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Gevrey functions and ultradistributions on compact Lie groups and homogeneous spaces ☆

Aparajita Dasgupta, Michael Ruzhansky *

Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom

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ABSTRACT

In this paper we give global characterisations of Gevrey–Roumieu and Gevrey–Beurling spaces of ultradifferentiable functions on compact Lie groups in terms of the representation theory of the group and the spectrum of the Laplace–Beltrami operator. Furthermore, we characterise their duals, the spaces of corresponding ultradistributions. For the latter, the proof is based on first obtaining the characterisation of their α-duals in the sense of Köthe and the theory of sequence spaces. We also give the corresponding characterisations on compact homogeneous spaces.

1. Introduction

The spaces of Gevrey ultradifferentiable functions are well-known on $\mathbb{R}^n$ and their characterisations exist on both the space-side and the Fourier transform side, leading to numerous applications in different areas. The aim of this paper is to obtain global characterisations of the spaces of Gevrey ultradifferentiable functions and of

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* Corresponding author.
E-mail addresses: a.dasgupta@imperial.ac.uk (A. Dasgupta), m.ruzhansky@imperial.ac.uk (M. Ruzhansky).

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the spaces of ultradistributions using the eigenvalues of the Laplace–Beltrami operator $\mathcal{L}_G$ (e.g. Casimir element) on the compact Lie group G. We treat both the cases of Gevrey–Roumieu and Gevrey–Beurling functions, and the corresponding spaces of ultradistributions, which are their topological duals with respect to their inductive and projective limit topologies, respectively.

If M is a compact homogeneous space, let G be its motion group and H a stationary subgroup at some point, so that $M \simeq G/H$. Our results on the motion group G will yield the corresponding characterisations for Gevrey functions and ultradistributions on the homogeneous space M. Typical examples are the real spheres $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$, complex spheres (and complex projective spaces) $\mathbb{CP}^n = \mathrm{SU}(n+1)/\mathrm{SU}(n)$, or quaternionic projective spaces $\mathbb{HP}^n$.

Working in local coordinates and treating G as a manifold the Gevrey–Roumieu class $\gamma_s(G)$, $s \geqslant 1$, is the space of functions $\phi \in C^\infty(G)$ such that in every local coordinate chart its local representative, say $\psi \in C^\infty(\mathbb{R}^n)$, is such that there exist constants $A > 0$ and $C > 0$ such that for all multi-indices α, we have that

$$\left|\partial^\alpha \psi(x)\right| \leqslant C A^{|\alpha|} (\alpha!)^s$$

holds for all $x \in \mathbb{R}^n$. By the chain rule one readily sees that this class is invariantly defined on (the analytic manifold) G for $s \geqslant 1$. For $s = 1$ we obtain the class of analytic functions. This behaviour can be characterised on the Fourier side by being equivalent to the condition that there exist $B > 0$ and $K > 0$ such that

$$\left|\widehat{\psi}(\eta)\right| \leqslant K e^{-B\langle\eta\rangle^{1/s}}$$

holds for all $\eta \in \mathbb{R}^n$. We refer to Komatsu [8] for the extensive analysis of these spaces and their duals in $\mathbb{R}^n$. However, such a local point of view does not tell us about the global properties of ϕ such as its relation to the geometric or spectral properties of the group G, and this is the aim of this paper. The characterisations that we give are global, i.e. they do not refer to the localisation of the spaces, but are expressed in terms of the behaviour of the global Fourier transform and the properties of the global Fourier coefficients. Characterisation of this type for analytic functions on compact analytic manifolds has been obtained by Seeley [15].

Such global characterisations will be useful for applications. For example, the Cauchy problem for the wave equation

$$\partial_t^2 u - a(t)\mathcal{L}_G u = 0 \tag{1.1}$$

is well-posed, in general, only in Gevrey spaces, if $a(t)$ becomes zero at some points. However, in local coordinates (1.1) becomes a second order equation with space-dependent coefficients and lower order terms. In this case comprehensive well-posedness results are not available even on $\mathbb{R}^n$, in general.[1] At the same time, in terms of the group Fourier transform Eq. (1.1) is basically with constant coefficients, and the global characterisation of Gevrey spaces together with an energy inequality for (1.1) yield the well-posedness result. We will address this and other applications elsewhere, but we note that in these problems both types of Gevrey spaces appear naturally, see e.g. [5] for the Gevrey–Roumieu ultradifferentiable and Gevrey–Beurling ultradistributional well-posedness of weakly hyperbolic partial differential equations in the Euclidean space.

In Section 2 we will fix the notation and formulate our results. We will also recall known (easy) characterisations for other spaces, such as spaces of smooth functions, distributions, or Sobolev spaces over L^2. Characterisation for Besov spaces, as well as for a number of other function spaces (Triebel–Lizorkin, Wiener and Beurling) will appear in [11]. The proof for the characterisation of Gevrey spaces will rely on the harmonic analysis on the group, the family of spaces $\ell^p(\widehat{G})$ on the unitary dual introduced in [13], and to some extent on the analysis of globally defined matrix-valued symbols of pseudo-differential operators developed in [13,14]. Our analysis of ultradistributions will rely on the theory of sequence spaces (echelon and co-echelon spaces), see e.g. Köthe [9], Ruckle [12]. Thus, we will first give characterisations of the so-called α-duals of the Gevrey spaces and then show that α-duals and topological duals coincide. We also prove that both types of Gevrey spaces are perfect spaces, i.e. the α-dual of its α-dual is the original space. This is done in Section 4, and the ultradistributions are treated in Section 5.

We note that the case of the periodic Gevrey spaces, which can be viewed as spaces on the torus $\mathbb{T}^n$, has been characterised by the Fourier coefficients in [17]. However, that paper stopped short of characterising the topological duals (i.e. the corresponding ultradistributions), so already in this case our characterisation in Theorem 2.5 appears to be new.

On torus, such a characterisation of Gevrey spaces is important for the analysis of dynamical systems, see e.g. [4]. We also mention characterisations of the Gelfand–Shilov spaces and their duals on $\mathbb{R}^n$ in terms of the short-time Fourier transform, see Gröchenig and Zimmerman [7] and Toft [18].

We note that compared to the proof of the corresponding results on $\mathbb{R}^n$, here we are in position to use the theory of sequence spaces, but then we still have to show that the duals we obtain in this way coincide with the topological duals of the spaces.

In the estimates throughout the paper the constants will be denoted by letter C which may change value even in the same formula. If we want to emphasise the change of the constant, we may use letters like C', A_1, etc.

[1] The result of Bronshtein [1] holds but is, in general, not optimal for some types of equations or does not hold for low regularity $a(t)$. More results are obtained by Nishitani [10].

2. Results

We first fix the notation and recall known characterisations of several spaces. We refer to [13] for details on the following constructions.

Let G be a compact Lie group of dimension n. Let $\widehat{G}$ denote the set of (equivalence classes of) continuous irreducible unitary representations of G. Since G is compact, $\widehat{G}$ is discrete. For $[\xi] \in \widehat{G}$, by choosing a basis in the representation space of ξ, we can view ξ as a matrix-valued function $\xi : G \to \mathbb{C}^{d_\xi \times d_\xi}$, where d_ξ is the dimension of the representation space of ξ. For $f \in L^1(G)$ we define its group Fourier transform at ξ by

$$\widehat{f}(\xi) := \int_G f(x)\xi(x)^* \, dx,$$

where dx is the normalised Haar measure on G. The Peter–Weyl theorem implies the Fourier inversion formula

$$f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}\big(\xi(x)\widehat{f}(\xi)\big). \tag{2.1}$$

For each $[\xi] \in \widehat{G}$, the matrix elements of ξ are the eigenfunctions for the Laplace–Beltrami operator $\mathcal{L}_G$ with the same eigenvalue which we denote by $-\lambda^2_{[\xi]}$, so that

$$-\mathcal{L}_G \xi_{ij}(x) = \lambda^2_{[\xi]} \xi_{ij}(x) \quad \text{for all } 1 \leqslant i, j \leqslant d_\xi.$$

Different spaces on the Lie group G can be characterised in terms of comparing the Fourier coefficients of functions with powers of the eigenvalues of the Laplace–Beltrami operator. We denote

$$\langle\xi\rangle := \big(1 + \lambda^2_{[\xi]}\big)^{1/2},$$

the eigenvalues of the elliptic first-order pseudo-differential operator $(I - \mathcal{L}_G)^{1/2}$.

Then, it is easy to see that $f \in C^\infty(G)$ if and only if for every $M > 0$ there exists $C > 0$ such that

$$\big\|\widehat{f}(\xi)\big\|_{\mathtt{HS}} \leqslant C\langle\xi\rangle^{-M},$$

and $u \in \mathcal{D}'(G)$ if and only if there exist $M > 0$ and $C > 0$ such that

$$\big\|\widehat{u}(\xi)\big\|_{\mathtt{HS}} \leqslant C\langle\xi\rangle^{M},$$

where we define

$$\widehat{u}(\xi)_{ij} := u(\overline{\xi_{ji}}), \quad 1 \leqslant i, j \leqslant d_\xi,$$

(see [3] for a slightly different description).

