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Geometry of Undecidable Systems

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Geometric properties of undecidable systems are numerically investigated. As an undecidable set, the halting set of the universal Turing machine is chosen, whose geometric representation is shown to have a different structure on an arbitrarily small scale, and is constructed non-uniformly in time. The set's structure has a fractal boundary dimension converging to the spatial dimension, which gives a geometric characterization of the undecidability.

Unpredictability in deterministic dynamical systems has been understood in terms of chaos. 'Undecidability', however, is a different concept from unpredictability. A problem is undecidable if there is no algorithm (i.e., finitely described procedure which is guaranteed to stop) to answer it. Although undecidability originated in computation theory, it is the most fundamental problem in computation processes and is also essential to understand the computational process in our brain and in machines in general. Then it is natural to ask how such undecidability appears in dynamical systems and how it is characterized. This class of problems is of general interest in cellular automata,[1] neural networks, DNA sequences, and dynamical systems theory in general. Furthermore, it is an interesting question whether there is a new class of dynamical process, different from chaos, corresponding to undecidability.

To understand the nature of dynamical process with undecidability, we choose the Turing machine (TM), introduced in 1936 as one of the computation models.[2)-4)] The total state (instantaneous description) of a TM is completely determined by specifying its internal state, its tape and the position of its head. If these three are determined at one time, then these three at the next step are uniquely determined. Thus a TM can be regarded as a deterministic dynamical system. Indeed there have been some studies[5),6)] in which a TM is embedded in some nonlinear dynamical system (e.g., in Ref. 5), the TM's internal state and tape are embedded in a two-dimensional space, and the motion of the TM corresponds to an application of a piecewise-linear map on that space). When we regard a TM as a dynamical system, a halting state corresponds to an attractor. Similarly a halting set (an accepted language) of a TM corresponds to a basin of attraction.

In this paper we investigate halting sets of TMs, and in particular of universal Turing machines (i.e., universal language, denoted as UL). By embedding a whole initial tape into a pair of real numbers, the halting sets are mapped into a unit square of a two-dimensional space. With these, we attempt to capture geometric properties of the halting sets to understand geometric aspects of undecidability. Instead of ordinary approaches in computation theory, experimental approaches (numerical calculations) from dynamical systems studies are adopted to shed new light on this subject. We also explore a new type of dynamical systems property realized in TMs.

With the above embedding, the geometric representation of the *UL* will be shown to have different (i.e., non-self-similar) fine structures on an arbitrarily small scale and is constructed non-uniformly in time, as is also seen in the long-time tail in the halting time distribution, in contrast with the construction process of ordinary fractals (i.e., transient chaos). The structure of the set is shown to have a fractal boundary dimension asymptotically approaching two, or in other words, the uncertainty exponent is zero, which is a geometric characterization of undecidability. The relation between unpredictability and undecidability will also be discussed.

Given an arbitrary TM and an initial tape, the problem whether the TM starting from the initial tape will halt or not is undecidable. Even if we choose a universal Turing machine (UTM), the halting problem is still undecidable. The set of initial tapes on which a UTM will eventually halt (i.e. a halting set of UTM) is called a universal language (*UL*), which is known as a recursively enumerable set but not a recursive set.[3]

Now we briefly describe Minsky's UTM,[4] which we mainly treat in this paper. Minsky's UTM has eight internal states including a halting state {q1, q2, ..., q7, halting state} and a bi-infinite tape with tape alphabet {y, 0, 1, A}. The symbol '0' of the tape alphabet is also used as a blank symbol. The symbol 'y' is read by the head of Minsky's UTM at the beginning of computations.*) Though we mainly treat Minsky's UTM in this paper, it should be noted that treating a specific 'U'TM is equal to treating all TMs.

Now we explain the code which maps an initial tape of a halting set to a point of a unit square of a two-dimensional space. Although we explain the code in Minsky's UTM's case, similar codes exist for other TMs. First we transform the tape alphabet {y, 0, 1, A} of an initial tape into {3, 0, 1, 2}, respectively. (Other choices of transformation from a symbol sequence to a number meet with the same results below.) Then we divide the initial tape into three parts, one cell which is read by the head at the beginning of the computation (where the symbol 'y' was written), and the right and left sides of the tape from this cell. Then we represent the right and left sides of the tape by a pair of real numbers given by the base-4 expansions of decimals, respectively. Finally, this pair is put on a unit square, where the right side of the tape corresponds to the horizontal axis, and the left side to the vertical axis.**)

*) There are further restrictions on initial tapes to use Minsky's UTM "properly". Still, the halting problem of Minsky's UTM on the initial tapes which we treat in our paper, i.e., all bi-infinite symbol sequences with 'y' at the center (which of course include "proper" initial tapes), is an undecidable one. Since our main concern is dynamical and geometrical aspects of the undecidability of the halting problem, we do not need to exclude tapes that do not use the TM properly.

