Geometry of Transformable Metamaterials Inspired by Modular Origami

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Modular origami is a type of origami where multiple pieces of paper are folded into modules, and these modules are then interlocked with each other forming an assembly. Some of them turn out to be capable of large-scale shape transformation, making them ideal to create metamaterials with tuned mechanical properties. In this paper, we carry out a fundamental research on two-dimensional (2D) transformable assemblies inspired by modular origami. Using mathematical tiling and patterns and mechanism analysis, we are able to develop various structures consisting of interconnected quadrilateral modules. Due to the existence of 4R linkages within the assemblies, they become transformable, and can be compactly packaged. Moreover, by the introduction of paired modules, we are able to adjust the expansion ratio of the pattern. Moreover, we also show that transformable patterns with higher mobility exist for other polygonal modules. The design flexibility among these structures makes them ideal to be used for creation of truly programmable metamaterials. [DOI: 10.1115/1.4038969]

Introduction

Recently, there has been a surge of interest in creating mechanical metamaterials using origami which for certain origami patterns could be folded into structures with unique mechanical properties that are uncommon among existing conventional materials. Most of the prior research on origami metamaterials mainly focus on stacking layers of sheets folded by a crease pattern known as the Miura-ori to obtain particular features such as a negative Poisson's ratio [1–6]. Currently, a few attempts have emerged using other origami techniques. For instance, Johannes et al. reported a transformable metamaterial utilizing a basic material unit made of paper strips that has three degrees-of-freedom (DOF) and can deform into several specific shapes [7]. Yang and Silverberg suggested a design strategy using repetitive kirigami pattern for mechanical metamaterial [8]. By varying the geometrical parameters, the metamaterial is able to form various shapes. These are inspiring directions to create next-generation smart materials.

In this research, our attention is drawn upon creating two-dimensional (2D) metamaterials inspired by the modular origami, which is a type of origami where multiple pieces of paper are folded into a module, and these modules are then interlinked with each other, resulting in a structure that cannot be obtained by stacking folded sheets. Most of the modular origami assemblies are rigid structures, but some could be made transformable, i.e., they can shift from one configuration to another. Figures 1(*a*) and 1(*b*) show two such examples: a ball capable of being easily pressed to a plane or clasp to a stick, whereas the other a cube assembly that can be closely packed together or expanded to a porous grid [7,9,10]. These two objects are essential mechanisms. The transformable feature in both assemblies is particularly valuable when the concepts are used to create programmable engineering metamaterials. Under external load such materials can be soft initially when they undergo mechanism motions and become stiff subsequently when the mechanism motions cease and modules are compacted together [11]. If any other modules were used in places of the polyhedron modules, e.g., the origami crash box [12], the behavior of these structures could be tuned further so that they may, for example, exhibit better energy absorption capacity. In other words, the overall mechanical behavior of the structures can be made responsive to external loading conditions. These structures and materials based on such structures can find their usage in automotive, aerospace structures, and body armors.

The focus of this paper is on the geometrical aspect of the 2D modular origami objects such as that shown in Fig. 1(*b*). This particular structure was first investigated by Ron Resch back in 1977 [13]. Inspired by its geometrical features, we have embarked on a mission to investigate whether there are other transformable assemblies similar to the cube assembly. To do so, we have used a mathematic technique known as tiling and patterns [14]. If the cubes and voids in the cube assembly are treated as 2D shapes, the assembly can be modeled as a tiling pattern in which both the modules and voids are seen as tiles. Then, the cube assembly is regarded as an edge to edge tiling of two shapes: squares and rhombuses. Different from conventional tiling of two shapes, the shape of rhombic void changes during the transformation (Fig. 1(*b*)).

To model the transformation kinematically, we treat the cubes (or other shaped modules) as rigid bodies and the voids as planar four-bar linkages (4*R* linkage). Hence, the assembly of nine cubes could be treated as nine rigid bodies interlinked by four 4*R* linkages. Following this path, we can then analyze this type of assemblies using the theory of mechanisms [15–17]. Moreover, we replace the cubes with quadrilaterals, and we are interested in the mobility of assemblies with only quadrilateral modules connected by 4*R* linkages. Our objective is to explore the transformable features of these arrangements with different modules.

The layout of the paper is as follows: First, we derive the geometrical conditions under which the assemblies of quadrilateral modules are transformable without overlap. Then, in second section, we analyze the orientation and duality of the assemblies during transformation. This is followed by a discussion on arrangements consisting of more than two types of modules (different sizes or shapes). In particular, a special module with star shapes is presented, which can greatly increase the folding ratio. The transformation principle of this star assembly and the expand rate calculation are provided. Finally, we demonstrate that there exist many more transformable assemblies derived from early analysis, that leads to the conclusion of the paper and thoughts on future work.

Transformable Tilings by Quadrilaterals

The arrangement shown in Fig. 1(*b*) consists of interlinked rigid square modules forming a number of 4*R* linkages. Now, we

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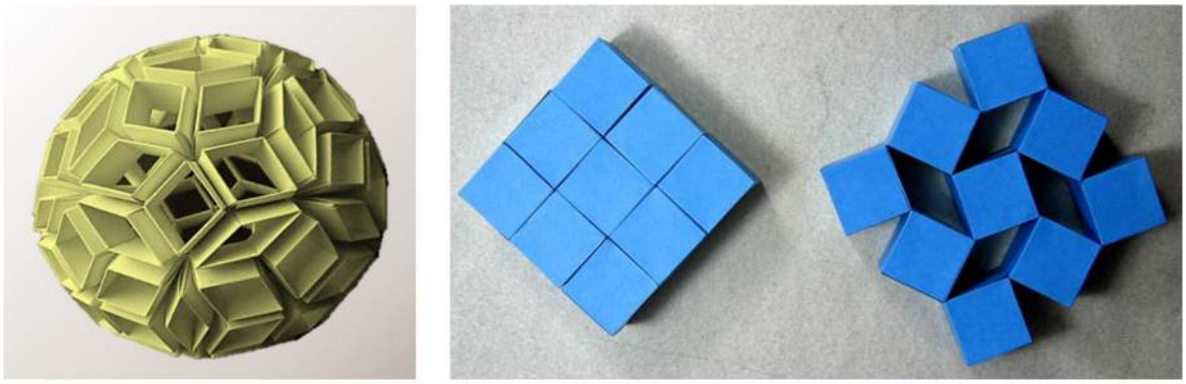


(*a*) (*b*)

Fig. 1 Modular origami models: (*a*) a snapology ball and (*b*) an interlinked cube assembly

replace each of the square modules with an arbitrary quadrilateral whose side lengths are a, b, c, and d. A suitable geometrical condition has to be imposed to the voids in order for the voids to vanish when the rigid modules are compactly packed together. This geometrical condition is that the sum of two adjacent side lengths must be equal to the sum of the other two side lengths. Under this condition, there are a number of tiling possibilities, which we shall discuss hereafter.