For this and other occasions, we can write this as $\widehat{u}(\xi) = u(\xi^*)$ in the matrix notation. The appearance of the Hilbert–Schmidt norm is natural in view of the Plancherel identity

$$(f, g)_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}\big(\widehat{f}(\xi)\widehat{g}(\xi)^*\big),$$

so that

$$\|f\|_{L^2(G)} = \left(\sum_{[\xi]\in\widehat{G}} d_\xi \big\|\widehat{f}(\xi)\big\|_{\mathtt{HS}}^2\right)^{1/2} =: \|\widehat{f}\|_{\ell^2(\widehat{G})}$$

can be taken as the definition of the Hilbert space $\ell^2(\widehat{G})$. Here, of course, $\|A\|_{\mathtt{HS}} = \sqrt{\operatorname{Tr}(AA^*)}$. It is convenient to use the sequence space

$$\Sigma = \big\{\sigma = \big(\sigma(\xi)\big)_{[\xi]\in\widehat{G}}:\ \sigma(\xi) \in \mathbb{C}^{d_\xi\times d_\xi}\big\}.$$

In [13], the authors introduced a family of spaces $\ell^p(\widehat{G})$, $1 \leqslant p < \infty$, by saying that $\sigma \in \Sigma$ belongs to $\ell^p(\widehat{G})$ if the norm

$$\|\sigma\|_{\ell^p(\widehat{G})} := \left(\sum_{[\xi]\in\widehat{G}} d_\xi^{p(\frac{2}{p}-\frac{1}{2})} \big\|\sigma(\xi)\big\|_{\mathtt{HS}}^p\right)^{1/p}$$

if finite. There is also the space $\ell^\infty(\widehat{G})$ for which the norm

$$\|\sigma\|_{\ell^\infty(\widehat{G})} := \sup_{[\xi]\in\widehat{G}} d_\xi^{-\frac{1}{2}} \big\|\sigma(\xi)\big\|_{\mathtt{HS}} \tag{2.2}$$

is finite. These are interpolation spaces for which the Hausdorff–Young inequality holds, in particular, we have

$$\|\widehat{f}\|_{\ell^\infty(\widehat{G})} \leqslant \|f\|_{L^1(G)} \quad \text{and} \quad \big\|\mathscr{F}^{-1}\sigma\big\|_{L^\infty(G)} \leqslant \|\sigma\|_{\ell^1(\widehat{G})}, \tag{2.3}$$

with

$$\big(\mathscr{F}^{-1}\sigma\big)(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}\big(\xi(x)\sigma(\xi)\big).$$

We refer to [13, Chapter 10] for further details on these spaces. Usual Sobolev spaces on G as a manifold, defined by localisations, can be also characterised by the global condition

$$f \in H^t(G) \quad \text{if and only if} \quad \langle\xi\rangle^t \widehat{f}(\xi) \in \ell^2(\widehat{G}). \tag{2.4}$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we define $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. We will adopt the convention that $0! = 1$ and $0^0 = 1$.

Let $X_1, \dots, X_n$ be a basis of the Lie algebra of G, normalised in some way, e.g. with respect to the Killing form. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we define the left-invariant differential operator of order $|\alpha|$, $\partial^\alpha := Y_1 \cdots Y_{|\alpha|}$, with $Y_j \in \{X_1, \dots, X_n\}$, $1 \leqslant j \leqslant |\alpha|$, and $\sum_{j: Y_j = X_k} 1 = \alpha_k$ for every $1 \leqslant k \leqslant n$. It means that ∂^α is a composition of left-invariant derivatives with respect to vectors $X_1, \dots, X_n$, such that each X_k enters ∂^α exactly α_k times. There is a small abuse of notation here since we do not specify in the notation ∂^α the order of vectors $X_1, \dots, X_n$ entering in ∂^α, but this will not be important for the arguments in the paper. The reason we define ∂^α in this way is to take care of the non-commutativity of left-invariant differential operators corresponding to the vector fields X_k.

We will distinguish between two families of Sobolev spaces over L^2. The first one is defined by $H^t(G) = \{f \in L^2(G)\colon (I - \mathcal{L}_G)^{t/2} f \in L^2(G)\}$ with the norm

$$\|f\|_{H^t(G)} := \left\|(I - \mathcal{L}_G)^{t/2} f\right\|_{L^2(G)} = \left\|\langle\xi\rangle^t \widehat{f}(\xi)\right\|_{\ell^2(\widehat{G})}. \tag{2.5}$$

The second one is defined for $k \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ by

$$W^{k,2} = \left\{f \in L^2(G)\colon \|f\|_{W^{k,2}} := \sum_{|\alpha| \leqslant k} \left\|\partial^\alpha f\right\|_{L^2(G)} < \infty\right\}.$$

Obviously, $H^k \simeq W^{k,2}$ for any $k \in \mathbb{N}_0$ but for us the relation between norms will be of importance, especially as k will tend to infinity.

Let $0 < s < \infty$. We first fix the notation for the Gevrey spaces and then formulate the results. In the definitions below we allow any $s > 0$, and the characterisation of α-duals in the sequel will still hold. However, when dealing with ultradistributions we will be restricting to $s \geqslant 1$.

Definition 2.1. Gevrey–Roumieu(R) class $\gamma_s(G)$ is the space of functions $\phi \in C^\infty(G)$ for which there exist constants $A > 0$ and $C > 0$ such that for all multi-indices α, we have

$$\left\|\partial^\alpha \phi\right\|_{L^\infty} \equiv \sup_{x \in G} \left|\partial^\alpha \phi(x)\right| \leqslant C A^{|\alpha|} (\alpha!)^s. \tag{2.6}$$

Functions $\phi \in \gamma_s(G)$ are called *ultradifferentiable functions of Gevrey–Roumieu class of order s*.

For $s = 1$ we obtain the space of analytic functions, and for $s > 1$ the space of Gevrey–Roumieu functions on G considered as a manifold, by saying that the function is in the Gevrey–Roumieu class locally in every coordinate chart. The same is true for the other Gevrey space:

Definition 2.2. Gevrey–Beurling(B) class $\gamma_{(s)}(G)$ is the space of functions $\phi \in C^\infty(G)$ such that for every $A > 0$ there exists $C_A > 0$ so that for all multi-indices α, we have

$$\left\|\partial^\alpha \phi\right\|_{L^\infty} \equiv \sup_{x\in G} \left|\partial^\alpha f(x)\right| \leqslant C_A A^{|\alpha|}(\alpha!)^s.$$

Functions $\phi \in \gamma_{(s)}(G)$ are called *ultradifferentiable functions of Gevrey–Beurling class of order* s.

The following theorem is our main result on Gevrey functions on compact Lie groups.

Theorem 2.3. *Let* $0 < s < \infty$.

(**R**) *We have* $\phi \in \gamma_s(G)$ *if and only if there exist* $B > 0$ *and* $K > 0$ *such that*

$$\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant K e^{-B\langle\xi\rangle^{1/s}} \tag{2.7}$$

holds for all $[\xi] \in \widehat{G}$.

(**B**) *We have* $\phi \in \gamma_{(s)}(G)$ *if and only if for every* $B > 0$ *there exists* $K_B > 0$ *such that*

$$\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant K_B e^{-B\langle\xi\rangle^{1/s}} \tag{2.8}$$

holds for all $[\xi] \in \widehat{G}$.

Expressions appearing in the definitions can be taken as seminorms, and the spaces are equipped with the inductive and projective topologies, respectively.[2] We now turn to ultradistributions.

Definition 2.4. The space of continuous linear functionals on $\gamma_s(G)$(or $\gamma_{(s)}(G)$) is called *the space of ultradistributions* and is denoted by $\gamma_s'(G)$(or $\gamma_{(s)}'(G)$), respectively.

For any $v \in \gamma_s'(G)$(or $\gamma_{(s)}'(G)$), for $[\xi] \in \widehat{G}$, we define the Fourier coefficients

$$\widehat{v}(\xi) := \left\langle v, \xi^*\right\rangle \equiv v\left(\xi^*\right).$$

These are well-defined since G is compact and hence $\xi(x)$ are actually analytic.

The following theorem is our main result on ultradistributions on compact Lie groups.

Theorem 2.5. *Let* $1 \leqslant s < \infty$.

[2] See also Definition 5.1 for an equivalent formulation.

(**R**) *We have $v \in \gamma_s'(G)$ if and only if for every $B > 0$ there exists $K_B > 0$ such that*

$$\left\|\widehat{v}(\xi)\right\|_{\mathtt{HS}} \leqslant K_B e^{B\langle\xi\rangle^{\frac{1}{s}}} \tag{2.9}$$

holds for all $[\xi] \in \widehat{G}$.

(**B**) *We have $v \in \gamma_{(s)}'(G)$ if and only if there exist $B > 0$ and $K_B > 0$ such that (2.9) holds for all $[\xi] \in \widehat{G}$.*

The proof of Theorem 2.5 follows from the characterisation of α-duals of[3] the Gevrey spaces in Theorem 4.2 and the equivalence of the topological duals and α-duals in Theorem 5.2.

The result on groups implies the corresponding characterisation on compact homogeneous spaces M. First we fix the notation. Let G be a compact motion group of M and let H be the stationary subgroup of some point. Alternatively, we can start with a compact Lie group G with a closed subgroup H. The homogeneous space $M = G/H$ is an analytic manifold in a canonical way (see, for example, Bruhat [2] or Stein [16] as textbooks on this subject). We normalise measures so that the measure on H is a probability one. Typical examples are the spheres $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ or complex spheres $\mathbb{CS}^n = \mathrm{SU}(n+1)/\mathrm{SU}(n)$.

We denote by $\widehat{G}_0$ the subset of $\widehat{G}$ of representations that are class I with respect to the subgroup H. This means that $[\xi] \in \widehat{G}_0$ if ξ has at least one non-zero invariant vector a with respect to H, i.e. that

$$\xi(h)a = a \quad \text{for all } h \in H.$$

Let $\mathcal{H}_\xi$ denote the representation space of ξ, i.e. $\xi(x) : \mathcal{H}_\xi \to \mathcal{H}_\xi$, and let $\mathcal{B}_\xi$ be the space of these invariant vectors. Let

$$k_\xi := \dim \mathcal{B}_\xi.$$

We fix an orthonormal basis of $\mathcal{H}_\xi$ so that its first k_ξ vectors are the basis of B_ξ. The matrix elements $\xi_{ij}(x)$, $1 \leqslant j \leqslant k_\xi$, are invariant under the right shifts by H. We refer to Vilenkin and Klimyk [19] for the details of these constructions.

We can identify Gevrey functions on $M = G/H$ with Gevrey functions on G which are constant on left cosets with respect to H. Here we will restrict to $s \geqslant 1$ to see the equivalence of spaces using their localisation. This identification gives rise to the corresponding identification of ultradistributions. Thus, for a function $f \in \gamma_s(M)$ we can recover it by the Fourier series of its canonical lifting $\widetilde{f}(g) := f(gH)$ to G, $\widetilde{f} \in \gamma_s(G)$, and the Fourier coefficients satisfy $\widehat{\widetilde{f}}(\xi) = 0$ for all representations with $[\xi] \notin \widehat{G}_0$. Also, for class I representations $[\xi] \in \widehat{G}_0$ we have $\widehat{\widetilde{f}}(\xi)_{ij} = 0$ for $i > k_\xi$.

[3] The characterisation of α-duals is actually valid for all $0 < s < \infty$.

