Gerber-Shiu distribution at Parisian ruin for Lévy insurance risk processes

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Abstract

Inspired by works of Landriault et al. [10, 11], we study the Gerber-Shiu distribution at Parisian ruin with exponential implementation delays for a spectrally negative Lévy insurance risk process. To be more specific, we study the so-called Gerber-Shiu distribution for a ruin model where at each time the surplus process goes negative, an independent exponential clock with rate $q > 0$ is started. If the clock rings before the surplus becomes positive again then the insurance company is ruined. Our methodology uses excursion theory for spectrally negative Lévy processes and relies on the theory of so-called scale functions. In particular, our results extend recent results of Landriault et al. [10, 11].

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1 Introduction and main results

Originally motivated by pricing American claims, Gerber and Shiu [7, 8] introduced in risk theory a function that jointly penalizes the present value of the time of ruin, the surplus before ruin and the deficit after ruin for Cramér-Lundberg-type processes. Since then this expected discounted penalty function, by now known as the Gerber-Shiu function, has been deeply studied. Recently, Biffis and Kyprianou [2] characterized a generalized version of this function in the setting of processes with stationary and independent increments with no positive jumps, also known as

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spectrally negative Lévy processes, using scale functions. In the current actuarial setting, we refer to the latter class of processes as Lévy insurance risk processes.

In the traditional ruin theory literature, if the surplus becomes negative, the company is ruined and has to go out of business. Here, we distinguish between *being ruin* and *going out of business*, where the probability of *going out of business* is a function of the level of negative surplus. The idea of this notion of *going out of business* comes from the observation that in some industries, companies can continue doing business even though they are technically ruined (see [10] for more motivation). In this paper, our definition of *going out of business* is related to so-called Parisian ruin. The idea of this type of actuarial ruin has been introduced by A. Dassios and S. Wu [6], where they consider the application of an implementation delay in the recognition of an insurer's capital insufficiency. More precisely, they assume that ruin occurs if the excursion below the critical threshold level is longer than a deterministic time. It is worth pointing out that this definition of ruin is referred to as Parisian ruin due to its ties with Parisian options (see Chesney et al. [3]).

In [6], the analysis of the probability of Parisian ruin is done in the context of the classical Cramér-Lundberg model. More recently, Landriault et al. [10, 11] and Loeffen et al. [12] considered the idea of Parisian ruin with respectively a stochastic implementation delay and a deterministic implementation delay, but in the more general setup of Lévy insurance risk models. In [10], the authors assume that the deterministic delay is replaced by a stochastic grace period with a pre-specified distribution, but they restrict themselves to the study of a Lévy insurance risk process with paths of bounded variation; explicit results are obtained in the case the delay is exponentially distributed. The model with deterministic delays has also been studied in the Lévy setup by Czarna and Palmowski [5] and by Czarna [4].

In this paper, we study the Gerber-Shiu distribution at Parisian ruin for general Lévy insurance risk processes, when the implementation delay is exponentially distributed. Since the Lévy insurance risk process does not jump at the time when Parisian ruin occurs, the Gerber-Shiu function that we present here only considers the discounted value of the surplus at ruin. Our results extend those of Landriault et al. [10], in the exponential case, by simultaneously considering more general ruin-related quantities and Lévy insurance risk processes of unbounded and bounded variation. Our approach is based on a *heuristic* idea presented in [11] and which consists in *marking* the excursions away from zero of the underlying surplus process. We will fill this gap and provide a rigorous definition of the time of Parisian ruin. Our main contribution is an explicit and compact expression, expressed in terms of the scale functions of the process, for the Gerber-Shiu distribution at Parisian ruin. From our results, we easily deduce the probability of Parisian ruin originally obtained by Landriault et al. [10, 11].

The rest of the paper is organized as follows. In the remainder of Section 1, we introduce Lévy insurance risk processes and their associated scale functions. We also provide some interesting fluctuation identities that will be useful for the sequel. We also introduce, formally speaking, the notion of Parisian ruin in terms of the excursions away from 0 of the Lévy insurance risk process and finally we provide the main results of this paper. Section 2 is devoted to the proofs of the main results. Finally in Section 3, we recover the results that appear in Landriault et al. [10, 11] and provide the arguments to include the running infimum or/and the running supremum of the surplus in the Gerber-Shiu function.

1.1 Lévy insurance risk processes

In what follows, we assume that $X = (X_t, t \geq 0)$ is a spectrally negative Lévy process with no monotone paths (i.e. we exclude the case of the negative of a subordinator) defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $x \in \mathbb{R}$ denote by $\mathbb{P}_x$ the law of X when it is started at x and write for convenience $\mathbb{P}$ in place of $\mathbb{P}_0$. Accordingly, we shall write $\mathbb{E}_x$ and $\mathbb{E}$ for the associated expectation operators. It is well known that the Laplace exponent $\psi : [0, \infty) \to \mathbb{R}$ of X, defined by

$$\psi(\lambda) := \log \mathbb{E}\left[e^{\lambda X_1}\right], \quad \lambda \geq 0,$$

is given by the well-known Lévy-Khintchine formula

$$\psi(\lambda) = \gamma\lambda + \frac{\sigma^2}{2}\lambda^2 - \int_{(0,\infty)} \left(1 - e^{-\lambda x} - \lambda x \mathbf{1}_{\{x<1\}}\right)\Pi(\mathrm{d}x),$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty,$$

which is called the Lévy measure of X. Even though X only has negative jumps, for convenience we choose the Lévy measure to have only mass on the positive instead of the negative half line.

It is also known that X has paths of bounded variation if and only if

$$\sigma = 0 \quad \text{and} \quad \int_{(0,1)} x\,\Pi(\mathrm{d}x) < \infty.$$

In this case X can be written as $X_t = ct - S_t$, $t \geq 0$, where $c = \gamma + \int_{(0,1)} x\Pi(\mathrm{d}x)$ and $(S_t, t \geq 0)$ is a driftless subordinator. Note that necessarily $c > 0$, since we have ruled out the case that X has monotone paths. In this case its Laplace exponent is given by

$$\psi(\lambda) = \log \mathbb{E}\left[e^{\lambda X_1}\right] = c\lambda - \int_{(1,\infty)} \left(1 - e^{-\lambda x}\right)\Pi(\mathrm{d}x).$$

The reader is referred to Bertoin [1] and Kyprianou [9] for a complete introduction to the theory of Lévy processes.