Example 1. First consider an arrangement of the rigid modules shown in Fig. 2. This pattern is made of three shapes: the rigid quadrilateral module and two quadrilateral voids: one with side lengths a, a, c, and c and the other b, b, d, and d. The quadrilateral modules can be closely packed once the voids vanish under this circumstance, should the tiling pattern be transformable. The angles for the quadrilateral module and voids are shown in Fig. 2. Note that the two opposite angles of the voids, marked as α and ϕ_1, are always identical because of the shape of the voids.

Next, we shall find the conditions under which this tiling is transformable.

Geometrically there are

$$\begin{aligned} \theta_1+\theta_2+\theta_3+\theta_4 &= 2\pi \\ 2\phi_1+\phi_2+\phi_3 &= 2\pi \\ 2\alpha+\beta_1+\beta_2 &= 2\pi \end{aligned} \tag{2.1}$$

At each point where quadrilateral modules are connected, there must be

$$\begin{aligned} \theta_1+\theta_3+\phi_3+\alpha &= 2\pi \\ \theta_1+\theta_2+\phi_1+\beta_2 &= 2\pi \\ \theta_2+\theta_4+\phi_2+\alpha &= 2\pi \\ \theta_3+\theta_4+\phi_1+\beta_1 &= 2\pi \end{aligned} \tag{2.2}$$

Therefore, we have

$$\theta_4 = 2\pi-\theta_1-\theta_2-\theta_3 \tag{2.3}$$

$$\phi_1 = \alpha \tag{2.4}$$

$$\phi_2 = \theta_1+\theta_3-\alpha \tag{2.5}$$

$$\phi_3 = 2\pi-\theta_1-\theta_3-\alpha \tag{2.6}$$

$$\beta_1 = \theta_1+\theta_2-\alpha \tag{2.7}$$

and

$$\beta_2 = 2\pi-\theta_1-\theta_2-\alpha \tag{2.8}$$

The edge lengths are also related by the following relationships:

$$a = \frac{\sin(\theta_1+\theta_2+\theta_3)}{\sin\theta_3}\left(\frac{d\sin\theta_1}{\sin(\theta_1+\theta_2)}-c\right)+\frac{d\sin\theta_2}{\sin(\theta_1+\theta_2)} \tag{2.9}$$

$$b = \frac{d\sin\theta_1 - c\sin(\theta_1+\theta_2)}{\sin\theta_3} \tag{2.10}$$

$$a\sin\frac{\phi_3}{2} = c\sin\frac{\phi_2}{2} \tag{2.11}$$

and

$$b\sin\frac{\beta_1}{2} = d\sin\frac{\beta_2}{2} \tag{2.12}$$

The latter two equations are obtained by considering the geometry of the voids. Now substituting Eqs. (2.3)–(2.10) into Eqs. (2.11) and (2.12), there are

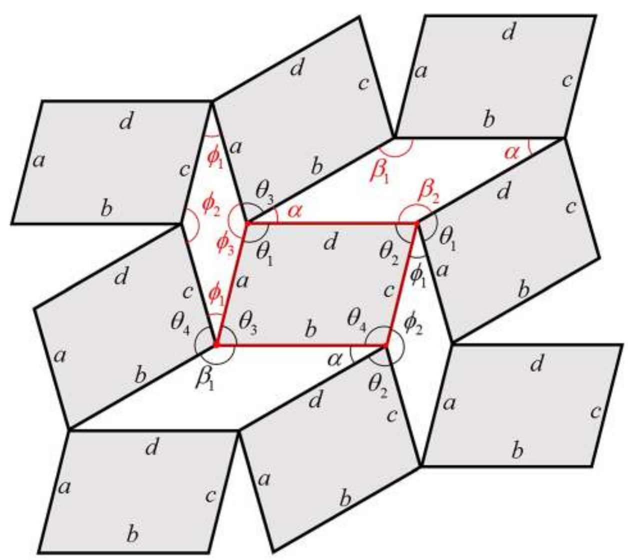


Fig. 2 An arrangement of identical quadrilateral modules

$$
\sin\frac{\theta_1+\theta_3+\alpha}{2}\sin\frac{\theta_1+\theta_2-\alpha}{2}\left[\begin{array}{l}\dfrac{\sin\theta_1}{\tan\theta_3}\sin(\theta_1+\theta_2)+\\ \sin\theta_1\cos(\theta_1+\theta_2)+\sin\theta_2\end{array}\right]
$$

$$
=\left[\begin{array}{l}\sin\theta_1\sin\dfrac{\theta_1+\theta_2-\alpha}{2}-\\ \sin\theta_3\sin\dfrac{\theta_1+\theta_2+\alpha}{2}\end{array}\right]
\times\left[\begin{array}{l}\sin\dfrac{\theta_1+\theta_3-\alpha}{2}+\\ \sin\dfrac{\theta_1+\theta_3+\alpha}{2}\left(\dfrac{\sin(\theta_1+\theta_2)}{\tan\theta_3}+\cos(\theta_1+\theta_2)\right)\end{array}\right] \tag{2.13}
$$

If $\theta_1+\theta_2=\pi$ and $\theta_1+\theta_3=\pi$, both sides of Eq. (2.13) become zero regardless of α. This indicates that under this condition, Eq. (2.13) holds for any α. In other words, α is not uniquely determined by θ_1, θ_2, and θ_3. That is to say, this assembly becomes transformable if the quadrilateral modules are a parallelogram. Note that the rigid modules can be a square, rectangle, or rhombus for all of them that are parallelograms.

From Eqs. (2.4), (2.5), and (2.7), we have $\theta_1+\theta_3=\phi_1+\phi_2$, and $\theta_1+\theta_2=\alpha+\beta_1$. Hence, $\phi_1+\phi_2=\pi$ and $\alpha+\beta_1=\pi$ because $\theta_1+\theta_2=\pi$ and $\theta_1+\theta_3=\pi$. This indicates that only when the quadrilateral voids are made of parallelograms can the tiling pattern be transformable. This conclusion prompts us to consider arrangements with only parallelogram voids, which are presented next.

