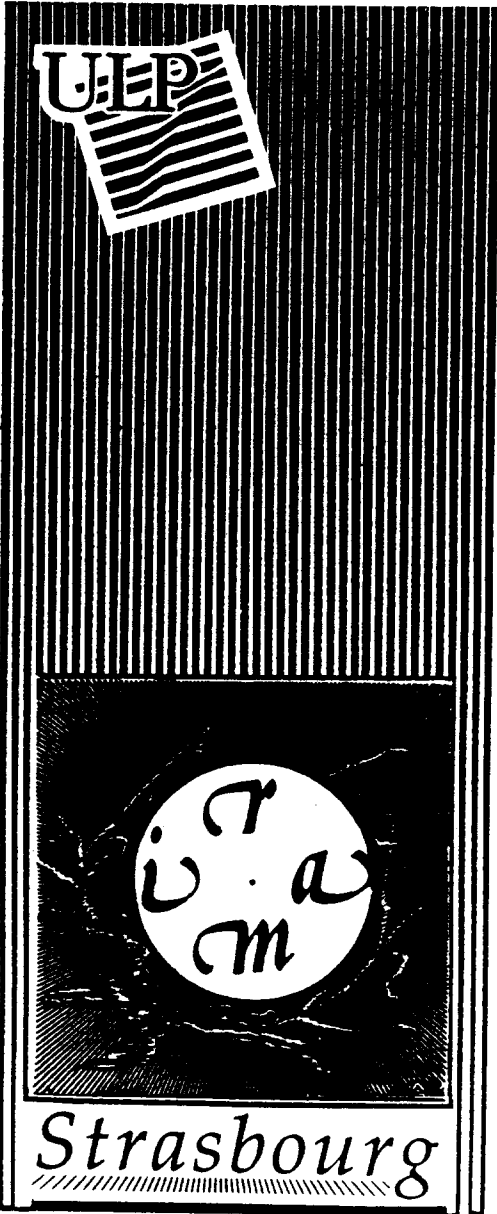
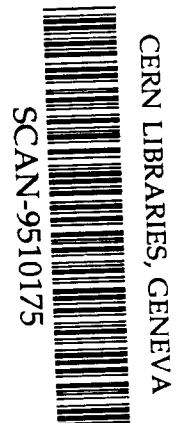


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GEVREY SEPARATION OF FAST AND SLOW VARIABLES

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# GEVREY SEPARATION OF FAST AND SLOW VARIABLES

Jean-Pierre RAMIS, Reinhard SCHÄPFKE

Abstract. — We consider (non necessarily conservative) perturbations of a one phase Hamiltonian system written with action-angle variables

$$(0) \quad \dot{x} = 0, \quad \dot{\varphi} = \omega(x),$$

where  $\omega \geq c > 0$  is real analytic, of the form

$$(1) \quad \dot{x} = \varepsilon f(x, \varphi, \varepsilon), \quad \dot{\varphi} = \omega(x) + \varepsilon g(x, \varphi, \varepsilon),$$

where  $f, g$  are real analytic in all the variables and  $2\pi$ -periodic in  $\varphi$ . More generally we consider systems similar to system (1) with  $x \in \mathbf{R}^m$ . It is well known that, using an iterated averaging process, it is possible to eliminate “formally in  $\varepsilon$ ” the phase  $\varphi$  by a formal transformation tangent to the identity

$$(\hat{T}) \quad x = y + \varepsilon \hat{U}(y, \psi, \varepsilon), \quad \varphi = \psi + \varepsilon \hat{V}(y, \psi, \varepsilon),$$

where  $\hat{U}, \hat{V}$  are  $2\pi$ -periodic in  $\psi$ .

Then one gets a formal autonomous system

$$(2) \quad \dot{y} = \varepsilon \hat{F}(y, \varepsilon), \quad \dot{\psi} = \omega(x) + \varepsilon \hat{G}(y, \varepsilon).$$

Fixing the normalization  $\hat{U}(y, 0, \varepsilon) = 0, \hat{V}(y, 0, \varepsilon) = 0$ , we prove that the transform  $\hat{T}$  is Gevrey 1 in  $\varepsilon$ . As an application, we give a new proof of a result of NEISHSTADT : it is possible to represent the formal transformation  $\hat{T}$  by an actual transformation  $T$  (admitting  $\hat{T}$  as its asymptotic expansion) such that the transformation  $T$  reduces the system (1) to a system which is autonomous up to perturbations which are exponentially small in  $\varepsilon$ . It is possible to use a cut-off “at the smallest term” like NEISHSTADT but we prefer to use an incomplete Laplace transform. Then we get for  $T$  a nice dependence in  $\varepsilon$  and we improve Neishstadt’s result. We will also give some similar improvements for the basic adiabatic invariants theory.

Our main statement generalizes a theorem of D. SAUZIN conjectured by P. LOCHAK.

## 1. Introduction.

The study of small perturbations (conservative or not) of integrable Hamiltonian systems is, following Henri POINCARÉ, one of the most important problems of mechanics. When one tries to obtain “normal forms” for the perturbed equations by elimination of *fast variables*, one gets in general *divergences* in the small parameter of perturbation. Following a classical prejudice the only source of these divergences is the *small denominators* phenomenon. In fact this is false, as it was already noticed by POINCARÉ, and there exist another source of divergence. Divergence occurs also in one-phase systems, and in this case there is no small denominators phenomenon! We can quote POINCARÉ : *Ce qui empêche la convergence, ce n'est donc pas la présence de petits diviseurs s'introduisant*

par l'intégration, mais celle de grands multiplicateurs s'introduisant par la différentiation [P 1] (Divergence des séries. 212, p. 392)<sup>(1)</sup>. In his study of the rapidly forced pendulum, whose equation is

$$\ddot{x} = \sin x + \mu\varphi(x)\sin \frac{t}{\varepsilon},$$

H. POINCARÉ gives an explicit description of a divergence coming from *great multipliers* : he computes an explicit minoration of a power series expansion  $S_1 = \sum T_p \mu^{p/2} : T_{p+2} > 2(p!)$  [P 1] (225, p. 457). Such a divergence is called a *Gevrey divergence*. Following a principle developed systematically by the first author (and later by a lot of mathematicians...) Gevrey divergences are always related to exponentially small corrections [Ra 1,2]. Such corrections appear in Poincaré's study of the perturbed pendulum ([P 1] (225, p. 461), [P 2]) but it seems that he did not notice the relation between the Gevrey divergence of the power series  $S_1$  and the exponentially small splitting of the separatrices. (For a modern study of Poincaré's problem cf. [Sau 2], [DS].) Recently Pierre LOCHAK proposed a systematical study of Gevrey divergences related to normal forms of small perturbations of integrable systems in relation with works of NEISHSTADT and NEKHOROSHEV [L]. In these works [N 1,2, 3, 4], [Ne 1,2] (related to questions like adiabatic invariants or exponentially long time of stability) appear some relations between cut off processes similar to the *summation at the smallest term* and exponentially small corrections. Some years ago the first author remarked that the exponential precision of such cut off processes is the "signature" of some Gevrey divergence [Ra 1]. Therefore it is reasonable to try to get systematically Gevrey estimates in such questions (and in related problems [BGG], [LST], [LM], [GP]). A first test problem was proposed by P. LOCHAK [L] and was affirmatively answered recently by D. SAUZIN [Sau 1].

In the present paper we will give a shorter proof and a generalization of Sauzin's result : we will prove that the *formal* elimination of the rapidly rotating phasis in a small *analytic* perturbation of an *analytic* system with one phase is Gevrey 1. As an application we will give a new proof of an important result of Neishstadt and improve this result. In order to get this improvement we will replace the "*least term*" cut off by an *incomplete Laplace transform*. Then we will obtain a nicer dependence in the small parameter of perturbation : in Neishstadt's work this dependence is piecewise algebraic with infinitely many jumps, in our work it is Gevrey one on a closed real interval beginning at the origin and real analytic on

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<sup>(1)</sup> In the case of two degrees of freedom Poincaré relates small divisors and great multipliers : cf. [P1], p. 392-93 : "On peut aussi présenter la chose d'une autre manière..."

the complementary of the origin. (In fact we even get a nice extension to the complex domain...) If the initial system and the small perturbation are Hamiltonian, then we will get formal and actual Gevrey one symplectic transformations.

Using our results we can give a slightly more precise description of the adiabatic invariance of the “action” for a slow analytic variation of an analytic system with one degree of freedom. (We built new adiabatic invariants, depending nicely on all the variables, which are exponentially stable during an exponentially long time.)

## 2. Formal reduction.

We recall (cf. [N 5]) that slow-fast systems are systems of differential equations of the form

$$\begin{aligned}\dot{x} &= \varepsilon g(u, x, \varepsilon), \\ \dot{u} &= f(u, x, \varepsilon),\end{aligned}$$

where  $\varepsilon > 0$  is a small parameter. Variables  $u$  are called *fast variables* and variables  $x$  are called *slow* ones. For  $\varepsilon = 0$  we get the *unperturbed system*. If the fast variables are angle variables  $\varphi$  (phases) on the  $m$ -torus  $\mathbb{R}^m / (2\pi\mathbb{Z})^m$ , the system is called a system with *fastly rotating phase*. It has the form

$$\begin{aligned}\dot{x} &= \varepsilon f(x, \varphi, \varepsilon) \\ \dot{\varphi} &= \omega(x) + \varepsilon g(x, \varphi, \varepsilon).\end{aligned}$$

Sometimes one denotes  $x = I$  the slow variables (cf. below).

Our aim is, using the averaging principle, to find changes of coordinates which permit us to eliminate the fast phase from the equations of perturbed motion, that is to separate the slow from the rapid motion. In this process we will use two operators : the *averaging operator*  $\langle \rangle^\psi$  and the *integration operator*  $\{ \}^\psi$ . All along this paper we limit ourselves to *single-frequency* systems, therefore we will define these two operators only in this case.

Let  $\omega(y) > 0$  be a given function (the frequency). Let  $h(y, \psi)$  be a function which is  $2\pi$  periodic in  $\psi \in \mathbb{R}$ . We expand it in a Fourier series

$$h(y, \psi) = h_0(y) + \sum_{k \in \mathbb{Z}^*} h_k(y) e^{ik\psi}.$$

We denote  $\langle h \rangle^\psi = h_0(y)$  and  $\{h\}^\psi = \sum_{k \in \mathbb{Z}^*} \frac{h_k}{ik\omega} e^{ik\psi}$ .

In multi-frequencies problem denominators  $ik\omega$  are replaced by  $i \langle k, \omega \rangle$ . Then they can vanish or be very small and the situation is very complicated for *arithmetical* reasons.

We have

$$\langle h \rangle^\psi = \frac{1}{2\pi} \int_0^{2\pi} h(y, u) du$$

and

$$\{h\}^\psi = \frac{1}{\omega} \left( \int_0^\psi h(y, u) du - \psi \langle h \rangle^\psi \right).$$

We have

$$\frac{\partial}{\partial \psi} \omega \{h\}^\psi = h - \langle h \rangle^\psi$$

$$\{h\}^\psi(y, 0) = 0.$$

The function  $\{h\}^\psi(y, \psi)$  is  $2\pi$ -periodic in  $\psi$ .

We will recall the procedure for eliminating *formally* fast variables in the single frequency systems (cf. [AKN], Ch. 5, 1.2, p. 142-143). We will describe it first in the *non hamiltonian* case. The hamiltonian case will be treated in section 7.

Limiting ourselves to the one-phase case, we start from the equations describing the perturbed motions

$$\dot{x} = \varepsilon f(x, \varphi, \varepsilon)$$

$$\dot{\varphi} = \omega(x) + \varepsilon g(x, \varphi, \varepsilon).$$

We have  $x = (x_1, \dots, x_m) \in \mathbf{R}^m$ ,  $\varphi \in \mathbf{R}$ . The functions  $f$  and  $g$  are  $2\pi$  periodic in  $\varphi$ . The variables  $x_j$  are the *slow* variables and the variable  $\varphi$  is the *fast variable* (the *fast rotating phase*). We suppose that  $f, g$  are real analytic in all the arguments.

We search a change of coordinates  $(\hat{T})$ , *formal* in  $\varepsilon$  and tangent to the identity

$$x = y + \varepsilon \hat{U}(y, \psi) = y + \varepsilon U_1(y, \psi) + \varepsilon^2 U_2(y, \psi) + \dots$$

$$\varphi = \psi + \varepsilon \hat{V}(y, \psi) = \psi + \varepsilon V_1(y, \psi) + \varepsilon^2 V_2(y, \psi) + \dots$$

so that the right hand sides of the transformed equations (which are formal in  $\varepsilon$ ) will not contain the fast phase  $\psi$

$$\dot{y} = \varepsilon \hat{F}(y, \varepsilon) = \varepsilon F_0(y) + \varepsilon^2 F_1(y) + \dots$$

$$\dot{\psi} = \omega(y) + \varepsilon \hat{G}(y, \varepsilon) = \omega(y) + \varepsilon G_0(y) + \varepsilon^2 G_1(y) + \dots$$

We search the  $U_n$ 's and the  $V_n$ 's analytic in all the arguments on a same domain (independent of  $n$ ) and  $2\pi$  periodic in  $\psi$ . Moreover we impose  $U_n(y, 0) = 0$  and  $V_n(y, 0) = 0$ .