**) There are three reasons for this specific choice of codes. First, there is a natural correspondence between distance in the two-dimensional plane and the degree of the influence in the tape. Indeed, the smaller bit corresponds to the cell on the tape farther from the initial 'y' position, and it has a weaker influence on whether a TM halts. Second, this encoding is a finite procedure. If description on the tape is finite, encoding is done in finite computation time. (Thus, this code cannot do universal computation.) Third, this code is general.[7] If we consider a two-dimensional piecewise-linear map which is equivalent to a TM, the geometric representation of its halting set by this code is (one part of) the basin of the map.[5]

The geometry of UL, acquired by using this code, has a different structure on an arbitrarily small scale. If UL had a geometric structure only up to a finite scale (e.g., classical geometric sets like line segments and circular disks), we could use this fact to decide whether a given point is included in UL, i.e., to answer the halting problem of UTM. But that would contradict the undecidability of the halting problem of UTM. Similarly if UL had a self-similar structure (i.e., the same structure) on an arbitrarily small scale, like a Cantor set, it would again contradict undecidability of the halting problem of UTM for the same reason. Hence the geometry of UL must have different fine structures on arbitrarily small scales.

In the following, we numerically study the UL (the halting set of Minsky's UTM). Of course, it is impossible to take infinite time for the numerical calculations. Thus we treat the set of initial tapes on which Minsky's UTM will halt within a given finite step n, denoted as $UL(n)$.

To see the change in the structure, we study the time course of the construction of UL. In Fig. 1, we have plotted three sets, $UL(39)$, i.e., the set halted within 39 steps, $UL(52) \setminus UL(39)$, i.e., that halted between 40 and 52 steps, and $UL(500) \setminus UL(52)$. These time steps are chosen so that approximately the same number of points are contained in each.

As is seen in Fig. 1, UL is constructed non-uniformly in time, in contrast with uniform construction in ordinary fractals,[8] where it is possible to predict which points will be included in, or excluded from the set during the construction of the set with time. An example is the construction of the complement of the Cantor set, i.e., the set of initial points which eventually leave the interval [0, 1] by the map $3x$ ($x < 1/2$), $3x - 2$ ($x \geq 1/2$). However, when a set is constructed non-uniformly in time, it is difficult to predict if a point is included in the set. Note that this non-uniform construction of UL is a geometric aspect of the undecidability of the halting problem of UTMs.

In the construction of the complement of the Cantor set, the fraction of the

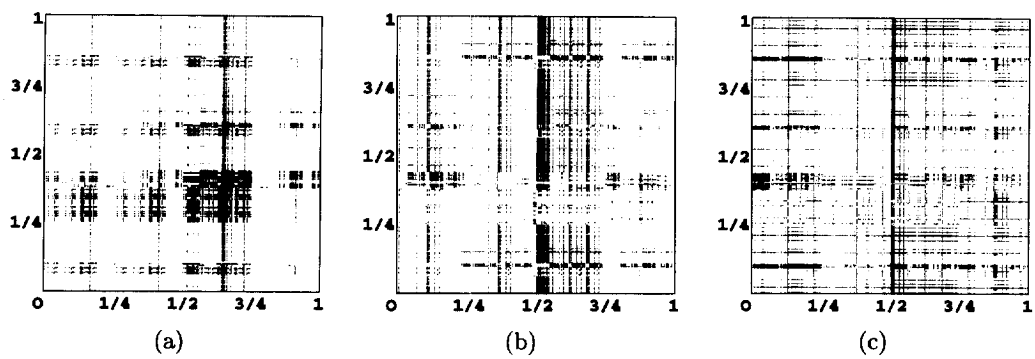

Fig. 1. The time course of the construction of UL. (a) $UL(39)$, i.e., the set halted within 39 steps (dots correspond to initial tapes halted within 39 steps). (b) $UL(52) \setminus UL(39)$, i.e., that halted between 40 and 52 steps. (c) $UL(500) \setminus UL(52)$. Instead of random sampling of the tapes, initial tapes on the grid point $(4^{-5}i, 4^{-5}j)$ are examined (i.e., those with blank symbols('0's) for cells farther than 5 sites from the initial 'y' position), to show the temporal construction clearly.

points which leave [0, 1] decays exponentially with time n when initial points are scattered uniformly in [0, 1]. In general, the distribution of the transient time decays exponentially for transient chaos.[9] On the other hand, in the case of our UL, the fraction of the initial points halting with computation time n is found to decay according to a power law or slower.[10] Thus the construction of UL is slower than a transient process in chaos (i.e., the construction of ordinary fractal sets).

