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GGS-GROUPS: ORDER OF CONGRUENCE QUOTIENTS AND HAUSDORFF DIMENSION

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Abstract. If G is a GGS-group defined over a p-adic tree, where p is an odd prime, we calculate the order of the congruence quotients $G_n = G/\mathrm{Stab}_G(n)$ for every n. If G is defined by the vector $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$, the determination of the order of G_n is split into three cases, according to whether $\mathbf{e}$ is non-symmetric, non-constant symmetric, or constant. The formulas that we obtain only depend on p, n, and the rank of the circulant matrix whose first row is $\mathbf{e}$. As a consequence of these formulas, we also obtain the Hausdorff dimension of the closures of all GGS-groups over the p-adic tree.

1. Introduction

Subgroups of the group of automorphisms of a regular rooted tree have turned out to be a source of many interesting examples in group theory. Particular attention has been given to the so-called Grigorchuk groups and to the Gupta-Sidki group, introduced in [10] and [12], respectively. The second of the Grigorchuk groups and the Gupta-Sidki group are particular instances of the family of *GGS-groups* (GGS after Grigorchuk, Gupta, and Sidki, a term coined by Gilbert Baumslag), to which this paper is devoted. We work over the p-adic tree, where p is an odd prime, and we determine the order of all congruence quotients of GGS-groups; these are the automorphism groups induced by GGS-groups on the finite trees which are obtained by truncating the p-adic tree at every level. As a consequence, we also obtain the Hausdorff dimension of the closures of GGS-groups.

Before defining GGS-groups and stating our main results, it is convenient to recall some concepts from the theory of automorphisms of rooted trees. If $m \geq 2$ is an integer and $X = \{1, \ldots, m\}$, the m-adic tree $\mathcal{T}$ is the tree whose set of vertices is the free monoid X^*, where a word u is a descendant of v if $u = vx$ for some $x \in X$. If we consider only words of length $\leq n$, then we have a finite tree $\mathcal{T}_n$, which we refer to as *the tree $\mathcal{T}$ truncated at level n*. The group $\mathrm{Aut}\,\mathcal{T}$ of all automorphisms of $\mathcal{T}$ is a profinite group with respect to the topology induced by the filtration of the level stabilizers $\mathrm{Stab}(n)$, and we have $\mathrm{Aut}\,\mathcal{T} \cong \varprojlim_n \mathrm{Aut}\,\mathcal{T}_n$. The stabilizer $\mathrm{Stab}(n)$ of the nth level of $\mathcal{T}$ is the normal subgroup of $\mathrm{Aut}\,\mathcal{T}$ consisting of all automorphisms leaving fixed all words of length n (and, consequently, also all vertices of $\mathcal{T}_n$). These stabilizers can be considered as natural congruence subgroups for $\mathrm{Aut}\,\mathcal{T}$. If G is a subgroup of $\mathrm{Aut}\,\mathcal{T}$ and we put $\mathrm{Stab}_G(n) = \mathrm{Stab}(n) \cap G$, then we refer to the quotient $G_n = G/\mathrm{Stab}_G(n)$ as the *nth congruence quotient* of G. Since the kernel

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of the action of G on $\mathcal{T}_n$ is $\mathrm{Stab}_G(n)$, it follows that G_n can be naturally seen as a subgroup of $\mathrm{Aut}\,\mathcal{T}_n$.

As is usual nowadays in group theory, we will write the composition of two maps f and g (where we apply first f and then g) in a symmetric group by juxtaposition, i.e. as fg, rather than $g \circ f$. This applies in particular to automorphism groups of trees. However, we will write the image of u under f as $f(u)$, instead of $(u)f$ or u^f. Thus we have $(fg)(u) = g(f(u))$.

If an automorphism g fixes a vertex u, then the restriction of g to the subtree hanging from u induces an automorphism g_u of $\mathcal{T}$. In particular, if $g \in \mathrm{Stab}(1)$, then g_i is defined for every $i = 1, \ldots, m$, and we can consider the map

$$\begin{array}{rcccc} \psi & : & \mathrm{Stab}(1) & \longrightarrow & \mathrm{Aut}\,\mathcal{T} \times \overset{m}{\cdots} \times \mathrm{Aut}\,\mathcal{T} \\ & & g & \longmapsto & (g_1, \ldots, g_m). \end{array}$$

Clearly, ψ is a group isomorphism.