A key element of the forthcoming analysis relies on the theory of so-called scale functions for spectrally negative Lévy processes. We therefore devote some time in this section reminding the reader of some fundamental properties of scale functions. For each $q \geq 0$, define $W^{(q)} : \mathbb{R} \to [0, \infty)$, such that $W^{(q)}(x) = 0$ for all $x < 0$ and on $[0, \infty)$ is the unique continuous function with Laplace transform

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\lambda) - q}, \quad \lambda > \Phi(q),$$

where $\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$ which is well defined and finite for all $q \geq 0$, since ψ is a strictly convex function satisfying $\psi(0) = 0$ and $\psi(\infty) = \infty$. The initial value of $W^{(q)}$ is known to be

$$W^{(q)}(0) = \begin{cases} 1/c & \text{when } \sigma = 0 \text{ and } \int_{(0,1)} x\,\Pi(\mathrm{d}x) < \infty, \\ 0 & \text{otherwise}, \end{cases}$$

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where we used the following definition: $W^{(q)}(0) = \lim_{x \downarrow 0} W^{(q)}(x)$. For convenience, we write W instead of $W^{(0)}$. Associated to the functions $W^{(q)}$ are the functions $Z^{(q)} : \mathbb{R} \to [1, \infty)$ defined by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) \mathrm{d}y, \qquad q \geq 0.$$

Together, the functions $W^{(q)}$ and $Z^{(q)}$ are collectively known as q-scale functions and predominantly appear in almost all fluctuation identities for spectrally negative Lévy processes.

The theorem below is a collection of known fluctuation identities which will be used along this work. See, for example, Chapter 8 of [9] for proofs and the origin of these identities.

Theorem 1. *Let X be a spectrally negative Lévy process and let*

$$\tau_a^+ = \inf\{t > 0 : X_t > a\} \qquad \text{and} \qquad \tau_0^- = \inf\{t > 0 : X_t < 0\}.$$

(i) For $q \geq 0$ and $x \leq a$

$$\mathbb{E}_x\left[e^{-q\tau_a^+} 1_{\{\tau_0^- > \tau_a^+\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}. \tag{1.1}$$

(ii) For any $a > 0, x, y \in [0, a], q \geq 0$

$$\int_0^\infty e^{-qt} \mathbb{P}_x\left(X_t \in \mathrm{d}y, t < \tau_a^+ \wedge \tau_0^-\right) \mathrm{d}t = \left\{\frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y)\right\} \mathrm{d}y. \tag{1.2}$$

Finally, we recall the following two useful identities taken from [13]: for $p, q \geq 0$ and $x \in \mathbb{R}$, we have

$$(q-p) \int_0^x W^{(p)}(x-y)W^{(q)}(y) \mathrm{d}y = W^{(q)}(x) - W^{(p)}(x) \tag{1.3}$$

and, for $p, q \geq 0$ and $y \leq a \leq x \leq b$, we have

$$\mathbb{E}_x\left[e^{-p\tau_a^-} W^{(q)}(X_{\tau_a^-} - y)\mathbf{1}_{\{\tau_a^- < \tau_b^+\}}\right] = W^{(q)}(x-y) - (q-p)\int_a^x W^{(p)}(x-z)W^{(q)}(z-y)\mathrm{d}z$$

$$- \frac{W^{(p)}(x-a)}{W^{(p)}(b-a)}\left(W^{(q)}(b-y) - (q-p)\int_a^b W^{(p)}(b-z)W^{(q)}(z-y)\mathrm{d}z\right). \tag{1.4}$$

1.2 Parisian ruin with exponential implementation delays

In what follows, we assume that the underlying Lévy insurance risk process X satisfies the *net profit condition*, i.e.

$$\mathbb{E}[X_1] = \psi'(0+) > 0. \tag{1.5}$$

We first give a descriptive definition of the time of Parisian ruin, here denoted by τ_q, using Itô's excursion theory for spectrally negative Lévy processes away from zero. In order to do so, we *mark* the Poisson point process of excursions away from zero with independent copies of a generic exponential random variable $\mathbf{e}_q$ with parameter $q > 0$. We will refer to them as implementation clocks. If the length of the negative part of a given excursion away from 0 is less than its associated implementation clock, then such excursion is neglected as far as ruin is concerned. More precisely,

we assume that ruin occurs at the first time that an implementation clock rings before the end of its corresponding excursion. It is worth pointing out that the time to ruin τ_q is properly defined when there are countably many drops below 0, this is the case when the Lévy insurance risk processes has paths of bounded variation. The probability to ruin in the latter case was studied by Landriault et al. [10].

In this paper, we are interested in the case when the Lévy insurance risk processes X has paths of unbounded variation. Our method uses a limiting argument which is motivated by the work of Loeffen et al. [12]. Let $\varepsilon > 0$ and consider the path of X up to the next time that the process returns to 0 after reaching the level $-\varepsilon$, i.e.

$$(X_t, 0 \le t \le \tau_0^{+,\varepsilon}) \qquad \text{where} \qquad \tau_0^{+,\varepsilon} = \inf\{t > \tau_{-\varepsilon}^- : X_t > 0\}.$$

Let $\tau_{-\varepsilon}^{-,1} := \tau_{-\varepsilon}^-$ and $\tau_0^{+,1} := \tau_0^{+,\varepsilon}$. Recursively, we define two sequences of stopping times $(\tau_{-\varepsilon}^{-,k})_{k \ge 1}$ and $(\tau_0^{+,k})_{k \ge 1}$ as follows: for $k \ge 2$,

$$\tau_{-\varepsilon}^{-,k} = \inf\{t > \tau_0^{+,k-1} : X_t < -\varepsilon\} \qquad \text{and} \qquad \tau_0^{+,k} = \inf\{t > \tau_{-\varepsilon}^{-,k} : X_t > 0\}.$$

Note that $X_{\tau_0^{+,k}} = 0$ for each $k \ge 1$. Hence, from the Markov property, we observe that $Y^{(k)} = (X_t, \tau_0^{+,k-1} \le t \le \tau_0^{+,k})$, for $k \ge 2$, are $\mathbb{P}_0$-independent copies of $(X_t, 0 \le t \le \tau_0^{+,\varepsilon})$. We call $(Y^{(k)})_{k \ge 1}$ the ε-excursions of X away from 0. Observe that under the net profit condition (see Equation (1.5), we necessarily have a finite number of these ε-excursions, almost surely. We also observe that the limiting case, i.e. when ε goes to 0, corresponds to the usual excursion of X away from 0. To avoid confusions, we call the limiting case a 0-excursion.