With this, we can write the Fourier series of f (or of $\widetilde{f}$, but as we said, from now on we will identify these and denote both by f) in terms of the spherical functions ξ_{ij} of the representations ξ, $[\xi] \in \widehat{G}_0$, with respect to the subgroup H. Namely, the Fourier series (2.1) becomes

$$f(x) = \sum_{[\xi]\in\widehat{G}_0} d_\xi \sum_{i=1}^{d_\xi} \sum_{j=1}^{k_\xi} \widehat{f}(\xi)_{ji}\xi_{ij}(x). \tag{2.10}$$

In view of this, we will say that *the collection of Fourier coefficients* $\{\widehat{\phi}(\xi)_{ij}\colon [\xi] \in \widehat{G},\ 1 \leqslant i,j \leqslant d_\xi\}$ *is of class I with respect to* H if $\widehat{\phi}(\xi)_{ij} = 0$ whenever $[\xi] \notin \widehat{G}_0$ or $i > k_\xi$. By the above discussion, if the collection of Fourier coefficients is of class I with respect to H, then the expressions (2.1) and (2.10) coincide and yield a function f such that $f(xh) = f(h)$ for all $h \in H$, so that this function becomes a function on the homogeneous space G/H. The same applies to (ultra)distributions with the standard distributional interpretation. With these identifications, Theorem 2.3 immediately implies

Theorem 2.6. *Let* $1 \leqslant s < \infty$.

(R) *We have* $\phi \in \gamma_s(G/H)$ *if and only if its Fourier coefficients are of class I with respect to* H *and, moreover, there exist* $B > 0$ *and* $K > 0$ *such that*

$$\|\widehat{\phi}(\xi)\|_{\mathtt{HS}} \leqslant K e^{-B\langle\xi\rangle^{1/s}} \tag{2.11}$$

holds for all $[\xi] \in \widehat{G}_0$.

(B) *We have* $\phi \in \gamma_{(s)}(G)$ *if and only if its Fourier coefficients are of class I with respect to* H *and, moreover, for every* $B > 0$ *there exists* $K_B > 0$ *such that*

$$\|\widehat{\phi}(\xi)\|_{\mathtt{HS}} \leqslant K_B e^{-B\langle\xi\rangle^{1/s}} \tag{2.12}$$

holds for all $[\xi] \in \widehat{G}_0$.

It would be possible to extend Theorem 2.6 to the range $0 < s < \infty$ by adopting Definition 2.1 starting with a frame of vector fields on M, but instead of obtaining the result immediately from Theorem 2.3 we would have to go again through arguments similar to those used to prove Theorem 2.3. Since we are interested in characterising the standard invariantly defined Gevrey spaces we decided not to lengthen the proof in this way. On the other hand, it is also possible to prove the characterisations on homogeneous spaces G/H first and then obtain those on the group G by taking H to be trivial. However, some steps would become more technical since we would have to deal with frames of vector fields instead of the basis of left-invariant vector fields on G, and elements of the symbolic calculus used in the proof would become more complicated.

We also have the ultradistributional result following from Theorem 2.5.

Theorem 2.7. *Let $1 \leqslant s < \infty$.*

(**R**) *We have $v \in \gamma'_s(G/H)$ if and only if its Fourier coefficients are of class I with respect to H and, moreover, for every $B > 0$ there exists $K_B > 0$ such that*

$$\|\widehat{v}(\xi)\|_{\mathtt{HS}} \leqslant K_B e^{B\langle\xi\rangle^{\frac{1}{s}}} \tag{2.13}$$

holds for all $[\xi] \in \widehat{G}_0$.

(**B**) *We have $v \in \gamma'_{(s)}(G/H)$ if and only if its Fourier coefficients are of class I with respect to H and, moreover, there exist $B > 0$ and $K_B > 0$ such that (2.13) holds for all $[\xi] \in \widehat{G}_0$.*

Finally, we remark that in the harmonic analysis on compact Lie groups sometimes another version of $\ell^p(\widehat{G})$ spaces appears using Schatten p-norms. However, in the context of Gevrey spaces and ultradistributions eventual results hold for all such norms. Indeed, given our results with the Hilbert–Schmidt norm, by an argument similar to that of Lemma 3.2 below, we can put any Schatten norm $\|\cdot\|_{S_p}$, $1 \leqslant p \leqslant \infty$, instead of the Hilbert–Schmidt norm $\|\cdot\|_{\mathtt{HS}}$ in any of our characterisations and they still continue to hold.

3. Gevrey classes on compact Lie groups

We will need two relations between dimensions of representations and the eigenvalues of the Laplace–Beltrami operator. On one hand, it follows from the Weyl character formula that

$$d_\xi \leqslant C\langle\xi\rangle^{\frac{n-\operatorname{rank} G}{2}} \leqslant C\langle\xi\rangle^{\frac{n}{2}} \tag{3.1}$$

(where $n = \dim G$), with the latter[4] also following directly from the Weyl asymptotic formula for the eigenvalue counting function for $\mathcal{L}_G$, see e.g. [13, Prop. 10.3.19]. This implies, in particular, that for any $0 \leqslant p < \infty$ and any $s > 0$ and $B > 0$ we have

$$\sup_{[\xi]\in\widehat{G}} d_\xi^p e^{-B\langle\xi\rangle^{1/s}} < \infty. \tag{3.2}$$

Also, see [20] for other relations. On the other hand, the following simple statement about the convergence for the series relating eigenvalue and dimensions will be useful for us:

Lemma 3.1. *We have $\sum_{[\xi]\in\widehat{G}} d_\xi^2 \langle\xi\rangle^{-2t} < \infty$ if and only if $t > \frac{n}{2}$.*

[4] Namely, the inequality $d_\xi \leqslant C\langle\xi\rangle^{\frac{n}{2}}$.

Proof. We notice that for the δ-distribution at the unit element of the group, $\widehat{\delta}(\xi) = I_{d_\xi}$ is the identity matrix of size $d_\xi \times d_\xi$. Hence, in view of (2.4) and (2.5), we can write

$$\sum_{[\xi]\in\widehat{G}} d_\xi^2 \langle\xi\rangle^{-2t} = \sum_{[\xi]\in\widehat{G}} d_\xi \langle\xi\rangle^{-2t} \big\|\widehat{\delta}(\xi)\big\|_{\mathtt{HS}}^2 = \big\|(I - \mathcal{L}_G)^{-t/2}\delta\big\|_{L^2(G)}^2 = \|\delta\|_{H^{-t}(G)}^2.$$

By using the localisation of $H^{-t}(G)$, we see that this is finite if and only if $t > n/2$. □

We denote by $\widehat{G}_*$ the set of representations from $\widehat{G}$ excluding the trivial representation. For $[\xi] \in \widehat{G}$, we denote $|\xi| := \lambda_\xi \geqslant 0$, the eigenvalue of the operator $(-\mathcal{L}_G)^{1/2}$ corresponding to the representation ξ. For $[\xi] \in \widehat{G}_*$ we have $|\xi| > 0$ (see e.g. [6]), and for $[\xi] \in \widehat{G}\backslash\widehat{G}_*$, ξ is trivial and we have $|\xi| = 0$. From the definition, we have $|\xi| \leqslant \langle\xi\rangle$. On the other hand, let $\lambda_1^2 > 0$ be the smallest positive eigenvalue of $-\mathcal{L}_G$. Then, for $[\xi] \in \widehat{G}_*$ we have $\lambda_\xi \geqslant \lambda_1$, implying

$$1 + \lambda_\xi^2 \leqslant \left(\frac{1}{\lambda_1^2} + 1\right)\lambda_\xi^2,$$

so that altogether we record the inequalities

$$|\xi| \leqslant \langle\xi\rangle \leqslant \left(1 + \frac{1}{\lambda_1^2}\right)^{1/2} |\xi|, \quad \text{for all } [\xi] \in \widehat{G}_*. \tag{3.3}$$

We will need the following simple lemma which we prove for completeness. Let $a \in \mathbb{C}^{d\times d}$ be a matrix, and for $1 \leqslant p < \infty$ we denote by $\ell^p(\mathbb{C})$ the space of such matrices with the norm

$$\|a\|_{\ell^p(\mathbb{C})} := \left(\sum_{i,j=1}^{d} |a_{ij}|^p\right)^{1/p},$$

and for $p = \infty$,

$$\|a\|_{\ell^\infty(\mathbb{C})} := \sup_{1\leqslant i,j\leqslant d} |a_{ij}|.$$

We note that $\|a\|_{\ell^2(\mathbb{C})} = \|a\|_{\mathtt{HS}}$. We adopt the usual convention $\frac{c}{\infty} = 0$ for any $c \in \mathbb{R}$.

Lemma 3.2. *Let $1 \leqslant p < q \leqslant \infty$ and let $a \in \mathbb{C}^{d\times d}$. Then we have*

$$\|a\|_{\ell^p(\mathbb{C})} \leqslant d^{2(\frac{1}{p}-\frac{1}{q})}\|a\|_{\ell^q(\mathbb{C})} \quad \textit{and} \quad \|a\|_{\ell^q(\mathbb{C})} \leqslant d^{\frac{2}{q}}\|a\|_{\ell^p(\mathbb{C})}. \tag{3.4}$$

Proof. For $q < \infty$, we apply Hölder's inequality with $r = \frac{q}{p}$ and $r' = \frac{q}{q-p}$ to get

$$\|a\|_{\ell^p(\mathbb{C})}^p = \sum_{i,j=1}^{d} |a_{ij}|^p \leqslant \left(\sum_{i,j=1}^{d} |a_{ij}|^{pr}\right)^{1/r} \left(\sum_{i,j=1}^{d} 1\right)^{1/r'} = \|a\|_{\ell^q(\mathbb{C})}^p d^{2\frac{q-p}{q}},$$

implying (3.4) for this range. Conversely, we have

$$\|a\|^q_{\ell^q(\mathbb{C})} = \sum_{i,j=1}^{d} |a_{ij}|^q \leqslant \sum_{i,j=1}^{d} \|a\|^q_{\ell^p(\mathbb{C})} = d^2\|a\|^q_{\ell^p(\mathbb{C})},$$

proving the other part of (3.4) for this range. For $q = \infty$, we have $\|a\|_{\ell^p(\mathbb{C})} \leqslant (\sum_{i,j=1}^{d} \|a\|^p_{\ell^\infty(\mathbb{C})})^{1/p} \leqslant \|a\|_{\ell^\infty(\mathbb{C})} d^{2/p}$. Conversely, we have trivially $\|a\|_{\ell^\infty(\mathbb{C})} \leqslant \|a\|_{\ell^p(\mathbb{C})}$, completing the proof. □

We observe that the Gevrey spaces can be described in terms of L^2-norms, and this will be useful to us in the sequel.