On the other hand, any $g \in \mathrm{Aut}\,\mathcal{T}$ can be completely determined by describing how g sends the descendants of every vertex u to the descendants of $g(u)$. This can be done by indicating, for every $x \in X$, the element $\alpha(x) \in X$ such that $g(ux) = g(u)\alpha(x)$. Then α is a permutation of X, which we call the *label* of g at u, and we denote it by $g_{(u)}$. The set of all labels of g constitutes the *portrait* of g. Thus g is determined by its portrait. We have the following rules for labels under composition and inversion:

$$(fg)_{(u)} = f_{(u)} g_{(f(u))} \quad \text{and} \quad (f^{-1})_{(u)} = (f_{(f^{-1}(u))})^{-1}. \tag{1}$$

An important automorphism of $\mathcal{T}$ is the automorphism that permutes the m subtrees hanging from the root rigidly according to the permutation $(1\ 2\ \ldots\ m)$. This is called a *rooted automorphism* and will be denoted by the letter a. Since a has order m, it makes sense to write a^k for $k \in \mathbb{Z}/m\mathbb{Z}$. Now, given a non-zero vector $\mathbf{e} = (e_1, \ldots, e_{m-1}) \in (\mathbb{Z}/m\mathbb{Z})^{m-1}$, we can recursively define an automorphism b of $\mathcal{T}$ via

$$\psi(b) = (a^{e_1}, \ldots, a^{e_{m-1}}, b).$$

We say that the subgroup $G = \langle a, b \rangle$ of $\mathrm{Aut}\,\mathcal{T}$ is the *GGS-group* corresponding to the *defining vector* $\mathbf{e}$. If $m = 2$, then there is only one GGS-group, which is isomorphic to D_∞, the infinite dihedral group. The second Grigorchuk group is obtained by choosing $m = 4$ and $\mathbf{e} = (1, 0, 1)$, and the Gupta-Sidki group arises for m equal to an odd prime and $\mathbf{e} = (1, -1, 0, \ldots, 0)$. The groups corresponding to $\mathbf{e} = (1, 0, \ldots, 0)$ and arbitrary m have also deserved special attention. In the case $m = 3$, this group was introduced by Fabrykowski and Gupta in [8]. As a reference for GGS-groups, the reader can consult Section 2.3 of the monograph [5] by Bartholdi, Grigorchuk, and Šunić, the habilitation thesis [15] of Rozhkov, or the papers [19] by Vovkivsky and [13, 14] by Pervova.

Little is known about the orders of the congruence quotients G_n when G is a GGS-group. As already mentioned, if $m = 2$, then G is infinite dihedral. We have $\psi(b) = (a, b)$, and then by direct calculation $\psi((ab)^2) = (ba, ab)$ (see also Section 6 of [11]). It readily follows that

$$\log_2 |G_n| = n + 1, \quad \text{for every } n \geq 2.$$

Hence we may always assume that $m \geq 3$, as far as the problem of determining $|G_n|$ is concerned. To the best of our knowledge, the only other cases of GGS-groups for which the order of G_n has been explicitly determined for every n correspond to $m = 3$. For the Gupta-Sidki group, Sidki himself (see [16]) proved that

$$\log_3 |G_n| = 2 \cdot 3^{n-2} + 1, \quad \text{for every } n \geq 2.$$

On the other hand, for $\mathbf{e} = (1,1)$, Bartholdi and Grigorchuk showed in [4] that