It is important to note that each ε-excursion ends with a 0-excursion that reaches the level $-\varepsilon$. For each $k \ge 1$, we denote by $\mathbf{e}_q^k$ the implementation clock of the last 0-excursion of $Y^{(k)}$. Once the last 0-excursion of $Y^{(k)}$ reaches the level $-\varepsilon$, we start the implementation clock, if the duration of the clock is greater than the time it takes to the last 0-excursion to reach the level zero this ε-excursion is neglected as far as ruin is concerned. On the other hand if the duration of the clock is less than such time we assume that the approximated ruin event occurs at the moment the clock rings. More precisely, the approximated ruin time τ_q^ε is defined as follows (i.e. as in [10])

$$\tau_q^\varepsilon := \tau_{-\varepsilon}^{-,k_q^\varepsilon} + \mathbf{e}_q^{k_q^\varepsilon},$$

where

$$k_q^\varepsilon = \inf\{k \ge 1 : \tau_{-\varepsilon}^{-,k} + \mathbf{e}_q^k < \tau_0^{+,k}\}.$$

Let $e^\dagger = (e_t^\dagger, t \ge 0)$ be a realization of the ε-excursion away from zero in which ruin occurs. We denote by $\zeta^\dagger$ and $h^\dagger$ for the length of the negative part of $e^\dagger$ and its depth, respectively. Since ruin occurs along this 0-excursion, note that we necessarily have

$$\zeta^\dagger > \mathbf{e}_q^\dagger > 0 \quad \text{and} \quad h^\dagger > 0, \tag{1.6}$$

where $\mathbf{e}_q^\dagger$ denotes its corresponding implementation clock. We also denote by $l^\dagger$ the left-end point of the 0-excursion $e^\dagger$, i.e. $l^\dagger = \sup\{0 < s \le \tau_q : X_t = 0\}$, and we consider

$$\rho_{-\varepsilon}^\dagger = \inf\{t \ge 0 : e_t^\dagger < -\varepsilon\}$$

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for $\varepsilon \geq 0$, the first time at which the excursion $e^\dagger$ is below the level $-\varepsilon$.

We first remark that for any $\varepsilon > 0$, $\tau_q^\varepsilon \geq \tau_q$. Indeed, if we denote by $e^{k_q^\varepsilon} = (e_t^{k_q^\varepsilon}, t \geq 0)$ the excursion away from zero in which the approximated ruin event occurs, and by $\zeta^{k_q^\varepsilon}$ and $l^{k_q^\varepsilon}$ for the length of the negative part of $e^{k_q^\varepsilon}$ and its left-end point, respectively. Then by definition of τ_q^ε, we have

$$\zeta^{k_q^\varepsilon} \geq \tau_0^{+,k_q^\varepsilon} - \tau_{-\varepsilon}^{-,k_q^\varepsilon} > \mathbf{e}_q^{k_q^\varepsilon}.$$

Therefore recalling that the approximated ruin event occurs at the first excursion away from zero that reaches level $-\varepsilon$ and whose length is greater than its mark then

$$\tau_q^\varepsilon = \tau_{-\varepsilon}^{-,k_q^\varepsilon} + \mathbf{e}_q^{k_q^\varepsilon} \geq l^{k_q^\varepsilon} + \rho_0^{k_q^\varepsilon} + \mathbf{e}_q^{k_q^\varepsilon} \geq \tau_q,$$

where $\rho_0^{k_q^\varepsilon} = \inf\{t \geq 0 : X_{t+l^{k_q^\varepsilon}} < 0\}$.

It is important to note that (1.6) implies that we can find $\varepsilon' > 0$ small enough such that

$$h^\dagger > \varepsilon' \quad \text{and} \quad \zeta^\dagger - \mathbf{e}_q^\dagger > \rho_{-\varepsilon'}^\dagger - \rho_0^\dagger,$$

which implies that $\zeta^\dagger - (\rho_{-\varepsilon'}^\dagger - \rho_0^\dagger) > \mathbf{e}_q^\dagger$. This in addition to the fact that $\tau_q^{\varepsilon'} \geq \tau_q$, implies that it must hold that the excursions $e^\dagger$ and $e^{k_q^{\varepsilon'}}$ are the same and therefore the approximating ruin time is given by $\tau_q^{\varepsilon'} = l^\dagger + \rho_{-\varepsilon}^\dagger + \mathbf{e}_q^\dagger$, while the Parisian ruin time satisfies $\tau_q = l^\dagger + \rho_0^\dagger + \mathbf{e}_q$. This implies that

$$\tau_q^{\varepsilon'} - \tau_q = \rho_{-\varepsilon}^\dagger - \rho_0^\dagger,$$

which converge to zero $\mathbb{P}$-a.s., as ε' goes to zero. In other words,

$$\tau_q^\varepsilon \xrightarrow[\varepsilon\downarrow 0]{} \tau_q, \quad \mathbb{P}\text{-a.s.} \tag{1.7}$$

1.3 Main results

In this section, we are interested in computing different Gerber-Shiu functions for a Lévy insurance risk process subject to Parisian ruin, as defined in the previous section. To do so, we first identify the Gerber-Shiu distribution. It is important to point out that in all the results in this subsection the net profit condition is not necessary.

Before going any further, let's define two auxiliary functions. First, for $p \geq 0$ and $q \in \mathbb{R}$ such that $p + q \geq 0$ and for $x \in \mathbb{R}$, define as in [13] the function

$$\mathcal{H}^{(p,q)}(x) = e^{\Phi(p)x}\left(1 + q\int_0^x e^{-\Phi(p)y}W^{(p+q)}(y)\mathrm{d}y\right).$$

We further introduce, for $\theta, q \geq 0$, $x > 0$ and $y \in [-x, \infty)$, the function

$$g(\theta, q, x, y) = W^{(\theta+q)}(x+y) - q\int_0^x W^{(\theta)}(x-z)W^{(\theta+q)}(z+y)\mathrm{d}z. \tag{1.8}$$

Note that g is of the same form as $\mathcal{W}_a^{(p,q)}$ in [13].

Here is the main result of this paper.