Example 2. Consider now an arrangement with a set of nine quadrilateral modules shown in Fig. 3(*a*). The central one is set as a reference module. This is a tiling pattern made of five shapes: the quadrilateral modules, two types of rhombic voids with side lengths a and c, respectively, and two types of parallelogram voids with side lengths b and d. The parallelogram voids are classified as two types because their angles could be different.

Now take α as an input, the following angular relationships can be established geometrically:

$$\theta_4=2\pi-\theta_1-\theta_2-\theta_3 \tag{2.14}$$

$$\phi_1=2\pi-\theta_1-\theta_3-\alpha \tag{2.15}$$

$$\phi_2=\theta_1+\theta_3+\alpha-\pi \tag{2.16}$$

$$\eta=\pi-\alpha \tag{2.17}$$

$$\beta_1=3\pi-2\theta_1-2\theta_3-\alpha \tag{2.18}$$

$$\beta_2=2\theta_1+2\theta_3+\alpha-2\pi \tag{2.19}$$

$$\gamma_1=2\pi-\theta_1-\theta_3-\alpha=\phi_1 \tag{2.20}$$

and

$$\gamma_2=\theta_1+\theta_3+\alpha-\pi=\phi_2 \tag{2.21}$$

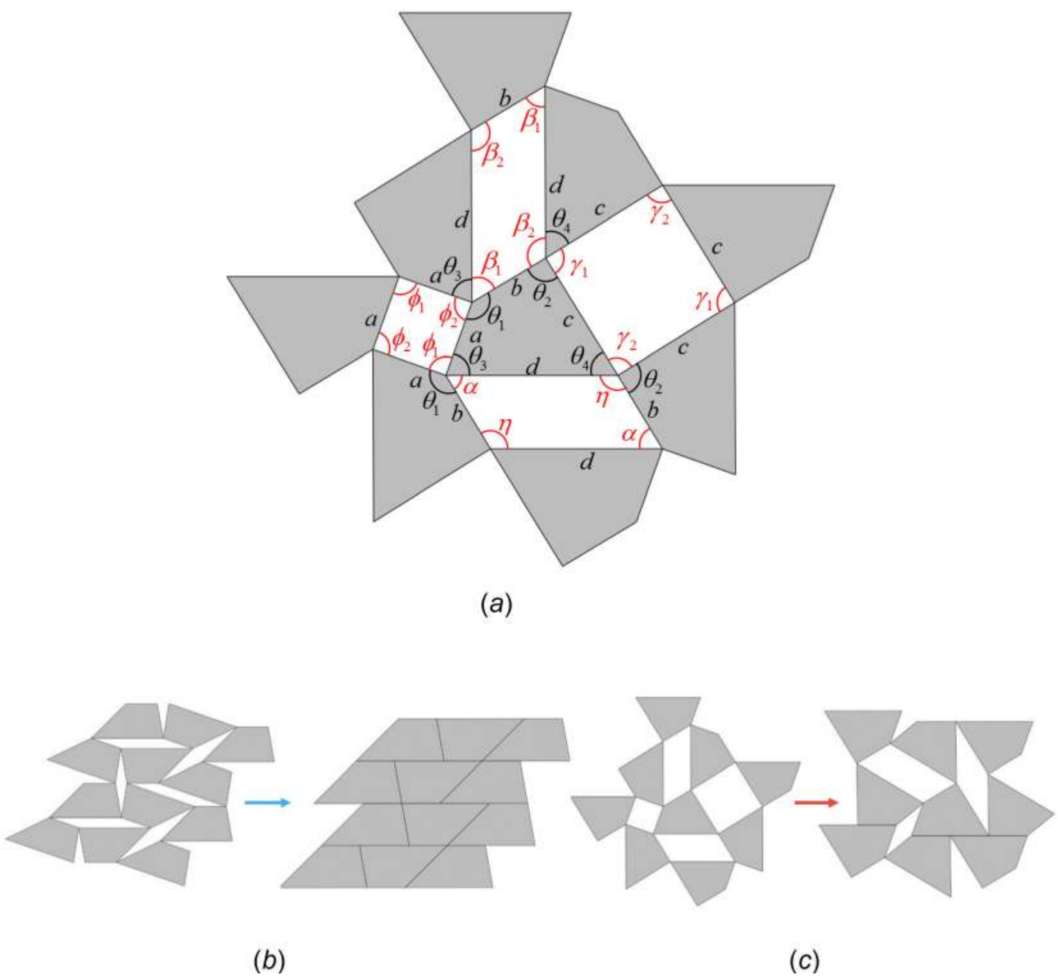


Fig. 3 The second arrangement of identical quadrilateral modules: (*a*) mathematical model and assemblies consisting of (*b*) trapezium and (*c*) arbitrary quadrilateral modules

The quadrilaterals units must not overlap with each other, so the interior angles of the parallelogram voids are only allowed to be between 0 and π. Applying $\phi_1, \phi_2, \beta_1, \beta_2, \eta \in [0, \pi]$ to Eqs. (2.15)–(2.19) yields

$$\begin{aligned} &0 \leq \alpha \leq \pi \\ &\pi - \theta_1 - \theta_2 \leq \alpha \leq 2\pi - \theta_1 - \theta_2 \\ &2\pi - 2\theta_1 - 2\theta_2 \leq \alpha \leq 3\pi - 2\theta_1 - 2\theta_2 \end{aligned} \tag{2.22}$$

With further simplification we obtain the range of the quadrilateral angles, which is

$$\pi \leq \theta_1 + \theta_2 \leq \frac{3}{2}\pi \tag{2.23}$$

Where this condition not met, the quadrilateral would overlap during transformation. The parallelogram voids completely close when $\alpha = 0$. If the rhombic voids close at the same time, $\phi_1 = 2\pi - \theta_1 - \theta_2 - 0 = \pi$, and thus

$$\theta_1 + \theta_2 = \pi \tag{2.24}$$

This indicates that the assembly based on this arrangement can only be compactly packed when two opposed edges in the quadrilateral are parallel to each other, i.e., the quadrilateral has to become a trapezium. Figure 3(*b*) shows such a trapezium assembly, whereas a general case consisting of arbitrary quadrilaterals is given in Fig. 3(*c*) that cannot be close packed.

Example 3. The third arrangement is shown in Fig. 4(*a*). Again only nine quadrilaterals are used, and the pattern consists of three shapes: a quadrilateral and two parallelograms voids.