We set  $\omega(y + \Delta) - \omega(y) = {}^t h(y, \Delta) \Delta$ .

**Proposition 2.1.** – *The above problem admits a unique solution given by recursive formulae*

$$\begin{aligned}
 F_n + \omega \frac{\partial}{\partial \psi} U_n &= R_n(f; (F_j, G_j, U_j, V_j, \frac{\partial}{\partial \psi} U_j, \frac{\partial}{\partial y} U_j)_{j=1, \dots, n-1}) \\
 F_n(y) &= \langle R_n(y, \psi) \rangle^\psi \\
 U_n(y, \psi) &= \{R_n(y, \psi)\}^\psi \\
 G_n + \omega \frac{\partial}{\partial \psi} V_n &= S_n(g, h; (F_j, G_j, U_j, V_j, \frac{\partial}{\partial \psi} V_j, \frac{\partial}{\partial y} V_j)_{j=1, \dots, n-1}; U_n) \\
 G_n(y) &= \langle S_n(y, \psi) \rangle^\psi \\
 V_n(y, \psi) &= \{S_n(y, \psi)\}^\psi.
 \end{aligned}$$

The proof of the above result can be found in [AKN], Ch. 5, 1.2. As we need more information about how to find  $R_n$  and  $S_n$ , we give a proof here too.

As we want that the series  $x(t, \varepsilon)$  and  $\varphi(t, \varepsilon)$  satisfy the perturbed equations if  $y(t, \varepsilon)$  and  $\psi(t, \varepsilon)$  satisfy the formal reduced equations, we want

$$\frac{d}{dt}(y + \varepsilon \hat{U}(y, \psi, \varepsilon)) = \varepsilon f(y + \varepsilon \hat{U}(y, \psi, \varepsilon), \psi + \varepsilon \hat{V}(y, \psi, \varepsilon), \varepsilon)$$

$$\frac{d}{dt}(\psi + \varepsilon \hat{V}(y, \psi, \varepsilon)) =$$

$$\omega(y + \varepsilon \hat{U}(y, \psi, \varepsilon)) + \varepsilon g(y + \varepsilon \hat{U}(y, \psi, \varepsilon), \psi + \varepsilon \hat{V}(y, \psi, \varepsilon), \varepsilon).$$

Using the equations for  $y$  and  $\psi$ , with  $\hat{F}$  and  $\hat{G}$  to be determined too, we obtain the transformation equations

$$\hat{F} + \omega \frac{\partial \hat{U}}{\partial \psi} = f(y + \varepsilon \hat{U}, \psi + \varepsilon \hat{V}, \varepsilon) - \varepsilon \hat{G} \frac{\partial \hat{U}}{\partial \psi} - \varepsilon \frac{\partial \hat{U}}{\partial y} \hat{F}$$

(TE)

$$\hat{G} + \omega \frac{\partial \hat{V}}{\partial \psi} = h(y, \varepsilon, \hat{U}) \hat{U} + g(y + \varepsilon \hat{U}, \psi + \varepsilon \hat{V}, \varepsilon) - \varepsilon \hat{G} \frac{\partial \hat{V}}{\partial \psi} - \varepsilon \frac{\partial \hat{V}}{\partial y} \hat{F}.$$

Comparing the expansions in powers of  $\varepsilon$  of right and left hand sides of the above equations we obtain the proposition 2.1.

### 3. Nagumo norms.

In this paper we will make an essential use of a family of norms for functions holomorphic on a disc. Our approach is related to a method of majoration of Mireille CANALIS-DURAND [CD] based on a Malgrange lemma [Ma 1]. It is also strongly based upon an idea of the second author [Sc 2], who used independently a Walter lemma [Wa]. In fact Malgrange and Walter lemmas are improvements of an older result of M. Nagumo [Na] (2, p. 42), so we will name such results *Nagumo lemmas*<sup>(1)</sup>.

Let  $r > 0$ . For functions  $f$  holomorphic in the disc  $\{|x| < r\} \subset \mathbb{C}$  we introduce the family of norms

$$\|f\|_l = \sup_{|x| < r} |f(x)|(r - |x|)^l.$$

We have a Nagumo lemma [W].

**Lemma 3.1.** — *If  $f$  is holomorphic in the open disc  $\{|x| < r\}$  then*

$$\|f'\|_{l+1} \leq e(l+1)\|f\|_l.$$

This lemma follows easily from Cauchy's formula.

For vector or matrix valued functions, we will denote by  $|\cdot|$  the maximum of the maximum norms of the entries. If  $F(y)$  (resp.  $F(y, \psi)$ ) is a vector valued or a matrix valued function holomorphic on a polydisc  $W$  centered at the origin in  $\mathbb{C}^m$  (resp. holomorphic on  $W \times \{\psi \in \mathbb{C}/|\operatorname{Im} \psi| < \delta\}$  and periodic in  $\psi$ ) we set  $\|F\|_l = \sup_{|y| < r} |F(y)|(r - |y|)^l$  (resp.  $\|F\|'_l = \sup_{|y| < r, |\operatorname{Im} \psi| < \delta} |F(y, \psi)|(r - |y|)^l$ ).

**Lemma 3.2.** —

(i) *If  $f, g$  are scalar valued functions holomorphic on the open polydisc  $\{|y| < r\} \subset \mathbb{C}^m$ , then*

$$(a) \|fg\|_{l+s} \leq \|f\|_l \|g\|_s$$

$$(b) \left\| \frac{\partial}{\partial y_j} f \right\|_{l+1} \leq e(l+1) \|f\|_l.$$

*If  $f, g$  are scalar valued functions holomorphic on  $\{|y| < r\} \times \{\psi \in \mathbb{C}/|\operatorname{Im} \psi| < \delta\}$  and periodic in  $\psi$ , then*

$$(a') \|fg\|'_{l+s} \leq \|f\|'_l \|g\|'_s$$

$$(b') \left\| \frac{\partial}{\partial y_j} f \right\|'_{l+1} \leq e(l+1) \|f\|'_l.$$

(1) After a first diffusion of this paper, B. Malgrange signaled to the first author that in fact Nagumo's method is similar to Maurice Gevrey's "Méthode des contours successifs" : cf. [G], p. 270–272 ((27) p. 272 and (31') p. 272).



(ii) Let  $g$  and  $F$  be respectively a vector valued and a matrix valued function (in  $\mathbb{C}^d$  and the complex (square  $d$ ) matrices respectively), holomorphic on the open polydisc  $\{|y| < r\} \subset \mathbb{C}^m$  (or more generally on  $\{|y| < r\} \times \{\psi \in \mathbb{C}/\text{Im } \psi| < \delta\}$  and periodic in  $\psi$ ). Then  $\|Fg\|_{l+s} \leq d\|F\|_l\|g\|_s$ , and  $\|Fg\|'_{l+s} \leq d\|F\|'_l\|g\|'_s$ .

We introduce the following notations. Let  $\hat{X} = \sum_{\ell=0}^{+\infty} X_\ell(y, \psi)\varepsilon^\ell$ , where the  $X_\ell$ 's are holomorphic on  $\{|y| < r\} \times \{\psi \in \mathbb{C}/\text{Im } \psi| < \delta\}$  and periodic in  $\psi$ , be a formal power series expansion. Let  $p = \sum_{\ell=0}^{+\infty} p_\ell z^\ell \in \mathbb{R}[[z]]$  be a real formal power series expansion. We define

$$\begin{aligned} X \ll p \text{ by } \|X_\ell\|_\ell &\leq p_\ell, \quad X \ll_1 p \text{ by } \|X_\ell\|_\ell \leq \ell! p_\ell \\ X \ll' p \text{ by } \|X_\ell\|'_\ell &\leq p_\ell, \quad X \ll'_1 p \text{ by } \|X_\ell\|'_\ell \leq \ell! p_\ell. \end{aligned}$$

We denote by  $E_{r,\delta,d,\ell}$  the space of holomorphic functions  $f = f(y, \psi)$  on  $\{|y| < r\} \times \{\psi \in \mathbb{C}/\text{Im } \psi| < \delta\}$ , with values in  $\mathbb{C}^d$ , which are  $2\pi$ -periodic in  $\psi$ , whose Nagumo norm  $\|f\|'_\ell$  is finite. This space is endowed with the norm  $\| \cdot \|'_\ell$ .

We set  $T_1(f) = \langle f \rangle^\varepsilon$ ,  $T_2(f) = \{f\}^\psi$  and  $T_3(f) = \frac{\partial}{\partial \psi} \{f\}^\psi$ . Then the operators  $T_1, T_2, T_3$  are continuous endomorphisms of the normed spaces  $E_{r,\delta,d,\ell}$ , uniformly with respect to  $\ell \in \mathbb{N}$ .

**Lemma 3.3.** — *There exists positive constants  $K = K(r, \delta) > 0$ , independent of  $\ell \in \mathbb{N}$  and  $f \in E_{r,\delta,d,\ell}$  such that*

$$\|T_1(f)\|'_\ell \leq K\|f\|'_\ell \quad \|T_2(f)\|'_\ell \leq K\|f\|'_\ell.$$

#### 4. A particular case.

In order to explain the principle of our method we will first apply it to Sauzin's case [Sau 1]. Daniel Sauzin proof is based upon explicit estimates mimicking Mircille Canalis-Durand explicit estimates in [CD]. In [CDRSS] we generalize Mircille Canalis-Durand work. As [CD], this generalization makes a systematical use of the operators  $\delta : f \mapsto f(0)$  (Dirac operator) and  $A : f \mapsto \frac{f(x)-f(0)}{x}$  (left translation operator). It is clear that in Sauzin work these operators are replaced respectively by the operators  $\langle \rangle$  (averaging operator) and  $\{ \}$  (integration operator) of averaging theory that we recalled above<sup>(1)</sup>. Then, in order to solve Sauzin's problem, we can mimick the majorant method of [CDRSS] : we replace the original problem of separation of fast and slow variables by the study of a formal power series solution of a germ of *irregular singular* non linear analytic differential equation.

<sup>(1)</sup> Here we consider 1-periodic functions, so we modify slightly the definitions.

Let  $W$  be an open set in  $\mathbf{R}^m$ . Let  $f : W \times \mathbf{R} \rightarrow \mathbf{R}^m$  be a function which is real-analytic in  $x$ ,  $C^1$  in  $(x, t)$  and 1-periodic in  $t$ . We consider the differential equation

$$(1) \quad \frac{dx}{dt} = \varepsilon f(x, t),$$

where  $\varepsilon > 0$  is a “small parameter”.

The problem is to conjugate this differential equation to an *autonomous* differential equation

$$(2) \quad \frac{dy}{dt} = \varepsilon F(y, \varepsilon),$$

by a change of variables tangent to the identity

$$x = y + \varepsilon U(y, t, \varepsilon),$$

where  $U$  is 1-periodic in  $t$ .

We do not know how to do that with actual functions (and when this is possible...), so we will give only a *formal* solution. We will work locally on  $W$ . Then we can suppose that  $0 \in W$ . We will search  $U = \hat{U}$  as a formal power series  $\hat{U}(y, t, \varepsilon) = \sum_{n=0}^{+\infty} U_n(y, t) \varepsilon^n$ , where the  $U_n$ 's are real-analytic in  $y$  and  $C^1$  in  $(y, t)$  on a *same* domain  $W' \times \mathbf{R}$ , where  $W'$  is an open neighborhood of 0 in  $W$ . Then  $F = \hat{F}$  will be analytic in  $y$  and formal in  $\varepsilon : \hat{F}(y, \varepsilon) = \sum_{n=0}^{+\infty} F_n(y) \varepsilon^n$ , where the  $F_n$ 's are real-analytic on  $W'$ .

**Theorem 4.1.** —

(i) *The problem admits a unique formal solution  $(\hat{U}, \hat{F})$  such that  $U_n(y, 0) = 0$ , for every  $n \in \mathbf{N}$  and every  $y \in W'$ .*

(ii) *If the open set  $W'$  is sufficiently small, then the power series expansions  $\hat{U}$  and  $\hat{F}$  of (i) are Gevrey 1 in  $\varepsilon$ , uniformly for  $(y, t) \in W' \times \mathbf{R}$  and for  $y \in W'$  respectively.*

*Proof.* As in proposition 2.1, we get

$$\begin{aligned} \hat{F} + \varepsilon \frac{\partial \hat{U}}{\partial y} \hat{F} + \frac{\partial \hat{U}}{\partial t} &= f(y + \varepsilon \hat{U}, t) \\ (3) \quad \hat{F} + \frac{\partial \hat{U}}{\partial t} &= f(y + \varepsilon \hat{U}, t) - \varepsilon \frac{\partial \hat{U}}{\partial y} \hat{F} \end{aligned}$$

and the recursion formulae

$$F_n + \frac{\partial U_n}{\partial t} = R_n,$$

where  $R_n = R_n(y, t)$  is obtained from  $f$  and the  $F_p$ 's,  $U_p$ 's with  $p < n$ .