Lemma 3.3. *We have $\phi \in \gamma_s(G)$ if and only if there exist constants $A > 0$ and $C > 0$ such that for all multi-indices α we have*

$$\|\partial^\alpha \phi\|_{L^2} \leqslant CA^{|\alpha|}(\alpha!)^s. \tag{3.5}$$

We also have $\phi \in \gamma_{(s)}(G)$ if and only if for every $A > 0$ there exists $C_A > 0$ such that for all multi-indices α we have

$$\|\partial^\alpha \phi\|_{L^2} \leqslant C_A A^{|\alpha|}(\alpha!)^s.$$

Proof. We prove the Gevrey–Roumieu case (R) as the Gevrey–Beurling case (B) is similar. For $\phi \in \gamma_s(G)$, (3.5) follows in view of the continuous embedding $L^\infty(G) \subset L^2(G)$ with $\|f\|_{L^2} \leqslant \|f\|_{L^\infty}$ since the measure is normalised.

Now suppose that for $\phi \in C^\infty(G)$ we have (3.5). In view of (2.3), and using Lemma 3.1 with an integer $k > n/2$, we obtain[5]

$$\begin{aligned}
\|\phi\|_{L^\infty} &\leqslant \sum_{[\xi]\in\widehat{G}} d_\xi^{3/2} \|\widehat{\phi}(\xi)\|_{\mathtt{HS}} \\
&\leqslant \Big(\sum_{[\xi]\in\widehat{G}} d_\xi \|\widehat{\phi}(\xi)\|^2_{\mathtt{HS}} \langle\xi\rangle^{2k}\Big)^{1/2} \Big(\sum_{[\xi]\in\widehat{G}} d_\xi^2 \langle\xi\rangle^{-2k}\Big)^{1/2} \\
&\leqslant C\|(I-\mathcal{L}_G)^{k/2}\phi\|_{L^2} \\
&\leqslant C_k \sum_{|\beta|\leqslant k} \|\partial^\beta \phi\|_{L^2},
\end{aligned}$$

with constant C_k depending only on G. Consequently we also have

$$\|\partial^\alpha \phi\|_{L^\infty} \leqslant C_k \sum_{|\beta|\leqslant k} \|\partial^{\alpha+\beta}\phi\|_{L^2}. \tag{3.6}$$

[5] Note that this can be adopted to give a simple proof of the Sobolev embedding theorem.

Using the inequalities

$$\alpha! \leqslant |\alpha|!, \quad |\alpha|! \leqslant n^{|\alpha|}\alpha! \quad \text{and} \quad \big(|\alpha|+k\big)! \leqslant 2^{|\alpha|+k}k!|\alpha|!, \tag{3.7}$$

in view of (3.6) and (3.5) we get

$$\begin{aligned}
\big\|\partial^\alpha\phi\big\|_{L^\infty} &\leqslant C_k A^{|\alpha|+k}\sum_{|\beta|\leqslant k}\big((\alpha+\beta)!\big)^s \\
&\leqslant C_k A^{|\alpha|+k}\sum_{|\beta|\leqslant k}\big(\big(|\alpha|+k\big)!\big)^s \\
&\leqslant C_k' A^{|\alpha|+k}\big(2^{|\alpha|+k}k!\big)^s\big(|\alpha|!\big)^s \\
&\leqslant C_k'' A_1^{|\alpha|}\big(n^{|\alpha|}\alpha!\big)^s \\
&\leqslant C_k'' A_2^{|\alpha|}(\alpha!)^s,
\end{aligned}$$

with constants C_k'' and A_2 independent of α, implying that $\phi \in \gamma_s(G)$ and completing the proof. □

The following proposition prepares the possibility to passing to the conditions formulated on the Fourier transform side.

Proposition 3.4. *We have $\phi \in \gamma_s(G)$ if and only if there exist constants $A > 0$ and $C > 0$ such that*

$$\big\|(-\mathcal{L}_G)^k\phi\big\|_{L^\infty} \leqslant CA^{2k}\big((2k)!\big)^s \tag{3.8}$$

holds for all $k \in \mathbb{N}_0$. Also, $\phi \in \gamma_{(s)}(G)$ if and only if for every $A > 0$ there exists $C_A > 0$ such that for all $k \in \mathbb{N}_0$ we have

$$\big\|(-\mathcal{L}_G)^k\phi\big\|_{L^\infty} \leqslant C_A A^{2k}\big((2k)!\big)^s.$$

Proof. We prove the Gevrey–Roumieu case (3.8) and indicate small additions to the argument for $\gamma_{(s)}(G)$. Thus, let $\phi \in \gamma_s(G)$. Recall that by the definition there exist some $A > 0$, $C > 0$ such that for all multi-indices α we have

$$\big\|\partial^\alpha\phi\big\|_{L^\infty} = \sup_{x\in G}\big|\partial^\alpha\phi(x)\big| \leqslant CA^{|\alpha|}(\alpha!)^s.$$

We will use the fact that for the compact Lie group G the Laplace–Beltrami operator $\mathcal{L}_G$ is given by $\mathcal{L}_G = X_1^2 + X_2^2 + \cdots + X_n^2$, where X_i, $i = 1, 2, \ldots, n$, is a set of left-invariant vector fields corresponding to a normalised basis of the Lie algebra of G. Then by the multinomial theorem[6] and using (3.7), with $Y_j \in \{X_1, \ldots, X_n\}$, $1 \leqslant j \leqslant |\alpha|$, we can estimate

$$\begin{aligned}
\left|(-\mathcal{L}_G)^k \phi(x)\right| &\leqslant C \sum_{|\alpha|=k} \frac{k!}{\alpha!} \left|Y_1^2 \ldots Y_{|\alpha|}^2 \phi(x)\right| \\
&\leqslant C \sum_{|\alpha|=k} \frac{k!}{\alpha!} \left[(2|\alpha|)!\right]^s A^{2|\alpha|} \\
&\leqslant C A^{2k} \left[(2k)!\right]^s \sum_{|\alpha|=k} \frac{k! n^{|\alpha|}}{|\alpha|!} \\
&\leqslant C_1 A^{2k} \left[(2k)!\right]^s n^k k^{n-1} \\
&\leqslant C_2 A_1^{2k} \left[(2k)!\right]^s, \qquad (3.9)
\end{aligned}$$

with $A_1 = 2nA$, implying (3.8). For the Gevrey–Beurling case $\gamma_{(s)}(G)$, we observe that we can obtain any $A_1 > 0$ in (3.9) by using $A = \frac{A_1}{2n}$ in the Gevrey estimates for $\phi \in \gamma_{(s)}(G)$.

Conversely, suppose $\phi \in C^\infty(G)$ is such that the inequalities (3.8) hold. First we note that for $|\alpha| = 0$ the estimate (2.7) follows from (3.8) with $k = 0$, so that we can assume $|\alpha| > 0$.

Following [14], we define the symbol of ∂^α to be $\sigma_{\partial^\alpha}(\xi) = \xi(x)^* \partial^\alpha \xi(x)$, and we have $\sigma_{\partial^\alpha}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ is independent of x since ∂^α is left-invariant. For the in-depth analysis of symbols and symbolic calculus for general operators on G we refer to [13,14] but we will use only basic things here. In particular, we have

$$\partial^\alpha \phi(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}\left(\xi(x) \sigma_{\partial^\alpha}(\xi) \widehat{\phi}(\xi)\right).$$

First we calculate the operator norm $\|\sigma_{\partial^\alpha}(\xi)\|_{op}$ of the matrix multiplication by $\sigma_{\partial^\alpha}(\xi)$. Since $\partial^\alpha = Y_1 \cdots Y_{|\alpha|}$ and $Y_j \in \{X_1, \ldots, X_n\}$ are all left-invariant, we have $\sigma_{\partial^\alpha} = \sigma_{Y_1} \cdots \sigma_{Y_{|\alpha|}}$, so that we get

$$\left\|\sigma_{\partial^\alpha}(\xi)\right\|_{op} \leqslant \left\|\sigma_{X_1}(\xi)\right\|_{op}^{\alpha_1} \cdots \left\|\sigma_{X_n}(\xi)\right\|_{op}^{\alpha_n}.$$

Now, since X_j are operators of the first order, one can show (see e.g. [14, Lemma 8.6], or [13, Section 10.9.1] for general arguments) that $\|\sigma_{X_j}(\xi)\|_{op} \leqslant C_j \langle \xi \rangle$ for some constants C_j, $j = 1, \ldots, n$. Let $C_0 := \sup_j C_j + 1$, then we have

$$\left\|\sigma_{\partial^\alpha}(\xi)\right\|_{op} \leqslant C_0^{|\alpha|} \langle \xi \rangle^{|\alpha|}. \qquad (3.10)$$

[6] The form in which we use it is adapted to non-commutativity of vector fields. Namely, although the coefficients are all equal to one in the non-commutative form, the multinomial coefficient appears once we make a choice for $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Let us define $\sigma_{P_\alpha} \in \Sigma$ by setting $\sigma_{P_\alpha}(\xi) := |\xi|^{-2k}\sigma_{\partial^\alpha}(\xi)$ for $[\xi] \in \widehat{G}_*$, and by $\sigma_{P_\alpha}(\xi) := 0$ for $[\xi] \in \widehat{G}\backslash\widehat{G}_*$. This gives the corresponding operator

$$(P_\alpha\phi)(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}\big(\xi(x)\sigma_{P_\alpha}(\xi)\widehat{\phi}(\xi)\big). \tag{3.11}$$

From (3.10) we obtain

$$\big\|\sigma_{P_\alpha}(\xi)\big\|_{op} \leqslant C_0^{|\alpha|}\langle\xi\rangle^{|\alpha|}|\xi|^{-2k} \quad \text{for all } [\xi] \in \widehat{G}_*. \tag{3.12}$$

Now, for $[\xi] \in \widehat{G}_*$, from (3.3) we have

$$|\xi|^{-2k} \leqslant C_1^{2k}\langle\xi\rangle^{-2k}, \quad C_1 = \left(1 + \frac{1}{\lambda_1^2}\right)^{1/2}.$$

Together with (3.12), and the trivial estimate for $[\xi] \in \widehat{G}\backslash\widehat{G}_*$, we obtain

$$\big\|\sigma_{P_\alpha}(\xi)\big\|_{op} \leqslant C_0^{|\alpha|}C_1^{2k}\langle\xi\rangle^{|\alpha|-2k} \quad \text{for all } [\xi] \in \widehat{G}. \tag{3.13}$$

