such that

$$r_{ii}(U) < \delta \ , \quad i = m+1,\ldots,s-1 \quad ;$$

and $\quad r_{ij}(U) < \delta \ , \quad i = s,..,n \ , \ j = m_{i-1}+1,..,m_i$.

We may split up (9) into $\int_{\Omega} + \int_{\Omega'}$, where Ω' is the complement of Ω in $\mathcal{U}(n)$. Using Fubini's theorem and the invariance of Haar measure, one may replace this integrand by

$$(2\pi)^{-n} \int_{[0,2\pi)^n} |\Sigma_{i=m+1}^{s-1} r_{ii} e^{i\theta_i} + \Sigma_{i=s}^{n} \Sigma_{j=m_{i-1}+1}^{m_i} r_{ij} e^{i\theta_j} | d\theta_1 \ldots d\theta_m$$

and so by Lemma 5.2.7,

$$\begin{aligned} \int_{\Omega} &\leq \int_{\Omega} (\pi^{1/2}/2 + \epsilon) \Delta \sigma \, dU \\ &\leq (\pi^{1/2}/2 + \epsilon) \{ \int_{\Omega} \Delta^2 \, dU \}^{1/2} \text{ , by Cauchy-Schwarz and (10)} \\ &\leq \pi^{1/2}/2 + \epsilon \quad , \end{aligned}$$

using Lemma 5.2.8 (i). This together with (2) and (8) proves the theorem providing we can show that $\int_{\Omega'}$ is small. Now by

Cauchy-Schwarz and Lemma 5.2.8 (i) again,

$$\int_{\Omega'} \le \sigma^2 \, \mathbb{P}(\Omega')^{1/2}$$

$$\le \{\mathbb{P}(\Delta<\tfrac12)+\Sigma_{i=m+1}^{s-1} \mathbb{P}(r_{ii} \, \Delta > \delta/2)+\Sigma_{i=s}^{n}\Sigma_{j=m_{i-1}+1}^{m_i} \mathbb{P}(r_{ij} \, \Delta > \delta/2)\}^{1/2}.$$

Using Chebychev's inequality and Lemma 5.2.8 (i) we see that

$$\mathbb{P}(\Delta<\tfrac12) = \mathbb{P}((1-\Delta^2)^2 > 9/16)$$

$$\le 2 \int_{\mathcal{U}(n)} (1 - 2\,\Delta^2 + \Delta^4) \, dU$$

$$= 2 \, [\int_{\mathcal{U}(n)} \Delta^4 \, dU - 1] \quad .$$

Expanding Δ^4 and applying Lemma 5.2.8 (ii) yields

$$\int_{\mathcal{U}(n)} \Delta^4 \, dU = n^2/(n^2-1) + \sigma^{-4} \, n/(n+1) \, [(\Sigma_{i=m+1}^{s-1} \lambda_i^4) \, (n-2)/(n-1) + (\Sigma_{i=s}^{n} \lambda_i^4) \, (1/[n^{1/2} \lambda_i]-1/(n+1))]$$

$$\le n^2/(n^2-1) + 16 \, C^4/n + 16/(C-1) \quad ,$$

by (3) and (10) .

By Chebychev's inequality and Lemma 5.2.8 (ii) we obtain

$$\mathbb{P} \, (\, |u_{ii}| > \sigma \, \delta/(2 \, n^{1/2} \lambda_i) \,) \le 128 \, C^4/(\delta^4 \, (n^2+n)) \quad ,$$

for $i=m+1,\ldots,s-1$; also for $i = s,\ldots,n$, and $j = m_{i-1}+1,\ldots,m_i$

$$\mathbb{P} \, (\, |u_{ij}| > \sigma \, \delta \, [n^{1/2} \lambda_i]^{1/2}/(n^{1/2} \lambda_i) \,) \le 128 \, \lambda_i^4 \, / \, (\delta^4 \, [n^{1/2} \lambda_i]^2) \quad .$$

Thus we have finally,

$$\int_{\Omega'} \le \{2/(n^2-1) + 32 \, C^4/n + 32/(C-1) + 128 \, C^4/(\delta \, n) + 128 \, \delta^{-4}/(C-1)\}^{1/2}$$

and so by choosing first C , and then n , large enough we can ensure

$$n^{1/2} \int_{\mathcal{U}(n)} | < A \, , \, U > | \, dU \le \pi^{1/2}/2 + \epsilon \quad .$$

This completes the proof of the theorem. □

5.2.9 **REMARK.** The proof above shows that the convolution operator

$$C(\mathcal{U}(n)) \to L_1(\mathcal{U}(n)) : f \mapsto \mathrm{Tr} * f$$

has a norm which is asymptotically bounded above by $\pi/(4\, n)$.

5.2.10 **REMARK.** The construction above was on a commutative C^*-algebra. Thus in the commutative case one has that the best constant lies between $4/\pi$ and $K_G^{\mathbb{C}}$. Is this constant equal to $K_G^{\mathbb{C}}$? In the general case is $K_{tcb} = 2$?

5.2.11 **REMARK.** This construction originated in Grothendieck's construction [**Gr2**] yielding a lower bound for his constant; however there are some additional complications here which we had to overcome. I am indebted to A. M. Davie for suggesting this approach. A. M. Davie has modified Grothendieck's construction to improve the lower bound for Grothendieck's constant (unpublished); if one adapted this in the way the Grothendieck construction is adapted in the theorem above one should be able to improve the lower bound $4/\pi$ for K_{tcb} .

5.2.12 **COROLLARY.** *The least constant which suffices in the inequality*

$$\|T\|_{cbd} \le K \;\|T\| \quad ,$$

for all C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, and all linear maps $T : \mathcal{A} \to \mathcal{B}^*$ (*where* $\|\cdot\|_{cbd}$ *is the completely bounded norm of* [It]) *is not smaller than* $4/\pi$.

Proof. Almost identical to the proof above, but now define

$$< T(f) , g > = \iint f(U) \, g(V) \, (\Sigma_{i,j=1}^{n} u_{ij} \overline{v_{ji}}) \, dU \, dV \quad . \qquad \square$$

5.3 REPRESENTATION OF TRACIALLY COMPLETELY BOUNDED MAPS.

The results of the previous section might lead the reader to suppose that tracially completely bounded maps are just bounded maps in another guise. The first example below shows that the situation is more complicated than this.

5.3.1 EXAMPLE. A bounded trilinear map that is not tracially completely bounded. Let $\mathcal{H}$ be the infinite separable Hilbert space, with orthonormal basis $(e_n)_{n\in\mathbb{N}}$. Define a trilinear functional

$$\Psi : B(\mathcal{H}) \times B(\mathcal{H}) \times B(\mathcal{H}) \to \mathbb{C} : (R , S , T) \mapsto < R^t S^t T^t e_1 , e_1 > ,$$

where ^{t} is the transpose map relative to (e_n) . It is clear that Ψ is bounded, however if $X = [e_j \otimes e_i]_{i,j=1}^{n}$ then

$$\|X\| = \| X X^* \|^{1/2} = \| [e_j \otimes e_i] [e_j \otimes e_i] \|^{1/2}$$

$$= \| [\delta_{ij} \Sigma_{k=1}^{n} e_k \otimes e_k] \|^{1/2}$$

$$= 1 \quad ,$$

whereas

$$\Psi^n(X,X,X) = n^{-1} \Sigma_{i,j,k=1}^{n} < (e_i \otimes e_j) (e_j \otimes e_k) (e_k \otimes e_i) (e_1), e_1 >$$

$$= n^{-1} \Sigma_{i,j,k=1}^{n} < (e_i \otimes e_i) (e_1) , e_1 >$$

$$= n \quad .$$

Thus Ψ is not tracially completely bounded.

5.3.2 PROPOSITION. *Let* $\mathcal{A}_1, \ldots, \mathcal{A}_m$ *be* C^**-algebras. Define an m-linear functional* $\Psi : \mathcal{A}_1 \times \ldots \times \mathcal{A}_m \to \mathbb{C}$ *by*

$$\Psi(a_1,\ldots,a_m) = \mathrm{Tr}(T_1\ \theta_1(a_1)\ \ldots\ T_m\ \theta_m(a_m)) \quad ,$$

where Tr *is the trace on a Hilbert space* $\mathcal{H}$ *, where* $\theta_1, \ldots, \theta_m$ *are *-representations of* $\mathcal{A}_1, \ldots, \mathcal{A}_m$ *respectively on* $\mathcal{H}$ *, and where for* $i = 1,\ldots,m$ *the* T_i *are in the von Neumann - Schatten* p_i *class* [**Ri**], *where* $1 \le p_i \le \infty$ *and* $\Sigma_{i=1}^m 1/p_i = 1$. *Then* Ψ *is tracially completely bounded and*

$$\|\Psi\|_{tcb} \le \|T_1\|_{p_1} \cdots \|T_m\|_{p_m} \quad .$$

Moreover, every completely bounded m-linear functional $\mathcal{A}_1 \times \ldots \times \mathcal{A}_m \to \mathbb{C}$ *may be written in this form.*

