Geostatistics of Extremes

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Joint with

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Spatial extremes

Many environmental extremal problems are spatial in nature:

- precipitation
- avalanches
- storms
- sea levels
- heatwaves

Increasingly seen as important in insurance, climate, engineering . . .

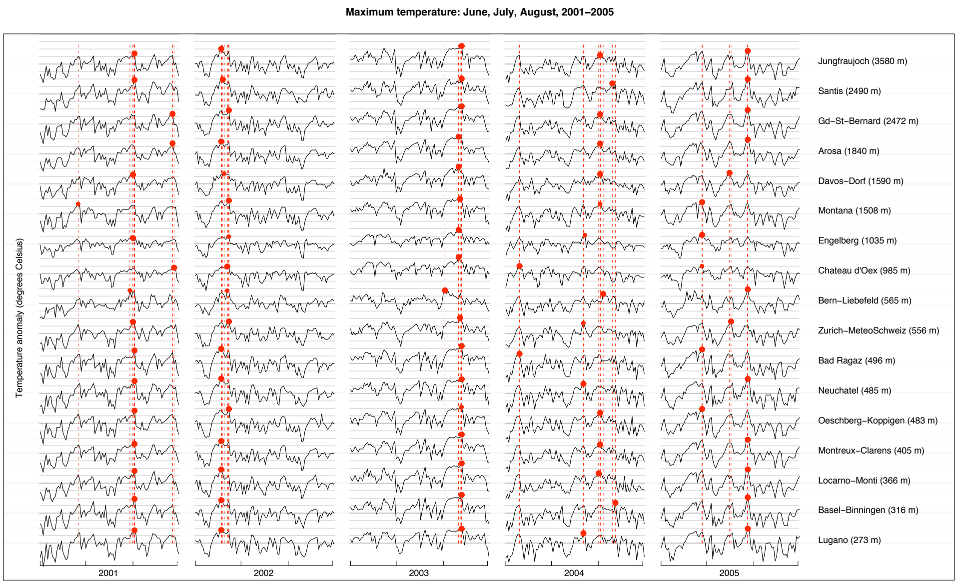
Possible goals:

- pointwise maps of quantiles (return levels)
- long-run prediction of events, for insurance/planning, e.g. floods
- short-range forecasting, e.g. avalanches, forest fires

Particularly important in mountain environments

http://extremes.epfl.ch EVA, Fort Collins, June 2009 – slide 3

Swiss summer temperatures 2001–2005



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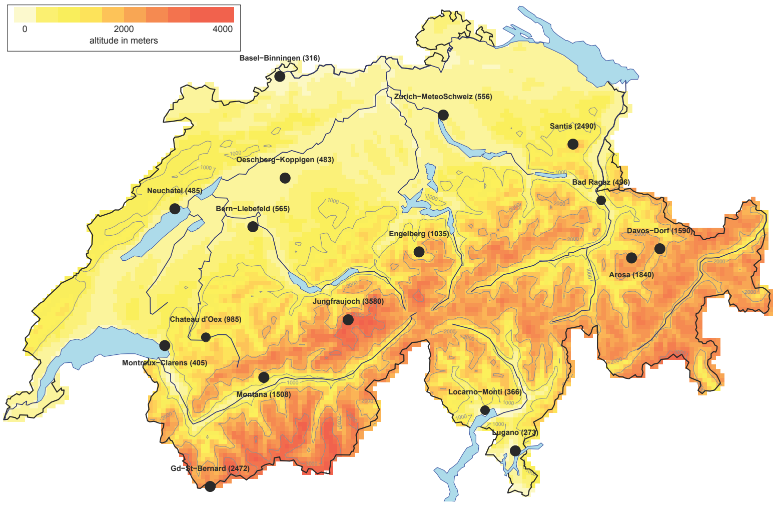
Sites



Problem formulation

- ☐ Seek to model extremes of process $Y(x)$ over spatial domain $\mathcal{X}$
- ☐ Data available are simultaneous time series at
  - sites $x_d \in \mathcal{X}_D = \{x_1, \ldots, x_D\}$ within $\mathcal{X}$
  - times $\mathcal{T} = \{t_1, \ldots, t_n\}$, for $d \in \{1, \ldots, D\}$
- ☐ Aim to compute distributions of quantities such as

$$R = \int_{\mathcal{X}} r(x) I\left\{Y(x) \geq y_{\text{danger}}\right\} dx$$

  where $r(x)$ is population at risk if $Y(x)$ exceeds some level y_{danger}, and the indicator $I(\cdot)$ shows where and when this happens.
- ☐ Example: population of elderly at risk from high temperatures in western Switzerland in summer 2020.