We want  $F_n = F_n(y)$  and  $U_n = U_n(y, t)$  1-periodic in  $t$ , with  $U_n(y, 0) = 0$ . Then we get, from the knowledge of  $R_n$ , a unique solution :

$$F_n(y) = \langle R_n(y, t) \rangle = \int_0^1 R_n(y, t) dt$$

$$U_n(y, t) = \{R_n(y, t)\} = \int_0^t R_n(y, \xi) d\xi - tF_n(y).$$

Now we fix a real positive number  $r > 0$ . We consider functions  $U(y, t)$ , where  $y = (y_1, \dots, y_m) \in \mathbb{C}^m$  are complex variables with  $|y| = \text{Sup } |y_i| < r$ , and  $t \in \mathbb{R}$  is arbitrary. We suppose that  $U$  is analytic in  $y$ ,  $\mathcal{C}^1$  in  $(y, t)$  and 1-periodic in  $t$ . We suppose that  $\frac{\partial U}{\partial t}$  is bounded on  $\{|y| < r\} \times \mathbb{R}$ . we will use Nagumo norms  $\|\cdot\|_l$ ;

If the function  $R(y, t)$  satisfies similar conditions, we can define operators  $T_1(R) = \langle R \rangle$  and  $T_2(R) = \{R\}$ . Then  $T_1(R)$  and  $T_2(R)$  will satisfy similar conditions. More precisely we get estimates

$$\|T_1(R)\|_l < K\|R\|_l, \quad \|T_2(R)\|_l < K\|R\|_l$$

where  $K > 0$  is independant of  $R$  and  $l$  (cf. lemma 3.2).

We set  $\hat{X} = \hat{F} + \frac{\partial}{\partial t} \hat{U}$ . We rewrite equation (3)

$$\hat{X} = f(y + \varepsilon T_2(\hat{X}), t) - \varepsilon \frac{\partial}{\partial y} T_2(\hat{X}) T_1(\hat{X}).$$

For a sufficiently small  $r > 0$ , the function  $f$  can be uniquely extended to a function defined on  $\{x \in \mathbb{C}^m / |x| < r\} \times \{t \in \mathbb{R}\}$  which is analytic in  $x$ ,  $\mathcal{C}^1$  in  $(x, t)$  and 1-periodic in  $t$ .

We set  $\hat{X} = \sum_{l=0}^{+\infty} X_l(y, t) \varepsilon^l$  and  $f(x, t) = \sum_k f_k(t) x^k$  (where  $k = (k_1, \dots, k_m)$  is a multi-index). If  $r' > 0$  is sufficiently small, then the 1-periodic functions  $f_k$  are bounded on  $\mathbb{R}$ , and the power series  $\sum_n \alpha_n r'^n$ , where  $\alpha_n = \sum_{|k|=n} \text{sup}_{t \in \mathbb{R}} |f_k(t)|$ , is convergent. We set  $\Phi(\xi) = \sum_n \alpha_n \xi^n$ .

We set  $p(z) = \sum_{l=0}^{+\infty} p_l z^l$ , where  $p_l \geq 0$ . We introduce a *majorizing differential equation*

$$p = \Phi(r + Kzp) + mK^2 zp \frac{d}{dz} (zp).$$

Using lemmas 3.1, 3.2 we prove easily the following majorization lemma.

**Lemma 4.3.** ---

(i) There exists a unique power series solution  $\hat{X} = \sum_{l=0}^{+\infty} X_l(y, t)\varepsilon^l$  of the equation

$$\hat{X} = f(y + \varepsilon T_2(\hat{X}), t) - \varepsilon \frac{\partial}{\partial y} T_2(\hat{X}) T_1(\hat{X}).$$

There exists explicit recursive formulae giving  $X_l$  from the knowledge of this differential equation and of the  $X_k$ 's for  $k < l$ .

(ii) There exists a unique power series solution  $\hat{p}(z) = \sum_{l=0}^{+\infty} p_l z^l$  of the real analytic differential equation

$$p = \Phi(r + Kzp) + mK^2 zp \frac{d}{dz}(zp).$$

There exists explicit recursive formulae giving  $p_l$  from the knowledge of this equation and of the  $p_k$ 's for  $k < l$ .

(iii) We have  $\hat{X} \ll \hat{p}$  (that is  $\|X_l\|_l \leq p_l$ , for every  $l \in \mathbb{N}$ ).

**Lemma 4.4.** — If  $p(z) = \sum_{l=0}^{+\infty} p_l z^l$  is a power series solution of the real analytic differential equation

$$(4) \quad p = \Phi(r + Kzp) + mK^2 zp \frac{d}{dz}(zp),$$

then  $p$  is Gevrey 1.

*Proof.* We rewrite (4) in the form  $G(z, p, dp/dz) = 0$ , where

$$G(z, p_0, p_1) = \Phi(r + Kzp_0) + mK^2(zp_0^2 + z^2p_0p_1) - p_0.$$

Then the linearization of  $G$  along the formal power series solution  $\hat{p}$  (cf. [Ma]) gives the operator

$$\hat{L} = \frac{\partial}{\partial p_1} G(z, \hat{p}, \frac{d}{dz}\hat{p}) \frac{d}{dz} + \frac{\partial}{\partial p_0} G(z, \hat{p}, \frac{d}{dz}\hat{p}) \in \mathbb{C}[[z]][[d/dz]].$$

We have  $\hat{L} = mK^2 z^2 p_0 \frac{d}{dz} - 1 + Kz\Phi'(r + Kzp_0) + mK^2 zp_0(2 + zp_1)$ .

The Newton polygon of  $\hat{L}$  has only one slope and this slope is  $\geq 1$ , therefore, using a Malgrange result [Ma 2], we see that  $p$  is Gevrey 1. That ends the proof of lemma 4.3 and of claim (ii) in theorem 4.1.

In section 5 we will prove a generalization of theorem 4.1. In our proof we will use majorizations  $\ll'_1$  in place of majorizations  $\ll$ . Then the majorizing differential equation will be replaced by an analytic implicit equation and we will no longer need Malgrange's result.

In the above case we could also rewrite the differential equation (4) as  $z^2 dp/dx = \Psi(z, p)$  and apply an elementary argument ([Si 2] Property A.2.1.8, p. 195).

**5. Gevrey asymptotic expansions and incomplete Laplace transform.**

In this part our purpose is to recall basic definitions and results about Gevrey estimates and Gevrey asymptotic expansions. The interested reader will find more details<sup>(1)</sup> in [Ra 1,2,3], [Ba], [Ma 3], [Si 2,3], [T], and an interesting introduction in [L] (Appendix 2, p. 127).

The idea of Gevrey asymptotic expansions in one complex variable is due to G. WATSON, at the beginning of 20th century. Later M. GEVREY introduced the concept of Gevrey estimates for functions depending on several variables (in relation with parabolic PDE). More recently, in the late seventies, the first author reintroduced and developed systematically Gevrey asymptotic expansions in relation with analytic ODE in the complex domain [Ra 1,3]. Following a suggestion of B. MALGRANGE (June 1979) he related systematically Gevrey asymptotic expansions with the *exponential precision* of two procedures of *approximate summation* of Gevrey formal power series : the *incomplete Laplace transform* and a “*least term*” *cut-off*. (For a detailed history of this subject cf [Ra 2].)

We will explain below that there exists a precise equivalence between the existence of a Gevrey asymptotic expansion and the exponential precision of a “least term” cut-off procedure<sup>(2)</sup>. This answers a question of P. LOCHAK (cf. [L], App. 2, comments following proposition 1, p. 128).

**Definition 5.1.** — *Let  $s, A > 0$  positive numbers. A formal power series  $\hat{f}(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]$  is said to be Gevrey of order  $s$  and type  $A$  if there are positive constants  $C, A, \alpha > 0$  such that*

$$|a_n| \leq CA^{sn} \Gamma(sn + \alpha).$$

If this condition is satisfied, we will denote  $\hat{f} \in \mathbb{C}[[x]]_{(s,A)}$ . We set  $\mathbb{C}[[x]]_s = \bigcup_{A>0} \mathbb{C}[[x]]_{(s,A)}$ .

Then  $\mathbb{C}[[x]]_s$  and  $\mathbb{C}[[x]]_{(s,A)}$  are subdifferential algebras of  $\mathbb{C}[[x]]$ .

**Definition 5.2.** — *Let  $V$  be an open sector of the complex plane whose vertex is at the origin. let  $\hat{f} = \sum_{n=0}^{+\infty} a_n x^n \in \mathbb{C}[[x]]$  be a formal power series. Let  $f$  be a function analytic on the sector  $V$ .*

(i) *We will say that  $f$  is asymptotic to  $\hat{f}$  on the sector  $V$  in Poincaré*

(1) Be careful with the changing notations... The original Gevrey order 2 becomes later Gevrey order 1, there are some changes in the definition of Borel transform, our  $\mathbb{C}[[x]]_{1/k}$  is in some papers denoted by  $\mathbb{C}[[x]]_k$  or  $\mathbb{C}\{x\}_{1/k}$ .

(2) There is a little bit “optimistic” statement in this direction in [Ra 1] : Théorème 2.4 (iv), p. 182.

sense if for every closed subsector  $W \subset V$  and every positive integer  $N \in \mathbf{N}^*$  there exists a positive constant  $C_{W,N} > 0$  such that for all  $x \in W$

$$|f(x) - \sum_{n=0}^{N-1} a_n x^n| \leq C_{W,N} |x|^N.$$

(ii) Let  $s, A > 0$  positive numbers. We will say that  $f$  is Gevrey- $s$  asymptotic of type  $A$  to  $\hat{f}$  on the sector  $V$  if there is a positive constant  $\alpha > 0$  and for every closed subsector  $W \subset V$  there is a positive constant  $C_W > 0$  such that for all  $x \in W$  and all  $N \in \mathbf{N}^*$

$$|f(x) - \sum_{n=0}^{N-1} a_n x^n| \leq C_W A^{sN} \Gamma(sN + \alpha) |x|^N.$$

If  $f$  is asymptotic to  $\hat{f}$ , we set  $J(f) = \hat{f}$ . By  $\mathcal{A}_{(s,A)}(V)$  we denote the space of all functions analytic on  $V$  having a Gevrey- $s$  asymptotic expansion of type  $A$  on the open sector  $V$ . We set  $\mathcal{A}_s = \bigcup_{A>0} \mathcal{A}_{(s,A)}$ .

**Proposition 5.3.** — *Let  $s = 1/k > 0$  be a positive number. Let  $V$  be an open sector. A function  $f \in \mathcal{A}_s(V)$  is infinitely flat at the origin if and only if it has an exponential decay of order  $\geq k$  uniformly on every closed subsector  $W \subset V$ , that is if, for every such subsector, there exists positive constants  $C_W, a_W$  such that*

$$|f(x)| < C_W e^{-a_W |x|^{-k}}.$$

This result is well known. (Cf. remark 5.10 below.)

We set  $\mathcal{A}^{\leq -k}(V) = \{f \in \mathcal{A}_{1/k}(V) / J(f) = 0\}$ .

**Definition 5.4.** — *We set  $\hat{B}(1) = \delta$  and, for  $n \geq 1$ ,  $\hat{B}(x^n) = \frac{t^{n-1}}{(n-1)!}$ . Then we define the formal Borel transform of  $\hat{f}(x) = \sum_{n \geq 0} a_n x^n \in \mathbf{C}[[x]]$  by  $\hat{B}(\hat{f})(t) = a_0 \delta + \sum_{n \geq 1} a_n \frac{t^{n-1}}{(n-1)!}$ .*

*Let  $\hat{\varphi} \in \mathbf{C}\{t\}$  be a convergent power series. Let  $R > 0$  be its radius of convergence. We denote its sum by  $\varphi$ . Choosing  $0 < r < R$  we introduce an incomplete Laplace transform along the real positive axis*

$$\mathcal{L}_{k,r}(\hat{\varphi})(x) = k \int_0^r \varphi(t) e^{-\frac{t}{x^k}} t^{k-1} dt.$$

**Proposition 5.5.** — *Let  $k > 0$  be a positive number. Let  $V$  be an open sector. Let  $f$  be an analytic function on  $V$  such that  $f \in \mathcal{A}(V)$ . Let  $\hat{f}$  be the asymptotic expansion of  $f$  (in Poincaré sense). We suppose that  $\hat{f} \in \mathbb{C}[[x]]_{1/k}$  and we denote by  $R > 0$  the radius of convergence of  $\hat{\mathcal{B}}(\hat{f} - f(0))$ . We choose a positive number  $r$  such that  $0 < r < R$ . Then the following conditions are equivalent :*

- (i)  $f \in \mathcal{A}_{1/k}(V)$ ,
- (ii)  $f - \mathcal{L}_{k,r}(\hat{\mathcal{B}}\hat{f}) \in \mathcal{A}^{\leq k}(V)$ .

We have clearly  $J(\mathcal{L}_{k,r}(\hat{\mathcal{B}}\hat{f})) = \hat{f}$ , and  $\hat{f}$  is the asymptotic expansion of  $\mathcal{L}_{k,r}(\hat{\mathcal{B}}\hat{f})$  in Gevrey  $1/k$  sense.

It is possible to give precise Gevrey estimates for the incomplete Laplace transform. We will limit ourselves to the case  $k = 1$  and to sectors  $V$  bisected by the real positive axis. We denote by  $2\delta_0 < \pi$  the opening of  $V$ . Let  $f \in \mathcal{A}_1(V)$ . We denote by  $R > 0$  the radius of convergence of  $\hat{\mathcal{B}}(\hat{f} - f(0))$ . We choose positive numbers  $r, A$  such that  $0 < r < R$  and  $Ar > 1$ . Let  $W \subset V$  be a closed subsector, bisected by  $\mathbb{R}^+$ , whose opening is  $2\delta < 2\delta_0$ . Then there exists a positive constant  $C_{A,\delta}$  such that

$$|\mathcal{L}_{k,r}(\hat{\mathcal{B}}\hat{f})(x) - \sum_{n=0}^N a_n x^n| \leq C_{A,\delta} (N+1)! \left(\frac{A}{\cos \delta}\right)^{N+1} |x|^{N+1}$$

for every  $N \in \mathbb{N}$  and every  $x \in W$ .