Geometrically the angular relationships are

$$\theta_4 = 2\pi - \theta_1 - \theta_2 - \theta_3 \tag{2.25}$$

$$\phi_1 = 2\pi - \theta_2 - \theta_3 - \alpha \tag{2.26}$$

$$\phi_2 = \theta_2 + \theta_3 + \alpha - \pi \tag{2.27}$$

$$\eta = \pi - \alpha \tag{2.28}$$

$$\beta_1 = \pi - \theta_1 + \theta_4 - \alpha \tag{2.29}$$

$$\beta_2 = \theta_1 - \theta_4 + \alpha \tag{2.30}$$

$$\gamma_1 = 2\theta_4 - \alpha \tag{2.31}$$

$$\gamma_2 = \pi + \alpha - 2\theta_4 \tag{2.32}$$

where α is treated as the input. Since $\phi_1, \phi_2, \beta_1, \beta_2, \eta, \gamma_1, \gamma_2 \in [0, \pi]$, applying it to Eqs. (2.26)–(2.32) gives

$$\begin{aligned} &0 \leq \alpha \leq \pi \\ &\theta_1 + \theta_4 - \pi \leq \alpha \leq \theta_1 + \theta_4 \\ &\theta_4 - \theta_1 \leq \alpha \leq \pi + \theta_4 - \theta_1 \\ &2\theta_4 - \pi \leq \alpha \leq 2\theta_4 \end{aligned} \tag{2.33}$$

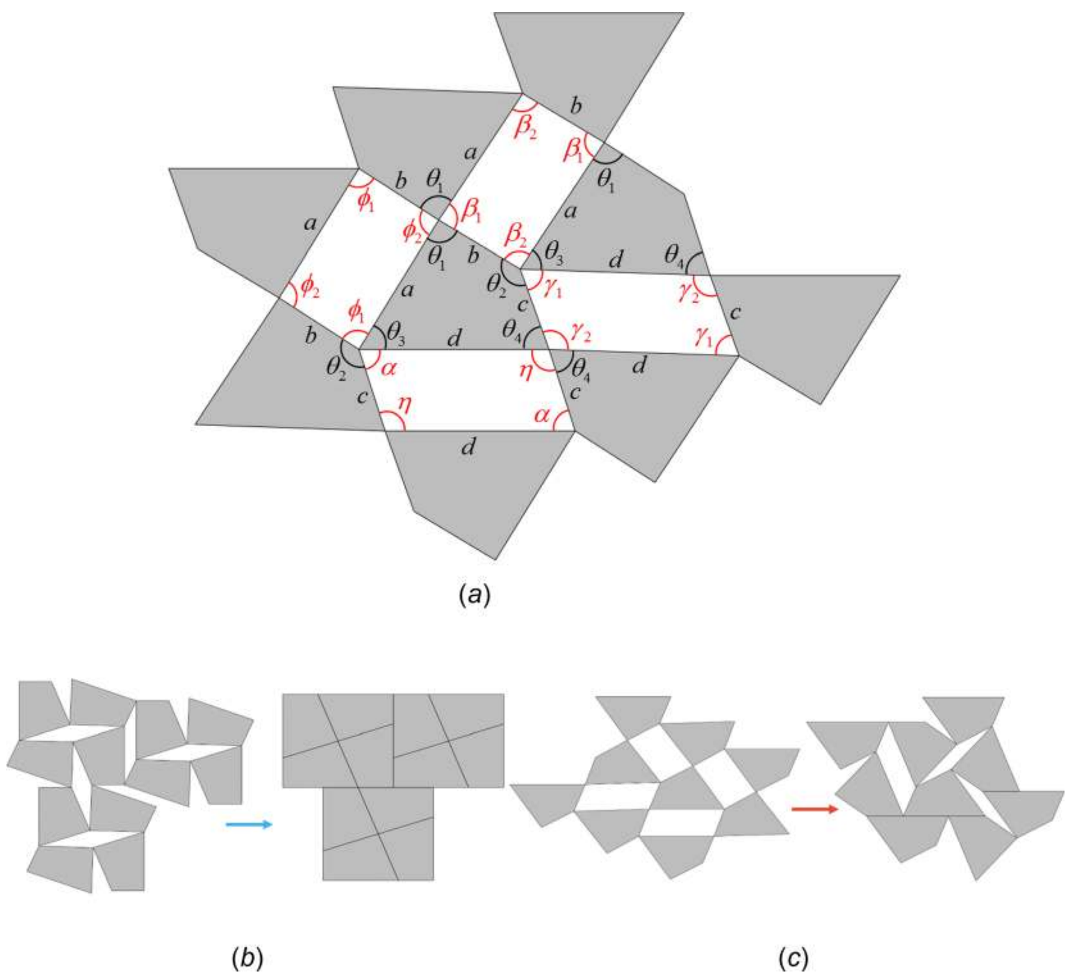


Fig. 4 The third arrangement of identical quadrilateral modules: (*a*) mathematical model and assemblies made from (*b*) quadrilateral modules with two opposite right angles and (*c*) general quadrilateral modules

Further simplification of the above inequalities yields

$$\begin{aligned} &0 \le \theta_1 + \theta_4 \le 2\pi \\ &|\theta_1 - \theta_4| \le \pi \\ &0 \le \theta_4 \le \pi \end{aligned} \tag{2.34}$$

In order to pack the modules compactly with this tiling pattern, there must be $\beta_1 = \pi$, $\gamma_1 = \pi$ when $\alpha = 0$, which gives

$$\theta_1 = \theta_4 = \frac{\pi}{2} \tag{2.35}$$

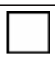
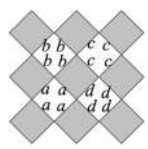
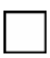
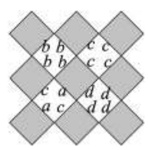
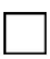
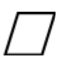
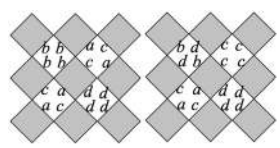
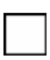
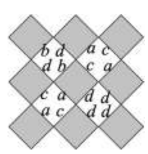
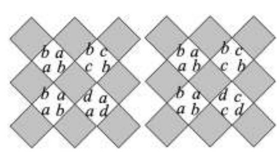
Hence, the quadrilateral must have two opposite right internal angles. Figure 4(*b*) shows such an example, and Fig. 4(*c*) is an assembly made from general quadrilateral modules that cannot fully pack.