We want to model the **joint behaviour** of maxima within $\mathcal{X}$, not just to tie together the marginal behaviour at different sites to produce contours of high quantiles.

Why do we need new methods?

- ☐ Geostatistics is well-developed and widely used for modelling spatial data, but it is mostly based on multivariate normal distributions, inappropriate for modelling tail behaviour
- ☐ The generalised extreme-value distribution (GEV) is used to model scalar extremes because of its **max-stability**, which gives a mathematical basis for extrapolation beyond the range of the data
- ☐ The natural models for spatial extremes are **max-stable processes**, which extend the GEV to spatial data, but
  - there are few models for max-stable processes, and even fewer applications
  - standard inferential tools (e.g. likelihood) can't be used
- ☐ Will discuss approaches to overcoming these difficulties—first steps only, applied to annual maximum series at different sites.

Geostatistics slide 8

Spatial extremes

Three main approaches:

- ☐ Gaussian anamorphosis
- ☐ latent processes
- ☐ max-stable processes

First give cartoon view of geostatistics.

Cartoon geostatistics

- ☐ Statistics of spatially-defined variables
- ☐ Mostly a multivariate normal theory: suppose that the variable of interest (annual maximum temperature) has a joint normal distribution, and that its values at different sites have some correlation function, depending on distance etc.
- ☐ Given data, we
  - remove (space-time) trends in mean and variance of data
  - transform residuals to standard normal margins
  - fit 'suitable' spatial/space-time correlation functions
  - make inferences using weighted least squares (kriging), likelihood, or Bayes (McMC)
  - make predictions using the fitted correlation function, then add back the estimated trends to obtain a map of predictions.
- ☐ See Diggle and Ribeiro (2007), Cressie (1993), lots of Bayesians

Gaussian anamorphosis

- ☐ Remove spatial and temporal trend by fitting GEV with annual maxima
- ☐ use this fit to transform maxima to Gaussianity
- ☐ apply standard geostatistics
- ☐ backtransformation to original data scale

Properties:

- $+$ easy using standard software
- $+$ Gaussianity not essential (could be uniform, or t_ν)
- $-$ distribution of joint extremes may be badly modelled because of properties of Gaussian model
- $+/-$ equivalent to use of copulas—see poster by **Simone Padoan**, who applies this approach to rainfall at 51 sites, using both standard and extremal copulas

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Latent processes

- ☐ Conditional on latent process $S(x)$, observations $Y(x)$, for $x \in \mathcal{X}$ follow an extremal distribution
- ☐ Examples:

$$Y(x) \quad | \quad S(x) = (\eta(x), \tau(x), \xi(x)) \overset{\text{ind}}{\sim} \text{GEV}\{y; S(x)\}, \quad S(x) \sim N_3\{\mu(x), \Omega(x)\}$$

$$Y(x) \quad | \quad S(x) = (\sigma(x), \xi(x)) \overset{\text{ind}}{\sim} \text{GPD}\{y; S(x)\}, \quad S(x) \sim N_2\{\mu(x), \Omega(x)\}$$

See Casson and Coles (1999, Extremes); Cooley et al. (2007, JASA); Fawcett and Walshaw (2006, Applied Statistics); Sang and Gelfand (2009, J. Ecol. Env. Statist.), etc.