Using (3.11) and the Plancherel identity, we estimate

$$\begin{aligned}
\big|P_\alpha\phi(x)\big| &\leqslant \sum_{[\xi]\in\widehat{G}} d_\xi\big\|\xi(x)\sigma_{P_\alpha}(\xi)\big\|_{\mathtt{HS}}\big\|\widehat{\phi}(\xi)\big\|_{\mathtt{HS}} \\
&\leqslant \left(\sum_{[\xi]\in\widehat{G}} d_\xi\big\|\widehat{\phi}(\xi)\big\|_{\mathtt{HS}}^2\right)^{1/2}\left(\sum_{[\xi]\in\widehat{G}} d_\xi\big\|\sigma_{P_\alpha}(\xi)\big\|_{op}^2\big\|\xi(x)\big\|_{\mathtt{HS}}^2\right)^{1/2} \\
&= \|\phi\|_{L^2}\left(\sum_{[\xi]\in\widehat{G}} d_\xi^2\big\|\sigma_{P_\alpha}(\xi)\big\|_{op}^2\right)^{1/2}.
\end{aligned}$$

From this and (3.13) we conclude that

$$\big|P_\alpha\phi(x)\big| \leqslant \|\phi\|_{L^2}C_0^{|\alpha|}C_1^{2k}\left(\sum_{[\xi]\in\widehat{G}} d_\xi^2\langle\xi\rangle^{-2(2k-|\alpha|)}\right)^{1/2}.$$

Now, in view of Lemma 3.1 the series on the right hand side converges provided that $2k - |\alpha| > n/2$. Therefore, for $2k - |\alpha| > n/2$ we obtain

$$\|P_\alpha\phi\|_{L^2} \leqslant CC_2^{2k}\|\phi\|_{L^2}, \tag{3.14}$$

with some C and $C_2 = C_0C_1$ independent of k and α. We note that here we used that $|\alpha| \leqslant 2k$ and that we can always have $C_0 \geqslant 1$.

We now observe that from the definition of σ_{P_α} we have

$$\sigma_{\partial^\alpha}(\xi) = \sigma_{P_\alpha}(\xi)|\xi|^{2k} \tag{3.15}$$

for all $[\xi] \in \widehat{G}_*$. On the other hand, since we assumed $|\alpha| \neq 0$, for $[\xi] \in \widehat{G} \backslash \widehat{G}_*$ we have $\sigma_{\partial^\alpha}(\xi) = \xi(x)^* \partial^\alpha \xi(x) = 0$, so that (3.15) holds true for all $[\xi] \in \widehat{G}$. This implies that in the operator sense, we have $\partial^\alpha = P_\alpha \circ (-\mathcal{L}_G)^k$. Therefore, from this relation and (3.14), for $|\alpha| < 2k - n/2$, we get

$$\begin{aligned}
\left\|\partial^\alpha \phi\right\|_{L^2}^2 &= \left\|P_\alpha \circ (-\mathcal{L}_G)^k \phi\right\|_{L^2}^2 \\
&\leqslant C C_2^{4k} \int\limits_G \left|(-\mathcal{L}_G)^k \phi(x)\right|^2 dx \\
&\leqslant C' C_2^{4k} A^{4k} \left((2k)!\right)^{2s} \\
&\leqslant C' A_1^{4k} \left((2k)!\right)^{2s},
\end{aligned}$$

where we have used the assumption (3.8), and with C' and $A_1 = C_2 A$ independent of k and α. Hence we have $\|\partial^\alpha \phi\|_{L^2} \leqslant C A_1^{2k}((2k)!)^s$ for all $|\alpha| < 2k - n/2$. Then, for every β, by the above argument, taking an integer k such that $|\beta| + 4n \geqslant 2k > |\beta| + n/2$, if $A_1 \geqslant 1$, we obtain

$$\left\|\partial^\beta \phi\right\|_{L^2} \leqslant C A_1^{|\beta|+4n} \left((|\beta| + 4n)!\right)^s \leqslant C' A_1^{|\beta|} \left(2^{|\beta|+4n} (4n)! |\beta|!\right)^s \leqslant C'' A_2^{|\beta|} (\beta!)^s,$$

in view of inequalities (3.7). By Lemma 3.3 it follows that $\phi \in \gamma_s(G)$.

If $A_1 < 1$ (in the case of $\gamma_{(s)}(G)$), we estimate

$$\left\|\partial^\beta \phi\right\|_{L^2} \leqslant C A_1^{|\beta|+n/2} \left((|\beta| + 4n)!\right)^s \leqslant C'' A_3^{|\beta|} (\beta!)^s$$

by a similar argument. The relation between constants, namely $A_1 = C_2 A$ and $A_3 = 2n A_1$, implies that the case of $\gamma_{(s)}(G)$ also holds true. □

We can now pass to the Fourier transform side.

Lemma 3.5. *For $\phi \in \gamma_s(G)$, there exist constants $C > 0$ and $A > 0$ such that*

$$\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant C d_\xi^{1/2} |\xi|^{-2m} A^{2m} \left((2m)!\right)^s \tag{3.16}$$

holds for all $m \in \mathbb{N}_0$ and $[\xi] \in \widehat{G}_*$. Also, for $\phi \in \gamma_{(s)}(G)$, for every $A > 0$ there exists $C_A > 0$ such that

$$\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant C_A d_\xi^{1/2} |\xi|^{-2m} A^{2m} \left((2m)!\right)^s$$

holds for all $m \in \mathbb{N}_0$ and $[\xi] \in \widehat{G}_*$.

Proof. We will treat the case γ_s since $\gamma_{(s)}$ is analogous. Using the fact that the Fourier transform is a bounded linear operator from $L^1(G)$ to $l^\infty(\widehat{G})$, see (2.3), and using Proposition 3.4, we obtain

$$\begin{aligned}\big\||\xi|^{2m}\widehat{\phi}(\xi)\big\|_{l^\infty(\widehat{G})} &\leqslant \int_G \big|(-\mathcal{L}_G)^m\phi(x)\big|\,dx\\ &\leqslant CA^{2m}\big((2m)!\big)^s\end{aligned}$$

for all $[\xi]\in\widehat{G}$ and $m\in\mathbb{N}_0$. Recalling the definition of $\ell^\infty(\widehat{G})$ in (2.2) we obtain (3.16). □

We can now prove Theorem 2.3.

Proof of Theorem 2.3. (**R**) "Only if" part.

Let $\phi\in\gamma_s(G)$. Using $k!\leqslant k^k$ and Lemma 3.5 we get

$$\big\|\widehat{\phi}(\xi)\big\|_{\mathtt{HS}} \leqslant Cd_\xi^{1/2}\inf_{2m\geqslant 0}|\xi|^{-2m}A^{2m}(2m)^{2ms} \tag{3.17}$$

for all $[\xi]\in\widehat{G}_*$. We will show that this implies the (sub-)exponential decay in (2.7). It is known that for $r>0$, we have the identity

$$\inf_{x>0} x^{sx}r^{-x} = e^{-(s/e)r^{1/s}}. \tag{3.18}$$

So for a given $r>0$ there exists some $x_0=x_0(r)>0$ such that

$$\inf_{x>0} x^{sx}\left(\frac{r}{8^s}\right)^{-x} = x_0^{sx_0}\left(\frac{r}{8^s}\right)^{-x_0}. \tag{3.19}$$

We will be interested in large r, in fact we will later set $r=\frac{|\xi|}{A}$, so we can assume that r is large. Suppose, we can take an even positive integer m_0 such that $m_0\leqslant x_0<m_0+2$. We also note that since in what follows we will be interested in estimating the infima from above, this assumption is not restrictive. Using the trivial inequalities

$$(m_0)^{sm_0}r^{-(m_0+2)}\leqslant x_0^{sx_0}r^{-x_0}, \quad r\geqslant 1,$$

and

$$(k+2)^{k+2}\leqslant 8^kk^k$$

for any $k\geqslant 2$, we obtain

$$(m_0+2)^{s(m_0+2)}r^{-(m_0+2)}\leqslant 8^{sm_0}m_0^{sm_0}r^{-(m_0+2)}\leqslant x_0^{sx_0}\left(\frac{r}{8^s}\right)^{-x_0}.$$

It follows from this, (3.18) and (3.19), that

$$\inf_{2m\geqslant 0}(2m)^{2sm}r^{-2m}\leqslant x_0^{sx_0}\left(\frac{r}{8^s}\right)^{-x_0}=e^{-(s/e)(\frac{r}{8^s})^{1/s}}. \tag{3.20}$$

Let now $r=\frac{|\xi|}{A}$. From (3.17) and (3.20) we obtain

$$\begin{aligned}\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}} &\leqslant Cd_\xi^{1/2}\inf_{2m\geqslant 0}\frac{A^{2m}}{|\xi|^{2m}}(2m)^{2ms}\\ &= Cd_\xi^{1/2}\inf_{2m\geqslant 0}r^{-2m}(2m)^{2ms}\\ &\leqslant Cd_\xi^{1/2}e^{-(s/e)(\frac{r}{8^s})^{1/s}}\\ &= Cd_\xi^{1/2}e^{-(s/e)\frac{|\xi|^{1/s}}{8A^{1/s}}}\\ &\leqslant Cd_\xi^{1/2}e^{-2B|\xi|^{1/s}},\end{aligned} \tag{3.21}$$

with $2B=\frac{s}{8e}\frac{1}{A^{1/s}}$. From (3.2) it follows that $d_\xi^{1/2}e^{-B|\xi|^{1/s}}\leqslant C$. Using (3.3), we obtain (2.7) for all $[\xi]\in\widehat{G}_*$. On the other hand, for trivial $[\xi]\in\widehat{G}\backslash\widehat{G}_*$ the estimate (2.7) is just the condition of the boundedness. This completes the proof of the "only if" part.