Theorem 2. *For $\theta, a, b \geq 0$, $x \in [-a, b)$ and $y \in [-a, 0]$, we have*

$$\mathbb{E}_x\left[e^{-\theta \tau_q}, X_{\tau_q} \in dy, \tau_q < \tau_b^+ \wedge \tau_{-a}^-\right] = q\left[\frac{g(\theta, q, x, a)}{g(\theta, q, b, a)} g(\theta, q, b, -y) - g(\theta, q, x, -y)\right] dy. \qquad (1.9)$$

Note that the above result can be written differently using the identity in Equation (1.3). More precisely, one can re-write $g(\theta, q, x, y)$ as follows:

$$g(\theta, q, x, y) = W^{(\theta)}(x + y) + q \int_0^y W^{(\theta)}(x + y - z) W^{(\theta + q)}(z) dz. \qquad (1.10)$$

By taking appropriate limits in Equation (1.9), either with definition of $g(\theta, q, x, y)$ given in (1.8) or in (1.10), one can obtain the following corollary:

Corollary 1. *For $\theta, a, b \geq 0$, then:*

1. for $x \geq -a$ and $y \in [-a, 0]$, we have

$$\mathbb{E}_x\left[e^{-\theta \tau_q}, X_{\tau_q} \in dy, \tau_q < \tau_{-a}^-\right] = q\left[\frac{g(\theta, q, x, a)}{\mathcal{H}^{(\theta,q)}(a)} \mathcal{H}^{(\theta,q)}(-y) - g(\theta, q, x, -y)\right] dy. \qquad (1.11)$$

2. for $x \leq b$ and $y \in (-\infty, 0]$, we have

$$\mathbb{E}_x\left[e^{-\theta \tau_q}, X_{\tau_q} \in dy, \tau_q < \tau_b^+\right] = q\left[\frac{\mathcal{H}^{(\theta+q,-q)}(x)}{\mathcal{H}^{(\theta+q,-q)}(b)} g(\theta, q, b, -y) - g(\theta, q, x, -y)\right] dy. \qquad (1.12)$$

3. for $x \in \mathbb{R}$ and $y \in (-\infty, 0]$, we have

$$\mathbb{E}_x\left[e^{-\theta \tau_q}, X_{\tau_q} \in dy, \tau_q < \infty\right]$$
$$= \left[\left(\Phi(\theta + q) - \Phi(\theta)\right)\mathcal{H}^{(\theta+q,-q)}(x)\mathcal{H}^{(\theta,q)}(-y) - qg(\theta, q, x, -y)\right] dy. \qquad (1.13)$$

2 Proofs

Proof of Theorem 2. Take $\varepsilon \in (0, a)$. We first compute

$$\mathbb{E}\left[e^{-\theta \tau_q^\varepsilon} f\left(-X_{\tau_q^\varepsilon}\right) \mathbf{1}_{\{\tau_q^\varepsilon < \tau_b^+ \wedge \tau_{-a}^-\}}\right]. \qquad (2.14)$$

Here, we express (2.14) in terms of the ε-excursions of X confined in the interval $[-a, b]$ and such that the time that each ε-excursion away from 0 spends below 0 after reaching the level $-\varepsilon$ is less than its associated implementation clock; subsequently, the first ε-excursion away from 0 that exits the interval $[-a, b]$ or such that the time that the ε-excursion spends below 0 after reaching the level$-\varepsilon$ is greater than its implementation clock. More precisely, let $(\xi_s^{i,\varepsilon}, 0 \leq s \leq \ell_i^\varepsilon)$ be the

i-th ε-excursion of X away from 0 confined in the interval $[-a, b]$ and such that $\ell_i^\varepsilon - \sigma_{-\varepsilon}^i \leq \mathbf{e}_q^i$, here ℓ_i^ε denotes the length of $\xi^{i,\varepsilon}$, and

$$\sigma_{-\varepsilon}^i = \inf\{s < \ell_i^\varepsilon : \xi_s^{i,\varepsilon} < -\varepsilon\}.$$

Similarly, let $(\xi_s^{*,\varepsilon}, 0 \leq s \leq \ell_*^\varepsilon)$ be the first ε-excursion of X away from 0 that exits the interval $[-a, b]$, or such that $\ell_*^\varepsilon - \sigma_{-\varepsilon}^* > \mathbf{e}_q^{k_q}$ where ℓ_*^ε is its length and

$$\sigma_{-\varepsilon}^* = \inf\{s < \ell_*^\varepsilon : \xi_s^{*,\varepsilon} < -\varepsilon\}.$$

We also define the infimum and supremum of the excursion $\xi^{*,\varepsilon}$, as follows

$$\underline{\xi}^{*,\varepsilon} = \inf_{s < \ell_*^\varepsilon} \xi_s^{*,\varepsilon} \qquad \text{and} \qquad \overline{\xi}^{*,\varepsilon} = \sup_{s < \ell_*^\varepsilon} \xi_s^{*,\varepsilon}.$$

From the strong Markov property, it is clear that the random variables $\left(\mathrm{e}^{-q\ell_i^\varepsilon}\right)_{i \geq 1}$ are iid and independent of

$$\Xi_{a,b}^{(*,\varepsilon)} := \mathrm{e}^{-\theta(\sigma_{-\varepsilon}^* + \mathbf{e}_q^{k_q})} f\left(-\xi_{\sigma_{-\varepsilon}^* + \mathbf{e}_q^{k_q}}^{*,\varepsilon}\right) \mathbf{1}_{\{\ell_*^\varepsilon < \infty\}} \mathbf{1}_{\{\overline{\xi}^{*,\varepsilon} \leq b\}} \mathbf{1}_{\{\underline{\xi}^{*,\varepsilon} \geq -a\}}.$$

Let $\zeta = \tau_0^{+,\varepsilon}$ and $p = \mathbb{P}(E)$, where

$$E = \left\{\sup_{t \leq \zeta} X_t \leq b, \inf_{t \leq \zeta} X_t \geq -a, \zeta - \tau_{-\varepsilon}^- \leq \mathbf{e}_q\right\}.$$

A standard description of ε-excursions of X away from 0 confined in the interval $[-a, b]$ with the amount of time spent below 0 after reaching the level $-\varepsilon$, less than an exponential time, dictates that the number of such ε-excursions is distributed according to an independent geometric random variable, say G_p, (supported on $\{0, 1, 2, \ldots\}$) with parameter p. Moreover, the random variables $\left(\mathrm{e}^{-q\ell_i^\varepsilon}\right)_{i \geq 1}$ have the same distribution as $\mathrm{e}^{-\theta\zeta}$ under the conditional law $\mathbb{P}(\cdot|E)$ and the random variable $\Xi_{a,b}^{(*,\varepsilon)}$ is equal in distribution to

$$\mathrm{e}^{-\theta(\tau_{-\varepsilon}^- + \mathbf{e}_q)} f\left(-X_{\tau_{-\varepsilon}^- + \mathbf{e}_q}\right) \mathbf{1}_{\{\inf_{t \leq \tau_{-\varepsilon}^- + \mathbf{e}_q} X_t \geq -a\}} \mathbf{1}_{\{\sup_{t \leq \tau_{-\varepsilon}^- + \mathbf{e}_q} X_t \leq b\}},$$