- ☐ **Properties**:
  - $+$ computationally feasible for large-scale problems using standard simulation techniques (Metropolis–Hastings algorithm, Gibbs sampling, ...);
  - $+$ possibility of estimating quantiles spatially
  - $-$ all extremal dependencies are incorporated through $S(x)$
  - $-$ marginal distributions are not extremal
  - $-$ episodic modelling/simulation difficult

  Talk by **Huiyan Sang** in this session

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Max-stable processes

- Consider joint distributions of maxima at sites $\mathcal{D} = \{x_1, \ldots, x_D\} \subset \mathcal{X}$
- Individually these maxima follow GEV distributions, and we use the marginal distributions to transform each to be unit Fréchet,

$$\Pr\left(\left[1+\xi\left(\frac{Y-\eta}{\tau}\right)\right]_+^{1/\xi} \le z\right) = \Pr(Z \le z) = \exp(-1/z), \quad z > 0,$$

  which is a special case of the GEV.
- Then we can write

$$\Pr(Z_1 \le z_1, \ldots, Z_D \le z_D) = \exp\left\{-V(z_1, \ldots, z_D)\right\},$$

  where the function V measures dependence among the different sites:
  - independence implies $V(z_1, \ldots, z_D) = 1/z_1 + \cdots + 1/z_D$
  - total dependence implies $V(z_1, \ldots, z_D) = \max(1/z_1, \cdots, 1/z_D)$
- Max-stability implies that for $k \in \mathbb{N}$,

$$\Pr(Z_1 \le z_1, \ldots, Z_D \le z_D)^k = \exp\left\{-kV(z_1, \ldots, z_D)\right\} = \exp\left\{-V(z_1/k, \ldots, z_D/k)\right\}.$$

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Exponent measure function

- The **exponent measure function** V
  - is homogeneous of order -1
  - satisfies $V(+\infty, \ldots, +\infty, z_d, +\infty, \ldots, +\infty) = 1/z_d$
- The **extremal coefficient** $\theta_{\mathcal{D}} = V(1, \ldots, 1) \in [1, D]$ summarises the degree of dependence among the maxima within $\mathcal{D}$
  - $\theta = 1$ implies that they are totally dependent
  - $\theta = D$ implies that they are independent
- Two problems:
  - need exponent measures V that are useful for spatial settings;
  - once we have them, we need to be able to fit them to data;
- Likelihood inference infeasible: to compute the joint density at $\{x_1, \ldots, X_D\}$ we must differentiate e^{-V} with respect to $z_1, \ldots, z_D$, leading to combinatorial explosion:

$$-V_1 e^{-V}, \quad (V_1V_2 - V_{12})e^{-V}, \quad (-V_1V_2V_3 + V_{12}V_3[3] - V_{123})e^{-V}, \quad \ldots$$

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Models: Spectral representation

- Two important classes of **spectral representations** of these processes, due to de Haan (1984), Schlather (2002)
- General form: Let $W(x)$ be a non-negative stationary process on $\mathbb{R}^p$ with $\mathrm{E}\{W(x)\} = 1$ at each x, and let Π be a Poisson process on $\mathbb{R}_+$ with intensity ds/s^2. If the $W_s(x)$ are independent copies of $W(x)$, for each $s \in \mathbb{R}_+$, then

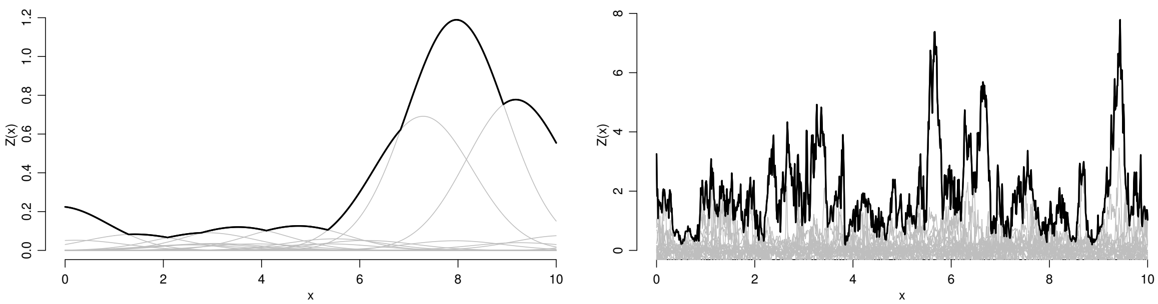
$$Z(x) = \max_{s \in \Pi} sW_s(x), \quad x \in \mathbb{R}^p,$$

  is a stationary max-stable random process with unit Frechét margins.
- If $z(x)$ is a well-behaved function on $\mathcal{X}$, then a point process argument yields that

$$\Pr\{Z(x) \le z(x), x \in \mathcal{X}\} = \exp\left(-\mathrm{E}\left[\sup_{x \in \mathcal{X}} \left\{\frac{W(x)}{z(x)}\right\}\right]\right).$$

