

GRADUATE STUDENT SERIES IN PHYSICS

Series Editor:

Professor Douglas F Brewer, MA, DPhil

Emeritus Professor of Experimental Physics, University of Sussex

GEOMETRY, TOPOLOGY AND PHYSICS

SECOND EDITION

MIKIO NAKAHARA

Department of Physics

Kinki University, Osaka, Japan



Taylor & Francis

Taylor & Francis Group

New York London

CONTENTS

Preface to the First Edition	xvii
Preface to the Second Edition	xix
How to Read this Book	xxi
Notation and Conventions	xxii
1 Quantum Physics	1
1.1 Analytical mechanics	1
1.1.1 Newtonian mechanics	1
1.1.2 Lagrangian formalism	2
1.1.3 Hamiltonian formalism	5
1.2 Canonical quantization	9
1.2.1 Hilbert space, bras and kets	9
1.2.2 Axioms of canonical quantization	10
1.2.3 Heisenberg equation, Heisenberg picture and Schrödinger picture	13
1.2.4 Wavefunction	13
1.2.5 Harmonic oscillator	17
1.3 Path integral quantization of a Bose particle	19
1.3.1 Path integral quantization	19
1.3.2 Imaginary time and partition function	26
1.3.3 Time-ordered product and generating functional	28
1.4 Harmonic oscillator	31
1.4.1 Transition amplitude	31
1.4.2 Partition function	35
1.5 Path integral quantization of a Fermi particle	38
1.5.1 Fermionic harmonic oscillator	39
1.5.2 Calculus of Grassmann numbers	40
1.5.3 Differentiation	41
1.5.4 Integration	42
1.5.5 Delta-function	43
1.5.6 Gaussian integral	44
1.5.7 Functional derivative	45
1.5.8 Complex conjugation	45
1.5.9 Coherent states and completeness relation	46

1.5.10	Partition function of a fermionic oscillator	47
1.6	Quantization of a scalar field	51
1.6.1	Free scalar field	51
1.6.2	Interacting scalar field	54
1.7	Quantization of a Dirac field	55
1.8	Gauge theories	56
1.8.1	Abelian gauge theories	56
1.8.2	Non-Abelian gauge theories	58
1.8.3	Higgs fields	60
1.9	Magnetic monopoles	60
1.9.1	Dirac monopole	61
1.9.2	The Wu–Yang monopole	62
1.9.3	Charge quantization	62
1.10	Instantons	63
1.10.1	Introduction	63
1.10.2	The (anti-)self-dual solution	64
	Problems	66
2	Mathematical Preliminaries	67
2.1	Maps	67
2.1.1	Definitions	67
2.1.2	Equivalence relation and equivalence class	70
2.2	Vector spaces	75
2.2.1	Vectors and vector spaces	75
2.2.2	Linear maps, images and kernels	76
2.2.3	Dual vector space	77
2.2.4	Inner product and adjoint	78
2.2.5	Tensors	80
2.3	Topological spaces	81
2.3.1	Definitions	81
2.3.2	Continuous maps	82
2.3.3	Neighbourhoods and Hausdorff spaces	82
2.3.4	Closed set	83
2.3.5	Compactness	83
2.3.6	Connectedness	85
2.4	Homeomorphisms and topological invariants	85
2.4.1	Homeomorphisms	85
2.4.2	Topological invariants	86
2.4.3	Homotopy type	88
2.4.4	Euler characteristic: an example	88
	Problems	91

3	Homology Groups	93
3.1	Abelian groups	93
3.1.1	Elementary group theory	93
3.1.2	Finitely generated Abelian groups and free Abelian groups	96
3.1.3	Cyclic groups	96
3.2	Simplexes and simplicial complexes	98
3.2.1	Simplexes	98
3.2.2	Simplicial complexes and polyhedra	99
3.3	Homology groups of simplicial complexes	100
3.3.1	Oriented simplexes	100
3.3.2	Chain group, cycle group and boundary group	102
3.3.3	Homology groups	106
3.3.4	Computation of $H_0(K)$	110
3.3.5	More homology computations	111
3.4	General properties of homology groups	117
3.4.1	Connectedness and homology groups	117
3.4.2	Structure of homology groups	118
3.4.3	Betti numbers and the Euler–Poincaré theorem	118
	Problems	120
4	Homotopy Groups	121
4.1	Fundamental groups	121
4.1.1	Basic ideas	121
4.1.2	Paths and loops	122
4.1.3	Homotopy	123
4.1.4	Fundamental groups	125
4.2	General properties of fundamental groups	127
4.2.1	Arcwise connectedness and fundamental groups	127
4.2.2	Homotopic invariance of fundamental groups	128
4.3	Examples of fundamental groups	131
4.3.1	Fundamental group of torus	133
4.4	Fundamental groups of polyhedra	134
4.4.1	Free groups and relations	134
4.4.2	Calculating fundamental groups of polyhedra	136
4.4.3	Relations between $H_1(K)$ and $\pi_1(K)$	144
4.5	Higher homotopy groups	145
4.5.1	Definitions	146
4.6	General properties of higher homotopy groups	148
4.6.1	Abelian nature of higher homotopy groups	148
4.6.2	Arcwise connectedness and higher homotopy groups	148
4.6.3	Homotopy invariance of higher homotopy groups	148
4.6.4	Higher homotopy groups of a product space	148
4.6.5	Universal covering spaces and higher homotopy groups	148
4.7	Examples of higher homotopy groups	150