Now we prove the "if" part. Suppose $\phi\in C^\infty(G)$ is such that (2.7) holds, i.e. we have

$$\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}}\leqslant Ke^{-B\langle\xi\rangle^{1/s}}.$$

The $\ell^1(\widehat{G})\to L^\infty(G)$ boundedness of the inverse Fourier transform in (2.3) implies

$$\begin{aligned}\left\|(-\mathcal{L}_G)^k\phi\right\|_{L^\infty(G)} &\leqslant \left\||\xi|^{2k}\widehat{\phi}\right\|_{\ell^1(\widehat{G})}\\ &= \sum_{[\xi]\in\widehat{G}}d_\xi^{3/2}|\xi|^{2k}\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}}\\ &\leqslant K\sum_{[\xi]\in\widehat{G}}d_\xi^{3/2}\langle\xi\rangle^{2k}e^{-B\langle\xi\rangle^{1/s}}\\ &\leqslant K\sum_{[\xi]\in\widehat{G}}d_\xi^{3/2}e^{\frac{-B\langle\xi\rangle^{1/s}}{2}}\left(\langle\xi\rangle^{2k}e^{\frac{-B\langle\xi\rangle^{1/s}}{2}}\right).\end{aligned} \tag{3.22}$$

Now we will use the following simple inequality, $\frac{t^N}{N!}\leqslant e^t$ for $t>0$. Setting later $m=2k$ and $a=\frac{B}{2}$, we estimate

$$(m!)^{-s}\langle\xi\rangle^m=\left(\frac{(a\langle\xi\rangle^{1/s})^m}{m!}\right)^s a^{-sm}\leqslant a^{-sm}e^{a\langle\xi\rangle^{1/s}},$$

which implies $e^{-\frac{B}{2}\langle\xi\rangle^{1/s}}\langle\xi\rangle^{2k} \leqslant A^{2k}((2k)!)^s$, with $A = a^{-s} = (2/B)^s$. Using this inequality and (3.22) we obtain

$$\left\|(-\mathcal{L}_G)^k\phi\right\|_{L^\infty} \leqslant K \sum_{[\xi]\in\widehat{G}} d_\xi^{3/2} e^{\frac{-B\langle\xi\rangle^{1/s}}{2}} A^{2k}\left((2k)!\right)^s \leqslant CA^{2k}\left((2k)!\right)^s \tag{3.23}$$

with $A = \frac{2^s}{B^s}$, where the convergence of the series in $[\xi]$ follows from Lemma 3.1. Therefore, $\phi \in \gamma_s(G)$ by Proposition 3.4.

(**B**) "Only if" part. Suppose $\phi \in \gamma_{(s)}(G)$. For any given $B > 0$ define A by solving $2B = (\frac{s}{8e})\frac{1}{A^{1/s}}$. By Lemma 3.5 there exists $K_B > 0$ such that

$$\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant K_B d_\xi^{1/2} \inf_{2m\geqslant 0} |\xi|^{-2m} A^{2m}(2m)^{2ms}.$$

Consequently, arguing as in case (**R**) we get (3.21), i.e.

$$\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant K_B d_\xi^{1/2} e^{-2B|\xi|^{1/s}}$$

for all $[\xi] \in \widehat{G}$. The same argument as in the case (**R**) now completes the proof.

"If" part. For a given $A > 0$ define $B > 0$ by solving $A = \frac{2^s}{B^s}$ and take C_A big enough as in the case of (**R**), so that we get

$$\left\|(-\mathcal{L}_G)^k\phi\right\|_{L^\infty} \leqslant C_A A^{2k}\left((2k)!\right)^s.$$

Therefore, $\phi \in \gamma_{(s)}(G)$ by Proposition 3.4. □

4. α-duals $\gamma_s(G)^\wedge$ and $\gamma_{(s)}(G)^\wedge$, for any s, $0 < s < \infty$

First we analyse α-duals of Gevrey spaces regarded as sequence spaces through their Fourier coefficients.

We can embed $\gamma_s(G)$(or $\gamma_{(s)}(G)$) in the sequence space Σ using the Fourier coefficients and Theorem 2.3. We denote the *α-dual* of such the sequence space $\gamma_s(G)$ (or $\gamma_{(s)}(G)$) as

$$\left[\gamma_s(G)\right]^\wedge := \left\{v = (v_\xi)_{[\xi]\in\widehat{G}} \in \Sigma\colon \sum_{[\xi]\in\widehat{G}} \sum_{i,j=1}^{d_\xi} \left|(v_\xi)_{ij}\right|\left|\widehat{\phi}(\xi)_{ij}\right| < \infty \text{ for all } \phi \in \gamma_s(G)\right\},$$

with a similar definition for $\gamma_{(s)}(G)$.

Lemma 4.1.

(**R**) *We have $v \in [\gamma_s(G)]^\wedge$ if and only if for every $B > 0$ we have the inequality*

$$\sum_{[\xi]\in\widehat{G}} e^{-B\langle\xi\rangle^{\frac{1}{s}}}\|v_\xi\|_{\mathtt{HS}} < \infty. \tag{4.1}$$

(**B**) *Also, we have $v \in [\gamma_{(s)}(G)]^\wedge$ if and only if there exists $B > 0$ such that the inequality (4.1) holds.*

The proof of this lemma in (R) and (B) cases will be different. For (R) we can show this directly, and for (B) we employ the theory of echelon spaces by Köthe [9].

Proof. (**R**) "Only if" part. Let $v \in [\gamma_s(G)]^\wedge$. For any $B > 0$, define ϕ by setting its Fourier coefficients to be $\widehat{\phi}(\xi)_{ij} := d_\xi e^{-B\langle\xi\rangle^{\frac{1}{s}}}$, so that $\|\widehat{\phi}(\xi)\|_{\mathtt{HS}} = d_\xi^2 e^{-B\langle\xi\rangle^{\frac{1}{s}}} \leqslant Ce^{-\frac{B}{2}\langle\xi\rangle^{\frac{1}{s}}}$ by (3.2), which implies that $\phi \in \gamma_s(G)$ by Theorem 2.3. Using Lemma 3.2, we obtain

$$\sum_{[\xi]\in\widehat{G}} e^{-B\langle\xi\rangle^{\frac{1}{s}}} \|v_\xi\|_{\mathtt{HS}} \leqslant \sum_{[\xi]\in\widehat{G}} d_\xi e^{-B\langle\xi\rangle^{\frac{1}{s}}} \|v_\xi\|_{\ell^1(\mathbb{C})} = \sum_{[\xi]\in\widehat{G}} \sum_{i,j=1}^{d_\xi} \left|(v_\xi)_{ij}\right| \left|\widehat{\phi}(\xi)_{ij}\right| < \infty$$

by the assumption $v \in [\gamma_s(G)]^\wedge$, proving the "only if" part.

"If" part. Let $\phi \in \gamma_s(G)$. Then by Theorem 2.3 there exist some $B > 0$ and $C > 0$ such that

$$\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant Ce^{-B\langle\xi\rangle^{\frac{1}{s}}},$$

which implies that

$$\sum_{[\xi]\in\widehat{G}} \sum_{i,j=1}^{d_\xi} \left|(v_\xi)_{ij}\right| \left|\widehat{\phi}(\xi)_{ij}\right| \leqslant \sum_{[\xi]\in\widehat{G}} \|v_\xi\|_{\mathtt{HS}} \left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant C \sum_{[\xi]\in\widehat{G}} e^{-B\langle\xi\rangle^{\frac{1}{s}}} \|v_\xi\|_{\mathtt{HS}} < \infty$$

is finite by the assumption (4.1). But this means that $v \in [\gamma_s(G)]^\wedge$.

(**B**) For any $B > 0$ we consider the so-called echelon space,

$$E_B := \left\{ v = (v_\xi) \in \Sigma \colon \sum_{[\xi]\in\widehat{G}} \sum_{i,j=1}^{d_\xi} e^{-B\langle\xi\rangle^{\frac{1}{s}}} \left|(v_\xi)_{ij}\right| < \infty \right\}.$$

Now, by diagonal transform we have $E_B \cong l^1$ and hence $\widehat{E_B} \cong l^\infty$, and it is easy to check that $\widehat{E_B}$ is given by

$$\widehat{E_B} = \left\{ w = (w_\xi) \in \Sigma \mid \exists K > 0 \colon \left|(w_\xi)_{ij}\right| \leqslant Ke^{-B\langle\xi\rangle^{1/s}} \text{ for all } 1 \leqslant i,j \leqslant d_\xi \right\}.$$

By Theorem 2.3 we know that $\phi \in \gamma_{(s)}(G)$ if and only if $(\widehat{\phi}(\xi))_{[\xi]\in\widehat{G}} \in \bigcap_{B>0} \widehat{E_B}$. Using Köthe's theory relating echelon and co-echelon spaces [9, Ch. 30.8], we have, consequently, that $v \in \gamma_{(s)}(G)^\wedge$ if and only if $(v_\xi)_{[\xi]\in\widehat{G}} \in \bigcup_{B>0} E_B$. But this means that for some $B > 0$

we have

$$\sum_{[\xi]\in\widehat{G}}\sum_{i,j=1}^{d_\xi} e^{-B\langle\xi\rangle^{\frac{1}{s}}}\left|(v_\xi)_{ij}\right| < \infty.$$

Finally, we observe that this is equivalent to (4.1) if we use Lemma 3.2 and (3.2). $\square$

We now give the characterisation for α-duals.

Theorem 4.2. *Let* $0 < s < \infty$.

(**R**) *We have* $v \in [\gamma_s(G)]^\wedge$ *if and only if for every* $B > 0$ *there exists* $K_B > 0$ *such that*

$$\|v_\xi\|_{\mathtt{HS}} \leqslant K_B e^{B\langle\xi\rangle^{\frac{1}{s}}} \tag{4.2}$$

holds for all $[\xi] \in \widehat{G}$.

(**B**) *We have* $v \in [\gamma_{(s)}(G)]^\wedge$ *if and only if there exist* $B > 0$ *and* $K_B > 0$ *such that* (4.2) *holds for all* $[\xi] \in \widehat{G}$.

Proof. We prove the case (R) only since the proof of (B) is similar. First we deal with "If" part. Let $v \in \Sigma$ be such that (4.2) holds for every $B > 0$. Let $\varphi \in \gamma_s(G)$. Then by Theorem 2.3 there exist some constants $A > 0$ and $C > 0$ such that $\|\widehat{\phi}(\xi)\|_{\mathtt{HS}} \leqslant Ce^{-A\langle\xi\rangle^{1/s}}$. Taking $B = A/2$ in (4.2) we get that

$$\sum_{[\xi]\in\widehat{G}}\sum_{i,j=1}^{d_\xi}\left|(v_\xi)_{ij}\right|\left|\widehat{\phi}(\xi)_{ij}\right| \leqslant \sum_{[\xi]\in\widehat{G}}\|v_\xi\|_{\mathtt{HS}}\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant CK_B\sum_{[\xi]\in\widehat{G}} e^{-\frac{A}{2}\langle\xi\rangle^{1/s}} < \infty,$$

so that $v \in [\gamma_s(G)]^\wedge$.