Models: Smith and Schlather

- de Haan (1984), Smith (1990): take $W(x) = g(x - T)$, where T is chosen randomly on $\mathcal{X}$ and g is a density function
- Interpretation: g is the shape of a storm, and T is its (random) centre, we observe the maximum of a number of random storms
- Schlather (2002): take $W(x)$ to be positive random process, such as $\sqrt{2\pi}\max\{\varepsilon(x), 0\}$, where $\varepsilon(x)$ is stationary Gaussian process with unit variance and correlation function $\rho(x)$.
- Interpretation: we observe the pointwise maximum of random processes $W_s(x)$
- Schlather (2002): as above, but restrict $W(x)$ to a random set $\mathcal{B}$.



Methods: Computation of V

- Use expression
$$\Pr\{Z(x) \le z(x), x \in \mathcal{X}\} = \exp\left(-\mathrm{E}\left[\sup_{x\in\mathcal{X}}\left\{\frac{W(x)}{z(x)}\right\}\right]\right).$$
to compute V for case $\#\mathcal{X} = D = 2$
- Schlather model: $\varepsilon_s(\cdot)$ stationary isotropic Gaussian processes with correlation $\rho(h) = \rho(x_1 - x_2)$, then
$$V(z_1, z_2) = \tfrac{1}{2}\left(\frac{1}{z_1} + \frac{1}{z_2}\right)\left(1 + \left[1 - 2\frac{\{\rho(h)+1\}z_1 z_2}{(z_1+z_2)^2}\right]^{1/2}\right)$$
- Corresponding extremal coefficient can only represent positive dependence—but most likely in practice
- Modified version can give independent extremes:
$$V(z_1, z_2) = \left(\frac{1}{z_1} + \frac{1}{z_2}\right)\left\{1 - \frac{\alpha(h)}{2}\left(1 - \left[1 - 2\frac{\{\rho(h)+1\}z_1 z_2}{(z_1+z_2)^2}\right]^{1/2}\right)\right\},$$
where $\alpha(h) = \mathrm{E}|\mathcal{B} \cap (h + \mathcal{B})|/\mathrm{E}|\mathcal{B}|$ lies in the unit interval.
- The extremal coefficient for this model can take any value in the interval $[1, 2]$.

Methods: Fitting and diagnostics

- For **fitting**
  - use bivariate marginal densities to compute **pairwise log likelihood**
$$\sum_{i>j}^{D} \log f(z_i, z_j; \theta),$$
constructed from all distinct disjoint pairs of observations.
  - If θ is identifiable from the pairwise marginal densities, then under mild regularity conditions the maximum pairwise likelihood estimator $\tilde{\theta}$ is consistent and asymptotically normal, and inferences can be performed (a bit painfully).
- For **diagnostics**
  - groupwise maxima
$$Z_{\mathcal{A}} = \max_{i\in\mathcal{A}} Z_i \sim \mathrm{Frechet}(\theta_{\mathcal{A}})$$
and can compare observed values with values simulated from a model
  - construct simulation envelopes to assess variability

Swiss summer temperature data

- ☐ Annual maximum temperature data at $D = 14$ Swiss sites, 1961–2006 (exclude Jungfraujoch, Lugano, and Locarno-Monti)
- ☐ **Marginal model**: simultaneously fit GEV to maxima $Y(x_d, t_j)$ with location

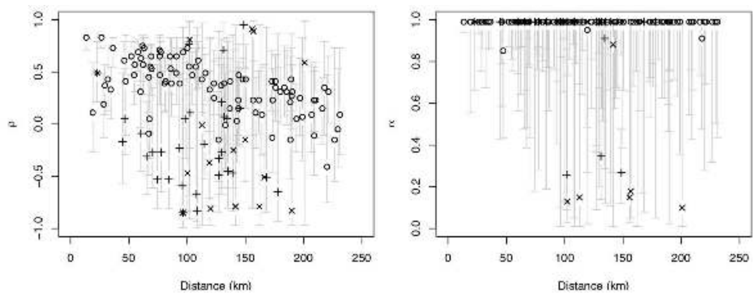
$$\eta_{d,t_j} \sim \text{alt}(x) + \text{alt}(x)^2 + \{\text{lon}(x) + \text{lat}(x)\}^2 + \text{time}, \quad \tau, \quad \xi, \quad d = 1, \ldots, 14, j = 1, \ldots, 46,$$