Summary and Discussion. A total of 56 tiling patterns of quadrilaterals exist [14], but most of them are not transformable even if parallelogram voids are introduced. By having parallelogram or rhombic voids, we have found ten arrangements of arbitrary quadrilaterals that are transformable. Let the edge lengths of a quadrilateral be a, b, c, and d, and express a rhombic void with edge length of a as a^4, and a parallelogram void with lengths of a and b as a^2b^2. We obtain Table 1 that summarizes all transformable arrangements. The schematic diagrams of part of the arrangements are also given in the table. Note that Example 1 section discussed that earlier is a special case of the first transformable type in the table, where the quadrilateral modules must be parallelograms, i.e., $a = c$, $b = d$, and the voids parameters become (a^4), (b^4), (a^4), and (b^4). Example 2 is actually type 3 in the table which refers to an arrangement with two squares and two parallelogram voids. And Example 3 belongs to type 10 in the table which has four parallelogram voids.

Alternative Shapes of Modules. The transformable assemblies shown so far include only some specific quadrilateral modules. One may wonder if they can be replaced by more general shapes, e.g., shapes with curved edges. The answer is positive. The transformable assemblies can accommodate such alternative modules if the following two conditions are met: First, the corners of a module at which the connections to its neighboring modules are located must be identical to those of the original quadrilateral module, and second, the edges of the alternative modules must piece together when they are compacted folded. Two examples that satisfy both conditions are given in Fig. 5.

To enable an alternative module to match with its identical neighbors while retracted, its shape must have certain symmetry. If the module is a mirror symmetry pattern, the assembly can only close in one direction. For instance, the fan-shaped module shown in Fig. 5(*a*) has mirror symmetry, the modules around the central module are only capable of rotation anticlockwise about the central module to 90 deg at most if we use the central one as a reference. At the fully expanded state, the mechanism cannot move any further, because the edges of modules in the other direction will collide with each other. However, if a module has rotational symmetry, the assembly will be able to close in both directions as demonstrated by the assembly of the propeller shaped modules in Fig. 5(*b*), because the module has rotational symmetry. These modules can transform to 180 deg and fold completely in both directions. We have used the three-dimensional printed physical models to validate these features.

Table 1 Arrangements of arbitrary quadrilaterals that are transformable

Number and shapes of the voids	Type and parameters of the voids	Schematic diagram of part of the arrangements
4 	(1) (a^4) (b^4) (c^4) (d^4)	
3  1 	(2) (a^2c^2) (b^4) (c^4) (d^4)	
2  2 	(3) (a^2c^2) (b^4) (c^2a^2) (d^4) (4) (a^2c^2) (b^2d^2) (c^4) (d^4)	
1  3 	(5) (a^2c^2) (b^2d^2) (c^2a^2) (d^4)	
4 	(6) (a^2c^2) (b^2d^2) (c^2a^2) (d^2b^2) (7) (a^2b^2) (b^2a^2) (c^2b^2) (d^2a^2) (8) (a^2b^2) (b^2a^2) (c^2b^2) (d^2c^2) (9) (a^2b^2) (b^2a^2) (c^2d^2) (d^2a^2) (10) (a^2b^2) (b^2a^2) (c^2d^2) (d^2c^2)	 ……

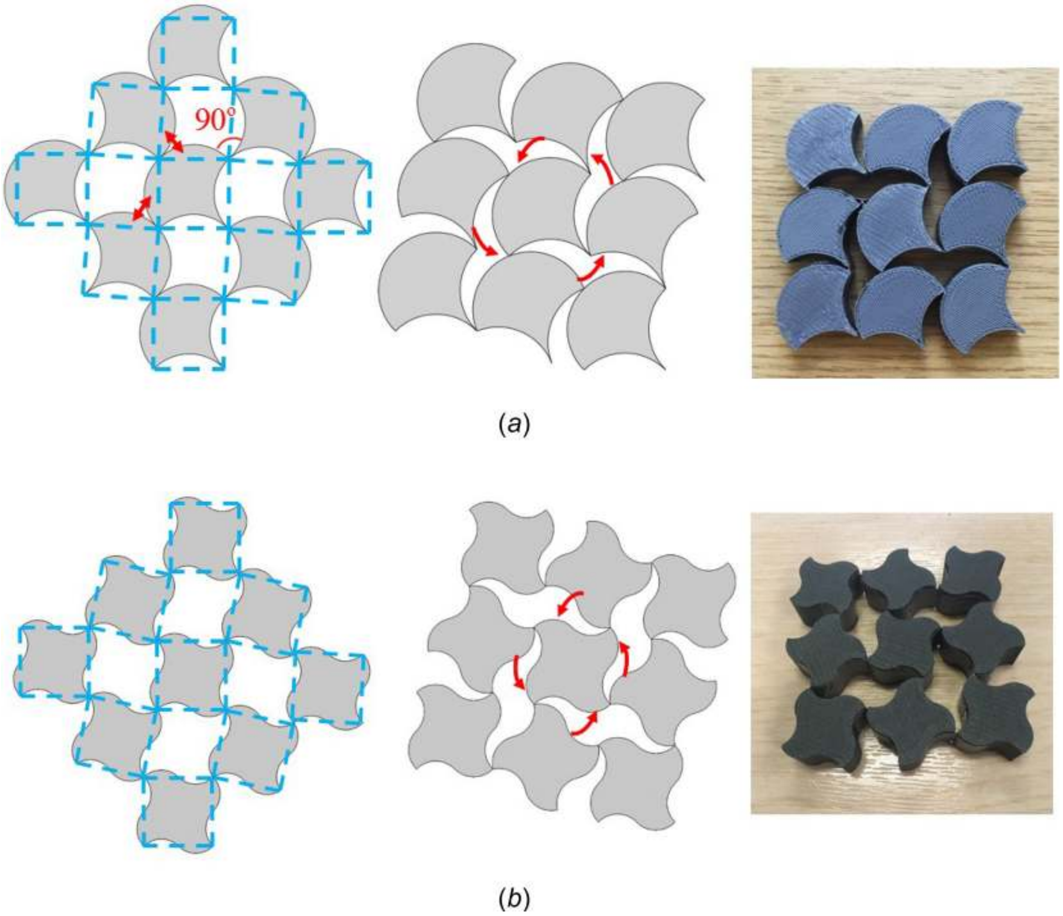


Fig. 5 Transformable assemblies made from alternative modules: (*a*) modules with mirror symmetry and (*b*) modules with rotational symmetry