**Proposition 5.6.** — *Let  $0 < \delta < \frac{\pi}{2}$*

*Consider the closed sector  $\bar{W}(\delta) = \{x \in \mathbb{C} / |\text{Arg } x| \leq \delta, |x| \geq 0\}$ . Consider also a convergent power series*

$$\hat{\varphi} = \sum_{n \geq 0} b_n t^n \in \mathbb{C}\{t\},$$

*whose radius of convergence is  $R > 0$ . Let  $C$  be a positive number such that  $0 < C < R$ .*

*Then, for every positive number  $r$  such that  $0 < r < R$ , there exists positive numbers  $K(C, \delta, r)$ ,  $a(C, \delta, r) > 0$  such that*

$$|\mathcal{L}_r(\hat{\varphi})(x) - \sum_{1 \leq n \leq C \cos \delta |x|^{-1}} n! b_n x^n| \leq K(C, \delta, r) e^{-a(C, \delta, r)|x|^{-1}}$$

*for  $x \in W(\delta)$ ,  $x \neq 0$ .*

*Proof.* We choose a positive number  $r_1$  such that  $C < r_1 < R$ . Then we have  $|\mathcal{L}_{r_1}(\hat{\varphi})(x) - \sum_{n=0}^N n! b_n x^n| \leq c_{C,\delta} (N+1)! \left(\frac{1}{C \cos \delta}\right)^{N+1} |x|^{N+1}$ .

We choose  $x \in W$  and we define uniquely a positive integer  $N = N(x)$  by  $(N+1)\frac{|x|}{C\cos\delta} \leq 1$  and  $(N+2)\frac{|x|}{C\cos\delta} > 1$ . We have the inequality  $\sqrt{m}e^{-m} \leq e^{-m/2}$ , for  $m \geq 1$ . Then, using Stirling formula, we get

$$|\mathcal{L}_{r_1}(\hat{\varphi})(x) - \sum_{n=0}^N n!b_n x^n| \leq c(C, \delta) \left( (N+1) \frac{|x|}{C\cos\delta} \right)^{N+1} e^{-\frac{N+1}{2}}$$

hence

$$|\mathcal{L}_{r_1}(\hat{\varphi})(x) - \sum_{n=0}^N n!b_n x^n| \leq c(C, \delta) e^{\frac{1}{2}} e^{-C\cos\delta|x|^{-1}}.$$

We complete the proof of proposition 5.6 by utilizing the fact that

$$\mathcal{L}_{k,r_1}(\hat{\varphi}) - \mathcal{L}_{k,r}(\hat{\varphi}) \in \mathcal{A}^{\leq -k}(V)$$

for an open sector  $V$  a little bit greater than  $W$ .

**Definition 5.7.** — *Let  $s = 1/k$ ,  $A > 0$  and  $V$  some open sector. We will say that a function  $f$  analytic on  $V$  has  $\hat{f}(x) = \sum_{n=0}^{+\infty} a_n x^n$  as a cut-off asymptotic of order  $s$  and type  $A$  if and only if there is a positive constant  $\alpha > 0$  and for each closed subsector  $W \subset V$  there is a positive constant  $C_W > 0$  such that*

$$|f(x) - \sum_{0 \leq n \leq kA^{-1}|x|^{-k}} a_n x^n| \leq C_W |x|^\alpha e^{-A^{-1}|x|^{-k}},$$

for all  $x \in W$ .

The set of all  $f \in \mathcal{O}(V)$  having such an asymptotic will be denoted by  $\mathcal{C}_{(s,A)}$ .

**Lemma 5.8.** —

(i) If  $f \in \mathcal{A}_{(s,A)}(V)$  then  $J(f) \in \mathbb{C}[[x]]_{(s,A)}$ .

(ii) If  $f \in \mathcal{C}_{(s,A)}(V)$  then  $J(f) \in \mathbb{C}[[x]]_{(s,A)}$ .

Claim (i) is well known. For (ii) observe that comparison of two inequalities (like in definition 5.7) for  $x$  such that  $A|x|^k = \frac{k}{N}$  and a variable  $\tilde{x}$  with slightly smaller absolute value yields in the limit  $\tilde{x} \rightarrow x$

$$|a_N||x^N| \leq C_W |x|^\alpha e^{-A^{-1}|x|^{-k}}$$

where  $A|x|^k = \frac{k}{N}$ . Hence we have for all  $N$

$$|a_N| \leq C_W A^{\frac{N-\alpha}{k}} \left( \frac{N}{k} \right)^{\frac{N-\alpha}{k}} e^{-\frac{N}{k}}$$



Using Stirling formula this yields that  $\hat{f} = J(f) \in \mathbb{C}[[x]]_{(s,A)}$ .

While it is obvious that  $\mathbb{C}[[x]]_{(s,A)} \subset \mathbb{C}[[x]]_{(s,B)}$  and  $\mathcal{A}_{(s,A)}(V) \subset \mathcal{A}_{(s,B)}(V)$  if  $B > A$ , the similar inclusion for  $\mathcal{C}_{(s,A)}(V)$  and  $\mathcal{C}_{(s,B)}(V)$  is not evident. But with the following theorem the question is unnecessary.

**Theorem 5.9** *A function  $f$  analytic on an open sector  $V$  has a Gevrey- $s$  asymptotic expansion of type  $A$  if and only if it has a cut-off asymptotic of order  $s$  and type  $A$  on that sector.*

**Remark 5.10.** A well known special case is the exponentially small behaviour of infinitely flat Gevrey- $s$  functions (cf. proposition 5.3 above). We will give the corresponding proof because it is similar to that of one of the implications of the theorem.

Let  $f \in \mathcal{A}_{(s,A)}(V)$  be such that  $J(f) = 0$ . then we can choose  $N$  in the estimate

$$|f(x)| \leq C_W A^{N/k} \Gamma\left(\frac{N}{k} + \alpha\right) |x|^N$$

optimal with respect to  $x \in W$ . Stirling formula yields that

$$\frac{N}{k} \text{Log } A + \left(\frac{N}{k} + \alpha - \frac{1}{2}\right) \text{Log } \frac{N}{k} - \frac{N}{k} + \frac{N}{k} \text{Log } |x|^k$$

has to be minimized. This is almost the case if  $\frac{N}{k} = A^{-1}|x|^{-k}$ . we get

$$|f(x)| \leq \tilde{C}_W |x|^{-k\alpha + \frac{k}{2}} e^{-A^{-1}|x|^{-k}}$$

which implies  $f \in \mathcal{C}_{(s,A)}(V)$ .

*Proof of theorem 5.9.* The inclusion  $\mathcal{A}_{(s,A)}(V) \subset \mathcal{C}_{(s,A)}(V)$  needs the same estimates as above and will be omitted. Now let  $f \in \mathcal{C}_{(s,A)}(V)$ . We set  $\hat{f} = J(f) = \sum a_n x^n \in \mathbb{C}[[x]]_{(s,A)}$ . Furthermore fix a closed subsector  $W \subset V$ . We have to show that there is  $C_W$  and  $\tilde{\alpha}$  with

$$|f(x)| \leq C_W A^{N/k} \Gamma\left(\frac{N}{k} + \tilde{\alpha}\right) |x|^N$$

for all  $x \in W$  and  $N \in \mathbb{N}$ . We consider  $x \neq 0$  and two cases.

*Case 1 :*  $kA^{-1}|x|^{-k} > N - 1$ . Here we find that

$$|f(x) - \sum_{n=0}^{N-1} a_n x^n| \leq C_1 |x|^\alpha e^{-A^{-1}|x|^{-k}} + \sum_{N < n < kA^{-1}|x|^{-k}} |a_n| |x|^n.$$

To estimate the right term times  $|x|^{-N}$ , we observe that the function

$$g(\sigma) = \sigma^{\frac{N-\alpha}{k}} e^{-A^{-1}\sigma}$$

reaches a maximum if  $\frac{N-\alpha}{k\sigma} = A^{-1}$  and there its value is

$$A^{\frac{N-\alpha}{k}} \left(\frac{N-\alpha}{k}\right)^{\frac{N-\alpha}{k}} e^{-\frac{N-\alpha}{k}} \leq C_2 \Gamma\left(\frac{N-\alpha}{k} - \frac{1}{2}\right) A^{N/k}.$$

Therefore we have

$$C_1 |x|^{-N+\alpha} e^{-A^{-1}|x|^{-k}} \leq C_1 C_2 \Gamma\left(\frac{N-\alpha}{k} - \frac{1}{2}\right) A^{N/k}$$

as needed.

Using  $\hat{f} \in \mathbb{C}[[x]]_{(s,A)}$ , we find for the sum

$$\begin{aligned} \sum_{N < n < kA^{-1}|x|^{-k}} |a_n| |x|^n &\leq C_3 \sum_{N < n < kA^{-1}|x|^{-k}} A^{n/k} \Gamma\left(\frac{n+\beta}{k}\right) |x|^n \leq \\ &\leq C_4 \sum_{N < n < kA^{-1}|x|^{-k}} A^{n/k} |x|^n \left(\frac{n}{k}\right)^{\frac{n+\beta}{k} - \frac{1}{2}} e^{-n/k} \leq \\ &\leq C_4 \left( \sum_{N < n < kA^{-1}|x|^{-k}} \left(\frac{n}{k}\right)^{\frac{n+\beta}{k} - \frac{1}{2}} e^{-n/k} \right) A^{N/k} |x|^N \end{aligned}$$

where we used  $A|x|^{-k} \frac{n}{k} < 1$ . The sum appearing on the right is less than

$$C_5 \int_N^{+\infty} \left(\frac{n}{k}\right)^{\frac{n+\beta}{k} - \frac{1}{2}} e^{-n/k} dn \leq C_6 \Gamma\left(\frac{N+\beta}{k} + \frac{1}{2}\right)$$

which concludes the proof in this case.  $\square$

The second case is similar and somewhat simpler. We will omit the details.

It is possible to give a little more precise statement than theorem 5.9 : looking carefully to our proof one sees that we lost only  $k/2$  on  $\alpha$ .

In many cases (for “sufficiently generic” solutions of a lot of analytic functional equations) it is possible to choose the parameters  $k$  and  $A$  such that the corresponding “least term” cut-off is “very near” of the *true* least term cut-off used by L. EULER and many followers. So in the applications the exponential precision of Euler’s cut-off is a very strong clue for the existence of Gevrey estimates.

In his work H. POINCARÉ insists in many places upon the *practical* value of divergent series occurring in celestial mechanics. He relates this practical value to some analogies between these series and Stirling series (studied

by Cauchy) and in particular to numerical efficacy of some least term cut-off process (cf. [P 1], Ch. VIII, 118, 119, p. 1-4, [P 2], p. 265<sup>(1)</sup>.)

So, using theorem 5.9, our main results in the present paper bring some precise mathematical justifications for Poincaré's statements.

### 6. Gevrey estimates.

In this section we want to prove that our formal series transformations  $\hat{U}$ ,  $\hat{V}$  as well as the right hand sides of our transformed equations  $\dot{y} = \varepsilon \hat{F}(y, \varepsilon)$ ,  $\dot{\psi} = \omega(y) + \varepsilon \hat{G}(y, \varepsilon)$  are Gevrey 1 with respect to  $\varepsilon$ . We will apply a majorant method to the transformation equations (TE) (obtained in section 2) using the Nagumo norms  $\| \cdot \|_1'$  and the lemmas of section 3 for the majorant relation  $\ll_1'$ . We recall the transformation equations

$$\hat{F} + \omega \frac{\partial \hat{U}}{\partial \psi} = f(y + \varepsilon \hat{U}, \psi + \varepsilon \hat{V}, \varepsilon) - \varepsilon \hat{G} \frac{\partial \hat{U}}{\partial \psi} - \varepsilon \frac{\partial \hat{U}}{\partial y} \hat{F}$$

(TE)

$$\hat{G} + \omega \frac{\partial \hat{V}}{\partial \psi} = {}^t h(y, \varepsilon, \hat{U}) \hat{U} + g(y + \varepsilon \hat{U}, \psi + \varepsilon \hat{V}, \varepsilon) - \varepsilon \hat{G} \frac{\partial \hat{V}}{\partial \psi} - \varepsilon \frac{\partial \hat{V}}{\partial y} \hat{F}$$

where we have omitted the arguments  $(y)$  of  $\omega$ ,  $(y, \varepsilon)$  of  $\hat{F}$  and  $\hat{G}$  and  $(y, \psi, \varepsilon)$  of  $\hat{U}$  and  $\hat{V}$  for the sake of brevity. The functions  $f$  and  $g$  are the ones given in the problem,  $h$  is some analytic function satisfying  $\omega(y + \Delta) - \omega(y) = {}^t h(y, \Delta) \Delta$  for sufficiently small  $\Delta$ .