4.8	Orders in condensed matter systems	153
4.8.1	Order parameter	153
4.8.2	Superfluid ^4He and superconductors	154
4.8.3	General consideration	157
4.9	Defects in nematic liquid crystals	159
4.9.1	Order parameter of nematic liquid crystals	159
4.9.2	Line defects in nematic liquid crystals	160
4.9.3	Point defects in nematic liquid crystals	161
4.9.4	Higher dimensional texture	162
4.10	Textures in superfluid $^3\text{He-A}$	163
4.10.1	Superfluid $^3\text{He-A}$	163
4.10.2	Line defects and non-singular vortices in $^3\text{He-A}$	165
4.10.3	Shankar monopole in $^3\text{He-A}$	166
	Problems	167
5	Manifolds	169
5.1	Manifolds	169
5.1.1	Heuristic introduction	169
5.1.2	Definitions	171
5.1.3	Examples	173
5.2	The calculus on manifolds	178
5.2.1	Differentiable maps	179
5.2.2	Vectors	181
5.2.3	One-forms	184
5.2.4	Tensors	185
5.2.5	Tensor fields	185
5.2.6	Induced maps	186
5.2.7	Submanifolds	188
5.3	Flows and Lie derivatives	188
5.3.1	One-parameter group of transformations	190
5.3.2	Lie derivatives	191
5.4	Differential forms	196
5.4.1	Definitions	196
5.4.2	Exterior derivatives	198
5.4.3	Interior product and Lie derivative of forms	201
5.5	Integration of differential forms	204
5.5.1	Orientation	204
5.5.2	Integration of forms	205
5.6	Lie groups and Lie algebras	207
5.6.1	Lie groups	207
5.6.2	Lie algebras	209
5.6.3	The one-parameter subgroup	212
5.6.4	Frames and structure equation	215
5.7	The action of Lie groups on manifolds	216

5.7.1	Definitions	216
5.7.2	Orbits and isotropy groups	219
5.7.3	Induced vector fields	223
5.7.4	The adjoint representation	224
	Problems	224
6	de Rham Cohomology Groups	226
6.1	Stokes' theorem	226
6.1.1	Preliminary consideration	226
6.1.2	Stokes' theorem	228
6.2	de Rham cohomology groups	230
6.2.1	Definitions	230
6.2.2	Duality of $H_r(M)$ and $H^r(M)$; de Rham's theorem	233
6.3	Poincaré's lemma	235
6.4	Structure of de Rham cohomology groups	237
6.4.1	Poincaré duality	237
6.4.2	Cohomology rings	238
6.4.3	The Künneth formula	238
6.4.4	Pullback of de Rham cohomology groups	240
6.4.5	Homotopy and $H^1(M)$	240
7	Riemannian Geometry	244
7.1	Riemannian manifolds and pseudo-Riemannian manifolds	244
7.1.1	Metric tensors	244
7.1.2	Induced metric	246
7.2	Parallel transport, connection and covariant derivative	247
7.2.1	Heuristic introduction	247
7.2.2	Affine connections	249
7.2.3	Parallel transport and geodesics	250
7.2.4	The covariant derivative of tensor fields	251
7.2.5	The transformation properties of connection coefficients	252
7.2.6	The metric connection	253
7.3	Curvature and torsion	254
7.3.1	Definitions	254
7.3.2	Geometrical meaning of the Riemann tensor and the torsion tensor	256
7.3.3	The Ricci tensor and the scalar curvature	260
7.4	Levi-Civita connections	261
7.4.1	The fundamental theorem	261
7.4.2	The Levi-Civita connection in the classical geometry of surfaces	262
7.4.3	Geodesics	263
7.4.4	The normal coordinate system	266
7.4.5	Riemann curvature tensor with Levi-Civita connection	268
7.5	Holonomy	271