$$\log_3 |G_n| = \frac{3^n + 2n + 3}{4}, \quad \text{for every } n \geq 2.$$

From now onwards, we assume that m is equal to an odd prime p, and so $\mathcal{T}$ stands for the p-adic tree. The first of our main results is the determination of the order of G_n for *all* GGS-groups under this assumption. Before giving the statement of the theorem, we introduce some notation. Given a vector $\mathbf{a} = (a_1, \ldots, a_n)$, we write $C(\mathbf{a})$ to denote the circulant matrix generated by $\mathbf{a}$, i.e. the matrix of size $n \times n$ whose first row is $\mathbf{a}$, and every other row is obtained from the previous one by applying a shift of length one to the right. In other words, the entries of $C(\mathbf{a})$ are $c_{ij} = a_{j-i+1}$, where a_k is defined for every integer k by reducing k modulo n to a number between 1 and n. If $\mathbf{e}$ is the defining vector of a GGS-group, then we write $C(\mathbf{e}, 0)$ for the circulant matrix $C(e_1, \ldots, e_{p-1}, 0)$ over $\mathbb{F}_p$. We say that $\mathbf{e}$ is *symmetric* if $e_i = e_{p-i}$ for all $i = 1, \ldots, p-1$.

Theorem A. *Let G be a GGS-group over the p-adic tree, where p is an odd prime, and let $\mathbf{e}$ be the defining vector of G. Then, for every $n \geq 2$, we have*

$$\log_p |G_n| = tp^{n-2} + 1 - \delta\,\frac{p^{n-2} - 1}{p-1} - \varepsilon\,\frac{p^{n-2} - (n-2)p + n - 3}{(p-1)^2},$$

where t is the rank of the circulant matrix $C(\mathbf{e}, 0)$,

$$\delta = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is symmetric,} \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \varepsilon = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is constant,} \\ 0, & \text{otherwise.} \end{cases}$$

If $\sigma = (1\,2\,\ldots\,p)$, then the automorphisms whose portrait consists only of powers of σ form a Sylow pro-p subgroup of $\operatorname{Aut}\mathcal{T}$, which we denote by Γ. Observe that, under the assumption $m = p$ that we have made, all GGS-groups are subgroups of Γ. According to Theorem 1 of [19], the requirement that $\mathbf{e}$ is non-zero implies that GGS-groups are infinite if $m = p$. Since they are countable groups, they cannot be closed in the pro-p group Γ. Our second main result is related to the Hausdorff dimension of the closures of GGS-groups.

The determination of the Hausdorff dimension of closed subgroups of Γ has received special attention in the last few years (see [2, 9, 17, 18]). The most natural choice is to calculate the Hausdorff dimension with respect to the metric induced by the filtration of Γ given by the level stabilizers $\operatorname{Stab}_\Gamma(n)$. In this case, it follows from a result of Abercrombie [1], and Barnea and Shalev [3], that the Hausdorff dimension of the closure $\overline{G}$ of a subgroup G of Γ is given by the following formula:

$$(2) \qquad \dim_\Gamma \overline{G} = \liminf_{n\to\infty} \frac{\log_p |G_n|}{\log_p |\Gamma_n|} = (p-1) \liminf_{n\to\infty} \frac{\log_p |G_n|}{p^n}.$$

As an immediate consequence of Theorem A, we get the Hausdorff dimension of the closure of any GGS-group.

Theorem B. *Let G be a GGS-group over the p-adic tree, where p is an odd prime, and let $\mathbf{e}$ be the defining vector of G. Then*

$$\dim_\Gamma \overline{G} = \frac{(p-1)t}{p^2} - \frac{\delta}{p^2} - \frac{\varepsilon}{(p-1)p^2},$$

where t is the rank of the circulant matrix $C(\mathbf{e}, 0)$,

$$\delta = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is symmetric,} \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \varepsilon = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is constant,} \\ 0, & \text{otherwise.} \end{cases}$$

Our proof of Theorem A relies on finding some kind of branch structure inside a GGS-group G. In particular, if $\mathbf{e}$ is not constant, we show that G is a regular branch group (see Section 3 for the definition). This result had been previously proved by Pervova and Rozhkov for *periodic* GGS-groups. On the other hand, it is worth mentioning that the theory of p-groups of maximal class also plays a crucial role in the proof of Theorem A, particularly in the case that $\mathbf{e}$ is constant.

Notation. We use the convention that $f^g = g^{-1}fg$ and $[f,g] = f^{-1}g^{-1}fg$. On the other hand, we denote the ith row and jth column of a matrix C by C_i and C^j, respectively.