Proof. Let Ψ be of the form described above. Then if $A_i \in M_n(\mathcal{A}_i)$ for $i = 1,\ldots,m$ we have

$$\begin{aligned}\Psi^n(A_1,\ldots,A_m) &= \tau_n \circ \Psi_n\ (A_1, \ldots, A_m) \\ &= \tau_n\ (\ (\mathrm{Tr})_n\ ((T_1 \otimes I_n)\ (\theta_1)_n(A_1)\ \ldots\ (T_m \otimes I_n)\ (\theta_m)_n(A_m))\ ) \\ &= n^{-1}\ \mathrm{Tr}(n)\ ((T_1 \otimes I_n)\ (\theta_1)_n(A_1)\ \ldots\ (T_m \otimes I_n)\ (\theta_m)_n(A_m)) \ ,\end{aligned}$$

where $\mathrm{Tr}(n)$ is the trace map on $\mathcal{H}^{(n)}$. Thus by well known von Neumann-Schatten class inequalities [**Ri**]

$$\begin{aligned}&|\Psi^n(A_1, \ldots, A_m)| \\ &\le n^{-1}\ \|T_1 \otimes I_n\|_{p_1}\ \|(\theta_1)_n(A_1)\ (T_2 \otimes I_n)\ \ldots\ (T_m \otimes I_n)\ (\theta_m)_n(A_m)\|_{p_1'} \\ &\le n^{-1}\ n^{1/p_1}\ \|T_1\|_{p_1}\ \|(\theta_1)_n(A_1)\|\ \|(T_2 \otimes I_n)\ldots(T_m \otimes I_n)\ (\theta_m)_n(A_m)\|_{p_1'}\end{aligned}$$

proceeding in this manner we obtain eventually

$$|\Psi^n(A_1,\ldots,A_m)| \le \|T_1\|_{p_1} \cdots \|T_m\|_{p_m}\ \|A_1\| \ldots \|A_m\| \quad .$$

Now suppose that $\Psi : \mathcal{A}_1 \times \ldots \times \mathcal{A}_m \to \mathbb{C}$ is completely bounded. By 3.1.11 and 3.1.9 we can find *-representations $\pi_1 , \ldots , \pi_m$ of $\mathcal{A}_1 , \ldots , \mathcal{A}_m$ on Hilbert spaces $\mathcal{H}_1 , \ldots , \mathcal{H}_m$ respectively, with $\mathcal{H}_1 = \mathcal{H}_m$, bridging operators $T_1 , \ldots , T_{m-1}$, and $\zeta , \eta \in \mathcal{H}_1$, such that

$$\Psi(a_1,\ldots,a_m) = \langle \pi_1(a_1)\, T_1 \ldots T_{m-1}\, \pi_m(a_m)\, \zeta , \eta \rangle \quad .$$

It is easy and standard [ChES] to adapt this representation so that a single Hilbert space is involved. Thus we may write

$$\Psi(a_1,\ldots,a_m) = \mathrm{Tr}\ (\ \pi_1(a_1)\, T_1 \ldots T_{m-1}\, \pi_m(a_m)\ (\zeta \otimes \eta)\ )\ ,$$

for $a_1 \in \mathcal{A}_1 , \ldots , a_m \in \mathcal{A}_m$. □

Perhaps it is possible to represent every tracially completely bounded functional in the form described in the proposition. This interesting class shares certain characteristics with the tracially completely bounded maps, for instance the trace Tr allows one to cyclically permute the variables, just as one may cyclically permute the indices (see remark in Section 5.1) in the expression for Ψ^n .

CHAPTER 6. SUBALGEBRAS OF C^*-ALGEBRAS.

In this chapter we consider Banach algebras which are isomorphic to a subalgebra of a C^*-algebra. The history of this subject probably began in the 1960's, with the study of *Q-algebras*. A Q-algebra is a commutative Banach algebra isomorphic to a quotient of a uniform algebra - the term is due to Varopoulos. B. Cole proved [**Wr2**] that such an algebra is isomorphic to a subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Many people have observed subsequently [Be] that his proof actually shows that a quotient of a subalgebra of a C^*-algebra is again a subalgebra of a C^*-algebra.

In 1972 A. M. Davie [**Da1**] gave a necessary and sufficient condition for a Banach algebra to be a Q-algebra, and shortly afterwards N. Th. Varopoulos [**Va3**] gave a characterization of Banach algebras which are isomorphic to a subalgebra of a C^*-algebra. Both proofs used properties of certain tensor norms; and both produced surprising examples of algebras which *were* Q-algebras or isomorphic to subalgebras of C^*-algebras, and algebras that were not. Subsequently T. K. Carne [**Ca3**] gave a proof of Varopoulos's result which displays more prominently the role of tensor norms (see also [**Ca2**]). A. M. Tonge [**Tn1,Tn2**] has produced other interesting results concerning the relationship with certain tensor norms.

In Section 6.1 we review some of the topics mentioned above in more detail. In Theorem 6.1.5 we show that if $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, and if α is either the projective, H', or Haagerup tensor norm, then $\mathcal{A} \otimes_\alpha \mathcal{B}$ is not isomorphic to a subalgebra of a C^*-algebra, unless $\mathcal{A}$ or $\mathcal{B}$ is finite dimensional. As a corollary to the method of the proof we obtain some estimates on how far the projective tensor product of two finite dimensional C^*-algebras is from being a subalgebra of a C^*-algebra. We show that the Haagerup tensor product of two operator spaces is represented isometrically on the space $B(\mathcal{H})$ of bounded operators on some Hilbert space, and that in a sense the Haagerup tensor product of two subalgebras of C^*-algebras is isometrically isomorphic to a subalgebra of $B(B(\mathcal{H}))$.

In 6.2 we consider operator spaces which are also Banach algebras. Following the characterization of operator spaces [**Ru**] as 'L^∞-matricial vector spaces' researchers in this area became interested in an abstract characterization of such 'matricial operator algebras' [**PnP**]. The author was made aware of this problem in conversations with E. G. Effros and V. I. Paulsen in 1987; and subsequently worked on the characterizations described in 6.2 with A. M. Sinclair.

It is easy to see that a completely bounded multiplication on an operator space satisfies Varopoulos's criterion, and consequently such a space is isomorphic to a subalgebra of a C^*-algebra. The difficulty lies in obtaining a 'complete isomorphism'. We study some examples which illustrate some of the problems if there is no identity of norm 1 for the algebra. Theorem 6.2.6, which was found by A. M. Sinclair, gives a characterization in the presence of an identity of norm 1 . We give some necessary and sufficient conditions in the general case, however these are not as desirable as one might wish. As a corollary we are able to generalize the aforementioned result of Cole to the operator space situation.

In [**PnP**] Paulsen and Power define three 'complete operator algebra tensor norms', and make some comments on the development of a theory of such norms. It is to be hoped that the characterizations above might play a role in this development.

6.1 OPERATOR ALGEBRAS.

6.1.1 ***Definition.*** We say a Banach algebra $\mathcal{A}$ is an *operator algebra* if there exists a Hilbert space $\mathcal{H}$ and a bicontinuous monomorphism of $\mathcal{A}$ into $B(\mathcal{H})$.

6.1.2 REMARK. Suppose that $\mathcal{A}$ is a Banach algebra with identity e , $\|e\| \geq 1$, and suppose that $\theta : \mathcal{A} \to B(\mathcal{H})$ is a bicontinuous homomorphism. Let $\mathcal{K}$ be the closed linear span in $\mathcal{H}$ of $\theta(\mathcal{A})(\mathcal{H})$. We obtain by restriction a bicontinuous homomorphism $\theta^{\sim} : \mathcal{A} \to B(\mathcal{K})$, with $\theta^{\sim}(e) = I_K$ and $\|\theta^{\sim}\| \leq \|\theta\|$, but now

$$1 \leq \|\theta^{\sim-1}\| \leq \|\theta(e)\| \; \|\theta^{-1}\| .$$

6.1.3 THEOREM (Cole [Wr2]). *Let* $\mathcal{A}$ *be an operator algebra, and suppose* I *is a closed two-sided ideal in* $\mathcal{A}$ *. Then* $\mathcal{A} / I$ *is an operator algebra.*

The following result of Varopoulos **[Va3,Ca3,Tn2]** gives a characterization of operator algebras:

6.1.4 THEOREM. *A Banach algebra* $\mathcal{A}$ *is an operator algebra if and only if the following condition is satisfied:*

There is a constant $K > 0$ *such that if* $f \in BALL(\mathcal{A}^*)$ *, and if* n *is a positive integer, then there exists a Hilbert space* $\mathcal{H}$, *elements* ζ *and* η *in* $\mathrm{BALL}(\mathcal{K})$, *and linear maps* $T_1,\ldots,T_n$ *of* $\mathcal{A}$ *into* $B(\mathcal{K})$, *each bounded by* K , *such that*

$$< f , a_1 \ldots a_n > = < T_1(a_1) \ldots T_n(a_n)\, \zeta , \eta > ,$$

for $a_1,\ldots,a_n \in \mathcal{A}$.

Proof. We prove the necessity only, the reader is referred to **[Va3]** or **[Ca3]** for the sufficiency. Let $\mathcal{A}$ be an operator algebra, let $\mathcal{K}$ be a Hilbert space, and let $\theta : \mathcal{A} \to B(\mathcal{K})$ be a bicontinuous homomorphism. Suppose $f \in \mathrm{BALL}(\mathcal{A}^*)$, then

$$g(S) = f\,(\theta^{-1}\,(S)) \qquad (S \in \theta(\mathcal{A}))$$

defines a bounded linear functional on $\theta(\mathcal{A})$, with $\|g\| \le \|\theta^{-1}\|$. Extend g to a functional $g^{\sim}$ on $B(\mathcal{K})$ with $\|g^{\sim}\| = \|g\|$. By Proposition 3.2.5 $g^{\sim}$ is completely bounded and $\|g^{\sim}\|_{cb} = \|g\|$. Thus by 3.1.7 there exists a Hilbert space $\mathcal{H}$, a *-representation π of $B(\mathcal{K})$ on $\mathcal{H}$, and ζ and $\eta \in \mathcal{H}$, such that $\|\zeta\|\,\|\eta\| \le \|\theta^{-1}\|$ and

$$< f , a > = g(\theta(a)) = < \pi(\theta(a))\, \zeta , \eta > ,$$

for $a \in \mathcal{A}$. Thus

$$< f , a_1 \ldots a_m > = < \|\zeta\|\,\|\eta\|\, \pi(\theta(a_1)) \ldots \pi(\theta(a_m))\, \|\zeta\|^{-1} \zeta , \|\eta\|^{-1} \eta >$$

for $a_1,\ldots,a_m \in \mathcal{A}$, and so the condition of the theorem is met. □

Let α be a Banach space tensor norm. We say a Banach algebra $\mathcal{A}$ is an α-algebra if the map

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A} : a_1 \otimes a_2 \mapsto a_1\, a_2$$

is continuous with respect to the α norm on $\mathcal{A} \otimes \mathcal{A}$.