but now under the conditional law $\mathbb{P}(\cdot|E^c)$. Then, it follows

$$\mathbb{E}\left[\mathrm{e}^{-\theta\tau_q^\varepsilon} f\left(-X_{\tau_q^\varepsilon}\right) \mathbf{1}_{\{\tau_q^\varepsilon < \tau_b^+ \wedge \tau_{-a}^-\}}\right]$$

$$= \mathbb{E}\left[\prod_{i=0}^{G_p} \mathrm{e}^{-\theta\ell_i^\varepsilon} \mathrm{e}^{-\theta(\sigma_{-\varepsilon}^* + \mathbf{e}_q^{k_q})} f\left(-\xi_{\sigma_{-\varepsilon}^* + \mathbf{e}_q^{k_q}}^{*,\varepsilon}\right) \mathbf{1}_{\{\ell_*^\varepsilon < \infty\}} \mathbf{1}_{\{\overline{\xi}^{*,\varepsilon} \leq b\}} \mathbf{1}_{\{\underline{\xi}^{*,\varepsilon} \geq -a\}}\right] \qquad (2.15)$$

$$= \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{-\theta\ell_1^\varepsilon}\right]^{G_p}\right] \mathbb{E}\left[\mathrm{e}^{-\theta(\sigma_{-\varepsilon}^* + \mathbf{e}_q^{k_q})} f\left(-\xi_{\sigma_{-\varepsilon}^* + \mathbf{e}_q^{k_q}}^{*,\varepsilon}\right) \mathbf{1}_{\{\ell_*^\varepsilon < \infty\}} \mathbf{1}_{\{\overline{\xi}^{*,\varepsilon} \leq b\}} \mathbf{1}_{\{\underline{\xi}^{*,\varepsilon} \geq -a\}}\right].$$

Recall that the generating function of the independent geometric random variable G_p satisfies,

$$F(s) = \frac{\overline{p}}{1 - ps}, \quad |s| < \frac{1}{p},$$

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where $\bar{p} = 1 - p$. Therefore, if we can make sure that $\mathbb{E}\left[e^{-\theta \ell_1^\varepsilon}\right] < 1/p$, then

$$\mathbb{E}\left[\mathbb{E}\left[e^{-\theta \ell_1^\varepsilon}\right]^{G_p}\right] = \frac{\bar{p}}{1 - p\mathbb{E}\left[e^{-\theta \ell_1^\varepsilon}\right]}. \tag{2.16}$$

Therefore, using (2.15) and (2.16), we have

$$\mathbb{E}\left[e^{-\theta \tau_q^\varepsilon} f\left(-X_{\tau_q^\varepsilon}\right) \mathbf{1}_{\{\tau_q^\varepsilon < \tau_b^+ \wedge \tau_{-a}^-\}}\right] = \frac{\bar{p}\mathbb{E}\left[\Xi_{a,b}^{(*,\varepsilon)}\right]}{1 - p\mathbb{E}\left[e^{-\theta \ell_1^\varepsilon}\right]}. \tag{2.17}$$

Taking account of the remarks in the previous paragraph and making use of the strong Markov property, we have

$$\mathbb{E}\left[e^{-\theta \ell_1^\varepsilon}\right] = \frac{1}{p}\mathbb{E}\left[e^{-\theta \tau_{-\varepsilon}^-} \mathbf{1}_{\{\tau_{-\varepsilon}^- < \tau_b^+ \wedge \tau_{-a}^-\}} \mathbb{E}_{X_{\tau_{-\varepsilon}^-}}\left[e^{-(\theta+q)\tau_0^+}; \tau_0^+ < \tau_{-a}^-\right]\right]$$

$$= \frac{1}{p}\mathbb{E}_\varepsilon\left[e^{-\theta \tau_0^-} \mathbf{1}_{\{\tau_0^- < \tau_{b+\varepsilon}^+\}} \frac{W^{(\theta+q)}(X_{\tau_0^-} - \varepsilon + a)}{W^{(\theta+q)}(a)}\right].$$

Note that we do not need the indicator function of $\{X_{\tau_0^-} - \varepsilon > -a\}$ since, on its complement, the scale function vanishes. Note also that it is now clear from the above computation that $\mathbb{E}\left[e^{-\theta \ell_1^\varepsilon}\right] < 1/p$. Using the identity in Equation (1.3), one can write

$$\mathbb{E}_\varepsilon\left[e^{-\theta \tau_0^-} W^{(\theta+q)}\left(X_{\tau_0^-} - \varepsilon + a\right) \mathbf{1}_{\{\tau_0^- < \tau_{b+\varepsilon}^+\}}\right]$$

$$= W^{(\theta+q)}(a) - q\int_{a-\varepsilon}^a W^{(\theta)}(a - z)W^{(\theta+q)}(z)\mathrm{d}z$$

$$- \frac{W^{(\theta)}(\varepsilon)}{W^{(\theta)}(b + \varepsilon)}\left(W^{(\theta+q)}(b + a) - q\int_{a-\varepsilon}^{b+a} W^{(\theta)}(b + a - z)W^{(\theta+q)}(z)\mathrm{d}z\right).$$

As a consequence,

$$1 - p\mathbb{E}\left[e^{-\theta \ell_1^\varepsilon}\right] = \frac{q}{W^{(\theta+q)}(a)}\int_{a-\varepsilon}^a W^{(\theta)}(a - z)W^{(\theta+q)}(z)\mathrm{d}z$$

$$+ \frac{W^{(\theta)}(\varepsilon)}{W^{(\theta+q)}(a)W^{(\theta)}(b + \varepsilon)}\left(W^{(\theta+q)}(b + a) - q\int_{a-\varepsilon}^{b+a} W^{(\theta)}(b + a - z)W^{(\theta+q)}(z)\mathrm{d}z\right).$$

Next, we compute the Laplace transform of $\Xi_{a,b}^{(*,\varepsilon)}$. Recalling that under $\mathbb{P}(\cdot|E^c)$ and on the event $\{\bar{\xi}^{*,\varepsilon} < b, \underline{\xi}^{*,\varepsilon} \geq -a\}$, we necessarily have that the excursion goes below the level $-\varepsilon$ and the exponential clock rings before the end of the excursion, i.e.