In order to find a majorant equation we introduce the new unknown functions  $\hat{X} = \hat{F} + \omega \frac{\partial \hat{U}}{\partial \psi}$  and  $\hat{Y} = \hat{G} + \omega \frac{\partial \hat{V}}{\partial \psi}$ .

We can recover  $\hat{F}, \hat{U}, \hat{G}, \hat{V}$  from  $\hat{X}, \hat{Y}$  by the equations

$$(6.1) \quad \hat{F} = \langle \hat{X} \rangle^\psi, \quad \hat{U} = \{ \hat{X} \}^\psi, \quad \hat{G} = \langle \hat{Y} \rangle^\psi, \quad \hat{V} = \{ \hat{Y} \}^\psi$$

where the operators  $\langle \rangle^\psi$  and  $\{ \}^\psi$  are those of section 2 and are applied to each coefficient in the formal  $\varepsilon$ -series in the argument. The equations for  $\hat{X}, \hat{Y}$  now involve these operators

$$\hat{X} = f(y + \varepsilon \{ \hat{X} \}^\psi, \psi + \varepsilon \{ \hat{Y} \}^\psi, \varepsilon) - \varepsilon \langle \hat{Y} \rangle^\psi \frac{\partial \{ \hat{X} \}^\psi}{\partial \psi} - \varepsilon \frac{\partial \{ \hat{X} \}^\psi}{\partial y} \langle \hat{X} \rangle^\psi$$

(1) "On sait en effet le parti qu'on peut tirer dans un calcul numérique de l'emploi des séries divergentes et la série fameuse de Stirling en est un exemple frappant. C'est grâce à une circonstance analogue que les développements usités en Mécanique céleste ont rendu déjà de si grands services et sont appelés à en rendre de plus grands encore."

(TTE)

$$\hat{Y} = {}^t h(y, \varepsilon \{\hat{X}\}^\psi) \{\hat{X}\}^\psi + g(y + \varepsilon \{\hat{X}\}^\psi, \psi + \varepsilon \{\hat{Y}\}^\psi, \varepsilon)$$

$$-\varepsilon \langle \hat{Y} \rangle^\psi \frac{\partial \{\hat{Y}\}^\psi}{\partial \psi} - \varepsilon \frac{\partial \{\hat{Y}\}^\psi}{\partial y} \langle \hat{X} \rangle^\psi.$$

We assume that  $\omega$  is analytic on the polydisc  $W = \{x \in \mathbf{C}^n / |x| < r_0\}$  and that  $f, g$  are analytic on the set  $W \times \{\varphi \in \mathbf{C} / |\operatorname{Im} \varphi| < \delta_0\} \times \{\varepsilon \in \mathbf{C} / |\varepsilon| < \varepsilon_0\}$  and  $2\pi$ -periodic in  $\varphi$ . To construct the majorant equation we expand

$$f(y + d, \varphi + \delta, \varepsilon) = \sum_{|k| \geq 0} \sum_{l=0}^{+\infty} \sum_{m=0}^{+\infty} f_{klm}(y, \psi) d^k \delta^l \varepsilon^m$$

if  $|y| + |d| < r_0$ ,  $|\operatorname{Im} \psi| < \delta_0$ ,  $|\operatorname{Im}(\psi + \delta)| < \delta_0$  and  $|\varepsilon| < \varepsilon_0$ .

Using the Nagumo norms  $\|\cdot\|'_\ell$  defined for the set  $\{|y| < r\} \times \{|\operatorname{Im} \psi| < \delta\}$ , with some  $r \in ]0, r_0[$  and  $\delta \in ]0, \delta_0[$ , we set

$$\hat{f}_{slm} = \sum_{|k|=s} \|f_{klm}\|'_0$$

and

$$\hat{f}(a, b, \varepsilon) = \sum_{s,l,m} \hat{f}_{slm} a^s b^l \varepsilon^m.$$

Similarly we define  $\hat{g}(a, b, \varepsilon)$  and  $\hat{h}(a)$  starting from  $g$  and  $h$ . The power series  $\hat{f}, \hat{g}, \hat{h}$  are convergent at least for  $|a| < r_0 - r$ ,  $|b| < \delta_0 - \delta$ ,  $|\varepsilon| < \varepsilon_0$ .

By lemma 3.2 and 3.3 we now find that  $\hat{A} \ll'_1 p$ ,  $\hat{B} \ll'_1 q$  for some series  $\hat{A} = \sum_1^\infty A_\ell(y, \psi) \varepsilon^\ell$ ,  $\hat{B} = \sum_1^\infty B_\ell(y, \psi) \varepsilon^\ell$  and  $p = \sum_1^{+\infty} p_\ell \varepsilon^\ell$ ,  $q = \sum_1^{+\infty} q_\ell \varepsilon^\ell$  implies that

$$f(y + \varepsilon \{A\}^\psi, \psi + \varepsilon \{B\}^\psi, \varepsilon) - \varepsilon \langle B \rangle^\psi \frac{\partial \{A\}^\psi}{\partial \psi} - \varepsilon \frac{\partial \{A\}^\psi}{\partial y} \langle A \rangle^\psi \ll'_1$$

$$\hat{f}(K\varepsilon p, K\varepsilon q, \varepsilon) + K^2 \varepsilon p q + K^2 n \varepsilon p^2$$

and

$${}^t h(y, \varepsilon \{A\}^\psi) \varepsilon \{A\}^\psi + g(y + \varepsilon \{A\}^\psi, \psi + \varepsilon \{B\}^\psi, \varepsilon) - \varepsilon \langle B \rangle^\psi \frac{\partial \{B\}^\psi}{\partial \psi} -$$

$$-\varepsilon \frac{\partial \{B\}^\psi}{\partial y} \langle A \rangle^\psi \ll'_1 K \hat{h}(K\varepsilon p) p + \hat{g}(K\varepsilon p, K\varepsilon q, \varepsilon) + K^2 \varepsilon q^2 + K^2 n \varepsilon p q.$$

Hence induction shows that the series solutions  $\hat{X}, \hat{Y}$  of the transformed system  $(TTE)$  satisfy  $\hat{X} \ll'_1 p$ ,  $\hat{Y} \ll'_1 q$  where  $p = \sum_1^{+\infty} p_\ell \varepsilon^\ell$ ,  $q = \sum_1^{+\infty} q_\ell \varepsilon^\ell$  is the *unique* power series solution of the *majorant system of equations*

$$\begin{aligned} p &= \hat{f}(K\varepsilon p, K\varepsilon q, \varepsilon) + K^2\varepsilon pq + K^2n\varepsilon p^2 \\ q &= K\hat{h}(K\varepsilon p) + \hat{g}(K\varepsilon p, K\varepsilon q, \varepsilon) + K^2\varepsilon q^2 + K^2n\varepsilon pq. \end{aligned}$$

If  $\varepsilon = 0$  then this system is satisfied by  $p = q = 0$ . In this point the assumptions of the implicit function theorem (for analytic functions) are clearly satisfied : the matrix of partial derivatives of the right hand side of the majorant equation with respect to  $p, q$  at  $\varepsilon = p = q = 0$  is  $M = \begin{pmatrix} 0 & 0 \\ K\hat{h}'(0) & 0 \end{pmatrix}$  and  $I - M$  is invertible. So the formal power series solution  $p, q$  is the one determined by the implicit function theorem, and so it is *convergent*.

From the definition of our majorant relation  $\ll'_1$  this implies that the formal solution  $\hat{X}, \hat{Y}$  of the system  $(TTE)$  is Gevrey one with respect to  $\varepsilon$ . Using lemma 3.2 and the equations (6.1) for recovering  $\hat{F}, \hat{G}, \hat{U}, \hat{V}$  from  $\hat{X}, \hat{Y}$  we find that the  $\varepsilon$ -power series  $\hat{F}, \hat{G}, \hat{U}, \hat{V}$  are Gevrey one, too

## 7. Gevrey asymptotic reduction and Neishtadt theorem.

If  $\delta > 0$ , we denote by  $\Sigma_\delta$  the complex strip :  $\Sigma_\delta = \mathbf{R} + i] - \delta, \delta[ \subset \mathbf{C}$ .

We start with a real analytic system (1) :

$$\dot{x} = \varepsilon f(x, \varphi, \varepsilon)$$

$$\dot{\varphi} = \omega(x) + \varepsilon g(x, \varphi, \varepsilon)$$

where  $f$  and  $g$  are analytic onto  $\Sigma_{\delta_0}$  and  $2\pi$  periodic in  $\varphi$  and  $\omega(x) \geq c > 0$ .

Our aim is to simplify this system using a transformation  $T$  :

$$x = y + \varepsilon U(y, \psi, \varepsilon)$$

$$\varphi = \psi + \varepsilon V(y, \psi, \varepsilon)$$

tangent to identity, where the functions  $U$  and  $V$  are  $2\pi$  periodic in  $\psi$ , complex analytic on the open set  $W' \times \Sigma_\delta \times W''$  (where  $W'$  is an open polydisc centered at the origin in the  $y$  plane,  $\delta > 0$ , and  $W''$  is an open sector, bisected by the positive real axis, whose vertex is at the origin in the  $\varepsilon$  plane), and admit a Gevrey one asymptotic expansion in  $\varepsilon$ , analytic and uniform on  $W' \times \Sigma_\delta$

We first built a formal transformation  $\hat{T}$  (theorem 4.1)

$$x = y + \varepsilon \hat{U}$$

$$\varphi = \psi + \varepsilon \hat{V}$$

where  $\hat{U}$  and  $\hat{V}$  are formal power series expansions in  $\varepsilon$  whose coefficients are complex analytic functions on a neighborhood of  $\bar{W}' \times \bar{\Sigma}_\delta$  in the  $(y, \psi)$  space  $\mathbf{C}^{m+1}$ . We can suppose that these series are Gevrey one uniformly for  $(y, \psi) \in \bar{W}' \times \bar{\Sigma}_\delta$ , where  $W'$  is an open polydisc centered at the origin in the  $y$  plane and  $\delta > 0$ . Then there exists  $\rho_0 > 0$  such that the radii of convergence of the Borel transforms  $\hat{\mathcal{B}}\hat{U}$  and  $\hat{\mathcal{B}}\hat{V}$  are  $\geq \rho_0$ , uniformly for  $(y, \psi) \in \bar{W}' \times \bar{\Sigma}_\delta$ .

Then we choose arbitrarily  $0 < \rho_1 < \rho_0$  (a good choice corresponds to  $\rho_0 - \rho_1$  "very small"). We get an incomplete Laplace transform defined by

$$\mathcal{L}_{\rho_1}(\varphi)(\varepsilon) = \int_0^{\rho_1} \varphi(u) e^{-\frac{u}{\varepsilon}} du.$$

We define a formal Borel transform  $\hat{\mathcal{B}}$  by its values on the monomials :  $\hat{\mathcal{B}}(\varepsilon^n)(u) = \frac{u^{n-1}}{(n-1)!}$ , for  $n > 0$ , and by  $\hat{\mathcal{B}}(1) = \delta$ .

We denote by  $\mathbf{U}$  and  $\mathbf{V}$  the respective sums (in  $u$ ) of the series  $\hat{\mathcal{B}}(\hat{U})$  and  $\hat{\mathcal{B}}(\hat{V})$ , and we set  $U = \mathcal{L}_{\rho_1}(\mathbf{U})$  and  $V = \mathcal{L}_{\rho_1}(\mathbf{V})$ . The functions  $U$  and  $V$  are complex analytic on the open set  $W' \times \Sigma_\delta \times W''$  and admit a Gevrey one asymptotic expansion in  $\varepsilon$ , analytic and uniform on  $W' \times \Sigma_\delta$ , where  $W''$  is an open sector, bisected by the positive real axis, whose vertex is at the origin in the  $\varepsilon$  plane and whose opening is  $2\delta''_0 < \pi$ .

The formal transformation  $\hat{T}$  transforms the system (1) into a formal autonomous system (2)

$$\dot{y} = \varepsilon \hat{F}(y, \varepsilon)$$

$$\dot{\psi} = \omega(y) + \varepsilon \hat{G}(y, \varepsilon)$$

where  $\hat{F}$  and  $\hat{G}$  are formal power series in  $\varepsilon$  whose coefficients are functions which are complex analytic on  $W'$  in the  $y$  space  $\mathbf{C}^m$ . These series are Gevrey one uniformly on  $W'$ . We denote by  $\mathbf{F}$  and  $\mathbf{G}$  the respective sums (in the variable  $u$ ) of the series<sup>(1)</sup>  $\hat{\mathcal{B}}(\hat{F})$  and  $\hat{\mathcal{B}}(\hat{G})$ , and we set  $F = \mathcal{L}_{\rho_1}(\mathbf{F})$  and  $G = \mathcal{L}_{\rho_1}(\mathbf{G})$ . The functions  $F$  and  $G$  are complex analytic on the open set  $W' \times W''$  and admits a Gevrey one asymptotic expansion in  $\varepsilon$ , analytic and uniform on  $W'$ .

The transformation  $T = (U, V)$  transforms the system (1) into a system (3)

$$\dot{y} = \varepsilon \mathcal{F}(y, \psi, \varepsilon)$$

$$\dot{\psi} = \omega(y) + \varepsilon \mathcal{G}(y, \psi, \varepsilon)$$

---

(1) It is easy to show that the radius of convergence of these series is  $\geq \rho_0$ .

where the functions  $\mathcal{F}$  and  $\mathcal{G}$  are  $2\pi$  periodic in  $\psi$ , complex analytic on the open set  $W' \times \Sigma_\delta \times W''$  and admit a Gevrey one asymptotic expansion in  $\varepsilon$ , analytic and uniform on  $W' \times \Sigma_\delta$ .