The necessity proof given in Theorem 6.1.4 above shows that every operator algebra is an H'-algebra, a result which appeared first in **[Ch]**. Tonge **[Tn2]** showed that every /H'-algebra is an operator algebra, where /H' is a related tensor norm. In **[Ca2]** Carne shows that if α is a Banach space tensor norm such that the class of operator algebra coincides with the class of α-algebras then α is equivalent to H' ; then he constructs an H'-algebra that is not an operator algebra. Recall that for C^*-algebras the norms γ and H' are equivalent (Theorem 4.2.5), and thus it is clear that every C^*-algebra is an H'-algebra. The following theorem shows that the H'-tensor product of two infinite dimensional C^*-algebras is never an operator algebra. I do not know if it is ever an H'-algebra.

6.1.5 THEOREM. *Suppose α is either the projective tensor norm, the Haagerup norm or the* H'*-tensor norm. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, then $\mathcal{A} \otimes_\alpha \mathcal{B}$ is an operator algebra if and only if $\mathcal{A}$ or $\mathcal{B}$ is finite dimensional.*

We shall need the following lemma, which the author was unable to find in the literature, although most of the ideas appear in **[DS]**. The proof given may not be the most direct one, however later we shall need some of the details contained in this particular proof.

6.1.6 LEMMA. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are commutative C^*-algebras, and $\mathcal{H}$ is a Hilbert space. Let ρ and σ be non-trivial bounded homomorphisms from $\mathcal{A}$ and $\mathcal{B}$ respectively into* $B(\mathcal{H})$ *with commuting ranges. Then there exists an invertible operator* T *on $\mathcal{H}$ with*

$$\|T\| \, \|T^{-1}\| \leq 81 \, \|\rho\|^2 \, \|\sigma\|^2$$

such that if $\rho^{\sim}(\cdot) = T\,\rho(\cdot)\,T^{-1}$ *and* $\sigma^{\sim}(\cdot) = T\,\sigma(\cdot)\,T^{-1}$ *then* $\rho^{\sim}$ *and* $\sigma^{\sim}$ *are* $*$*-representations of* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively on* $\mathcal{H}$ *.*

Proof of lemma. First notice that because of the existence of contractive approximate identities for C^*-algebras $\|\rho\|$ and $\|\sigma\| \geq 1$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are isometrically isomorphic to $C_0(X)$ and $C_0(Y)$, for locally compact Hausdorff spaces X and Y respectively. Suppose $\zeta , \eta \in \mathcal{H}$. By the Riesz representation theorem the functional on $C_0(X)$ given by

$$f \mapsto \langle \rho(f)\, \zeta , \eta \rangle$$

defines a regular Borel measure $\mu_{\zeta,\eta}$ on X , and

$$\| \mu_{\zeta,\eta} \| \leq \|\rho\| \, \|\zeta\| \, \|\eta\| \, .$$

We now define a regular bounded $B(\mathcal{H})$ - valued spectral measure (c. f. [**Hd**]) E by

$$\langle E(B)\, \zeta , \eta \rangle = \mu_{\zeta,\eta}(B) \quad ,$$

for Borel sets B of X . This is not a spectral measure in the sense of [**DS**] since E(X) is not necessarily $I_{\mathcal{H}}$. The important thing here is that E(B) is an idempotent in $B(\mathcal{H})$ for Borel sets B of X . We now have the following identity:

$$\langle \rho(f)\, \zeta , \eta \rangle = \langle \int_X f(x)\, E(dx)\, \zeta , \eta \rangle = \int_X f \, d\mu_{\zeta,\eta} \quad ,$$

for $f \in C_0(X)$ and $\zeta , \eta \in \mathcal{H}$.

Similarly we can find a regular bounded $B(\mathcal{H})$ - valued spectral measure F on Y , and an associated family of regular Borel measures $\{ \nu_{\zeta,\eta} \}_{\zeta,\eta\in\mathcal{H}}$ such that

$$< \sigma(g)\ \zeta\ ,\ \eta > = < \int_Y g(y)\ F(dy)\ \zeta\ ,\ \eta > = \int_Y g\ d\nu_{\zeta,\eta} \quad ,$$

for $g \in C_0(Y)$ and $\zeta\ ,\ \eta \in \mathcal{H}$. Using these identities one may verify that

$$E(B)\ F(C) = F(B)\ E(C) \quad ,$$

for Borel sets B and C of X and Y respectively.

Consider the family $\mathcal{E}$ of operators of the form $I_{\mathcal{H}} - 2\ E(B)$. This family is bounded by the constant $3\ \|\rho\|$; and since

$$(\ I_{\mathcal{H}} - 2\ E(B)\ )^2 = I_{\mathcal{H}}$$

$$(\ I_{\mathcal{H}} - 2\ E(B_1)\ )\ (\ I_{\mathcal{H}} - 2\ E(B_2)\ ) = I_{\mathcal{H}} - 2\ E(B_1\ \Delta\ B_2) \quad ,$$

where Δ is the usual set theoretic symmetric difference, we see that $\mathcal{E}$ is a bounded group of operators on $\mathcal{H}$. Similarly the family $\mathcal{F}$ of operators on $\mathcal{H}$ of form $I_{\mathcal{H}} - 2\ F(C)$, for Borel sets C of Y , forms a bounded group in $B(\mathcal{H})$. Then, since $\mathcal{E}$ and $\mathcal{F}$ commute, we see that $\mathcal{E}\ \mathcal{F}$ is a group of operators on $\mathcal{H}$ bounded by the constant $9\ \|\rho\|\ \|\sigma\|$.

We now have recourse to a theorem of Wermer [DS], which states that if $\mathcal{G}$ is a group of operators on a Hilbert space, which is bounded by the constant M , then there exists an invertible operator T on the Hilbert space, with $\|T\|\ \|T^{-1}\| \le M^2$, such that every operator S in $\mathcal{G}$ is similar via T to a unitary operator. Thus in our case there exists an invertible operator T on $\mathcal{H}$, with

$$\|T\|\ \|T^{-1}\| \le 81\ \|\rho\|^2\ \|\sigma\|^2 \quad ,$$

such that for Borel sets B and C of X and Y respectively, there exists unitary operators U_B and V_C with

$$T\ (\ I_{\mathcal{H}} - 2\ E(B)\ )\ T^{-1} = U_B \quad \text{and} \quad T\ (\ I_{\mathcal{H}} - 2\ F(C)\ )\ T^{-1} = V_C \quad .$$

Now $U_B^2 = I_{\mathcal{X}}$ and so $U_B = U_B^* = U_B^{-1}$; thus $T\,E(B)\,T^{-1}$ is the orthogonal projection $\frac{1}{2}(I_{\mathcal{X}} + U_B)$. Similarly $T\,F(C)\,T^{-1}$ is an orthogonal projection.

Defining $\rho^{\sim}$ and $\sigma^{\sim}$ by

$$\rho^{\sim}(\cdot) = T^{-1}\,\rho(\cdot)\,T \quad \text{and} \quad \sigma^{\sim}(\cdot) = T^{-1}\,\sigma(\cdot)\,T ,$$

we see that $\rho^{\sim}$ and $\sigma^{\sim}$ are homomorphisms of $C_0(X)$ and $C_0(Y)$ respectively into $B(\mathcal{X})$. The $B(\mathcal{X})$ - valued spectral measures corresponding to $\rho^{\sim}$ and $\sigma^{\sim}$ are in fact orthogonal projection valued, and hence $\rho^{\sim}$ and $\sigma^{\sim}$ are *-homomorphisms. □

If, in the statement of the lemma above, ρ (or σ) was contractive, then the proof would imply that it is a *-homomorphism already, in which case one could improve the bound on $\|T\|\,\|T^{-1}\|$.

6.1.7 COROLLARY. *Suppose* $\{ S_i \}_{i\in I}$ *and* $\{ T_j \}_{j\in J}$ *are families of idempotents in* $B(\mathcal{X})$ *such that*

(*i*) $S_i\,T_j = T_j\,S_i$ ($i \in I$ *and* $j \in J$) ,

(*ii*) $S_{i_1}\,S_{i_2} = T_{j_1}\,T_{j_2} = 0$ (*if* $i_1 \neq i_2$ *and* $j_1 \neq j_2$) , *and*

(*iii*) *if* $\alpha \in \mathrm{BALL}(\ell_n^{\infty})$ *then* $\Sigma_{k=1}^{n}\,\alpha_k\,S_{i_k}$ *and* $\Sigma_{k=1}^{n}\,\alpha_k\,T_{j_k}$ *are bounded independently of* n *or the choice of* α , $\{i_n\}$ *or* $\{j_n\}$.

Then there is a positive constant C *such that*

$$\| \Sigma_{k,l=1}^{n}\,\alpha_{kl}\,S_{i_k}T_{j_l} \| \leq C \max \{ |\alpha_{kl}| : 1 \leq k , l \leq n \} .$$

for all sequences $\{\alpha_{ij}\}$ *of complex numbers.*

Proof of corollary. The result follows either by the methods of the proof of 6.1.6 or by an elementary argument directly from the statement, after defining two homomorphisms from ℓ_n^∞ into $B(\mathcal{H})$. □

Proof of Theorem 6.1.5. The sufficiency is clear. For the necessity we suppose $\mathcal{A}$ and $\mathcal{B}$ are infinite dimensional, and that $\theta : \mathcal{A} \otimes_\alpha \mathcal{B} \to B(\mathcal{H})$ is a bicontinuous homomorphism. Choose a maximal abelian *-subalgebra (henceforth a 'masa') in $\mathcal{A}$, which we may take to be $C_0(X)$, for some locally compact Hausdorff space X ; similarly find a masa $C_0(Y)$ in $\mathcal{B}$. Now X and Y are infinite spaces, since masa's of infinite dimensional C^*-algebras are infinite dimensional ([**KR**] Exercise 4.6.12). If α was the Haagerup norm then the injectivity (Theorem 3.3.4) would enable us to assume without loss of generality that $\mathcal{A}$ and $\mathcal{B}$ are commutative.