7.6	Isometries and conformal transformations	273
7.6.1	Isometries	273
7.6.2	Conformal transformations	274
7.7	Killing vector fields and conformal Killing vector fields	279
7.7.1	Killing vector fields	279
7.7.2	Conformal Killing vector fields	282
7.8	Non-coordinate bases	283
7.8.1	Definitions	283
7.8.2	Cartan's structure equations	284
7.8.3	The local frame	285
7.8.4	The Levi-Civita connection in a non-coordinate basis	287
7.9	Differential forms and Hodge theory	289
7.9.1	Invariant volume elements	289
7.9.2	Duality transformations (Hodge star)	290
7.9.3	Inner products of r -forms	291
7.9.4	Adjoins of exterior derivatives	293
7.9.5	The Laplacian, harmonic forms and the Hodge decomposition theorem	294
7.9.6	Harmonic forms and de Rham cohomology groups	296
7.10	Aspects of general relativity	297
7.10.1	Introduction to general relativity	297
7.10.2	Einstein–Hilbert action	298
7.10.3	Spinors in curved spacetime	300
7.11	Bosonic string theory	302
7.11.1	The string action	303
7.11.2	Symmetries of the Polyakov strings	305
	Problems	307
8	Complex Manifolds	308
8.1	Complex manifolds	308
8.1.1	Definitions	308
8.1.2	Examples	309
8.2	Calculus on complex manifolds	315
8.2.1	Holomorphic maps	315
8.2.2	Complexifications	316
8.2.3	Almost complex structure	317
8.3	Complex differential forms	320
8.3.1	Complexification of real differential forms	320
8.3.2	Differential forms on complex manifolds	321
8.3.3	Dolbeault operators	322
8.4	Hermitian manifolds and Hermitian differential geometry	324
8.4.1	The Hermitian metric	325
8.4.2	Kähler form	326
8.4.3	Covariant derivatives	327

8.4.4	Torsion and curvature	329
8.5	Kähler manifolds and Kähler differential geometry	330
8.5.1	Definitions	330
8.5.2	Kähler geometry	334
8.5.3	The holonomy group of Kähler manifolds	335
8.6	Harmonic forms and $\bar{\partial}$ -cohomology groups	336
8.6.1	The adjoint operators ∂^\dagger and $\bar{\partial}^\dagger$	337
8.6.2	Laplacians and the Hodge theorem	338
8.6.3	Laplacians on a Kähler manifold	339
8.6.4	The Hodge numbers of Kähler manifolds	339
8.7	Almost complex manifolds	341
8.7.1	Definitions	342
8.8	Orbifolds	344
8.8.1	One-dimensional examples	344
8.8.2	Three-dimensional examples	346
9	Fibre Bundles	348
9.1	Tangent bundles	348
9.2	Fibre bundles	350
9.2.1	Definitions	350
9.2.2	Reconstruction of fibre bundles	353
9.2.3	Bundle maps	354
9.2.4	Equivalent bundles	355
9.2.5	Pullback bundles	355
9.2.6	Homotopy axiom	357
9.3	Vector bundles	357
9.3.1	Definitions and examples	357
9.3.2	Frames	359
9.3.3	Cotangent bundles and dual bundles	360
9.3.4	Sections of vector bundles	361
9.3.5	The product bundle and Whitney sum bundle	361
9.3.6	Tensor product bundles	363
9.4	Principal bundles	363
9.4.1	Definitions	363
9.4.2	Associated bundles	370
9.4.3	Triviality of bundles	372
	Problems	372
10	Connections on Fibre Bundles	374
10.1	Connections on principal bundles	374
10.1.1	Definitions	375
10.1.2	The connection one-form	376
10.1.3	The local connection form and gauge potential	377
10.1.4	Horizontal lift and parallel transport	381
10.2	Holonomy	384