"Only if" part. Let $v \in [\gamma_s(G)]^\wedge$ and let $B > 0$. Then by Lemma 4.1 we have that

$$\sum_{[\xi]\in\widehat{G}} e^{-B\langle\xi\rangle^{1/s}}\|v_\xi\|_{\mathtt{HS}} < \infty.$$

This implies that the exists a constant $K_B > 0$ such that $e^{-B\langle\xi\rangle^{1/s}}\|v(\xi)\|_{\mathtt{HS}} \leqslant K_B$, yielding (4.2). $\square$

We now want to show that the Gevrey spaces are perfect in the sense of Köthe. We define the α-dual of $[\gamma_s(G)]^\wedge$ as

$$\widehat{[\gamma_s(G)]}^\wedge = \left\{w = (w_\xi)_{[\xi]\in\widehat{G}} \in \Sigma : \sum_{[\xi]\in\widehat{G}}\sum_{i,j=1}^{d_\xi}\left|(w_\xi)_{ij}\right|\left|(v_\xi)_{ij}\right| < \infty \text{ for all } v \in \left[\gamma_s(G)\right]^\wedge\right\},$$

and similarly for $[\gamma_{(s)}(G)]^\wedge$. First, we prove the following lemma.

Lemma 4.3.

(**R**) *We have* $w \in \widehat{[\gamma_s(G)]}^{\wedge}$ *if and only if there exists* $B > 0$ *such that*

$$\sum_{[\xi]\in\widehat{G}} e^{B\langle\xi\rangle^{\frac{1}{s}}} \|w_\xi\|_{\mathtt{HS}} < \infty. \tag{4.3}$$

(**B**) *We have* $w \in \widehat{[\gamma_{(s)}(G)]}^{\wedge}$ *if and only if for every* $B > 0$ *the series* (4.3) *converges.*

Proof. We first show the Beurling case as it is more straightforward.

(**B**) "Only if" part. We assume that $w \in \widehat{[\gamma_{(s)}(G)]}^{\wedge}$. Let $B > 0$, and define $(v_\xi)_{ij} := d_\xi e^{B\langle\xi\rangle^{\frac{1}{s}}}$. Then $\|v_\xi\|_{\mathtt{HS}} = d_\xi^2 e^{B\langle\xi\rangle^{\frac{1}{s}}} \leqslant Ce^{2B\langle\xi\rangle^{\frac{1}{s}}}$ by (3.2), which implies $v \in [\gamma_{(s)}(G)]^{\wedge}$ by Theorem 4.2. Consequently, using Lemma 3.2 we can estimate

$$\sum_{[\xi]\in\widehat{G}} e^{B\langle\xi\rangle^{\frac{1}{s}}} \|w_\xi\|_{\mathtt{HS}} \leqslant \sum_{[\xi]\in\widehat{G}} d_\xi e^{B\langle\xi\rangle^{\frac{1}{s}}} \sum_{i,j=1}^{d_\xi} \left|(w_\xi)_{ij}\right| = \sum_{[\xi]\in\widehat{G}} \sum_{i,j=1}^{d_\xi} \left|(v_\xi)_{ij}\right|\left|(w_\xi)_{ij}\right| < \infty,$$

implying (4.3).

"If" part. Here we are given $w \in \Sigma$ such that for every $B > 0$ the series (4.3) converges. Let us take any $v \in [\gamma_{(s)}(G)]^{\wedge}$. By Theorem 4.2 there exist $B > 0$ and $K > 0$ such that $\|v_\xi\|_{\mathtt{HS}} \leqslant Ke^{B\langle\xi\rangle^{\frac{1}{s}}}$. Consequently, we can estimate

$$\sum_{[\xi]\in\widehat{G}} \sum_{i,j=1}^{d_\xi} \left|(v_\xi)_{ij}\right|\left|(w_\xi)_{ij}\right| \leqslant \sum_{[\xi]\in\widehat{G}} \|v_\xi\|_{\mathtt{HS}}\|w_\xi\|_{\mathtt{HS}} \leqslant K \sum_{[\xi]\in\widehat{G}} e^{B\langle\xi\rangle^{\frac{1}{s}}} \|w_\xi\|_{\mathtt{HS}} < \infty$$

by the assumption (4.3), which shows that $w \in \widehat{[\gamma_{(s)}(G)]}^{\wedge}$.

(**R**) For $B > 0$ we consider the echelon space

$$D_B := \left\{v = (v_\xi) \in \Sigma \;\middle|\; \exists K > 0\colon \left|(v_\xi)_{ij}\right| \leqslant Ke^{B\langle\xi\rangle^{1/s}} \text{ for all } 1 \leqslant i,j \leqslant d_\xi\right\}.$$

By diagonal transform we have $D_B \cong l^\infty$, and since l^∞ is a perfect sequence space, we have $\widehat{D_B} \cong l^1$, and it is given by

$$\widehat{D_B} = \left\{w = (w_\xi) \in \Sigma\colon \sum_{[\xi]\in\widehat{G}} \sum_{i,j=1}^{d_\xi} e^{B\langle\xi\rangle^{\frac{1}{s}}} \left|(w_\xi)_{ij}\right| < \infty\right\}.$$

By Theorem 4.2 we know that $\gamma_s(G)^{\wedge} = \bigcap_{B>0} D_B$, and hence $\widehat{[\gamma_s(G)]}^{\wedge} = \bigcup_{B>0} \widehat{D_B}$. This means that $w \in \widehat{[\gamma_s(G)]}^{\wedge}$ if and only if there exists $B > 0$ such that we have $\sum_{[\xi]\in\widehat{G}} \sum_{i,j=1}^{d_\xi} e^{2B\langle\xi\rangle^{\frac{1}{s}}} |(w_\xi)_{ij}| < \infty$. Consequently, by Lemma 3.2 we get

$$\sum_{[\xi]\in\widehat{G}} e^{B\langle\xi\rangle^{\frac{1}{s}}}\|w_\xi\|_{\mathtt{HS}} \leqslant \sum_{[\xi]\in\widehat{G}} d_\xi e^{B\langle\xi\rangle^{\frac{1}{s}}}\|w_\xi\|_{\ell^1(\mathbb{C})} \leqslant C\sum_{[\xi]\in\widehat{G}}\sum_{i,j=1}^{d_\xi} e^{2B\langle\xi\rangle^{\frac{1}{s}}}\left|(w_\xi)_{ij}\right| < \infty,$$

completing the proof of the "only if" part. Conversely, given (4.3) for some $2B > 0$, we have

$$\sum_{[\xi]\in\widehat{G}}\sum_{i,j=1}^{d_\xi} e^{B\langle\xi\rangle^{\frac{1}{s}}}\left|(w_\xi)_{ij}\right| \leqslant \sum_{[\xi]\in\widehat{G}} d_\xi e^{B\langle\xi\rangle^{\frac{1}{s}}}\|w_\xi\|_{\mathtt{HS}} \leqslant C\sum_{[\xi]\in\widehat{G}} e^{2B\langle\xi\rangle^{\frac{1}{s}}}\|w_\xi\|_{\mathtt{HS}} < \infty,$$

implying $w \in [\widehat{\gamma_s(G)}]^\wedge$. $\square$

Now we can show that the Gevrey spaces are perfect spaces (sometimes called Köthe spaces).

Theorem 4.4. *$\gamma_s(G)$ and $\gamma_{(s)}(G)$ are perfect spaces, that is, $\gamma_s(G) = [\widehat{\gamma_s(G)}]^\wedge$ and $\gamma_{(s)}(G) = [\widehat{\gamma_{(s)}(G)}]^\wedge$.*

Proof. We will show this for $\gamma_s(G)$ since the proof for $\gamma_{(s)}(G)$ is analogous. From the definition of $[\widehat{\gamma_s(G)}]^\wedge$ we have $\gamma_s(G) \subseteq [\widehat{\gamma_s(G)}]^\wedge$. We will prove the other direction, i.e., $[\widehat{\gamma_s(G)}]^\wedge \subseteq \gamma_s(G)$. Let $w = (w_\xi)_{[\xi]\in\widehat{G}} \in [\widehat{\gamma_s(G)}]^\wedge$ and define

$$\phi(x) := \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}\bigl(w_\xi \xi(x)\bigr).$$

The series makes sense due to Lemma 4.3, and we have $\|\widehat{\phi}(\xi)\|_{\mathtt{HS}} = \|w_\xi\|_{\mathtt{HS}}$. Now since $w \in [\widehat{\gamma_s(G)}]^\wedge$ by Lemma 4.3 there exists $B > 0$ such that $\sum_{[\xi]\in\widehat{G}} e^{B\langle\xi\rangle^{\frac{1}{s}}}\|w_\xi\|_{\mathtt{HS}} < \infty$, which implies that for some $C > 0$ we have

$$e^{B\langle\xi\rangle^{1/s}}\|w_\xi\|_{\mathtt{HS}} < C \quad\Rightarrow\quad \bigl\|\widehat{\phi}(\xi)\bigr\|_{\mathtt{HS}} \leqslant Ce^{-B\langle\xi\rangle^{1/s}}.$$

By Theorem 2.3 this implies $\phi \in \gamma_s(G)$. Hence $\gamma_s(G) = [\widehat{\gamma_s(G)}]^\wedge$, i.e. $\gamma_s(G)$ is a perfect space. $\square$

5. Ultradistributions $\gamma_s'(G)$ and $\gamma_{(s)}'(G)$

Here we investigate the Fourier coefficients criteria for spaces of ultradistributions. The space $\gamma_s'(G)$ (resp. $\gamma_{(s)}'(G)$) of the ultradistributions of order s is defined as the dual of $\gamma_s(G)$ (resp. $\gamma_{(s)}(G)$) endowed with the standard inductive limit topology of $\gamma_s(G)$ (resp. the projective limit topology of $\gamma_{(s)}(G)$).

Definition 5.1. The space $\gamma_s'(G)$(resp. $\gamma_{(s)}'(G)$) is the set of the linear forms u on $\gamma_s(G)$(resp. $\gamma_{(s)}(G)$) such that for every $\epsilon > 0$ there exists C_ϵ such that (resp. for some $\epsilon > 0$ and $C > 0$) we have

$$\left|u(\phi)\right| \leqslant C_\epsilon \sup_\alpha \epsilon^{|\alpha|}(\alpha!)^{-s} \sup_{x\in G}\left|(-\mathcal{L}_G)^{|\alpha|/2}\phi(x)\right|$$

holds for all $\phi \in \gamma_s(G)$ (resp. $\phi \in \gamma_{(s)}(G)$).