Now we study the dimension of UL. Since UL has positive Lebesgue measure (i.e., a fat fractal), its box-counting dimension is the same as the dimension of the space (i.e. $D_0 = 2$). Thus, for studying the fine structure of UL, we investigate the box-counting dimension of the boundary of UL (i.e. the exterior dimension[9),11)] of UL) numerically.

The definition of the box-counting dimension of a set S in N-dimensional space is equivalent to $D_0 = N - \lim_{\epsilon \to 0} \ln V[S(\epsilon)]/\ln \epsilon$, where $V[S(\epsilon)]$ is the N-dimensional volume of the set $S(\epsilon)$ created by fattening the original set by an amount ϵ.[9] Based on this definition, we estimate the box-counting dimension of the boundary of UL (the boundary dimension of UL).

Of course, we cannot treat UL directly since we have only a finite time for numerical calculation. Instead, we investigate the boundary dimension of $UL(n)$. Then we survey the asymptotic behavior of the boundary dimension of $UL(n)$ when n is increased (to infinity).

The detailed procedure is as follows. First we choose a point (X, Y) in a unit square of a two-dimensional space (i.e. a tape on which 'y' is written at the center) at random. Then we perturb this point (X, Y) to $(X+\epsilon, Y)$, $(X, Y+\epsilon)$ and $(X+\epsilon, Y+\epsilon)$, where $\epsilon = 4^{-i}$ (i.e., we perturb the ith digit of the right side or left side or both sides of the tape). Then we decide whether Minsky's UTM, starting from each of these four tapes (at the center), will halt within n steps or not (i.e., in $UL(n)$ or not). If all four points are in $UL(n)$ or none of them are in $UL(n)$, we regard there to be no boundary in the square (of length ϵ) made from these four points. Otherwise we regard that there is a boundary in the square. We repeat this procedure for a large number of points and evaluate the fraction of squares with a boundary, denoted as $f(\epsilon)$, which gives the estimation of $V[S(\epsilon)]$. Varying ϵ, we obtain the scaling of $f(\epsilon)$ with ϵ, and can evaluate $N - D_0$ (i.e. $2 - D_0$).

In Fig. 2, the log-log plot of $f(\epsilon)$ of $UL(n)$ with ϵ is shown for several values of n. It can be fit as $f(\epsilon) \sim \epsilon^{2-D_0}$ for small ϵ, from which one can obtain $2 - D_0$ for each n. Figure 3 displays the log-log plot of $2 - D_0$ versus the computation time n. It shows that $2 - D_0$ of the boundary of $UL(n)$ approaches zero roughly as $n^{-\frac{2}{3}}$ with the increase of the computation time n. In other words, when we increase n, the box-counting dimension of the boundary of $UL(n)$ approaches two, which is the same as the dimension of the space. Thus we can expect that the box-counting dimension of the boundary of UL is two.

Here we briefly refer to the uncertainty exponent α $(= N - D_0)$.[9),11)] Suppose there exists a set A in a certain N-dimensional space and our ability to determine the position of points has an uncertainty ϵ. S denotes the boundary of A. If we have to determine whether a given point belongs to A, the probability $f(\epsilon)$ of making a

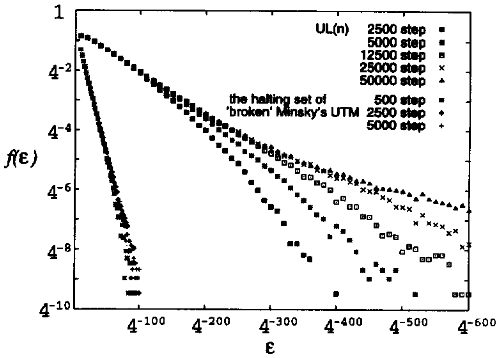

Fig. 2. Log-log plot of $f(\epsilon)$ (the fraction of squares with a boundary) of $UL(n)$ and the halting set of 'broken' Minsky's UTM (changing q7 as a new halting state to Minsky's UTM), with ϵ for several values of computation time n. The slope of $f(\epsilon)$ of $UL(n)$ becomes smaller with the increase of n, while that of the halting set of 'broken' Minsky's UTM does not show such asymptotic behavior with the increase of n.