2. General properties of GGS-groups

Throughout the paper, a and b denote the canonical generators of a GGS-group G, and $b_i = b^{a^i}$ for every integer i. Note that $b_i = b_j$ if $i \equiv j \pmod p$. The images of the elements b_i under the map ψ of the introduction can be easily described:

$$\begin{aligned} \psi(b_0) &= (a^{e_1}, a^{e_2}, \dots, a^{e_{p-1}}, b), \\ \psi(b_1) &= (b, a^{e_1}, \dots, a^{e_{p-2}}, a^{e_{p-1}}), \\ &\vdots \\ \psi(b_{p-1}) &= (a^{e_2}, a^{e_3}, \dots, b, a^{e_1}). \end{aligned} \tag{3}$$

We begin with some easy facts about GGS-groups.

Theorem 2.1. *If $G = \langle a, b \rangle$ is a GGS-group, then:*

(i) $\mathrm{Stab}_G(1) = \langle b \rangle^G = \langle b_0, \dots, b_{p-1} \rangle$ *and* $G = \langle a \rangle \ltimes \mathrm{Stab}_G(1)$.
(ii) $\mathrm{Stab}_G(2) \le G' \le \mathrm{Stab}_G(1)$.
(iii) $|G : G'| = p^2$ *and* $|G : \gamma_3(G)| = p^3$.

Proof. One can easily check the equalities in part (i). Thus $G/\mathrm{Stab}_G(1)$ is cyclic and $G' \le \mathrm{Stab}_G(1)$.

The quotient $G/G' = \langle aG', bG' \rangle$ is elementary abelian of order at most p^2. It follows that $G'/\gamma_3(G) = \langle [a,b]\gamma_3(G) \rangle$ has order at most p. If $G' = \gamma_3(G)$, then $\gamma_i(G) = G'$ for every $i \ge 3$. On the other hand, since G is residually a finite p-group, the intersection of all the $\gamma_i(G)$ is trivial. Consequently $G' = 1$, which is a contradiction, since $b^a \ne b$ by (3). We conclude that $|G' : \gamma_3(G)| = p$. Now, if $|G : G'| \le p$, then G/G' is cyclic, and $G' = \gamma_3(G)$. Hence we necessarily have $|G : G'| = p^2$, and (iii) follows.

It only remains to prove that $N = \mathrm{Stab}_G(2)$ is contained in G'. Since $|G : G'| = p^2$, it suffices to prove that $|G/N : (G/N)'| = p^2$. If $|G/N : (G/N)'| \le p$, then G/N, being a finite p-group, must be cyclic. This is a contradiction, since $\langle aN \rangle$ and $\langle bN \rangle$

are two different subgroups of order p in G/N. (Note that $\langle bN\rangle$ is contained in $\mathrm{Stab}_G(1)/N$ while $\langle aN\rangle$ is not.) $\square$

Now if $g \in \mathrm{Stab}_G(1)$, it readily follows from (3) and the previous theorem that $g_i \in G$ for all $i = 1, \dots, p$. Thus the image of $\mathrm{Stab}_G(1)$ under ψ is actually contained in $G \times \overset{p}{\cdots} \times G$, and so

$$\psi(\mathrm{Stab}_G(k)) \subseteq \mathrm{Stab}_G(k-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(k-1) \tag{4}$$

for all $k \geq 1$. Another important property of the map ψ is the following.

Proposition 2.2. *If G is a GGS-group, then the composition of ψ with the projection on any component is surjective from $\mathrm{Stab}_G(1)$ onto G.*

Proof. Let us fix a position $i \in \{1, \dots, p\}$, and let $j \in \{1, \dots, p-1\}$ be such that $e_j \neq 0$. It follows from (3) that $\psi(b_{i-j})$ and $\psi(b_i)$ have the entries a^{e_j} and b in the ith component. Since $G = \langle a, b\rangle = \langle a^{e_j}, b\rangle$, the result follows. $\square$

For every positive integer n, we can define an isomorphism ψ_n from the stabilizer of the first level in $\mathrm{Aut}\,\mathcal{T}_n$ to the direct product $\mathrm{Aut}\,\mathcal{T}_{n-1} \times \overset{p}{\cdots} \times \mathrm{Aut}\,\mathcal{T}_{n-1}$, in the same way as ψ is defined. Since G_n can be seen as a subgroup of $\mathrm{Aut}\,\mathcal{T}_n$, we can consider the restriction of ψ_n to $\mathrm{Stab}_{G_n}(1)$. It follows from (4) that

$$\psi_n(\mathrm{Stab}_{G_n}(k)) \subseteq \mathrm{Stab}_{G_{n-1}}(k-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_{G_{n-1}}(k-1).$$