10.2.1	Definitions	384
10.3	Curvature	385
10.3.1	Covariant derivatives in principal bundles	385
10.3.2	Curvature	386
10.3.3	Geometrical meaning of the curvature and the Ambrose–Singer theorem	388
10.3.4	Local form of the curvature	389
10.3.5	The Bianchi identity	390
10.4	The covariant derivative on associated vector bundles	391
10.4.1	The covariant derivative on associated bundles	391
10.4.2	A local expression for the covariant derivative	393
10.4.3	Curvature rederived	396
10.4.4	A connection which preserves the inner product	396
10.4.5	Holomorphic vector bundles and Hermitian inner products	397
10.5	Gauge theories	399
10.5.1	$U(1)$ gauge theory	399
10.5.2	The Dirac magnetic monopole	400
10.5.3	The Aharonov–Bohm effect	402
10.5.4	Yang–Mills theory	404
10.5.5	Instantons	405
10.6	Berry’s phase	409
10.6.1	Derivation of Berry’s phase	410
10.6.2	Berry’s phase, Berry’s connection and Berry’s curvature	411
	Problems	418
11	Characteristic Classes	419
11.1	Invariant polynomials and the Chern–Weil homomorphism	419
11.1.1	Invariant polynomials	420
11.2	Chern classes	426
11.2.1	Definitions	426
11.2.2	Properties of Chern classes	428
11.2.3	Splitting principle	429
11.2.4	Universal bundles and classifying spaces	430
11.3	Chern characters	431
11.3.1	Definitions	431
11.3.2	Properties of the Chern characters	434
11.3.3	Todd classes	435
11.4	Pontrjagin and Euler classes	436
11.4.1	Pontrjagin classes	436
11.4.2	Euler classes	439
11.4.3	Hirzebruch L -polynomial and \hat{A} -genus	442
11.5	Chern–Simons forms	443
11.5.1	Definition	443

11.5.2	The Chern–Simons form of the Chern character	444
11.5.3	Cartan’s homotopy operator and applications	445
11.6	Stiefel–Whitney classes	448
11.6.1	Spin bundles	449
11.6.2	Čech cohomology groups	449
11.6.3	Stiefel–Whitney classes	450
12	Index Theorems	453
12.1	Elliptic operators and Fredholm operators	453
12.1.1	Elliptic operators	454
12.1.2	Fredholm operators	456
12.1.3	Elliptic complexes	457
12.2	The Atiyah–Singer index theorem	459
12.2.1	Statement of the theorem	459
12.3	The de Rham complex	460
12.4	The Dolbeault complex	462
12.4.1	The twisted Dolbeault complex and the Hirzebruch–Riemann–Roch theorem	463
12.5	The signature complex	464
12.5.1	The Hirzebruch signature	464
12.5.2	The signature complex and the Hirzebruch signature theorem	465
12.6	Spin complexes	467
12.6.1	Dirac operator	468
12.6.2	Twisted spin complexes	471
12.7	The heat kernel and generalized ζ -functions	472
12.7.1	The heat kernel and index theorem	472
12.7.2	Spectral ζ -functions	475
12.8	The Atiyah–Patodi–Singer index theorem	477
12.8.1	η -invariant and spectral flow	477
12.8.2	The Atiyah–Patodi–Singer (APS) index theorem	478
12.9	Supersymmetric quantum mechanics	481
12.9.1	Clifford algebra and fermions	481
12.9.2	Supersymmetric quantum mechanics in flat space	482
12.9.3	Supersymmetric quantum mechanics in a general manifold	485
12.10	Supersymmetric proof of index theorem	487
12.10.1	The index	487
12.10.2	Path integral and index theorem	490
	Problems	500

13 Anomalies in Gauge Field Theories	501
13.1 Introduction	501
13.2 Abelian anomalies	503
13.2.1 Fujikawa's method	503
13.3 Non-Abelian anomalies	508
13.4 The Wess–Zumino consistency conditions	512
13.4.1 The Becchi–Rouet–Stora operator and the Faddeev–Popov ghost	512
13.4.2 The BRS operator, FP ghost and moduli space	513
13.4.3 The Wess–Zumino conditions	515
13.4.4 Descent equations and solutions of WZ conditions	515
13.5 Abelian anomalies <i>versus</i> non-Abelian anomalies	518
13.5.1 m dimensions <i>versus</i> $m + 2$ dimensions	520
13.6 The parity anomaly in odd-dimensional spaces	523
13.6.1 The parity anomaly	524
13.6.2 The dimensional ladder: 4–3–2	525
14 Bosonic String Theory	528
14.1 Differential geometry on Riemann surfaces	528
14.1.1 Metric and complex structure	528
14.1.2 Vectors, forms and tensors	529
14.1.3 Covariant derivatives	531
14.1.4 The Riemann–Roch theorem	533
14.2 Quantum theory of bosonic strings	535
14.2.1 Vacuum amplitude of Polyakov strings	535
14.2.2 Measures of integration	538
14.2.3 Complex tensor calculus and string measure	550
14.2.4 Moduli spaces of Riemann surfaces	554
14.3 One-loop amplitudes	555
14.3.1 Moduli spaces, CKV, Beltrami and quadratic differentials	555
14.3.2 The evaluation of determinants	557
References	560
Index	565