$$\bar{p}\mathbb{E}\left[\Xi_{a,b}^{(*,\varepsilon)}\right] = \mathbb{E}\left[e^{-\theta \tau_{-\varepsilon}^-}\mathbb{E}_{X_{\tau_{-\varepsilon}^-}}\left[e^{-\theta \mathbf{e}_q} f\left(-X_{\mathbf{e}_q}\right); \mathbf{e}_q < \tau_{-a}^- \wedge \tau_0^+\right]\mathbf{1}_{\{\tau_{-\varepsilon}^- < \tau_{-a}^- \wedge \tau_b^+\}}\right]$$

$$= q\int_{-a}^0 f(-y)\mathbb{E}\left[e^{-\theta \tau_{-\varepsilon}^-}\left\{\frac{W^{(\theta+q)}(X_{\tau_{-\varepsilon}^-} + a)W^{(\theta+q)}(-y)}{W^{(\theta+q)}(a)} - W^{(\theta+q)}(X_{\tau_{-\varepsilon}^-} - y)\right\}\mathbf{1}_{\{\tau_{-\varepsilon}^- < \tau_b^+\}}\right]\mathrm{d}y, \tag{2.18}$$

thanks to Fubini's theorem and identity (1.2) in Theorem 1. Using once more the identity in Equation (1.4) and rearranging the terms, one can write

$$\mathbb{E}\left[e^{-\theta\tau_{-\varepsilon}^-}\left\{\frac{W^{(\theta+q)}(X_{\tau_{-\varepsilon}^-}+a)W^{(\theta+q)}(-y)}{W^{(\theta+q)}(a)}-W^{(\theta+q)}(X_{\tau_{-\varepsilon}^-}-y)\right\}\mathbf{1}_{\{\tau_{-\varepsilon}^-<\tau_b^+\}}\right]$$

$$=\frac{W^{(\theta)}(\varepsilon)}{W^{(\theta)}(b+\varepsilon)}\left\{\left[W^{(\theta+q)}(b-y)-q\int_0^{b+\varepsilon}W^{(\theta)}(b+\varepsilon-z)W^{(\theta+q)}(z-y-\varepsilon)\mathrm{d}z\right]\right.$$

$$\left.-\frac{W^{(\theta+q)}(-y)}{W^{(\theta+q)}(a)}\left[W^{(\theta+q)}(b+a)-q\int_0^{b+\varepsilon}W^{(\theta)}(b+\varepsilon-z)W^{(\theta+q)}(z+a-\varepsilon)\mathrm{d}z\right]\right\}$$

$$+q\left\{\int_0^{\varepsilon}W^{(\theta)}(\varepsilon-z)W^{(\theta+q)}(z-y-\varepsilon)\mathrm{d}z\right.$$

$$\left.-\frac{W^{(\theta+q)}(-y)}{W^{(\theta+q)}(a)}\int_0^{\varepsilon}W^{(\theta)}(\varepsilon-z)W^{(\theta+q)}(z+a-\varepsilon)\mathrm{d}z\right\}.$$

Now we are interested in computing the limit of $\mathbb{E}\left[e^{-\theta\tau_q^{\varepsilon}}f\left(-X_{\tau_q^{\varepsilon}}\right)\mathbf{1}_{\{\tau_q^{\varepsilon}<\tau_b^+\wedge\tau_{-a}^-\}}\right]$, as given in Equation (2.17), when ε goes to 0. We use the above computations for the numerator and the denominator, and we divide both by $W^{(\theta)}(\varepsilon)$. First, we have

$$\frac{1-p\mathbb{E}\left[e^{-\theta\ell_1^{\varepsilon}}\right]}{W^{(\theta)}(\varepsilon)}=\frac{q}{W^{(\theta+q)}(a)}\frac{\int_{a-\varepsilon}^a W^{(\theta)}(a-z)W^{(\theta+q)}(z)\mathrm{d}z}{W^{(\theta)}(\varepsilon)}$$

$$+\frac{1}{W^{(\theta+q)}(a)W^{(\theta)}(b+\varepsilon)}\left(W^{\theta+q}(b+a)-q\int_{a-\varepsilon}^{b+a}W^{(\theta)}(b+a-z)W^{(\theta+q)}(z)\mathrm{d}z\right)$$

$$\xrightarrow[\varepsilon\downarrow0]{}\frac{1}{W^{(\theta+q)}(a)W^{(\theta)}(b)}\left(W^{(\theta+q)}(b+a)-q\int_a^{b+a}W^{(\theta)}(b+a-z)W^{(\theta+q)}(z)\mathrm{d}z\right).$$

Indeed, when the process has paths of bounded variation, we have

$$\frac{\int_{a-\varepsilon}^a W^{(\theta)}(a-z)W^{(\theta+q)}(z)\mathrm{d}z}{W^{(\theta)}(\varepsilon)}\xrightarrow[\varepsilon\downarrow0]{}\frac{0}{W^{(\theta)}(0)}=0,$$

while, when it has paths of unbounded variation, we have

$$\frac{1}{W^{(\theta)}(\varepsilon)/\varepsilon}\frac{\int_{a-\varepsilon}^a W^{(\theta)}(a-z)W^{(\theta+q)}(z)\mathrm{d}z}{\varepsilon}\xrightarrow[\varepsilon\downarrow0]{}\frac{W^{(\theta)}(0)W^{(\theta+q)}(a)}{W^{(\theta)\prime}(0)}=0.$$

Similarly, using Lebesgue's dominated convergence theorem, we have

$$\frac{\overline{p}\mathbb{E}\left[\Xi_{a,b}^{(*,\varepsilon)}\right]}{W^{(\theta)}(\varepsilon)}\xrightarrow[\varepsilon\downarrow0]{}q\int_{-a}^0\frac{f(-y)}{W^{(\theta)}(b)}\left\{\left[W^{(\theta+q)}(b-y)-q\int_0^b W^{(\theta)}(b-z)W^{(\theta+q)}(z-y)\mathrm{d}z\right]\right.$$

$$\left.-\frac{W^{(\theta+q)}(-y)}{W^{(\theta+q)}(a)}\left[W^{(\theta+q)}(b+a)-q\int_0^b W^{(\theta)}(b-z)W^{(\theta+q)}(z+a)\mathrm{d}z\right]\right\}\mathrm{d}y.$$

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Putting all the pieces together, we deduce

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[e^{-\theta\tau_q^\varepsilon}f\left(-X_{\tau_q^\varepsilon}\right)\mathbf{1}_{\{\tau_q^\varepsilon<\tau_b^+\wedge\tau_{-a}^-\}}\right]$$

$$= q\int_{-a}^0 f(-y)\left\{W^{(\theta+q)}(a)\frac{g(\theta,q,b,-y)}{g(\theta,q,b,a)}-W^{(\theta+q)}(-y)\right\}\mathrm{d}y,\quad(2.19)$$

where $g(\theta,q,x,y)$ is given as in (1.8).