Write ρ and σ for the induced homomorphisms from $C_0(X)$ and $C_0(Y)$ into $B(\mathcal{H})$. Since ρ and σ have commuting ranges we find ourselves in the situation of Lemma 6.1.6, and may choose E , F and T as in the lemma. Here $\|T\| \, \|T^{-1}\| \leq 81 \, \|\theta\|^4$.

Since X is locally compact we can choose a sequence $\{ f_n \}$ in $\mathrm{Ball}(C_0(X))_+$, and a sequence $\{ \delta_n \}$ in the maximal ideal space of $C_0(X)$ (evaluation at points in X), such that

$$f_i \, f_j = 0 \qquad (i \neq j) ,$$

and

$$< \delta_i \, , f_j > = \delta_{ij} \quad .$$

By the Hahn-Banach theorem we may extend each δ_n to a contractive functional φ_n on $\mathcal{A}$. Choose sequences $\{ g_n \}$ and $\{ \psi_n \}$ in $\mathcal{B}$ and $\mathcal{B}^*$ similarly.

Let a positive integer N be given. For $\{ \omega_{jk} \}_{j,k=1}^N$, a double sequence in $[0,2\pi)$, we have

$$\Sigma_{j,k=1}^{N} e^{i\omega_{jk}} \theta(f_j \otimes g_k) = \Sigma_{j,k=1}^{N} e^{i\omega_{jk}} \int_X f_j(x)\ E(dx) \int_Y g_k(y)\ F(dy).$$

Choose for $k = 1,\dots,N$ sequences $\{ f_{k_m} \}_{m=1}^{\infty}$ and $\{ g_{k_m} \}_{m=1}^{\infty}$ of positive simple functions converging uniformly to f_k and g_k respectively from below. For fixed $m \in \mathbb{N}$ we have

$$\Sigma_{j,k=1}^{N} e^{i\omega_{jk}} \int_X f_{j_m}(x)\ E(dx) \int_Y g_{k_m}(y)\ F(dy)$$
$$= T^{-1} (\Sigma_{j,k=1}^{N} e^{i\omega_{jk}}\ T \int_X f_{j_m}(x)\ E(dx)\ T^{-1}\ T \int_Y g_{k_m}(y)\ F(dy)\ T^{-1})\ T$$

and this last expression is bounded by $\|T^{-1}\|\ \|T\|$, using the fact that if $P_1 , \dots , P_n$ are orthogonal projections onto mutually orthogonal subspaces then

$$\| \Sigma_{i=1}^{n} \lambda_i P_i \| = \max \{|\lambda_1|,\dots,|\lambda_n|\} ,$$

for $\lambda_1,\dots,\lambda_n \in \mathbb{C}$. Thus in the limit as $m \to \infty$ we obtain

$$\| \Sigma_{j,k=1}^{N} e^{i\omega_{jk}} \theta(f_j \otimes g_k) \| \le \|T^{-1}\|\ \|T\| \le 81\ \|\theta\|^4 .$$

Now let $[u_{jk}]_{j,k=1}^{N}$ be a unitary matrix with $|u_{jk}| = N^{-1/2}$; for example let

$$u_{jk} = N^{-1/2} \exp(2\pi i(j-1)k/N) .$$

Define a functional

$$V = \Sigma_{j,k=1}^{N} u_{jk}\ \varphi_j \otimes \psi_k$$

on $\mathcal{A} \otimes_\alpha \mathcal{B}$. Now if $a \in \mathcal{A}$ and $b \in \mathcal{B}$ then

$$|V(a \otimes b)| = | \Sigma_{j,k=1}^{N} u_{jk}\ \varphi_j(a)\ \psi_k(b) |$$
$$= \{ \Sigma_{i=1}^{N} |\varphi_i(a)|^2 \}^{1/2} \{\Sigma_{i=1}^{N} |\psi_i(b)|^2 \}^{1/2}$$

$$\leq N \, \|a\| \, \|b\| \quad .$$

Thus if $\alpha = \gamma$ then $\|V\| \leq N$. If $\alpha = H'$ then Theorem 4.2.5 gives $\|V\| \leq 2N$. If $\alpha = \|\cdot\|_h$, then Theorem 4.2.3 gives $\|V\| \leq K_G N$ (recall we are assuming in this case that $\mathcal{A}$ and $\mathcal{B}$ are commutative).

However, if we choose ω_{kl} such that $u_{kl} \, e^{i\omega_{kl}} = N^{-\frac{1}{2}}$, then

$$\|V\| \geq |V(\Sigma_{k,l=1}^{N} e^{i\omega_{kl}} f_k \otimes g_l)| \,/\, \|\Sigma_{k,l=1}^{N} e^{i\omega_{kl}} \theta^{-1} \theta(f_k \otimes g_l)\|$$

$$\geq \|\theta^{-1}\|^{-1} N^{3/2} \,/\, (81 \|\theta\|^4) \quad ,$$

which is a contradiction, since N was chosen arbitrarily. □

The construction above gives a direct proof of Theorem 4.2.8:

6.1.8 COROLLARY. *The tensor norms γ and λ are equivalent on the tensor product $\mathcal{A} \otimes \mathcal{B}$ of two C^*-algebras if and only if $\mathcal{A}$ or $\mathcal{B}$ is finite dimensional.*

Proof. If γ is equivalent to λ on $\mathcal{A} \otimes \mathcal{B}$ then γ is equivalent to $\|\cdot\|_{min}$, and so $\mathcal{A} \otimes_\gamma \mathcal{B}$ is an operator algebra. An application of Theorem 6.1.5 concludes the proof. □

Suppose $\mathcal{A}$ is an operator algebra, and that $\theta : \mathcal{A} \to B(\mathcal{H})$ is a bicontinuous homomorphism. By Remark 6.1.2, if $\mathcal{A}$ possesses an identity then the associated unital homomorphism $\theta^\sim$ satisfies the condition $\|\theta^{\sim -1}\| \geq 1$. Notice of course that in any case $\|\theta\| \, \|\theta^{-1}\| \geq 1$; thus if $\|\theta\| \leq 1$ then $\|\theta^{-1}\| \geq 1$, and if $\|\theta^{-1}\| \leq 1$ then $\|\theta\| \geq 1$.

6.1.9 ***Definition.*** Let $\mathcal{A}$ be an operator algebra. Define the *non-expansive distance* $d^{ne}_{0A}(\mathcal{A})$ *of* $\mathcal{A}$ *from an operator algebra* to be the following expression:

$$\inf\{\|\theta\|\ \|\theta^{-1}\| : \text{bicontinuous homomorphisms } \theta : \mathcal{A} \to B(\mathcal{H}) \ ,\ \|\theta^{-1}\| \geq 1\}.$$

Define the *contractive distance* $d^{c}_{0A}(\mathcal{A})$ *of* $\mathcal{A}$ *from an operator algebra* to be the same expression, except now the infimum is taken over all contractive bicontinuous homomorphisms $\theta : \mathcal{A} \to B(\mathcal{H})$. Finally, define the *expansive distance* $d^{e}_{0A}(\mathcal{A})$ *of* $\mathcal{A}$ *from an operator algebra* to be:

$$\inf\{\|\theta\| : \text{bicontinuous homomorphisms } \theta : \mathcal{A} \to B(\mathcal{H}) \text{ with } \|\theta^{-1}\| \leq 1\}.$$

The next results gives some idea of how far the projective tensor product of two finite dimensional C^*-algebras is from being a subalgebra of some $B(\mathcal{H})$.

6.1.10 COROLLARY. *For* $n , m \in \mathbb{N}$ *, with* $n \leq m$ *, we have*

$$3^{-1}\ n^{1/8} \leq d^{ne}_{0A}(\ell^\infty_n \otimes_\gamma \ell^\infty_m) \leq (2\ n)^{1/2} \ ,$$

$$3^{-1}\ n^{1/8} \leq d^{e}_{0A}(\ell^\infty_n \otimes_\gamma \ell^\infty_m) \quad , \textit{ and}$$

$$1 \leq d^{c}_{0A}(\ell^\infty_n \otimes_\gamma \ell^\infty_m) \ / \ n^{1/2} \leq 2^{1/2} \ .$$

Proof. Let $\mathcal{A} = \ell^\infty_n$, let $\mathcal{B} = \ell^\infty_m$, and suppose that $\mathcal{H}$ is a Hilbert space and that $\theta : \mathcal{A} \otimes_\gamma \mathcal{B} \to B(\mathcal{H})$ is a bicontinuous homomorphism. Proceeding as in the proof of Theorem 6.1.5, we obtain the inequality:

$$n \geq (81\ \|\theta^{-1}\|\ \|\theta\|^4)^{-1}\ n^{3/2} \ .$$

Thus

$$\|\theta\|^4 \; \|\theta^{-1}\| \geq 81^{-1} \, n^{\frac{1}{2}} \quad ,$$

and so $d^{ne}_{0A}(\mathcal{A} \otimes_\gamma \mathcal{B})$ and $d^{e}_{0A}(\mathcal{A} \otimes_\gamma \mathcal{B})$ both exceed $3^{-1} \, n^{1/8}$.