The two functions  $F$  and  $\mathcal{F}$  (resp.  $G$  and  $\mathcal{G}$ ) have the same Gevrey asymptotic expansion in  $\varepsilon$ , therefore the two functions  $\alpha = \mathcal{F} - F$  and  $\beta = \mathcal{G} - G$  are infinitely flat in  $\varepsilon$ ,  $2\pi$  periodic in  $\psi$ , complex analytic on the open set  $W' \times \Sigma_\delta \times W''$  and admit a Gevrey one asymptotic expansion in  $\varepsilon$ , analytic and uniform on  $W' \times \Sigma_\delta$ . (More precisely they satisfy precised Gevrey estimates of order one and type  $\frac{1}{\rho_1 \cos \delta''_0}$ , cf. 5.) Therefore the two functions  $\alpha = \alpha(y, \psi, \varepsilon)$  and  $\beta = \beta(y, \psi, \varepsilon)$  are exponentially flat in  $\varepsilon$ . More precisely, for every  $0 < \rho < \rho_1$  and every  $0 \leq \delta'' \leq \delta''_0$  there exists a constant  $C(\rho, \delta'') > 0$  such that

$$|\alpha| < C(\rho, \delta'') e^{-\frac{\rho \cos \delta''}{|\varepsilon|}}$$

$$|\beta| < C(\rho, \delta'') e^{-\frac{\rho \cos \delta''}{|\varepsilon|}},$$

for  $|\text{Arg } x| \leq \delta''$ , uniformly on  $W' \times \Sigma_\delta$ .

We get an improvement of a theorem of Neishstadt ([N 4], Theorem 1, p. 134).

**Theorem 7.1** *Let  $W'_0$  (resp.  $W''_0$ ) be a neighborhood of the origin in the  $x$  space  $\mathbf{C}^m$  (resp. in the  $\varepsilon$  plane  $\mathbf{C}$ ). Let  $\delta_0 > 0$ . Let (1)*

$$\dot{x} = \varepsilon f(x, \varphi, \varepsilon)$$

$$\dot{\varphi} = \omega(x) + \varepsilon g(x, \varphi, \varepsilon)$$

be a real analytic system defined on  $W'_0 \times \Sigma_{\delta_0} \times W''_0$ , where  $f$  and  $g$  are  $2\pi$  periodic in  $\varphi$  and  $\omega(x) \geq c > 0$ .

Then there exists a coordinate transformation  $T$  :

$$x = y + \varepsilon U(y, \psi, \varepsilon)$$

$$\varphi = \psi + \varepsilon V(y, \psi, \varepsilon)$$

tangent to identity, where the functions  $U$  and  $V$  are  $2\pi$  periodic in  $\psi$ , complex analytic on the open set  $W' \times \Sigma_\delta \times W''$  and admit a Gevrey one asymptotic expansion in  $\varepsilon$ , analytic and uniform on  $W' \times \Sigma_\delta$  (where  $W'$  is an open polydisc centered at the origin in the  $y$  plane,  $\delta > 0$ , and  $W''$  is an open sector, bisected by the positive real axis, whose vertex is at the origin in the  $\varepsilon$  plane) which transforms the system (1) into a system

$$\dot{y} = \varepsilon(F(y, \varepsilon) + \alpha(y, \varphi, \varepsilon))$$

$$\dot{\varphi} = \omega(x) + \varepsilon(G(y, \varepsilon) + \beta(y, \varphi, \varepsilon)),$$

where the functions  $\alpha$  and  $\beta$  are uniformly exponentially small in  $\varepsilon$  : there exists two constants  $C > 0$ ,  $\rho > 0$  such that

$$|\alpha| + |\beta| < Ce^{-\frac{\rho}{|\alpha|}}$$

on  $W' \times \Sigma_\delta \times W''$ .

It is easy to derive the original Neishtadt statement from our formal Gevrey estimates and the “least term” cut-off procedure described in part 5.

### 8. Hamiltonian perturbations.

We will now eliminate *symplectically* the fast variable in a *single frequency Hamiltonian system* with “*slowly varying*” parameters. As above we will perform a *formal* elimination and we will prove that, modulo a natural normalization condition, this formal elimination is Gevrey 1. Then, using an incomplete Laplace transform, we will obtain an actual symplectic “quasi-elimination”.

The method is similar to the method used in section 3. In order to explain it we will begin with a very simple case<sup>(1)</sup>.

Let  $H_0(I)$  be the Hamiltonian of an unperturbed system with *one* degree of freedom in action-angle coordinates  $(I, \varphi) \in \mathbb{R}^2$ . The equations governing the unperturbed motion have the form

$$\dot{I} = 0, \quad \dot{\varphi} = \frac{\partial H_0}{\partial I}.$$

We set  $\omega(I) = \frac{\partial}{\partial I} H_0(I)$ . Assume  $\omega(I) \geq c > 0$ .

We introduce a small *Hamiltonian* perturbation. Then the perturbed motion is described by the Hamiltonian

$$H = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon)$$

which is supposed to be  $2\pi$  periodic in  $\varphi$ . The corresponding equations are

$$\dot{I} = -\varepsilon \frac{\partial H_1}{\partial \varphi}, \quad \dot{\varphi} = \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I}.$$

We will do a symplectic elimination of the fast variable using Lindstedt’s method [AKN] (Ch. 5, 2.2.A, p. 175). We work locally in the variable  $I$ .

---

<sup>(1)</sup> This case is perhaps too simple... We will build a convergent canonical transformation. In general the canonical transformation will be divergent and Gevrey one.



We search a formal symplectic<sup>(1)</sup> change of coordinates

$$\hat{T} : (I, \varphi) \mapsto (J, \psi),$$

tangent to the identity, such that the new (formal) Hamiltonian  $\hat{\mathcal{H}}$  will depend only of the slow variable :  $\hat{\mathcal{H}} = \hat{\mathcal{H}}(J, \varepsilon)$ .

We search a formal *generating function*<sup>(2)</sup>  $\hat{S} = J\varphi + \varepsilon\hat{S}(J, \varphi, \varepsilon)$ , where

$$\hat{S}(J, \varphi, \varepsilon) = S_1(J, \varphi) + \varepsilon S_2(J, \varphi) + \dots$$

for the symplectic change of variables  $\hat{T}$  and a formal Hamiltonian

$$\hat{\mathcal{H}}(J, \varepsilon) = \mathcal{H}_0(J) + \varepsilon\mathcal{H}_1(J, \varepsilon)$$

independent of  $\psi$ .

Then the transformation  $\hat{T}$  will be defined by

$$I = J + \varepsilon \frac{\partial \hat{S}}{\partial \varphi}$$

$$\varphi = \psi - \varepsilon \frac{\partial \hat{S}}{\partial J}.$$

The power series  $\hat{S}$  must be  $2\pi$  periodic in  $\varphi$ . We introduce the normalization condition  $\hat{S}(J, 0, \varepsilon) = 0$  (that is  $S_n(J, 0) = 0$  for every  $n \in \mathbb{N}^*$ ).

**Theorem 8.1.**

(i) *The above problem admits a unique formal solution  $(\hat{S}, \hat{\mathcal{H}})$  such that  $S_n(J, 0) = 0$  for every  $n \in \mathbb{N}^*$  and every  $J$  in a small open interval  $W'$  of the real line.*

(ii) *If  $W'$  is sufficiently small, then the power series expansions  $\hat{S}$  and  $\hat{\mathcal{H}}$  of (i) are convergent in  $\varepsilon$ , uniformly for  $(J, \varphi) \in W' \times \mathbb{R}$  and for  $J \in W'$  respectively.*

*Proof.* We want

$$(1) \quad \hat{\mathcal{H}}(J, \varepsilon) = H(I, \psi, \varepsilon) = H_0\left(J + \varepsilon \frac{\partial \hat{S}}{\partial \varphi}\right) + \varepsilon H_1\left(J + \varepsilon \frac{\partial \hat{S}}{\partial \varphi}, \varphi, \varepsilon\right).$$

---

<sup>(1)</sup> Here the dimension of the phase space is two, so symplectic is equivalent to area preserving.

<sup>(2)</sup> For basic definitions about canonical transformations and generating functions, cf. [LL], VII, 45.

We get the equations

$$-\mathcal{H}_0(J) = -H_0(J)$$

$$\omega(J) \frac{\partial S_1}{\partial \varphi} - \mathcal{H}_1(J) = -H_1(J, \varphi, 0)$$

.....

$$\omega(J) \frac{\partial S_n}{\partial \varphi} - \mathcal{H}_n(J) = R_n(J, \varphi)$$

where the  $R_n$ 's are polynomials in  $\frac{\partial}{\partial \varphi} S_1, \dots, \frac{\partial}{\partial \varphi} S_{n-1}$ .

With the conditions  $S_n(J, 0) = 0$ , we get a unique solution that we can compute by the recursive formulae :

$$\mathcal{H}_1 = \langle H_1 \rangle^\varphi, \quad S_1 = -\{H_1\}^\varphi$$

.....

$$\mathcal{H}_n = -\langle R_n \rangle^\varphi, \quad S_n = \{R_n\}^\varphi.$$

This ends the proof of claim (i).

We set  $\hat{\Sigma} = \hat{\Sigma}(J, \varphi, \varepsilon) = \omega \frac{\partial}{\partial \varphi} \hat{S} - \hat{\mathcal{H}} + H_0$ . We have  $\hat{\mathcal{H}} = -\langle \hat{\Sigma} \rangle^\varphi = -T_1(\hat{\Sigma})$  and  $\hat{S} = \{\hat{\Sigma}\}^\psi = T_2(\hat{\Sigma})$ . We set  $T_3(\hat{\Sigma}) = \hat{\Sigma} - T_1(\hat{\Sigma}) = \omega \frac{\partial}{\partial \varphi} \hat{S}$ .

From (1) we get an equation

$$\hat{\Sigma} = \Phi(J, \varepsilon T_3(\hat{\Sigma}), \varphi, \varepsilon).$$

Then, using a majorizing equation, we can prove easily claim (ii).

In the general situation the method is similar. We suppose now that the unperturbed system has  $n \geq 2$  degrees of freedom and *one phase* and that the corresponding Hamiltonian is  $H_0(I, Y, \varepsilon X)$ , where  $I \in \mathbf{R}$ ,  $X, Y \in \mathbf{R}^{n-1}$ . The phase is  $\varphi = \varphi(I, \psi, \varepsilon X) = \frac{\partial}{\partial I} H_0$ . (The case of an Hamiltonian  $H$  that "varies slowly with the time" is reduced to this formalism by the usual trick : we take the time  $t = X$  as a new coordinate and introduce the canonically conjugate momentum  $Y$ . Then the new Hamiltonian is  $H - Y$ .)

The perturbed Hamiltonian has the form

$$H = H_0(I, Y, \varepsilon X) + \varepsilon H_1(I, \varphi, Y, \varepsilon X, \varepsilon).$$

It is supposed to be  $2\pi$  periodic in  $\varphi$ .

The fast phase can be eliminated by a formal *symplectic* transformation of coordinates tangent to identity (cf. [AKN], ch. 5, 4.3, p. 207)

$$\hat{T} : (J, \psi, \eta, \xi) \mapsto (I, \varphi, Y, X).$$

We search a formal ( $2\pi$  periodic in  $\varphi$ ) generating function  $\hat{S} = J\varphi + \varepsilon\hat{S}(J, \varphi, \eta, \varepsilon X, \varepsilon)$ , where

$$\hat{S} = \hat{S}(J, \varphi, \eta, \varepsilon X, \varepsilon) = \sum_{n=1}^{+\infty} \varepsilon^n S_n(J, \varphi, \eta, \varepsilon X)$$

(where  $S$  is supposed to be  $2\pi$  periodic in  $\varphi$ ) and a new formal Hamiltonian  $\hat{\mathcal{H}} = \hat{\mathcal{H}}(J, \eta, \varepsilon X, \varepsilon) = \sum_{n=0}^{+\infty} \varepsilon^n \mathcal{H}_n(J, \eta, \varepsilon X)$ . We will impose the normalization condition

$$\hat{S}(J, 0, \eta, \varepsilon X, \varepsilon) = 0.$$

We get the equation

$$(2) \quad \hat{\mathcal{H}}(J, \eta, \varepsilon X + \varepsilon^2 \frac{\partial \hat{S}}{\partial \eta}, \varepsilon) = H(J + \varepsilon \frac{\partial \hat{S}}{\partial \varphi}, \varphi, \eta + \varepsilon^2 \frac{\partial \hat{S}}{\partial \varepsilon X}, \varepsilon X, \varepsilon).$$

We set as above  $\hat{\Sigma} = \hat{\Sigma}(J, \varphi, \eta, \varepsilon X, \varepsilon) = \omega \frac{\partial}{\partial \varphi} \hat{S} - \hat{\mathcal{H}} + H_0$ .

From (2) we get an equation

$$\hat{\Sigma} = \Phi(J, \varepsilon T_3(\hat{\Sigma}), \varphi, \eta + \varepsilon \frac{\partial T_2(\hat{\Sigma})}{\partial X}, \varepsilon X, \varepsilon).$$

Then, using Nagumo norms and a majorizing equation as in section 6, we can prove easily the following theorem.