Hence, from (1.7) we have that if f is a continuous and bounded function, we can use Lebesgue's dominated convergence theorem to conclude

$$\mathbb{E}\left[e^{-\theta\tau_q}f\left(-X_{\tau_q}\right)\mathbf{1}_{\{\tau_q<\tau_b^+\wedge\tau_{-a}^-\}}\right]=\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[e^{-\theta\tau_q^\varepsilon}f\left(-X_{\tau_q^\varepsilon}\right)\mathbf{1}_{\{\tau_q^\varepsilon<\tau_b^+\wedge\tau_{-a}^-\}}\right].$$

In order to prove the result when the process starts at $x>0$, we consider the first 0-excursion. Here, we have two possibilities when the process X goes below the level 0, either it touches 0 (coming from below) before the exponential clock rings, or the clock rings before the process X finishes its negative excursion. In the first case, once the process X returns to 0, we can start the procedure all over again. Hence, using the strong Markov property and the independence between the excursions, we obtain

$$\mathbb{E}_x\left[e^{-\theta\tau_q}f\left(-X_{\tau_q}\right)\mathbf{1}_{\{\tau_q<\tau_b^+\wedge\tau_{-a}^-\}}\right]$$

$$=\mathbb{E}_x\left[e^{-\theta\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[e^{-\theta\mathbf{e}_q}f\left(-X_{\mathbf{e}_q}\right)\mathbf{1}_{\{\mathbf{e}_q<\tau_{-a}^-\wedge\tau_0^+\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\wedge\tau_{-a}^-\}}\right]$$

$$+\mathbb{E}_x\left[e^{-\theta\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[e^{-\theta\tau_0^+};\mathbf{1}_{\{\tau_0^+<\tau_{-a}^-\wedge\mathbf{e}_q\}}\right]\mathbf{1}_{\{\tau_0^-<\tau_b^+\}}\right]\mathbb{E}_0\left[e^{-\theta\tau_q}f\left(-X_{\tau_q}\right)\mathbf{1}_{\{\tau_q<\tau_b^+\wedge\tau_{-a}^-\}}\right].$$

Using once again the identities in Equations (1.1), (1.2) and (1.4), and putting all the pieces together yield the result. $\qquad\square$

Proof of Corollary 1. The first two results in Equation (1.11) and Equation (1.12) follow by taking appropriate limits, i.e. letting a and b go to infinity in Equation (1.9) or Equation (1.10), and by using the following identity (see e.g. Exercice 8.5 in [9]): for $r\geq 0$ and $x\in\mathbb{R}$,

$$\lim_{c\to\infty}\frac{W^{(r)}(c-x)}{W^{(r)}(c)}=e^{-\Phi(r)x}.$$

The third part of the Corollary, i.e. Equation (1.13), is obtained by computing the following limit

$$\lim_{b\to\infty}\mathbb{E}_x\left[e^{-\theta\tau_q},X_{\tau_q}\in\mathrm{d}y,\tau_q<\tau_b^+\right]=\mathbb{E}_x\left[e^{-\theta\tau_q},X_{\tau_q}\in\mathrm{d}y,\tau_q<\infty\right],$$

and by noticing that

$$\lim_{b\to\infty}\frac{W^{(\theta)}(b)}{e^{\Phi(\theta+q)b}-q\int_0^b W^{(\theta)}(b-z)e^{\Phi(\theta+q)z}\mathrm{d}z}=\frac{\Phi(\theta+q)-\Phi(\theta)}{q}$$

and

$$\lim_{b\to\infty} \frac{W^{(\theta+q)}(b-y) - q\int_0^b W^{(\theta)}(b-z)W^{(\theta+q)}(z-y)\mathrm{d}z}{\mathrm{e}^{\Phi(\theta+q)b} - q\int_0^b W^{(\theta)}(b-z)\mathrm{e}^{\Phi(\theta+q)z}\mathrm{d}z}$$
$$= \frac{\Phi(\theta+q) - \Phi(\theta)}{q}\left(\mathrm{e}^{-\Phi(\theta)y} + q\int_0^{-y}\mathrm{e}^{-\Phi(\theta)(y+z)}W^{(\theta+q)}(z)\mathrm{d}z\right).$$

The implementation of the Lebesgue's dominated convergence theorem is similar to the arguments used in [13], so we omit them for the sake of brevity, leaving the details to the reader. $\square$

3 Concluding remarks and more general joint laws

In this section, we will show how to use the Gerber-Shiu distributions of Theorem 2 and Corollary 1 to compute specific Gerber-Shiu functions. Let's first consider the following Gerber-Shiu function: for $\lambda \geq 0$, consider

$$\mathbb{E}_x\left[\mathrm{e}^{-\theta\tau_q + \lambda X_{\tau_q}}; \tau_q < \tau_b^+\right] = \int_{-\infty}^0 \mathrm{e}^{\lambda y}\mathbb{E}_x\left[\mathrm{e}^{-\theta\tau_q}, X_{\tau_q} \in \mathrm{d}y, \tau_q < \tau_b^+\right],$$

where the *distribution* is given by Equation (1.12).

Since, for $\lambda > \Phi(\theta + q)$, we have

$$\int_0^\infty \mathrm{e}^{-\lambda y}g(\theta, q, x, y)\mathrm{d}y = \mathrm{e}^{\lambda x}\left(\frac{1}{\psi(\lambda) - \theta - q} - \int_0^x \mathrm{e}^{-\lambda y}W^{(\theta+q)}(y)\mathrm{d}y\right)$$
$$- q\int_0^x W^{(\theta)}(x - z)\left(\frac{1}{\psi(\lambda) - \theta - q} - \int_0^z \mathrm{e}^{-\lambda y}W^{(\theta+q)}(y)\mathrm{d}y\right)\mathrm{d}z,$$

then we obtain

$$\mathbb{E}_x\left[\mathrm{e}^{-\theta\tau_q + \lambda X_{\tau_q}}; \tau_q < \tau_b^+\right] = q\frac{\mathcal{H}^{(\theta+q,-q)}(x)}{\mathcal{H}^{(\theta+q,-q)}(b)}\left[\mathrm{e}^{\lambda b}\left(\frac{1}{\psi(\lambda) - \theta - q} - \int_0^b \mathrm{e}^{-\lambda y}W^{(\theta+q)}(y)\mathrm{d}y\right)\right.$$
$$\left. - q\int_0^b W^{(\theta)}(b - z)\left(\frac{1}{\psi(\lambda) - \theta - q} - \int_0^z \mathrm{e}^{-\lambda y}W^{(\theta+q)}(y)\mathrm{d}y\right)\mathrm{d}z\right]$$
$$- q\left[\mathrm{e}^{\lambda x}\left(\frac{1}{\psi(\lambda) - \theta - q} - \int_0^x \mathrm{e}^{-\lambda y}W^{(\theta+q)}(y)\mathrm{d}y\right)\right.$$
$$\left. - q\int_0^x W^{(\theta)}(x - z)\left(\frac{1}{\psi(\lambda) - \theta - q} - \int_0^z \mathrm{e}^{-\lambda y}W^{(\theta+q)}(y)\mathrm{d}y\right)\mathrm{d}z\right].$$