If θ was contractive, then by the remarks after Lemma 6.1.6 we actually obtain the inequality $\|\theta^{-1}\| \geq n^{\frac{1}{2}}$, which implies that

$$d^{c}_{oa}(\ell^\infty_n \otimes_\gamma \ell^\infty_m) \geq n^{\frac{1}{2}} \quad .$$

Indeed in this case it is easy to see that

$$\theta(\ell^\infty_n \otimes_\gamma \ell^\infty_m) = \ell^\infty_n \otimes_\lambda \ell^\infty_m$$

isometrically.

On the other hand, it is well known (see remark after 4.2.8) that the canonical contraction $\ell^\infty_n \otimes_\gamma \ell^\infty_m \to \ell^\infty_n \otimes_\lambda \ell^\infty_m$ has an inverse with norm dominated by $(2\,n)^{\frac{1}{2}}$. □

6.1.11 COROLLARY. *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are* C^**-algebras, with* $\mathcal{A}$ *finite dimensional and* $\mathcal{B}$ *infinite dimensional, then*

$$d^{ne}_{0A}(\mathcal{A} \otimes_\gamma \mathcal{B}) \geq 3^{-1} \; (\dim \mathcal{A})^{1/16} \quad ,$$

$$d^{e}_{0A}(\mathcal{A} \otimes_\gamma \mathcal{B}) \geq 3^{-1} \; (\dim \mathcal{A})^{1/16} \quad , \textit{and}$$

$$d^{c}_{0A}(\mathcal{A} \otimes_\gamma \mathcal{B}) \geq (\dim \mathcal{A})^{1/4} \quad .$$

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be as above, suppose $\mathcal{H}$ is a Hilbert space and suppose $\theta : \mathcal{A} \otimes_\gamma \mathcal{B} \to B(\mathcal{H})$ is a bicontinuous homomorphism. Write

$$\mathcal{A} = \mathcal{M}_{n_1} \oplus \ldots \oplus \mathcal{M}_{n_k} \quad ,$$

where $n_1,\ldots,n_k \in \mathbb{N}$, and put $m = n_1 + \ldots + n_k$. We proceed as in the proof of Theorem 6.1.5, but now choose $f_1,\ldots,f_m$ of the theorem to be the 'diagonal elements'

$$(0,\dots,0,e_{ii},0,\dots,0)$$

of $\mathcal{A}$. We obtain

$$m \geq (81\ \|\theta^{-1}\|\ \|\theta\|^4)^{-1}\ m^{3/2}\quad ,$$

and thus

$$\|\theta\|^4\ \|\theta^{-1}\| \geq 81^{-1}\ m^{\frac{1}{2}} \geq 81^{-1}\ (\dim \mathcal{A})^{1/4}\quad .$$

The case when θ is contractive follows as in the last corollary. □

It is shown in Theorem 4.2.1 that the Haagerup tensor product $\mathcal{A} \otimes_h \mathcal{B}$ of two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a Banach algebra. In addition, by the same theorem, there is a natural faithful representation of $\mathcal{A} \otimes_h \mathcal{B}$ on a Hilbert space. However by Theorem 6.1.5 we know that $\mathcal{A} \otimes_h \mathcal{B}$ is never an operator algebra, unless $\mathcal{A}$ or $\mathcal{B}$ is finite dimensional. Earlier Paulsen and Power [**PnP**] had noticed that there can exist no isometric homomorphism of $\mathcal{A} \otimes_h \mathcal{B}$ into the bounded operators on a Hilbert space. It is interesting to note that there often exist bicontinuous homomorphisms into $B(B(\mathcal{H}))$ for some Hilbert space $\mathcal{H}$ (see [**KaS**]).

6.1.12 THEOREM. *Suppose that* X *and* Y *are operator spaces contained in* C^**-algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *, and suppose that* $(\mathcal{A}_U, \rho_U, \mathcal{H}_U)$ *and* $(\mathcal{B}_U, \sigma_U, \mathcal{K}_U)$ *are the universal representations of* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively. There is an natural isometry*

$$\theta : X \otimes_h Y \to B(B(\mathcal{H}_U \oplus \mathcal{K}_U))$$

given by

$$\theta(a \otimes b)(T) = (\rho_U \oplus 0)(a)\ T\ (0 \oplus \sigma_U)(b) \quad ,$$

for $T \in B(\mathcal{H}_U \oplus \mathcal{K}_U)$.

If we give $\mathcal{A} \otimes_h \mathcal{B}$ *the multiplication*

$$(a \otimes b) \circ (c \otimes d) = (a\ c) \otimes (d\ b) \quad ,$$

for $a,c \in \mathcal{A}$ *and* $b,d \in \mathcal{B}$ *, then* $\mathcal{A} \otimes_h \mathcal{B}$ *is a Banach algebra with respect to* $\circ$ *, and with respect to this multiplication* $\theta : \mathcal{A} \otimes_h \mathcal{B} \to B(B(\mathcal{H}_U \oplus \mathcal{K}_U))$ *is an isometric homomorphism.*

Proof. Clearly θ is contractive. Let $u \in X \otimes_h Y$ be fixed, with $\|u\|_h = 1$. By the Hahn-Banach Theorem there is a contractive linear functional f on $X \otimes_h Y$ with

$$< f\ ,\ u > = 1 \quad .$$

By 3.3.3 there exists unital *-representations θ and π of $\mathcal{A}$ and $\mathcal{B}$ on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, a contractive linear operator $T : \mathcal{K} \to \mathcal{H}$, and $\zeta \in BALL(\mathcal{K})$ and $\eta \in BALL(\mathcal{H})$, such that

$$f(x \otimes y) = < \theta(x)\ T\ \pi(y)\ \zeta\ ,\ \eta > \quad ,$$

for $x \in X$, $y \in Y$.

Now since θ and π are subrepresentations of ρ_U and π_U respectively we may write

$$f(x \otimes y) = < \rho_U(x)\ T'\ \pi_U(y)\ \zeta'\ ,\ \eta' >$$

for some $T' \in BALL(B(\mathcal{K}_U,\mathcal{H}_U))$, $\zeta' \in BALL(\mathcal{K}_U)$, $\eta' \in BALL(\mathcal{H}_U)$. Now let S be the operator on $\mathcal{H}_U \oplus \mathcal{K}_U$ which equals T' on $\mathcal{K}_U$, and which annihilates $\mathcal{H}_U$. Then

$$\|\theta(u)\ (S)\| \ge |< \theta(u)(S)\ (0 \oplus \zeta')\ ,\ (\eta' \oplus 0) > |$$

$$= |\langle f , u \rangle|$$

$$= 1 ,$$

and so θ is an isometry.

That $\mathcal{A} \otimes_h \mathcal{B}$ is a normed algebra with the multiplication $\circ$ follows as in 4.2.1. The other statements of this proposition are obvious. □

6.2 MATRICIAL OPERATOR ALGEBRAS.

In this section we investigate some of the themes of 6.1 in the context of operator spaces.

6.2.1 ***Definition.*** Let $(X , \|\cdot\|_n)$ be a norm-closed operator space, and suppose X is also an algebra with multiplication m. We write such a space as a triple $(X , \|\cdot\|_n , m)$, or (X , m), or even X when there is no danger of confusion. We say that $(X , \|\cdot\|_n , m)$ is *completely bicontinuously isomorphic to an operator algebra* if there exist a Hilbert space $\mathcal{H}$, and a completely bicontinuous map $\theta : X \to B(\mathcal{H})$ with

$$\theta(m(x,y)) = \theta(x)\ \theta(y) \qquad (x , y \in X) .$$

In this case we say that $(X , \|\cdot\|_n , m)$ is a *matricial operator algebra* if θ is a complete isometry.

Notice that just as in the operator algebra situation, we may assume completely bicontinuous homomorphisms of complete operator algebras with identity are unital; but again the norm of the inverse mapping may change.

We would like to characterize matricial operator algebras, preferably up to complete isometry, but also up to complete bicontinuity. Clearly a necessary condition is that the multiplication m is completely bounded. Indeed, if X is an operator space, and if $\theta : X \to B(\mathcal{H})$ is a completely bicontinuous linear mapping onto a subalgebra of $B(\mathcal{H})$, then defining

$$m(x,y) = \theta^{-1}(\theta(x)\ \theta(y)) \quad ,$$

for $x,y \in X$, we obtain a completely bounded multiplication.

6.2.2 THEOREM. *If X is an operator space, and if m is a completely bounded multiplication on X , then X , with the multiplication m , is an operator algebra.*

Proof. By Theorem 3.3.3 the multiplication m satisfies the condition of Theorem 6.1.4. □

The preceding theorem shows that '$\|\cdot\|_h$ - matricial algebras', whatever this means, are operator algebras. The following two examples show that a completely bounded multiplication is not sufficient for a completely isometric characterization.

6.2.3 EXAMPLE. Let X be an operator space, realized on the Hilbert space $\mathcal{H}$, and choose $f \in BALL(X^*)$. Define

$$m(x,y) = f(x)\ y \quad ,$$

for $x,y \in X$. Then m is a completely contractive and associative multiplication. The algebra (X , m) has no identity unless X is one dimensional. If f is the zero functional then

(X, m) is indeed a matricial operator algebra, via the complete isometry $\theta : X \to \mathcal{M}_2(B(\mathcal{H}))$ given by

$$\theta(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} .$$

On the other hand, if X is the C^*-algebra ℓ_2^∞ , and if

$$f((\lambda_1,\lambda_2)) = \lambda_1 ,$$

for $(\lambda_1,\lambda_2) \in \ell_2^\infty$, then there is no isometric imbedding $\theta : X \to B(\mathcal{H})$. For if there were, and $\theta((1,0)) = P$, $\theta((0,1)) = T$, then P is a contractive idempotent and consequently an orthogonal projection onto a subspace of $\mathcal{H}$. The relations

$$P\,T = T\ ,\ T^2 = T\,P = 0$$

give

$$\| \lambda_1 P + \lambda_2 T \| = \{ |\lambda_1|^2 + |\lambda_2|^2 \}^{\frac{1}{2}} ,$$

which is a contradiction.