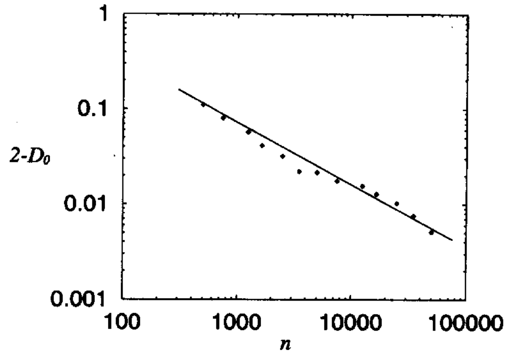

Fig. 3. Log-log plot of $2 - D_0$ (uncertainty exponent α) of $UL(n)$ versus the computation time n, obtained from Fig. 2. It approaches zero roughly as $n^{-\frac{2}{3}}$ with the increase of the computation time n.

mistake in such a determination is proportional to $V[S(\epsilon)]$. Thus, if the box-counting dimension of the boundary S is D_0, $f(\epsilon)$ is proportional to ϵ^{N-D_0} $(= \epsilon^{\alpha})$. If α is small, then a large decrease in ϵ leads to only a relatively small decrease in $f(\epsilon)$. Thus α is called an uncertainty exponent.

The above result indicates that the uncertainty exponent of the boundary of $UL(n)$ becomes smaller with the increase of computation time n (i.e., decrease of mistake in determination of $UL(n)$ is more difficult). The result that the uncertainty exponent of the boundary of UL is zero (i.e. $D_0 = 2$) indicates that the probability of making a mistake ($V[S(\epsilon)]$) does not depend on ϵ. This result is reasonable, since unlike chaotic behavior, the undecidability of the halting problem of UTM exists even if descriptions are known exactly. Thus the undecidability is explained from the viewpoint of the uncertainty exponent, i.e. the boundary dimension.

The boundary dimension of UL of other UTMs is also two. Since UTMs can imitate each other, the geometric representations of UL for any UTMs contain each other, and each has the same boundary dimension, two.

As a contrast, we study the boundary dimension of halting sets of TMs which are not UTM. In Fig. 2, the fraction of the boundary is also plotted for the halting set of a 'broken' Minsky's UTM, as an example of not UTM. A 'broken' Minsky's UTM is defined by changing one of q1, ..., q7 as a new halting state to Minsky's UTM. Because this TM has the same (number of) internal states and tape symbols as Minsky's UTM, it is appropriate as a contrast study. We have again investigated the boundary dimension of the 'broken' Minsky's UTM for several values of computation time n. Unlike the case of Minsky's UTM, $f(\epsilon)$ of the 'broken' Minsky's UTM shows no asymptotic approach of α to zero with the increase of n. Thus, the boundary dimension D_0 is estimated to be less than two even in the limit of $n \to \infty$. In terms of the uncertainty exponent, unlike UL, the probability of making a mistake ($V[S(\epsilon)]$) can be decreased to any amount, in principle, by a decrease in ϵ. This result indicates that not all halting sets of TM satisfy the condition the boundary dimension is equal to the space dimension, and also that our result for the boundary dimension of UL is not based on the property of our specific choice of the code.

Unpredictability in the context of chaos and undecidability are quite different concepts. They have not been treated in parallel from the same viewpoint. However, in terms of the boundary dimension (uncertainty exponent), they can be treated together. Unpredictability in the context of chaos is equivalent to the boundary dimension of basin of attraction satisfying $N-1 < D_0 < N$, where the probability of making a mistake can be decreased to any amount in principle by decreasing ϵ. However, when the boundary dimension $D_0 = N$, the probability cannot be decreased. As we have seen, undecidable sets belong to this case. Hence undecidability belongs to a general class of unpredictability not necessarily having chaotic instability (i.e., the class of boundary dimension $D_0 > N-1$, i.e. the boundary with a fine structure).

To sum up, we have characterized the geometric feature of undecidability as the boundary dimension converging to the space dimension. Details on geometric and dynamical systems features, as well as possible relationships with models of analog computation [12),*)] and riddled basin structure [13)] will be reported elsewhere. [10)]

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*) Note, however, in their formulation in Ref. 12), a fractal set (like a typical Julia set) is also undecidable, in contrast with our conclusion. We emphasize here the difference of our inclusion of precision, as is typically seen in the importance of the uncertainty exponent.