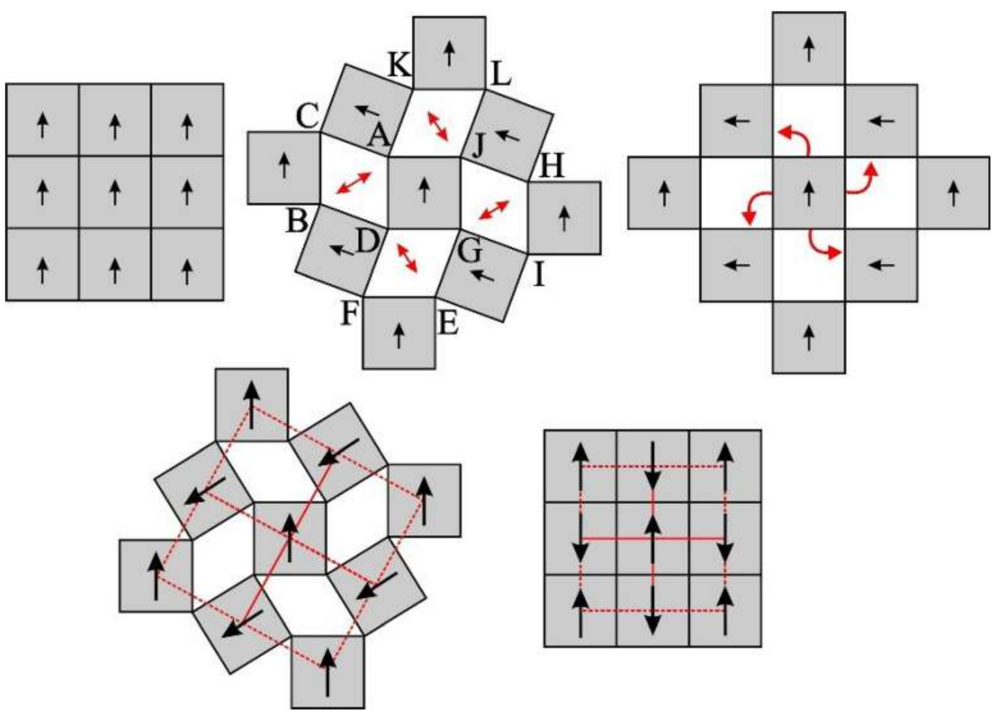


Fig. 6 Orientation of modules in a transformable assembly and its dual

Moreover, it is not necessary that all the modules must be identical. We shall discuss this in more detail later.

Orientation and Duality

So far, we have found a family of transformable assemblies. However, applying the Kutsbach criterion [18] to these assemblies yields no positive number. For instance, for the nine module assembly, the mobility m is

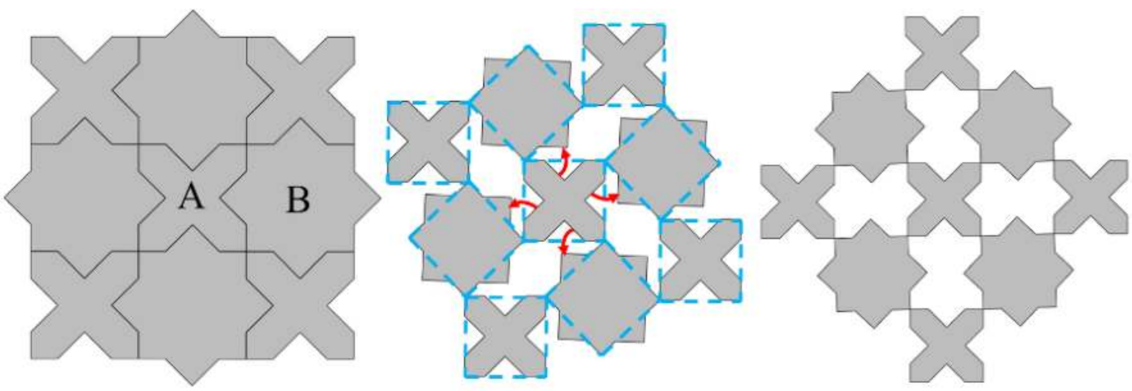
$$m = 3(n - j - 1) + \sum_{i=1}^{j} f_i = 3(9 - 12 - 1) + 12 = 0 \quad (3.1)$$

where n, j, and f_i are number of modules, the number of joints, and the degrees-of-freedom of each joint, respectively. This indicates that the assembly is actually an overconstrained mechanism, and the existence of mobility in it is due to the special geometry.

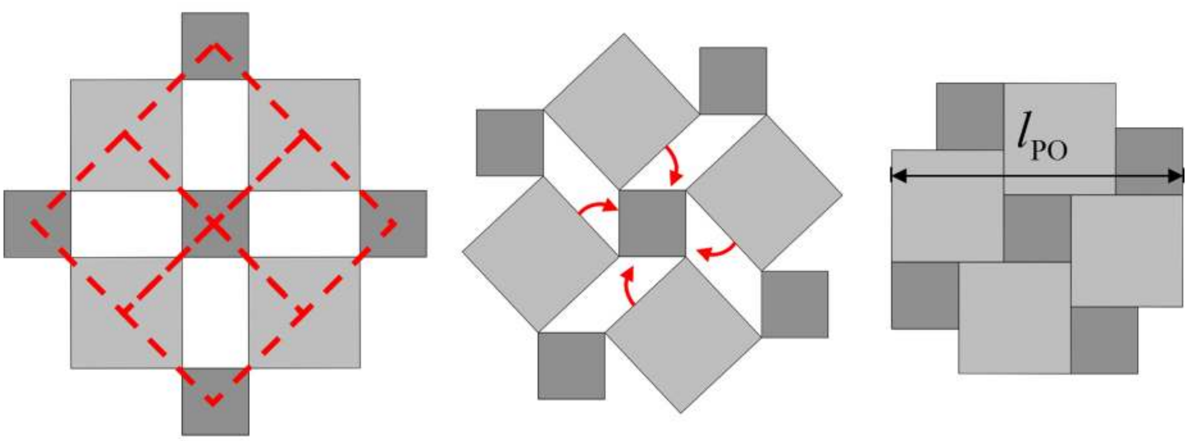
Intuitively the existence of mobility can be illustrated using the example shown in Fig. 6. When the modules are rotating anticlockwise around the central module, point B is moving away from A, which results in D getting closer to C but simultaneously apart from E. Accordingly, F is closer to G, but H moves away from G. In turn, I moves toward J, K is farther apart from J, and L is closer to A. The entire motion of four interlinked 4R linkages are synchronized with 1DOF. It can be shown that all of the transformable assemblies have only 1DOF.