A completely bicontinuous representation $\theta : X \to \mathcal{M}_2(B(\mathcal{H}))$ of any multiplication m of the type above, given by a functional f , can also be written down explicitly, namely

$$\theta(x) = \begin{bmatrix} f(x)I_{\mathcal{H}} & (x - f(x)I_{\mathcal{H}}) \\ 0 & 0 \end{bmatrix} .$$

6.2.4 EXAMPLE. Let n be a positive integer or ∞ , and let $\mathcal{M}_n$ be the C^*-algebra of bounded operators on $\mathbb{C}^n$. With respect to the usual basis of $\mathbb{C}^\infty$ we regard elements in $\mathcal{M}_\infty$ as infinite complex matrices. Consider the Banach algebra **[Va3]** $(\mathcal{M}_n , \circ)$, where $\circ$ is the Schur product

$$A \circ B = [\, a_{ij}\, b_{ij} \,] \qquad (A , B \in \mathcal{M}_n) .$$

With respect to the usual matricial norms on $\mathcal{M}_n$ the Schur multiplication can be shown to be completely contractive. However if $\theta : \mathcal{M}_n \to B(\mathcal{H})$ is a contraction, and a homomorphism with respect to $\circ$, then $\{ \theta(e_{ij}) \}$ is a double sequence of orthogonal projections onto mutually orthogonal subspaces of $\mathcal{H}$, and hence

$$\| \Sigma^n_{i,j=1} \lambda_{ij} \theta(e_{ij}) \| = \sup \{ |\lambda_{ij}| : 1 \le i,j \le n \} .$$

Thus θ could not be an isometry.

Of course if $n < \infty$ and if θ is the map taking a matrix A to an $n^2 \times n^2$ matrix with the a_{ij} on the main diagonal then θ is a homomorphism with respect to $\circ$, and $\|\theta\| \le 1$, $\|\theta^{-1}\| \le n$. In the case $n < \infty$ the multiplication $\circ$ has an identity of norm n . It is difficult to imagine a homomorphism θ from $(\mathcal{M}_n , \circ)$ into $B(\mathcal{H})$ with $\|\theta\| \, \|\theta^{-1}\| \le K$, where K is independent of n , however some such homomorphism must exist, since the condition of Theorem 6.1.4 shows **[Va3]** that $\mathcal{M}_\infty$ is an operator algebra, and of course $\mathcal{M}_n$ is algebraically embedded in $\mathcal{M}_\infty$ in a natural way.

6.2.5 EXAMPLE. Let $\mathcal{A}$ be a subspace of $B(\mathcal{H})$, and suppose there is an operator $V \in \mathrm{BALL}(B(\mathcal{H}))$ such that $\mathcal{A} V \mathcal{A} \subset \mathcal{A}$. Then we may define a bilinear mapping $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by

$$m(a,b) = a V b \quad ,$$

for $a,b \in \mathcal{A}$. It is clear that m is a completely contractive multiplication. The map $\theta : \mathcal{A} \to \mathcal{M}_2(B(\mathcal{H}))$ defined by

$$\theta(a) = \begin{bmatrix} a V & a (I - V V^*)^{1/2} \\ 0 & 0 \end{bmatrix}$$

is a completely isometric homomorphism with respect to the multiplication m .

More generally if X is an operator space, if θ is a completely bicontinuous linear mapping $X \to B(\mathcal{K})$, and if $\theta(X) \vee \theta(X) \subset \theta(X)$, then defining

$$m(x,y) = \theta^{-1}(\theta(x) \vee \theta(y)) \qquad (x,y \in X)$$

we obtain a completely bounded multiplication. One can construct, as in the last paragraph, a completely bicontinuous homomorphism from (X , m) into $\mathcal{M}_2(B(\mathcal{K}))$.

Examples 6.2.3 and 6.2.4 above suggest that the absence of a completely isometric homomorphism into $B(\mathcal{H})$ could be attributed to the lack of an identity of norm 1. This is in fact the case as the following theorem, found by A. M. Sinclair, shows:

6.2.6 THEOREM. *Let* X *be an operator space with a completely contractive multiplication* m *, and suppose there is an identity* e *for* m *, and* $\|e\| = 1$ *. Then* (X , m) *is a matricial operator algebra.*

Proof. Suppose X is a subspace of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Let L be the self-adjoint subspace of $\mathcal{M}_2(B(\mathcal{H}))$ consisting of elements of the form $\begin{bmatrix} 0 & x \\ y^* & 0 \end{bmatrix}$, where x and y are in X . Define a multiplication $m^\sim$ on L by

$$m^\sim \left(\begin{bmatrix} 0 & x_1 \\ y_1^* & 0 \end{bmatrix} , \begin{bmatrix} 0 & x_2 \\ y_2^* & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & m(x_1,x_2) \\ m(y_2,y_1)^* & 0 \end{bmatrix} ,$$

for x_1 , x_2 , y_1 and $y_2 \in X$. It is easy to see that $m^\sim$ is symmetric (Definition 3.1.8). Also $\begin{bmatrix} 0 & e \\ e^* & 0 \end{bmatrix}$ is an identity for L of norm 1. Further, the map taking an element x in X to the element $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ of L is a complete isometry. Thus to prove the statement of the theorem we may assume without loss of generality that X is self-adjoint in $B(\mathcal{H})$, and that m is symmetric.

Let $\mathcal{A}$ be the C^*-algebra generated by X in $B(\mathcal{H})$, and let $(\mathcal{A}_U, \pi_U, \mathcal{H}_U)$ be the universal representation of $\mathcal{A}$. Now since $\pi_U \circ m : X \times X \to B(\mathcal{H}_U)$ is a completely contractive bilinear map, it induces a completely contractive map $X \otimes_h X \to B(\mathcal{H}_U)$. By the injectivity of the Haagerup norm (Theorem 3.3.4) and the Arveson-Wittstock-Hahn-Banach Theorem (Theorem 3.2.6), the linear map induced by $\pi_U \circ m$ extends to a completely contractive map $\psi : \mathcal{A} \otimes_h \mathcal{A} \to B(\mathcal{H}_U)$. Then

$$m^{\sim} = \tfrac{1}{2} (\psi + \psi^*)$$

induces a symmetric completely contractive bilinear map $\mathcal{A} \times \mathcal{A} \to B(\mathcal{H}_U)$. The Christensen-Sinclair representation theorem (Theorem 3.1.10) allows us to choose a Hilbert space $\mathcal{K}$, a unital *-representation π of $\mathcal{A}$ on $\mathcal{K}$, an operator $U_1 \in BALL(B(\mathcal{H},\mathcal{K}))$, and a self-adjoint operator V_1 in $BALL(B(\mathcal{K}))$, such that

$$m^{\sim}(a \otimes b) = U_1^* \pi(a) V_1 \pi(b) U_1 ,$$

for $a, b \in \mathcal{A}$.

Since π is a sub-representation of π_U there exist an operator U and a self-adjoint operator V in $BALL(B(\mathcal{H}_U))$ such that

$$(\pi_U m)(a,b) = U^* \pi_U(a) V \pi_U(b) U \qquad (a, b \in X).$$

Without loss of generality, and for notational simplicity, we can replace X with $\pi_U(X)$, $\mathcal{H}$ with $\mathcal{H}_U$, and m with $\pi_U \circ m$; now

$$m(a,b) = U^* a V b U \qquad (a , b \in X) .$$

Define an operator Ψ in $B(B(\mathcal{H}))$ by

$$< \Psi(S) \zeta , \eta > = \text{LIM} < (U^*)^n S U^n \zeta , \eta > ,$$

for $S \in B(\mathcal{H})$ and $\zeta , \eta \in \mathcal{H}$; where LIM is a Banach limit on ℓ^∞ (see [Conw]). By the properties of Banach limits Ψ is completely positive. Also if $A , B \in \mathcal{M}_n(X)$ then

$$\begin{aligned} (U^* \otimes I_n)^k m_n(A,B)^* m_n(A,B) (U \otimes I_n)^k & \\ \leq (U^* \otimes I_n)^{k+1} B^* (V \otimes I_n) A^* A (V \otimes I_n) B (U \otimes I_n)^{k+1} & \\ \leq \|A\|^2 (U^* \otimes I_n)^{k+1} B^* B (U \otimes I_n)^{k+1} & , \end{aligned}$$

and thus

$$\Psi_n(m_n(A,B)^* m_n(A,B)) \leq \|A\|^2 \Psi_n(B^* B) . \qquad (1)$$

Also
$$\begin{aligned} B^* B &= m_n(e \otimes I_n,B)^* m_n(e \otimes I_n,B) \\ &\leq (U^* \otimes I_n) B^* B (U \otimes I_n) , \end{aligned}$$

and hence inductively, for $k = 1,2,\ldots$ we have

$$B^* B \leq (U^* \otimes I_n)^k B^* B (U \otimes I_n)^k .$$

Thus it is clear that

$$B^* B \leq \Psi_n(B^* B) . \qquad (2)$$

Now following the construction before Theorem 3.1.2 define a semi inner product on $X \otimes \mathcal{H}$ by

$$< a \otimes \zeta , b \otimes \eta > = < \Psi(b^* a) \zeta , \eta > .$$

Let $\mathcal{K}$ be the Hilbert space which is the completion of the quotient of $X \otimes \mathcal{H}$ by the subspace

$$\{ u \in X \otimes \mathcal{H} : < u , u > = 0 \}$$

with respect to the induced inner product. We shall write [u] for the coset of an element $u \in X \otimes \mathcal{H}$.