We can take the Laplace–Beltrami operator in Definition 5.1 because of the equivalence of norms given by Proposition 3.4.

We recall that for any $v \in \gamma_s'(G)$, for $[\xi] \in \widehat{G}$, we define the Fourier coefficients $\widehat{v}(\xi) := \langle v, \xi^*\rangle \equiv v(\xi^*)$.

We have the following theorem showing that topological and α-duals of Gevrey spaces coincide.

Theorem 5.2. *Let $1 \leqslant s < \infty$. Then $v \in \gamma_s'(G)$(resp. $\gamma_{(s)}'(G)$) if and only if $v \in \gamma_s(G)^\wedge$(resp. $\gamma_{(s)}(G)^\wedge$).*

Proof. (**R**) "If" part. Let $v \in \gamma_s(G)^\wedge$. For any $\phi \in \gamma_s(G)$ define

$$v(\phi) := \sum_{[\xi]\in\widehat{G}} d_\xi \operatorname{Tr}\bigl(\widehat{\phi}(\xi)v_\xi\bigr). \tag{5.1}$$

Since by Theorem 2.3 there exists some $B > 0$ such that $\|\widehat{\phi}(\xi)\|_{\mathtt{HS}} \leqslant Ce^{-B\langle\xi\rangle^{1/s}}$, we can estimate

$$\left|v(\phi)\right| \leqslant \sum_{[\xi]\in\widehat{G}} d_\xi\left|\operatorname{Tr}\bigl(\widehat{\phi}(\xi)v_\xi\bigr)\right| \leqslant \sum_{[\xi]\in\widehat{G}} d_\xi\left\|\widehat{\phi}(\xi)\right\|_{\mathtt{HS}}\|v_\xi\|_{\mathtt{HS}} \leqslant C\sum_{[\xi]\in\widehat{G}} d_\xi e^{-B\langle\xi\rangle^{1/s}}\|v_\xi\|_{\mathtt{HS}} < \infty$$

by Lemma 4.1 and (3.2). Therefore, $v(\phi)$ in (5.1) is a well-defined linear functional on $\gamma_s(G)$. It remains to check that v is continuous. Suppose $\phi_j \to \phi$ in $\gamma_s(G)$ as $j \to \infty$, that is, in view of Proposition 3.4, there is a constant $A > 0$ such that

$$\sup_\alpha A^{-|\alpha|}(\alpha!)^{-s}\sup_{x\in G}\left|(-\mathcal{L}_G)^{|\alpha|/2}\bigl(\phi_j(x) - \phi(x)\bigr)\right| \to 0$$

as $j \to \infty$. It follows that

$$\left\|(-\mathcal{L}_G)^{|\alpha|/2}(\phi_j - \phi)\right\|_\infty \leqslant C_j A^{|\alpha|}\bigl((|\alpha|)!\bigr)^s,$$

for a sequence $C_j \to 0$ as $j \to \infty$. From the proof of Theorem 2.3 it follows that we then have

$$\left\|\widehat{\phi_j}(\xi) - \widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant K_j e^{-B\langle\xi\rangle^{1/s}},$$

where $B > 0$ and $K_j \to 0$ as $j \to \infty$. Hence we can estimate

$$\begin{aligned}\left|v(\phi_j - \phi)\right| &\leqslant \sum_{[\xi]\in\widehat{G}} d_\xi \left\|\widehat{\phi_j}(\xi) - \widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \left\|v(\xi)\right\|_{\mathtt{HS}} \\ &\leqslant K_j \sum_{[\xi]\in\widehat{G}} d_\xi e^{-B\langle\xi\rangle^{1/s}} \|v_\xi\|_{\mathtt{HS}} \to 0\end{aligned}$$

as $j \to \infty$ since $K_j \to 0$ as $j \to \infty$ and $\sum_{[\xi]\in\widehat{G}} d_\xi e^{-B\langle\xi\rangle^{1/s}} \|v_\xi\|_{\mathtt{HS}} < \infty$ by Lemma 4.1 and (3.2). Therefore, we have $v \in \gamma_s'(G)$.

"Only if" part. Let us now take $v \in \gamma_s'(G)$. This means that for every $\epsilon > 0$ there exist C_ϵ such that

$$\left|v(\phi)\right| \leqslant C_\epsilon \sup_\alpha \epsilon^{|\alpha|}(\alpha!)^{-s} \sup_{x\in G}\left|(-\mathcal{L}_G)^{|\alpha|/2}\phi(x)\right|$$

holds for all $\phi \in \gamma_s(G)$. So then, in particular, we have

$$\begin{aligned}\left|v\left(\xi_{ij}^*\right)\right| &\leqslant C_\epsilon \sup_\alpha \epsilon^{|\alpha|}(\alpha!)^{-s} \sup_{x\in G}\left|(-\mathcal{L}_G)^{|\alpha|/2}\xi_{ij}^*(x)\right| \\ &= C_\epsilon \sup_\alpha \epsilon^{|\alpha|}(\alpha!)^{-s}|\xi|^{|\alpha|} \sup_{x\in G}\left|\xi_{ij}^*(x)\right| \\ &\leqslant C_\epsilon \sup_\alpha \epsilon^{|\alpha|}(\alpha!)^{-s}\langle\xi\rangle^{|\alpha|} \sup_{x\in G}\left\|\xi^*(x)\right\|_{\mathtt{HS}} \\ &= C_\epsilon \sup_\alpha \epsilon^{|\alpha|}(\alpha!)^{-s}\langle\xi\rangle^{|\alpha|} d_\xi^{1/2}.\end{aligned}$$

This implies

$$\left\|v\left(\xi^*\right)\right\|_{\mathtt{HS}} = \sqrt{\sum_{i,j=1}^{d_\xi}\left|v\left(\xi_{ij}^*\right)\right|^2} \leqslant C_\epsilon d_\xi^{3/2} \sup_\alpha \epsilon^{|\alpha|}(\alpha!)^{-s}\langle\xi\rangle^{|\alpha|}.$$

Setting $r = \epsilon\langle\xi\rangle$ and using inequalities

$$\alpha! \geqslant |\alpha|!n^{-|\alpha|} \text{ and } \left(\frac{(r^{1/s}n)^{|\alpha|}}{|\alpha|!}\right)^s \leqslant \left(e^{r^{1/s}n}\right)^s = e^{nsr^{1/s}},$$

we obtain

$$\begin{aligned}\left\|v\left(\xi^*\right)\right\|_{\mathtt{HS}} &\leqslant C_\epsilon d_\xi^{3/2} \sup_\alpha \left(rn^s\right)^{|\alpha|}\left(|\alpha|!\right)^{-s} \\ &\leqslant C_\epsilon d_\xi^{3/2} \sup_\alpha e^{nsr^{1/s}} \\ &= C_\epsilon d_\xi^{3/2} e^{ns\epsilon^{1/s}\langle\xi\rangle^{1/s}}\end{aligned} \tag{5.2}$$

for all $\epsilon > 0$. We now recall that $v(\xi^*) = \widehat{v}(\xi)$ and, therefore, with $v_\xi := \widehat{v}(\xi)$, we get $v \in \gamma_s(G)^\wedge$ by Theorem 4.2 and (3.2).

(**B**) This case is similar but we give the proof for completeness.

"If" part. Let $v \in \gamma_{(s)}(G)^\wedge$ and for any $\phi \in \gamma_{(s)}(G)$ define $v(\phi)$ by (5.1). By a similar argument to the case (R), it is a well-defined linear functional on $\gamma_{(s)}(G)$. To check the continuity, suppose $\phi_j \to \phi$ in $\gamma_{(s)}(G)$, that is, for every $A > 0$ we have

$$\sup_\alpha A^{-|\alpha|}(\alpha!)^{-s} \sup_{x\in G}\left|(-\mathcal{L}_G)^{|\alpha|/2}\big(\phi_j(x) - \phi(x)\big)\right| \to 0$$

as $j \to \infty$. It follows that

$$\left\|(-\mathcal{L}_G)^{|\alpha|/2}(\phi_j - \phi)\right\|_\infty \leqslant C_j A^{|\alpha|}\big((|\alpha|)!\big)^s,$$

for a sequence $C_j \to 0$ as $j \to \infty$, for every $A > 0$. From the proof of Theorem 2.3 it follows that for every $B > 0$ we have

$$\left\|\widehat{\phi_j}(\xi) - \widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \leqslant K_j e^{-B\langle\xi\rangle^{1/s}},$$

where $K_j \to 0$ as $j \to \infty$. Hence we can estimate

$$\begin{aligned}\left|v(\phi_j - \phi)\right| &\leqslant \sum_{[\xi]\in\widehat{G}} d_\xi \left\|\widehat{\phi_j}(\xi) - \widehat{\phi}(\xi)\right\|_{\mathtt{HS}} \|v_\xi\|_{\mathtt{HS}} \\ &\leqslant K_j \sum_{[\xi]\in\widehat{G}} d_\xi e^{-B\langle\xi\rangle^{1/s}} \|v_\xi\|_{\mathtt{HS}} \to 0\end{aligned}$$

as $j \to \infty$ since $K_j \to 0$ as $j \to \infty$, and where we now take $B > 0$ to be such that $\sum_{[\xi]\in\widehat{G}} d_\xi e^{-B\langle\xi\rangle^{1/s}} \|v_\xi\|_{\mathtt{HS}} < \infty$ by Lemma 4.1 and (3.2). Therefore, we have $v \in \gamma'_{(s)}(G)$.

"Only if" part. Let $v \in \gamma'_{(s)}(G)$. This means that there exists $\epsilon > 0$ and $C > 0$ such that

$$\left|v(\phi)\right| \leqslant C \sup_\alpha \epsilon^{|\alpha|}(\alpha!)^{-s} \sup_{x\in G}\left|(-\mathcal{L}_G)^{|\alpha|/2}\phi(x)\right|$$

holds for all $\phi \in \gamma_{(s)}(G)$. Then, proceeding as in the case (R), we obtain

$$\left\|v(\xi^*)\right\|_{\mathtt{HS}} \leqslant C d_\xi^{3/2} e^{ns\epsilon^{1/s}\langle\xi\rangle^{1/s}}, \tag{5.3}$$

i.e. $\|\widehat{v}(\xi)\|_{\mathtt{HS}} \leqslant C e^{\delta\langle\xi\rangle^{1/s}}$, for some $\delta > 0$. Hence $v \in \gamma_{(s)}(G)^\wedge$ by Theorem 4.2. $\square$

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