  where $\text{alt}(x)$, $\text{lat}(x)$, $\text{lon}(x)$ are altitude, latitude and longitude at site x, and time is time
- ☐ **Exploratory analysis**: estimate 'correlations' $\widehat{\rho}_{ij}$ and probabilities $\widehat{\alpha}_{ij}$ separately for each pair of sites
- ☐ **Spatial model**: use pairwise likelihood to fit stationary isotropic covariance function

$$\rho(h) = \gamma_1 \exp\{-(h/\gamma_2)^{\gamma_3}\}, \quad 0 \le \gamma_1 \le 1, \gamma_2, \gamma_3 > 0, \quad h > 0,$$

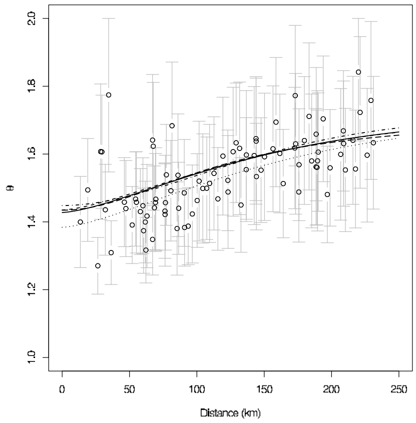
  for sites h apart, giving $\tilde{\gamma}_1 = 0.71\ (0.2)$, $\tilde{\gamma}_2 = 200\ (100)$km, $\gamma_3 = 1.5$;
- ☐ **Fit** using pairwise likelihood, but allow for timing (Stephenson & Tawn, 2005, Bka)
- ☐ **Risk analysis**: simulate max-stable random fields $Z^*(x)$ from fitted model, then transform back to 'real' scale
- ☐ **Generalisations**: no improvement with random set model or anisotropic covariances

Estimates of ρ and α



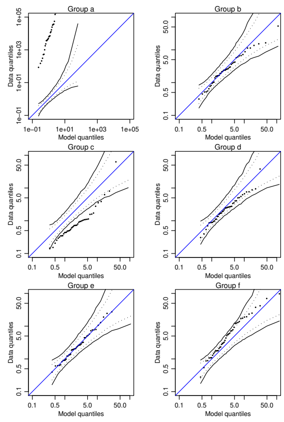
Maximum likelihood estimates of parameters ρ (left) and α (right) for 46 years of maximum temperatures observed at all distinct pairs of 17 sites in Switzerland, transformed to the unit Fréchet scale. $+$ denotes pairs with a site in the Tessin, $\times$ with a site in the Jungfraujoch.

Dependence of θ on distance



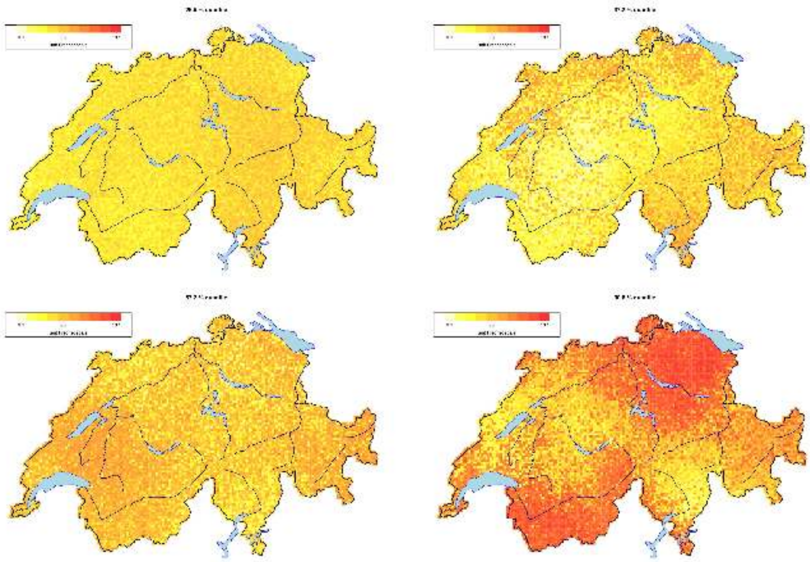
Fitted extremal coefficients for the pairs of sites, excluding Jungfraujoch, Locarno-Monti and Lugano, with Schlather–Tawn pairwise estimates and their standard errors, as a function of distance. Shown are curves for the exponential covariance with shape $\kappa = 1.5$; the Whittle–Matérn covariance with shape $\kappa = 1.5$ and the Cauchy covariance functions.