For $a, b_1, \ldots, b_n \in X$ and $\zeta_1, \ldots, \zeta_n \in \mathcal{H}$ we have

$$< \Sigma_{i=1}^n m(a,b_i) \otimes \zeta_i , \Sigma_{i=1}^n m(a,b_i) \otimes \zeta_i >$$

$$= \Sigma_{i,j=1}^n < \Psi(m(a,b_j)^* m(a,b_i)) \zeta_i , \zeta_j >$$

$$= < \Psi_n(m_n(a \otimes e_{11}, B)^* m_n(a \otimes e_{11}, B)) \underset{\sim}{\zeta} , \underset{\sim}{\zeta} > ,$$

where $B = \Sigma_{i=1}^n b_i \otimes e_{1i}$. Thus, by (1) we see

$$< \Sigma_{i=1}^n m(a,b_i) \otimes \zeta_i , \Sigma_{i=1}^n m(a,b_i) \otimes \zeta_i > \le \|a\|^2 < \Psi_n(B^* B) \underset{\sim}{\zeta} , \underset{\sim}{\zeta} >$$

$$= \|a\|^2 < \Sigma_{i=1}^n b_i \otimes \zeta_i , \Sigma_{i=1}^n b_i \otimes \zeta_i > .$$

This inequality allows us to define a mapping $\theta : X \to B(\mathcal{K})$ by

$$\theta(a) ([b \otimes \zeta]) = [m(a,b) \otimes \zeta] ,$$

for $a , b \in X$ and ζ in $\mathcal{H}$. It is clear that θ is a contractive homomomorphism. Indeed the matricial counterpart of the calculation above shows that θ is completely contractive. Also if $A \in \mathcal{M}_n(X)$, if $\underset{\sim}{\zeta} \in$ BALL $(\mathcal{H}^{(n)})$ and if ξ is the vector in BALL$(\mathcal{K}^{(n)})$ whose i'th component is $[e \otimes \zeta_i]$, then

$$\|\theta_n(A)\|_n^2 \ge \| \theta_n(A) (\xi) \|^2$$

$$= \Sigma_{i=1}^n < \Sigma_{j=1}^n a_{ij} \otimes \zeta_i , \Sigma_{j=1}^n a_{ij} \otimes \zeta_i >$$

$$= \Sigma_{i,j,k=1}^n < \Psi(a_{ik}^* a_{ij}) \zeta_j , \zeta_k >$$

$$= < \Psi_n(A^* A) \underset{\sim}{\zeta} , \underset{\sim}{\zeta} >$$

$$\ge < A^* A \underset{\sim}{\zeta} , \underset{\sim}{\zeta} >$$

$$= < A \underset{\sim}{\zeta} , A \underset{\sim}{\zeta} >$$

using (2) . Thus $\|\theta_n(A)\|_n \ge \|A\|_n$ and consequently θ is a complete isometry. □

If X is an algebra with respect to a multiplication m we shall write $m^{\sim}$ for the extension of m to $X \oplus \mathbb{C}$ given by

$$m^{\sim}(a \oplus \lambda, b \oplus \mu) = (m(a,b) + \lambda\, b + \mu\, a) \oplus \lambda\, \mu \quad ,$$

for a , $b \in X$ and λ , $\mu \in \mathbb{C}$. With this multiplication $X \oplus \mathbb{C}$ has an identity, namely $0 \oplus 1$.

6.2.7 COROLLARY. *Let* X *be an operator space with a completely contractive multiplication* m . *Then* (X , m) *is a matricial operator algebra if and only if there exists an* L^∞*-matricial structure* $\{ |\cdot|_n \}$ *for* $X \oplus \mathbb{C}$ *such that*

(*i*) $|A|_n = \|A\|_n \qquad (A \in M_n(X))$,

(*ii*) $| 0 \oplus 1 | = 1$, *and*

(*iii*) *the multiplication* $m^{\sim}$ *on* $X \oplus \mathbb{C}$ *extending* m *is completely contractive.*

The following result generalizes Theorem 6.1.3:

6.2.8 COROLLARY. *Let* $\mathcal{A}$ *be a matricial operator algebra, and suppose* I *is a closed two-sided ideal in* $\mathcal{A}$. *Then* $\mathcal{A} / I$ *with the quotient matricial norms is a matricial operator algebra.*

Proof. We may assume without loss of generality there exists a Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is a subalgebra of $B(\mathcal{H})$. Now apply Corollary 6.2.7. □

6.2.9 REMARK. The condition in Corollary 6.2.7 is less than desirable, but unfortunately we have not been able to improve on it. It would be of interest if one could obtain a proof mimicking the construction of Varopoulos giving the sufficiency in Theorem 6.1.4. Varopoulos uses Theorem 6.1.3 to construct a contractive monomorphism from a concrete operator algebra onto the algebra satisfying the condition of 6.1.4. However for this to succeed in our case we require an operator space version of the open mapping theorem.

The next result informs us that we can assume that the multiplication has an identity if we are interested only in a completely bicontinuous representation.

6.2.10 PROPOSITION. *Let* X *be an operator space with a completely contractive multiplication* m *. Then there exists an* L^{∞}*-matricial structure on* $X \oplus \mathbb{C}$ *such that the natural extension* $\tilde{m}$ *of* m *to* $X \oplus \mathbb{C}$ *is completely contractive, and the canonical embedding of* X *in* $X \oplus \mathbb{C}$ *is completely bicontinuous.*

Proof. Define an L^{∞}-matricial structure on $X \oplus \mathbb{C}$ by

$$| A \oplus \Lambda |_n = \max \{ \|A\|_n \, , \, \|\Lambda\| \} \quad ,$$

for $A \in \mathcal{M}_n(X)$ and $\Lambda \in \mathcal{M}_n$. With respect to this structure $\tilde{m}$ is completely bounded, with $|\tilde{m}|_{cb} = \kappa$, say. Then

$$|\cdot|_n' = \kappa \, |\cdot|_n$$

defines a new L^{∞}-matricial structure on $X \oplus \mathbb{C}$ with respect to which $\tilde{m}$ is completely contractive. Note that the identity of $X \oplus \mathbb{C}$ does not have norm 1. □

6.2.11 REMARK. Let X be an operator space, represented on $B(\mathcal{H})$, with a completely contractive multiplication m, and suppose there exists an identity e for m, $\|e\| \geq 1$. Let $\mathcal{A}$ be the C^*-algebra generated by X in $B(\mathcal{H})$. Then following the proof of 6.2.6 we can assume X self-adjoint and m symmetric and write

$$\pi(m(a,b)) = U^* \pi(a) V \pi(b) U \qquad (a, b \in X)$$

for some Hilbert space $\mathcal{H}$, some representation π of $\mathcal{A}$ on $\mathcal{H}$, and some operator U and self-adjoint operator V in $B(\mathcal{H})$. If these objects can be chosen such that $\|(V \pi(e) U)^n\|$ is bounded uniformly by some positive constant K, then (X, m) is completely bicontinuously isomorphic to an operator algebra. In fact in this case there exists a Hilbert space $\mathcal{K}$, and a unital completely bicontinuous and completely contractive linear mapping θ from X into $B(\mathcal{K})$, which is a homomorphism with respect to m.

7. APPENDIX.

7.1 ***Definition.*** A *non-trivial invariant subspace* of an operator T on a Banach space X is a proper closed linear subspace E of X such that $E \neq \{0\}$ and $T(E) \subset E$. The subspace E is said to be *hyperinvariant* for T if E is an invariant subspace for every operator on X that commutes with T.

Throughout what follows the set $[0,1)$ is taken to be identified in the usual way as a topological group with $\mathbb{T}$, the unit circle in the complex plane. In this appendix we give a sufficient condition for an operator on $L^2[0,1)$ composed of a multiplication operator and a translation to possess an invariant subspace. In fact all that follows is valid for $L^p[0,1)$, $1 \leq p \leq \infty$.

More specifically, let α be a fixed number in $[0,1)$ and let φ be a fixed non-zero continuous function on $[0,1)$. This implies that $\varphi(0) = \varphi(1^-)$. Define an operator T on $L^2[0,1)$ by

$$Tf(x) = \varphi(x)\ f(x + \alpha) \qquad (x \in [0,1))$$

for each $f \in L^2[0,1)$. Here addition is modulo 1 of course. Thus if M_φ is the multiplication operator on $L^2[0,1)$

$$M_\varphi f(x) = \varphi(x)\ f(x) \qquad (x \in [0,1)) ,$$

and if S_α is the translation operator

$$S_\alpha f(x) = f(x + \alpha) \qquad (x \in [0,1)) ,$$

then we have

$$T = M_\varphi S_\alpha \quad .$$

This operator is related to a class of operators introduced by Bishop as candidates for operators possibly not possessing invariant subspaces. Subsequently almost all of these have been shown by A. M. Davie [**Da2**] to have hyperinvariant subspaces.

7.2 ***Definition.*** An irrational number ℓ is called a *Liouville number* if for each natural number n there exist integers p and q with $q \geq 2$ such that

$$|\ell - p/q| < q^{-n} \quad .$$

One can [**Ox**] show that the set of Liouville numbers is dense in $\mathbb{R}$ but has s-dimensional Hausdorff (and consequently also Lebesgue) measure zero for all $s > 0$.

We shall need the following theorem:

7.3 THEOREM (Wermer [Wr1,CoF]). *Let* X *be a Banach space and suppose* R *is an invertible operator on* X *satisfying the following two conditions:*

(i) the spectrum of R *contains more than one point, and*

(ii) $\Sigma_{n=-\infty}^{\infty} \log \|R^n\| / (1 + n^2) < \infty$.

Then R *possesses a non-trivial hyperinvariant subspace.*

The result we give below asserts that the operator T defined above possesses an invariant subspace provided that α is not a Liouville number and provided that φ is sufficiently smooth. For a bounded function $g : [0,1) \to \mathbb{C}$ the *modulus of continuity* ω_g of g

is defined to be

$$\omega_g(\delta) = \sup \{ \, |g(x) - g(x')| : |x-x'| \le \delta \, \} \quad ,$$

for $\delta \ge 0$. This is an increasing function of $\delta \ge 0$. Note that ω_g is not quite the usual modulus of continuity **[Zy]** since the subtraction above on $[0,1)$ is modulo 1 ; however if $g(1^-) = g(0)$ then ω_g is bounded above and below by a constant multiple of the usual modulus of continuity.