**Theorem 8.2.** —

(i) *The above problem admits a unique formal solution  $(\hat{S}, \hat{\mathcal{H}})$  such that  $S_n(J, 0, \eta, \varepsilon X) = 0$  for every  $n \in \mathbb{N}^*$  and every  $(J, \eta, X)$  in a small open subset  $W'$  of  $\mathbb{R}^{2n-1}$ .*

(ii) *If the open subsets  $W' \subset \mathbb{R}^{2n-1}$ ,  $W'' \subset \mathbb{R}^{2n-1}$  are sufficiently small, then the power series expansions  $\hat{S}$  and  $\hat{\mathcal{H}}$  of (i) are Gevrey one in  $\varepsilon$ , uniformly for  $(J, \eta, \varepsilon X, \varphi) \in W' \times \mathbb{R}$  and for  $(J, \eta, \varepsilon \xi) \in W''$  respectively.*

There exists a conservative version of theorem 7.1.

**Theorem 8.3** *Let  $H = H_0(I, Y, \varepsilon X) + \varepsilon H_1(I, \varphi, Y, \varepsilon X, \varepsilon)$ . be a real analytic Hamiltonian with a fast phase*

*Then there exists a symplectic transformation of coordinates tangent to identity*

$$T : (J, \psi, \eta, \xi, \varepsilon) \mapsto (I, \varphi, Y, X, \varepsilon),$$

*which is complex analytic in the variables  $(J, \psi, \eta, \varepsilon \xi, \varepsilon)$  (for  $\varepsilon \in W''$ , where  $W''$  is an open sector bisected by the real positive axis) and admits an*

analytic and uniform Gevrey one asymptotic expansion in  $\varepsilon$ , transforming the Hamiltonian  $H$  into a Hamiltonian

$$\mathcal{H} = \mathcal{H}_0(J, \eta, \varepsilon\xi) + \varepsilon(\mathcal{H}_1(J, \eta, \varepsilon\xi, \varepsilon) + \mu(J, \psi, \eta, \varepsilon\xi, \varepsilon))$$

where the function  $\mu$  is uniformly exponentially small in  $\varepsilon$  : there exists two constants  $C > 0$ ,  $\rho > 0$  such that

$$|\mu| < Ce^{-\frac{\rho}{|\varepsilon|}}.$$

As for theorem 7.1, it is possible to get precise estimates for the constant  $\rho$ . (We could derive them from the knowledge of the radius of convergence of  $\hat{B}\hat{T}$ .)

### 9. Application to adiabatic invariants.

First we recall some basic definitions (cf.[AKN], Ch. 5, 4.1, p. 200).

We consider a Hamiltonian system with *one degree of freedom* whose parameters change *slowly*, i. e. the Hamiltonian  $E$  of the system has the form  $E = E(p, q; \lambda)$ , where  $\lambda = \lambda(\tau)$ ,  $\tau = \varepsilon t$  ( $\varepsilon$  “small”). We suppose that the Hamiltonian  $E$  is real analytic in all the variables and that the function  $\lambda(\tau)$  is real analytic.

We will assume that for each fixed value of  $\lambda$  the Hamiltonian  $E(p, q; \lambda)$  has *closed* phase curves on which the frequency of the motion is different from zero. Then we can introduce action-angle variables for the system with *fixed*  $\lambda$  :

$$I = I(p, q; \lambda), \varphi = \varphi(p, q; \lambda).$$

We get a Hamiltonian  $H = H_0(I, \lambda) + \varepsilon H_1(I, \varphi, \lambda)$  which is  $2\pi$  periodic in  $\varphi$  and we have

$$\begin{aligned} \dot{I} &= -\varepsilon \frac{\partial H_1}{\partial \varphi} \\ \dot{\varphi} &= \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I}. \end{aligned}$$

The simplest example is an harmonic oscillator whose parameters  $a$  and  $b$  vary slowly. The corresponding Hamiltonian is  $E = \frac{ap^2}{2} + \frac{bq^2}{2}$ , and  $\lambda = (a, b)$ . The closed phase curves are ellipses defined by  $E = h > 0$ . Their areas are  $A = \pi \sqrt{\frac{2h}{a}} \sqrt{\frac{2h}{b}} = \frac{2\pi h}{\sqrt{ab}}$ . We set  $I = A/2\pi$ .

For a pendulum whose length change slowly, we have  $a = 1/m$  and  $b = m\omega^2$ . Then  $I = h/\omega$ .

We can take time  $t = X$  as a new coordinate and introduce the canonical conjugate momentum  $Y$ . Then the case of a Hamiltonian system with

one degree of freedom varying slowly appears as a particular case of a Hamiltonian system with  $n \geq 2$  degrees of freedom associated to an Hamiltonian  $\mathcal{E}(p, q, Y, \varepsilon X)$  whose variation in all but one coordinate (that is  $q$ ) is *slow* ( $\mathcal{E}(p, q, Y, \varepsilon X) = E(p, q, \lambda(\varepsilon X)) - Y$ ). Setting  $\varepsilon X = \text{constant}$  and  $Y = \text{constant}$  we get the *unperturbed* system. We suppose that in the phase portrait of the unperturbed system there are closed trajectories on which the motion has non zero frequency and we write the Hamiltonian using action-angle coordinates. We get

$$H = H_0(I, Y, \varepsilon X) + \varepsilon H_1(I, \varphi, Y, \varepsilon X, \varepsilon).$$

Then it is well known that the “action”  $I$  is an *adiabatic invariant* during an exponentially long time (cf. [AKN], Ch. 5, 4.1, p. 200, and [Wa 1, 2] for the simple oscillator case). Our purpose is to derive this result from the Gevrey estimates that we have proved above. Classically one introduces a *formal first integral*  $\hat{J}(I, \varphi, Y, \varepsilon X, \varepsilon)$  of the system. We will prove that this formal first integral is Gevrey one in the last variable  $\varepsilon$ . Then we can “sum” it using an incomplete Laplace transform and we get an adiabatic invariant  $J(I, \varphi, Y, \varepsilon X, \varepsilon)$  which changes only by an exponentially small quantity during an exponentially long time and which satisfies  $I = J + O(\varepsilon)$ . The adiabatic invariance of the “action”  $I$  follows. Comparing with the classical approach the new fact is the regularity in  $\varepsilon$  of the adiabatic invariant  $J$ . This invariant depends analytically of  $(I, \varphi, Y, \varepsilon X)$ , is Gevrey one in  $\varepsilon$  and analytic in  $\varepsilon$  for  $\varepsilon \neq 0$ .

We start with a *real analytic* Hamiltonian  $H = H_0(I, Y, \varepsilon X) + \varepsilon H_1(I, \varphi, Y, \varepsilon X, \varepsilon)$  which is  $2\pi$  periodic in the fast phase  $\varphi$ .

Then (theorem 8.2) there exists a formal symplectic transformation of coordinates tangent to identity

$$\hat{T} : (J, \psi, \eta, \xi, \varepsilon) \mapsto (I, \varphi, Y, X, \varepsilon)$$

eliminating the fast phase. We denote by  $\rho_0 > 0$  the radius of convergence of the formal Borel transformed  $\hat{B}\hat{T}$  of  $\hat{T}$  (uniformly in  $(J, \psi, \eta, \varepsilon\xi)$ ). Then it is easy to check that the radius of convergence of  $\hat{B}\hat{J}(I, \varphi, Y, \varepsilon X, u)$  in  $u$  is at least  $\rho_0$ .

We choose a positive number  $\rho_1$  with  $0 < \rho_1 < \rho_0$  and we perform an incomplete Laplace transform of  $\hat{B}\hat{J}(I, \varphi, Y, \varepsilon X, u)$ . We set  $J = \mathcal{L}_{\rho_1} \hat{B}\hat{J}$ . Then  $J = J(I, \varphi, Y, \varepsilon X, \varepsilon)$  admits  $\hat{J}$  as an asymptotic expansion in the last variable  $\varepsilon$ . More precisely the function  $J$  admits  $\hat{J}$  as an asymptotic expansion in  $\varepsilon$  in Gevrey one sense in any sector  $W''$ , bisected by the positive real axis, whose opening is  $2\delta''_0 < \pi$ . We set  $\hat{\mathcal{H}} = H\hat{T}$ .

The formal power series expansion  $\hat{J}$  is a formal prime integral :  $\frac{d}{dt} \hat{J} = -\frac{\partial \hat{\mathcal{H}}}{\partial \psi} = 0$ .

We set  $\mathcal{H} = \mathcal{L}_{\rho_1} \hat{\mathcal{B}}\mathcal{H}$ . Then  $\dot{J}$  and  $-\frac{\partial \mathcal{H}}{\partial \psi}$  differ by a function which is infinitely flat in  $\varepsilon$ . Therefore  $\dot{J}$  is Gevrey one and infinitely flat in  $\varepsilon$ .

It is easy to check that we have in fact *precised* Gevrey estimates of order one and type  $\frac{1}{\rho_1 \cos \delta^{n_0}}$  (cf. section 5). Therefore, if  $\varepsilon$  is real,  $J = O(e^{-\frac{\varepsilon}{\rho_1}})$  for every  $0 < \rho < \rho_1$ .

Then  $J$  undergoes only exponentially small oscillations during the time  $\theta(\rho_0) = e^{\frac{\varepsilon}{\rho_1}}$  for every  $0 < \rho < \rho_1$ .

Finally  $I$  undergoes only small oscillations of order  $\varepsilon$  during the time  $\theta(\rho) = e^{\frac{\varepsilon}{\rho}}$ , for every  $0 < \rho < \rho_0$  : for every  $0 < \rho < \rho_0$  there exists a constant  $C(\rho) > 0$  such that

$$|I(t)| < C(\rho)\varepsilon$$

for every  $0 < t < \theta(\rho)$ .

**Proposition 9.1.** *Let  $H = H_0(I, Y, \varepsilon X) + \varepsilon H_1(I, \varphi, Y, \varepsilon X, \varepsilon)$  be a real analytic Hamiltonian which is  $2\pi$  periodic in the fast phase  $\varphi$  (obtained from a single frequency Hamiltonian system with slowly varying parameters using action-angle variables).*

*We denote by  $\rho_0 > 0$  the radius of convergence of the formal Borel transform  $\hat{\mathcal{B}}\hat{T}$  of  $\hat{T}$ , where  $\hat{T}$  is the formal symplectic transformation of coordinates eliminating the fast phase built in theorem 8.2. Then the "action"  $I$  undergoes only small oscillations of order  $\varepsilon$  during the time  $\theta(\rho) = e^{\frac{\varepsilon}{\rho}}$ , for every  $0 < \rho < \rho_0$  : for every  $0 < \rho < \rho_0$  there exists a constant  $C(\rho) > 0$  such that*

$$|I(t)| < C(\rho)\varepsilon$$

for every  $0 < t < e^{\frac{\varepsilon}{\rho}}$ .

## 10. Remarks and open problems.

The idea to get Gevrey estimates for the formal power series arising in reducing to *normal forms* complex *analytic* dynamical systems at a *singular point* is quite old for some particular cases. It has been systematically developed by the first author in collaboration with J. MARTINET and by J. ECALLE in the early eighties [Ma 4]. At the same time the first author conjectured also that Gevrey estimates appears in some *singularly perturbed* problems (in relation with duck phenomena [CD 1,2,3], [CDRSS]). The first systematic theorem about Gevrey estimates for singular perturbations of analytic differential equations is due to Y. SIBUYA [Si 1,3], [CD 5,6], [CDRSS]. Similar ideas for *perturbations of Hamiltonian systems* can be found in [GDFGS], [GP]... but, as far as we know, it is only in [L] that appears a systematic program of research in this direction (cf.

[L], p. 128-30). The present work goes clearly in the direction suggested by LOCHAK : ...*let us return to canonical perturbation theory. The overall idea would be to clarify the Gevrey properties, if any, of the various series which appear classically in normal form theory,...* The message is simply that it might be interesting to dig a little deeper, exploring the analyticity properties in  $\varepsilon$ ....

As it is suggested in [L] it would be interesting to investigate several frequencies problems. In that case appears a blend of two sources of divergence, that is *small denominators* and *big multipliers*. Apparently there is a “coupling” between these two sources : *The index  $k$  of the Gevrey spaces...and the exponent  $\tau$  of the Diophantine condition are closely related* [L], p. 129. (Cf also [Sim].)

Another interesting question is the problem of *optimality* of small corrections in theorems 7.1 and 8.3. This question is related to the (more difficult) problem of optimality of estimate of conservation time of adiabatic invariants. More precisely we can ask the following questions. When is it possible to completely eliminate these corrections (and to get perpetual adiabatic invariants)? When is it possible to improve the *type estimate* in the exponential estimate? Is it possible to get optimal surexponential estimates like  $\leq Ce^{-\varepsilon^{1/\tau}}$  ?... We do not know any serious answer to such questions. (There are some remarks about the *unimprovability of the estimates* in [N 4], 5, p. 138 or in [AKN], p. 148.)

We think that our technics (i. e. Gevrey estimates of power series and incomplete Laplace transform) are better than Neishstadt's (i. e. more or less cut-off at “least term”) for future research in this direction. Here we will limit ourselves to some remarks in order to justify this affirmation.