It is also interesting to note that the orientations of the modules may alter when they rotate around each other. To examine it, we mark each module with an arrow as shown in Fig. 6 with all the arrows pointing upward at the start when the modules are closely packed. While rotating anticlockwise around the central module, the directions of the arrows on those modules that are not directly connected to the central module remain unchanged, indicating they translate without any rotation. The rest of the modules turns. In the fully opened configuration, the arrows on the translating modules remain the same, whereas the turning modules have rotated by 90 deg. In the final configuration, the modules are closely packed together again. The turning modules have now rotated by 180 deg, while the translating modules never change their orientations. The orientation variation is a particularly useful feature as it could be used to design programmable acoustic or electromagnetic metamaterials. The switch between geometric states could be used to alter the wave directions [19].

While the orientation changes with motion, the dual shape of the tiling pattern remains the same no matter how it transforms. Two patterns are called a dual to each other if it is possible to set a one-to-one correspondence between the modules, edges, and vertices of the first tiling and those of the second one [20]. The way to draw a dual of an arrangement is to connect the center of each module with those of its neighbor's. Take the arrangement given in Fig. 6 as an example. It can be shown that the dual of this tiling with square modules is the same as itself. The dual of a



(*a*)



(*b*)

Fig. 7 Paired modules based on the arrangement of type 1: (*a*) jigsaw puzzle pair and (*b*) pair of square modules with different sizes

transformable tiling is a basic topology feature can be used to distinguish the tiling from others.

Paired Modules

Earlier we touched upon the alternative shapes of the modules, e.g., those shown in Fig. 5, but all the modules are kept identical. It is however to have paired modules that can satisfy the two conditions set in the Alternative Shapes of Modules section.

Figure 7 shows two examples of paired modules that is based on type 1 of Table 1. Figure 7(*a*) is a jigsaw puzzle-paired module where shapes A and B are called a pair, and the parameters of type 1 pattern are given as a = b=c = d. Draw lines between each of the four connection points of either of the two modules, and the original shapes of the modules are obtained. The motion of the pattern is exactly as same as that of the type 1. Despite that the two modules forming the pair have different shapes, they bit into each other in the fully compact configuration.

Mathematical tiling shows that there exist tiling patterns consisting of two similar quadrilaterals of different sizes. We can also produce transformable assemblies with two types of modules with different sizes. The example shown in Fig. 7(*b*) is a tiling pattern of two types of square tiles: one big and one small. The squares are connected according to type 1. Because of the different edge length, the voids are parallelograms instead of rhombuses. When the pattern is fully expanded, the parallelogram voids become rectangles instead of squares. Its dual is the same as the type 1 (Fig. 7(*b*)), so the motion of this pattern is also the same as previously discussed.

The concepts of having paired modules or similar modules of different sizes can be merged to produce assembly with unique features. For instance, if we use a special pair of modules, one being a four-pointed star whereas the other of a square, a tiling pattern shown in Fig. 8(*a*) is obtained based on the type 1. The star and square modules form a grid once fully open, and the former tightly surround the latter when fully folded. This has resulted in a transformable tiling pattern with an expansion ratio much greater than that of the one made from square modules. The expansion ratio is calculated as follows.

The start and square are defined by parameters a and α. The fully expanded length can be obtained as

$$l_E = \sqrt{2}a + 4\sqrt{2}a\sin\frac{\alpha}{2} \tag{4.1}$$

When fully packed without any voids, the length becomes

$$l_{PS} = 4a\sin\frac{\alpha}{2} + a\sin\frac{\alpha}{2}\left(1-\tan\frac{\pi-\alpha}{2}\right) \tag{4.2}$$

Thus, the expansion ratio is

$$r = \frac{l_E}{l_{PS}} = \frac{\sqrt{2}+4\sqrt{2}\sin\frac{\alpha}{2}}{4\sin\frac{\alpha}{2}+\sin\frac{\alpha}{2}\left(1-\tan\frac{\pi-\alpha}{2}\right)} \tag{4.3}$$

Because $\alpha \in [(\pi/2), \pi]$, the maximum and minimum values of r are $0.5+\sqrt{2}$ and $\sqrt{2}$, respectively. If we replace the star module with a square, as shown previously in Fig. 7(*b*), then α becomes π (denoted by α_o), the expanded length is $5\sqrt{2}a$, and the packaged length l_{PO} in Fig. 7(*b*) becomes

$$l_{PO} = 4a\sin\frac{\alpha_O}{2} + a\sin\frac{\alpha_O}{2}\left(1-\tan\frac{\pi-\alpha_O}{2}\right) = 4a + a = 5a \tag{4.4}$$

The expansion ratio then is

$$r_O = \sqrt{2} \tag{4.5}$$

Therefore, for the paired square tiling pattern, the expansion ratio is fixed to $\sqrt{2}$, but using the star tiling pattern, the expansion ratio is tuned by varying the thickness of the star arm. The value of r in star tiling and paired square tiling is given in Fig. 8(*b*). It can be concluded that, the "slimmer" the star is, the larger the expansion ratio. This star and square design can be adopted to other types of transformable arrangements.

Other Transformable Assemblies

The paper so far confines to assemblies made from quadrilateral modules and their alternatives, each of which has four connection points. However, there are also mathematical tiling patterns that are made from shapes other than quadrilaterals. Some of these can also be used to engineer 1DOF transformable assemblies provided that the voids are kept as 4*R* linkages. For example, the 3636 tiling pattern with hexagons and triangles shown in Fig. 9(*a*) can be made to a transformable assembly if the rhombuses are made as voids and the triangles and hexagons as modules. It contains rhombus voids. The tiling pattern with triangle, hexagon and