Further, if we analytically extend (in terms of λ) the previous expression and take the limit when λ decreases to zero, then we get

$$\mathbb{E}_x\left[\mathrm{e}^{-\theta\tau_q}; \tau_q < \tau_b^+\right] = \frac{q}{\theta+q}\frac{\mathcal{H}^{(\theta+q,-q)}(x)}{\mathcal{H}^{(\theta+q,-q)}(b)}\left[-Z^{(\theta+q)}(b) + q\int_0^b W^{(\theta)}(b - z)Z^{(\theta+q)}(z)\mathrm{d}z\right]$$
$$- \frac{q}{\theta+q}\left[-Z^{\theta+q}(x) + q\int_0^x W^{(\theta)}(x - z)Z^{(\theta+q)}(z)\mathrm{d}z\right].$$

Similarly as for Equation (1.3), one can show that (see Equation (6) in [13]), for $p, q \geq 0$ and $x \in \mathbb{R}$, we have

$$(q - p) \int_0^x W^{(p)}(x - y) Z^{(q)}(y) \mathrm{d}y = Z^{(q)}(x) - Z^{(p)}(x).$$

Consequently, the Laplace transform of the time to ruin before the surplus exceed the level b is given by

$$\mathbb{E}_x \left[e^{-\theta \tau_q}; \tau_q < \tau_b^+ \right] = \frac{q}{\theta + q} \left(Z^{(\theta)}(x) - \frac{\mathcal{H}^{(\theta+q,-q)}(x)}{\mathcal{H}^{(\theta+q,-q)}(b)} Z^{(\theta)}(b) \right). \tag{3.20}$$

The above identity extends the result of Landriaut et al., see Lemma 2.2 in [10], in the case of exponential implementation delays and when the insurance risk process X has paths of bounded variation. We observe that the function $\mathcal{H}^{(\theta+q,-q)}$ is the same as the function $H_d^{(\theta)}$ defined in section 2.2 in [10].

Next, we are interested in computing the probability of Parisian ruin in the case when the net profit condition (1.5) is satisfied. To this end, let us take limits as $b \uparrow \infty$ in (3.20), and noticing that

$$\lim_{b \to \infty} \frac{W^{(\theta)}(b)}{e^{\Phi(\theta+q)b} - q \int_0^b W^{(\theta)}(b - z) e^{\Phi(\theta+q)z} \mathrm{d}z} = \frac{\Phi(\theta + q) - \Phi(\theta)}{q},$$

and

$$\lim_{b \to \infty} \frac{Z^{(\theta)}(b)}{W^{(\theta)}(b)} = \frac{\theta}{\Phi(\theta)},$$

we deduce

$$\mathbb{E}_x \left[e^{-\theta \tau_q}; \tau_q < \infty \right] = \frac{\theta \left(\Phi(\theta) - \Phi(\theta + q) \right)}{\Phi(\theta)(\theta + q)} \mathcal{H}^{(\theta+q,-q)}(x) + \frac{q}{\theta + q} Z^{(\theta)}(x).$$

Finally, if we assume that X satisfies the net profit condition, which yields that $\Phi(0) = 0$, and if we take the limit when θ decreases to zero in the previous expression, then we get

$$\mathbb{P}_x \left(\tau_q < \infty \right) = 1 - \psi'(0+) \frac{\Phi(q)}{q} \mathcal{H}^{(q,-q)}(x),$$

which agrees with Theorem 1 and Corollary 1 in [11] since we have the following identity (using a change of variable and an integration by parts):

$$\mathcal{H}^{(q,-q)}(x) = q \int_0^\infty e^{-\Phi(q)y} W(x + y) \mathrm{d}y.$$

In particular, when $x = 0$, we have

$$\mathbb{P} \left(\tau_q < \infty \right) = 1 - \psi'(0+) \frac{\Phi(q)}{q},$$

as in [11].

We finish this manuscript with an explanation of how to us the Gerber-Shiu function given in (1.9) in order to get more interesting identities. Our aim is to compute explicitly the the Gerber-Shiu function that take into account the running supremum of the surplus.

First we make the following observation, using (1.9) we can obtain the Gerber-Shiu measure of the process on the interval $[a, b]$. With this measure, we are able to obtain information about the running supremum of the process as follows

$$
\mathbb{E}\left[e^{-\theta\tau_r}; X_{\tau_r} \in \mathrm{d}y, \tau_r < \tau_b^+ \wedge \tau_{-a}^-\right] = \mathbb{E}\left[e^{-\theta\tau_r}; X_{\tau_r} \in \mathrm{d}y, \overline{X}_{\tau_r} \leq b, \tau_r < \tau_{-a}^-\right]
$$

$$
= q\left[\frac{g(\theta, q, x, a)}{g(\theta, q, b, a)}g(\theta, q, b, -y) - g(\theta, q, x, -y)\right]\mathbf{1}_{\{b \geq 0\}}\mathbf{1}_{\{-a < y < 0\}}\mathrm{d}y, \quad (3.21)
$$

where $\overline{X}_t = \sup_{s \in [0, t]} X_s$. For simplicity, we denote by $K(\theta, y, a, b)$ for the right-hand side of (3.21). Observe that $K(\theta, y, a, b)$ is differentiable with respect to the variable b almost everywhere, implying

$$
\int_0^b \frac{\partial}{\partial z}K(\theta, y, a, z)\mathbf{1}_{\{-a < y < 0\}}\mathrm{d}z = K(\theta, y, a, b)\mathbf{1}_{\{-a < y < 0\}}
$$

$$
= \mathbb{E}\left[e^{-\theta\tau_r}; X_{\tau_r} \in \mathrm{d}y, \overline{X}_{\tau_r} < b, \tau_r < \tau_{-a}^-\right],
$$

where we have used the fact that $K(\theta, y, a, 0) = 0$. The last part of the computation consist in obtaining in close form $\frac{\partial}{\partial b}K(\theta, y, a, b)\mathbf{1}_{\{-a < y < 0\}}$, we leave the details to the reader.

We note that it is possible to obtain a more general form of the Gerber-Shiu measure that takes into account the law of the process and its running infimum (and also its supremum) up to time to Parisian ruin, by differentiating $K(\theta, y, a, b)$ with respect to a (and b). For the sake of brevity the explicit form of this joint law is left to the reader.

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