We denote by  $R > 0$  the radius of convergence of the transformation  $(\hat{B}\hat{U}, \hat{B}\hat{V})$  in the *Borel plane*. We denote by  $\mathbf{U}, \mathbf{V}$  the respective sums of  $\hat{B}\hat{U}, \hat{B}\hat{V}$ . It can happen that the functions  $\mathbf{U}, \mathbf{V}$  extend analytically along the real positive axis *out of the convergence disc*  $D(0, R)$ . Then we can compute incomplete Laplace transforms  $U = \mathcal{L}_r \mathbf{U}, V = \mathcal{L}_r \mathbf{V}$ , where  $r > R$  is arbitrary. In that case we get a more precise “incarnation”  $(U, V)$  of the formal transformation  $(\hat{U}, \hat{V})$ . (The precision grows with  $r$ .) The ideal situation is in some sense the following : the functions  $\mathbf{U}, \mathbf{V}$  can be extended analytically all along the positive real axis with a growth at most exponential of order one along  $\mathbf{R}^+$ . (This hypothesis is slightly more general than Borel summability or 1-summability.) Then we can set  $U = \mathcal{L} \mathbf{U} = \int_0^{+\infty} \mathbf{U}(t) e^{-t/r} dt, V = \mathcal{L} \mathbf{V}$ . Using standard summability theory, it is easy to prove that we have  $\alpha = \beta = 0$  in theorem 7.1 : we have eliminated the exponentially small corrections.

There are interesting intermediate situations. If  $\mathbf{U}, \mathbf{V}$  can be extended out of the convergence disc along the real positive axis but admits a first

singularity at a distance  $r_0 > R$  on  $\mathbb{R}^+$ , then we can choose positive numbers  $\rho, \rho'$  with  $0 < \rho < \rho' < r_0$  and perform an incomplete Laplace transform  $\mathcal{L}_{\rho'}$ . Then, using Lemma 10.1 (cf. below), we get estimates  $|\alpha| + |\beta| < C_{\rho} e^{-\frac{\rho}{\varepsilon}}$ , for  $\varepsilon \in \mathbb{R}$ ,  $\varepsilon > 0$ .

Let  $\rho > 0$ . We denote by  $E_{\rho} = \mathcal{C}([0, \rho]; \mathbb{C})$  the space of continuous functions on the closed interval  $[0, \rho]$  with values in  $\mathbb{C}$ . We endow it with the  $L^{\infty}$  norm  $M(f) = \text{Sup}_{t \in [0, \rho]} |f(t)|$ . We can define the convolution of two functions  $f, g \in E_{\rho}$  and interpret it as a function of  $E_{\rho}$  :  $f * g(t) = \int_0^t f(u)g(t-u)du$ .

We set  $\mathcal{L}_{\rho}(f)(\varepsilon) = \int_0^{\rho} f(t)e^{-t/\varepsilon} dt$ .

**Lemma 10.1.** *Let  $\rho > 0$ .*

(i)  $\mathcal{L}_{\rho}(1)(\varepsilon) = \varepsilon - \varepsilon e^{-\rho/\varepsilon}$ .

(ii)  $\mathcal{L}_{\rho}$  is  $\mathbb{C}$ -linear.

(iii)  $\mathcal{L}_{\rho}(f * g)(\varepsilon) = \mathcal{L}_{\rho}(f)(\varepsilon)\mathcal{L}_{\rho}(g)(\varepsilon) = \gamma(\varepsilon)$

with  $\gamma(\varepsilon) \leq e^{-\rho/\varepsilon} \rho^2 M(f)M(g)$ .

*Proof.* Cf. [CD 2], Propositions 6, 8 (p. 11). (M. CANALIS-DURAND hypothesis on the function spaces are less general but her proofs remain clearly valid in our case.)

The above estimates on the small corrections are not necessarily optimal. If  $U, V$  are *resurgent*<sup>(1)</sup>, then it is perhaps possible to perform a *median sum* (in Ecalle's sense : cf. [MR], p. 358, [E]), i. e. to compute a Laplace transform "across the singularities" along the positive real axis. If it works, then we will get strictly better estimates (at least we get the following : for every positive number  $\rho > 0$  there exists a positive number  $C_{\rho} > 0$  such that  $|\alpha| + |\beta| < C_{\rho} e^{-\frac{\rho}{\varepsilon}}$ , for  $\varepsilon \in \mathbb{R}$ ,  $\varepsilon > 0$ ). Using this method it is perhaps even possible to eliminate completely the small corrections in some resurgent<sup>(2)</sup> cases. Anyway computation of the *exact* estimate of the rate of decay of the optimal small corrections or computation of the *exact conservation time* of adiabatic invariants are apparently related to delicate questions like a very precise analysis of the resurgence and of the growth of the functions  $U, V$  along the real axis in the Borel plane.

It is "well known" that it is not allway possible to eliminate the small exponential corrections in some *linear* situations. This is related to the *parametric resonance* phenomena. The corresponding modulated

(1) It is reasonable to think that these series are resurgent in some "simple cases" by analogy with similar problems [Sau 2,3]. However we think that it is no longer true for the general analytic case : cf. below.

(2) For an introduction to Ecalle's resurgence theory cf. [CNP 1,2].



oscillators are defined by equations

$$\ddot{x} + \omega^2(1 + \eta g(\varepsilon t)) = 0$$

where  $\omega, \eta$  are positive numbers such that  $\omega > 0$ ,  $0 \leq \eta < 1$  and  $g$  is a  $2\pi$ -periodic real analytic function such that  $\|g\| = 1$ .

We will give now some simple explicit examples of slowly varying Hamiltonian systems where the *exact* conservation time of the corresponding adiabatic invariant is exponentially small *with a finite type* in the parameter  $\varepsilon$ . It follows that in such cases it is impossible to eliminate the exponentially small corrections<sup>(1)</sup>. Therefore the corresponding formal symplectic normalizing transformations  $\hat{T}$  are “not summable” in the direction of the positive real axis<sup>(2)</sup>. (Probably such transformations are not resurgent...). We will use an analysis of [LM], S.3, p. 205-210 and well known facts about Hill’s equations [H 1.2], [MW] (cf. also [A 3]).

**Proposition 10.2.** *Let*

$$Q(u) = \frac{1}{2} \sum_{j=-s}^s b_j e^{2ju}$$

*be a finite Fourier series, where  $b_0 = 0$ ,  $b_s \neq 0$ ,  $s > 0$ ,  $b_{-s} = \bar{b}_s$ . Let  $\omega > 0$  be a real number.*

*Let*

$$(1) \frac{d^2 x}{d\tau^2} + \left(\frac{\omega}{\varepsilon}\right)^2 (1 + f_q(\tau))x = 0$$

*be the differential equation such that the corresponding equation obtained by a Liouville transformation (cf. [MW]) is*

$$(2) \frac{d^2 z}{du^2} + \left(\frac{\omega^2}{\varepsilon^2} + qQ(u)\right)z = 0. \quad *$$

*Then there exists  $q_0 > 0$  such that, for  $0 < q < q_0$ , we have  $\|f_q\| \leq \eta < 1$  and there exist positive constants  $c(q) > 0$  and  $\gamma(q) > 0$  such that the “action”  $I$  associated to the differential equation*

$$(3) \ddot{x} + \omega^2(1 + f_q(\varepsilon t))x = 0$$

<sup>(1)</sup> If it is possible to eliminate the exponentially small corrections using a symplectic transformation, then the corresponding transformation  $T$  will give a true first integral  $J$  and therefore  $I$  must be a perpetual adiabatic invariant.

<sup>(2)</sup> Allowing complex values of the sum of the formal transformation  $\hat{T}$  and using the same method, we can prove that  $\hat{T}$  is not Borel summable or more generally not multisummable in Martinet-Ramis sense [MR], [Ba], nor accelerosummable in Ecalle’s sense [E]) along a direction  $d$  such that  $\text{Arg } d \in ]-\frac{\pi}{2}, \frac{\pi}{2}[$

does not remain an adiabatic invariant for  $t > c(q)e^{\gamma(q)/\varepsilon}$ .

Using a result of [H 1] (Theorem, p. 251), we verify that there exists constants  $1 > q_0 > 0$ ,  $a > 0$  (independent of  $n$  and  $q$ ) such that the length of the  $n$ -th instability interval of (2) satisfies

$$|\nu'_n - \nu''_n| \geq aq^{[n/s]},$$

where  $\nu = \frac{\omega}{\varepsilon}$ .

Then there exists  $b(q) > 0$  and  $\alpha(q) > 0$  (independent of  $n$ ) such that

$$|\nu'_n - \nu''_n| \geq b(q)e^{-\alpha(q)n}.$$

The function  $Q$  is  $2\pi$  periodic. The function  $f_q$  is  $2\pi$  periodic (in  $\tau$ ). We denote by  $P$  the Poincaré map associated to  $\tau \mapsto \tau + 2\pi$ . This map is area preserving. The boundaries of the instability intervals  $[\nu'_n, \nu''_n]$  are given by  $|\Delta(P)| = 2$ , where  $\Delta(P)$  is the trace of  $P$ .

The period of the unperturbed motion corresponding to equation (3) is  $T_0 = \frac{2\pi}{\omega}$ . The period in  $t$  of (3) is  $T_\varepsilon = \frac{2\pi}{\varepsilon}$ . Therefore the  $n$ -th instability interval corresponds to  $\nu(\varepsilon) = \frac{T_\varepsilon}{T_0} = \frac{\omega T_\varepsilon}{2\pi} = \frac{\omega}{\varepsilon} \approx n/2$  (cf. [H1], [LM]), that is to  $\varepsilon \approx 2\omega/n$ . For  $\nu(\varepsilon) \in [\nu'_n, \nu''_n]$ , we denote by  $e^{\frac{\sigma_\varepsilon \tau}{2\pi}} \psi(\tau)$  the unstable eigensolution. (The function  $\psi$  is  $2\pi$ -periodic in  $\tau$ .) We have  $x = e^{\frac{\sigma_\varepsilon \tau}{2\pi}} \psi(\tau) = e^{\frac{\sigma_\varepsilon \varepsilon t}{2\pi}} \psi(\varepsilon t)$ . We fix the integer  $n$  and we suppose that  $\varepsilon$  varies such that  $\varepsilon \approx 2\omega/n$ . If the time interval  $t_1$  is an integer multiple of  $2\pi/\varepsilon$ , then by the transformation  $t \mapsto t + t_1$ , the unstable solution is multiplied by the factor  $e^{\sigma_\varepsilon \varepsilon t_1}$  and the "action"  $I$  (which is quadratic in  $x, \dot{x}$ ) is multiplied by the factor  $e^{2\sigma_\varepsilon \varepsilon t_1}$ . Then the adiabatic invariance of  $I$  will break for  $t_1$  of the order of  $\frac{1}{\sigma_\varepsilon}$ . The smaller is  $\sigma_\varepsilon$  the sooner the breaking will occur. We set  $\sigma(n) = \text{Sup}_{\nu(\varepsilon) \in [\nu'_n, \nu''_n]} \sigma_\varepsilon$ . Then the adiabatic invariance will be surely broken for  $t_1 \gg \frac{1}{\sigma(n)}$  and a good choice of  $\varepsilon$ .

Using [LM], 8.3, p. 210, we get  $\sigma(n) \approx \pi \delta_n$ , where  $\delta_n = \nu''_n - \nu'_n$  is the length of the  $n$ -th instability interval. Therefore the adiabatic invariance will be broken for

$$t_1 \gg \frac{1}{2b(q)} e^{\alpha(q)n} \approx \frac{1}{2b(q)} e^{2\omega\alpha(q)/\varepsilon}.$$

Then we can choose any  $\gamma(q) > 2\omega\alpha(q)$  and we obtain the required estimate for some  $c(q) > 0$  sufficiently big.

Writing the equation of the oscillator as a stationary one dimensional Schrödinger equation (where the small parameter  $\varepsilon$  plays the role of Planck's reduced constant  $\hbar$ ) [LM], we get

$$-\varepsilon^2 d^2 \psi / dx^2 - \omega^2(x) \psi = 0.$$

For the study of stationary one dimensional Schrödinger equations with polynomial potentials there exists a resurgent approach initiated by A. Voros [DDP]. Similarly in our problem it is perhaps possible to use a *resurgent WKB method* when  $f$  is *polynomial in  $\cos \epsilon t$* .

As a conclusion we hope that we have convinced the reader of the interest of developing *truly complex methods in perturbation theory* (cf. [LM], p. 221).

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# How to connect to the preprint-server of the Institut de Recherche Mathématique Avancée Strasbourg

The best way is to use Mosaic or any WWW client-program (like Netscape, Macweb, XMosaic, etc.). The URL of this site is

*http://galois.u-strasbg.fr/* (french home page)

or

*http://galois.u-strasbg.fr/home.html* (english home page).

Another way is to use a ftp command in a UNIX-like environment (Telnet on Macintosh, any command-tool on X-Windows, etc.). The command

*ftp galois.u-strasbg.fr*

will connect you to the server (login : anonymous, password : your e-mail address). The preprints of the year 19XX are located in the directory

*/preprint/szbXX*

Once you found the file you want, you can get a copy by entering first

*binary*

The command

*get filename*

will transfer the file to your current local directory. You can leave ftp with

*bye*

when you are finished.