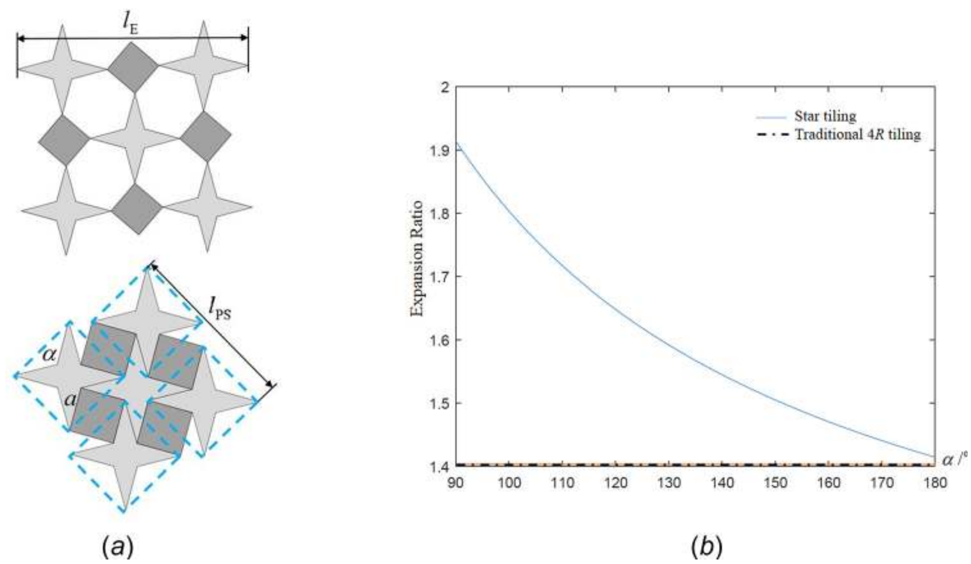


Fig. 8 Comparison between an assembly with paired modules and that with square modules: (*a*) parameters of two assemblies and (*b*) their respective expansion ratios

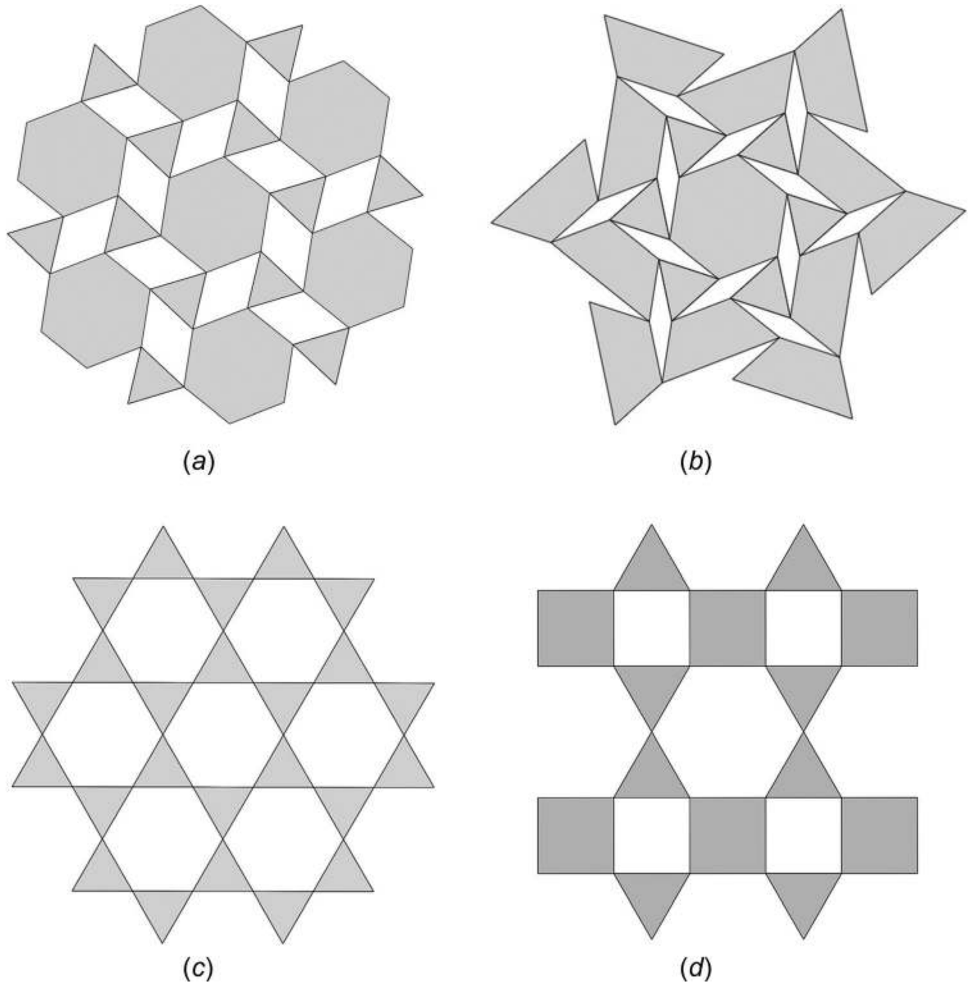


Fig. 9 Other transformable tiling patterns with (*a*) triangles and hexagons, (*b*) triangles, trapezium, and hexagons, (*c*) triangles, and (*d*) squares and triangles

trapezium shapes in Fig. 9(*b*) can also be made to a transformable assembly if we insert rhombus and treat the triangles, hexagon, and trapeziums as modules.

If the transformation is the main design objective whereas the number of DOF is less of a concern, other shaped voids could be used. The triangle tiling pattern shown in Fig. 9(*c*) has voids which are 6R linkages, each of which has 3DOFs. The assembly would have many DOFs. An assembly based on the tiling pattern shown in Fig. 9(*d*) would also have many DOFs, because the voids are either 4*R* or 6*R* linkages.

Conclusion

Inspired by the modular origami, we have discovered a family of transformable planar structures that can be used to construct mechanical metamaterial. Utilizing the mathematic tiling patterns coupled with mechanism theory, we manage to greatly expend the family of the transformable assemblies. Our achievements are summarized as follows:

(1) A family of 1DOF transformable assemblies consisting of quadrilateral modules are found, which has brought much diversity to known transformable arrangements.
(2) It has been found that during the transformation, the orientation will be changed, and the size of the dual varies to reflect the overall dimension change of tiling patterns.
(3) It is found that the identical quadrilateral modules could be replace by alternative or paired modules, which maintains the ability of transformation and compact folding. Paired modules in different sizes and shapes are also discussed in detail. A special type of star module that has a larger expandable ratio is provided.
(4) Tentatively, we have also discussed transformable tiling patterns using other polygons and discussed them according to the planar linkages inhibited within the patterns.

Despite that our attention here is paid on 2D patterns, they can be conveniently placed one on top of another to form transformable three-dimensional metamaterials just like the retractable structures outlined in Ref. [18]. In the on-going research, we are exploring the transformable tiling patterns that include more shapes, may not be completely packaged and have multiple DOFs to complete the classification of all transformable patterns. Moreover, we have been working on the exact design of modules that could provide favorable mechanical properties after the transformation ceases. Real metamaterials based on our findings are under construction as well.

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