Groupwise diagnostics

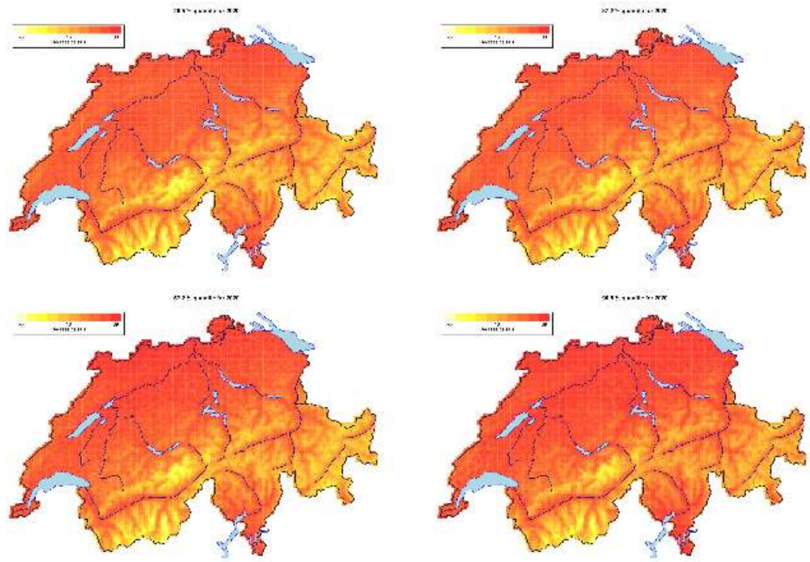


Comparison of groupwise annual maxima with data simulated from the fitted model. In each panel the outer band is a 95% overall confidence band and the inner one a 95% pointwise confidence band. The groups of sites are: (a) Jungfraujoch; (b) Engelberg, Grand-St-Bernard, Montana; (c) Locaro-Monti, Lugano; (d) Bern-Liebefeld, Chateau d'Oex, Montreux-Clarens, Neuchâtel; (e) Basel-Binningen, Oeschberg-Koppingen, Zurich-MeteoSchweiz; (f) Arosa, Bad-Ragaz, Davos-Dorf, Santis.

Simulated random fields, Fréchet scale



Simulated summers for 2020



Discussion

- Approach based on max-stable processes
  - + properly accounts for mathematical properties of multivariate extremes
  - + can incorporate geostatistics in a flexible way
  - + can be used for (correct!) episodic prediction through simulation from the fitted model
  - – can be awkward to build in spatial variation in marginal parameters
  - – is less standard to fit and to use
- Next on the agenda:
  - – R library **SpatialExtremes** for fitting such models—**Mathieu Ribatet** workshop yesterday
  - – more difficult applications (snow—**Juliette Blanchet** talk)
  - – non-stationary max-stable processes
  - – spatio-temporal models
  - – peaks over threshold modelling

"... Red sky in the morning, potential extreme weather event"

Thank you!

Risk, Rare Events, and Extremes

- Six-month programme of research on modelling of extremes for complex problems, with focus on spatial and spatio-temporal aspects
- Aim to bring together climate and environmental scientists, and statisticians interested in modelling extremes (visitors, . . .)
- Bernoulli Interdisciplinary Centre, EPF Lausanne, July–December 2009
- Workshops
  - July 13–17: Spatial extremes and applications
  - September 14–18: High-dimensional extremes
  - November 9–11: Spatio-temporal extremes and applications
  - November 12–13: Final conference
- **http://extremes.epfl.ch**