Suppose g is a fixed non-vanishing complex valued function on $[0,1)$, with $g(0) = g(1^-)$, such that g and g^{-1} are bounded. Let ω_g be the modulus of continuity of g . Notice that if $|x - x'| \le t$ then

$$|g(x)| \le |g(x')| + \omega_g(t) \quad ,$$

and consequently

$$\begin{aligned} \big| \log|g(x)| - \log|g(x')| \big| &= \big| \log(|g(x)|/|g(x')|) \big| \\ &\le \log(1 + \|g^{-1}\|_\infty \, \omega_g(t)) \\ &\le \|g^{-1}\|_\infty \, \omega_g(t) \quad . \end{aligned}$$

Thus it is clear that the modulus of continuity of $\log|g|$ is dominated by a constant multiple of the modulus of continuity of $|g|$.

7.4 THEOREM. *Let* $\alpha \in [0,1)$. *Let* φ *be a fixed non-vanishing continuous complex valued function on* $[0,1)$ (*with* $\varphi(0) = \varphi(1^-)$). *The operator* T *on* $L^2[0,1)$ *defined by*

$$Tf(x) = \varphi(x) \; f(x + \alpha) \qquad (\, x \in [0,1) \,) \quad ,$$

where the addition is modulo 1 , *possesses an invariant subspace* *provided that* α *is not a Liouville number and provided that the modulus of continuity* ω *of* φ *, or even of* $\log|\varphi|$ *, satisfies*

$$\int_0^1 \omega(t) \,/\, t \, dt < \infty \quad .$$

If, in addition, α *is irrational then* T *possesses a hyperinvariant subspace.*

Proof. If α is rational then we do not need the smoothness condition for φ : if $\alpha = p/q$, for some $p,q \in \mathbb{N}$, then the space of functions which are zero on alternate intervals of length $(2q)^{-1}$ is an invariant subspace for T .

Assume henceforth then that α is irrational, and put $\psi = \log|\varphi|$. By the remark immediately before the statement of the theorem we may as well assume the integral condition holds for the modulus of continuity ω of ψ . If n is a non-negative integer and we put

$$\varphi_n = \Pi_{k=0}^{n-1} (S_\alpha^k \varphi)$$

then we have $T^n = M_{\varphi_n} \circ S_\alpha^n$ and so $\|T^n\| = \|\varphi_n\|_\infty$. Similarly if n is a negative integer we have $\|T^n\| = \|\varphi_{-n}^{-1}\|_\infty$. Thus as $n \to \infty$

$$\log \|T^n\| \,/\, n = \sup \{ \, (n^{-1} \Sigma_{k=0}^{n-1} S_\alpha^k(\psi)) \; (x) \; : \; x \in [0,1) \, \}$$
$$\to \int_0^1 \psi \, dt$$

by the uniform ergodic theorem (see **[Pa]** 1.1), if not by more elementary considerations. We may conclude from this that r(T) , the spectral radius of T , satisfies

$$r(T) = \exp(\int_0^1 \psi \, dt) \quad .$$

If M is the unitary multiplication operator on $L^2[0,1)$ given by

$$Mf(x) = e^{2\pi i x} f(x) \qquad (x \in [0,1)) ,$$

for $f \in L^2[0,1)$, then it is easy to see that

$$T - e^{2\pi i\alpha} \lambda I = e^{2\pi i\alpha} M (T - \lambda I) M^{-1}$$

for any $\lambda \in \mathbb{C}$. This shows that the spectrum of T is invariant under rotation by $e^{2\pi i\alpha}$, and so certainly contains more than one point. Indeed it is easy to see that the spectrum $\sigma(T)$ is the circle of radius $r(T)$, centred at 0 , but we shall not explicitly need this fact. Normalize the operator T by setting

$$R = r(T)^{-1} T \quad .$$

This is equivalent to scaling φ by a constant.

For a bounded function $g : [0,1) \to \mathbb{C}$ let us write $D(g,n)$ for the discrepancy

$$D(g,n) = \sup \{ \mid (n^{-1} \Sigma_{k=0}^{n-1} S_\alpha^k g)(x) - \int_0^1 g \, dt \mid : x \in [0,1) \} \quad .$$

We now appeal to Wermers Theorem (7.3 above) to deduce that the operator R has a non-trivial hyperinvariant subspace if

$$\Sigma_{n=1}^\infty n^{-1} \sup \{ (n^{-1} \Sigma_{k=0}^{n-1} S_\alpha^k g)(x) - \int_0^1 g \, dt : x \in [0,1) \} < \infty$$

for $g = \psi$ *and* $g = -\psi$. We may rewrite this condition as

$$\Sigma_{n=1}^\infty n^{-1} D(\psi,n) < \infty \quad . \qquad (*)$$

Now since α is not a Liouville number by elementary number theory (see [**Da2**]) there exists K , $N \in \mathbb{N}$ such that if n is a positive integer greater than N then there exists p , $q \in \mathbb{N}$, with p and q coprime, such that both

$$n^{1/K} \le q \le n^{1/2}$$

and

$$|\alpha - p/q| \le q^{-2}$$

hold. For such n we may write $n = rq + s$, for some non-negative integers r and s, with $s < q$. We obtain

$$|n^{-1} \Sigma_{k=0}^{n-1} \psi(x + k\alpha) - \int_0^1 \psi \, dt|$$

$$\leq n^{-1} |\Sigma_{k=rq}^{n-1} \psi(x + k\alpha)| + |(n^{-1} - (rq)^{-1}) \Sigma_{k=0}^{rq-1} \psi(x + k\alpha)| +$$

$$|(rq)^{-1} \Sigma_{k=0}^{rq-1} \psi(x + k\alpha) - \int_0^1 \psi \, dt|$$

$$\leq 2 \, (q/n) \, \|\psi\|_\infty + |(rq)^{-1} \Sigma_{k=0}^{rq-1} \psi(x + k\alpha) - \int_0^1 \psi \, dt|$$

$$\leq r^{-1} \Sigma_{j=o}^{r-1} | q^{-1} \Sigma_{k=0}^{q-1} \psi(x + jq\alpha + k\alpha) - \int_0^1 \psi \, dt | + O(n^{-\frac{1}{2}}) .$$

As an integer a runs from 1 to q, the number $a\,p/q$ assumes each of the values $0, 1/q, \ldots, (q-1)/q$ in some order (modulo 1 of course). Since $|a\,\alpha - a\,p/q| \leq q^{-1}$ the following assertion is clear: for each $x \in [0,1)$ there is a partition of $[0,1)$ into disjoint intervals $I_0, \ldots, I_{q-1}$ each of length q^{-1} such that each of the q numbers $x, x + \alpha, \ldots, x + (q-1)\alpha$ may be associated with a unique interval $I_0, \ldots, I_{q-1}$ respectively which it lies within a distance of q^{-1} from.

By the mean value theorem we may for each $k = 0,\ldots,(q-1)$ choose $\xi_k \in I_k$ such that

$$|I_k|^{-1} \int_{I_k} \psi \, dt = \psi(\xi_k) .$$

Then
$$| \psi(x + k\alpha) - |I_k|^{-1} \int_{I_k} \psi \, dt | \leq \omega(2/q) ,$$

where ω is the modulus of continuity of ψ, and so

$$|q^{-1} \Sigma_{k=0}^{q-1} \psi(x + k\alpha) - \int_0^1 \psi \, dt| \leq q^{-1} \Sigma_{k=0}^{q-1} |\psi(x + k\alpha) - q \int_{I_k} \psi \, dt|$$

$$\leq \omega(2/q) .$$

Thus for any $x \in [0,1)$ we see that

$$r^{-1} \Sigma_{j=o}^{r-1} \mid q^{-1} \Sigma_{k=0}^{q-1} \psi(x + jq\alpha + k\alpha) - \int_0^1 \psi \, dt \mid \leq \omega(2/q)$$

and so ,

$$D(\psi,n) \leq \omega(2/q) + O(n^{-\frac{1}{2}})$$
$$\leq \omega(2\, n^{-1/K}) + O(n^{-\frac{1}{2}}) .$$

Consequently (*) is satisfied if

$$\Sigma_{n=1}^{\infty} n^{-1} \omega(2\, n^{-1/K}) < \infty .$$

which proves the theorem after an application of the integral test of elementary undergraduate analysis. □

7.5 **REMARK.** It would be of interest if one could enlarge the set of numbers α or the set of functions φ for which the result holds. It is probably possible to use the method of [**Da2**] to extend this result to the case when φ is permitted to assume the value 0 .

For $s > 0$ let Λ_s be the *Hölder class* [**Zy**]: the class of those bounded complex valued functions g on $[0,1)$ for which there exists a constant $C > 0$ such that the modulus of continuity ω of g satisfies

$$\omega(\delta) \leq C\, \delta^s \qquad (\delta \geq 0) .$$

7.6 **COROLLARY.** *The operator* T *defined above possesses an invariant subspace if* α *is not a Liouville number and if the function* φ *, or even* $\log|\varphi|$ *, is in the Hölder class* Λ_s *for some* $s > 0$ ..

Proof. If $\varphi \in \Lambda_s$ then $\log|\varphi| \in \Lambda_s$, by the remark above Theorem 7.4. Thus if either φ or $\log|\varphi|$ is in Λ_s for some $s > 0$ and if ω is the modulus of continuity of $\log|\varphi|$ then

$$\int_0^1 \omega(t) \,/\, t \; dt < \infty \quad .$$

An application of Theorem 7.4 completes the proof. □

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