Obviously, G_1 is of order p, generated by the image $\overline{a}$ of a. Next we deal with G_2. Let us write $\tilde{g}$ for the image of an element $g \in G$ in G_2. Since $G_2 = \langle \tilde{a}\rangle \ltimes \mathrm{Stab}_{G_2}(1)$, it suffices to understand $\mathrm{Stab}_{G_2}(1) = \langle \tilde{b}_0, \dots, \tilde{b}_{p-1}\rangle$. Observe that ψ_2 sends $\mathrm{Stab}_{G_2}(1)$ into $G_1 \times \overset{p}{\cdots} \times G_1$, which can be identified with $\mathbb{F}_p^p$ under the linear map

$$(\overline{a}^{i_1}, \dots, \overline{a}^{i_p}) \longmapsto (i_1, \dots, i_p).$$

This allows us to consider $\mathrm{Stab}_{G_2}(1)$ as a vector space over $\mathbb{F}_p$.

Before analyzing G_2 in the next theorem, we need the following lemma (see Exercise 4 in Section 1 of the book [6]) about finite p-groups of maximal class, which will also be used at some other places in the paper.

Lemma 2.3. *Let P be a finite p-group such that $|P : P'| = p^2$. If P has an abelian maximal subgroup A, then P is a group of maximal class. Furthermore, if $g_0 \in P \smallsetminus A$, then:*

(i) *If $a \in A \smallsetminus \gamma_2(P)$, then $\gamma_2(P)/\gamma_3(P)$ is generated by the image of $[a, g_0]$.*
(ii) *If $i \geq 2$ and $a \in \gamma_i(P) \smallsetminus \gamma_{i+1}(P)$, then $\gamma_{i+1}(P)/\gamma_{i+2}(P)$ is generated by the image of $[a, g_0]$.*

Theorem 2.4. *Let G be a GGS-group with defining vector $\mathbf{e}$, and put $C = C(\mathbf{e}, 0)$. Then:*

(i) *The dimension of $\mathrm{Stab}_{G_2}(1)$ coincides with the rank t of C.*
(ii) *G_2 is a p-group of maximal class of order p^{t+1}.*

Proof. (i) If $\tilde{g} \in \mathrm{Stab}_{G_2}(1)$ and $\psi_2(\tilde{g}) = (\overline{a}^{i_1}, \dots, \overline{a}^{i_p})$, where we consider the exponents $i_1, \dots, i_p$ as elements of $\mathbb{F}_p$, we define

$$\Psi_2(\tilde{g}) = (i_1, \dots, i_p) \in \mathbb{F}_p^p.$$

Observe that Ψ_2 is injective.

By (3),

$$\Psi_2(\tilde{b}_0) = (e_1, e_2, \dots, e_{p-1}, 0) = (\mathbf{e}, 0)$$

coincides with the first row of C. Since the components of the rest of the b_i are obtained by cyclically permuting those of b_0, and since $C = C(\mathbf{e}, 0)$, it follows that $\Psi_2(\tilde{b}_i)$ is the $(i+1)$st row of C. Thus the dimension of $\operatorname{Stab}_{G_2}(1)$ coincides with the dimension of the subspace of $\mathbb{F}_p^p$ generated by the rows of C, i.e. with the rank t of the matrix C.

(ii) We have

$$|G_2| = |G_2 : \operatorname{Stab}_{G_2}(1)||\operatorname{Stab}_{G_2}(1)| = p \cdot p^t = p^{t+1}.$$

On the other hand, it follows from (ii) and (iii) of Theorem 2.1 that $|G_2 : G_2'| = p^2$. Since $\operatorname{Stab}_{G_2}(1)$ is an abelian maximal subgroup of G_2, we conclude from Lemma 2.3 that G_2 is a p-group of maximal class. □

As a consequence, we can improve part (ii) of Theorem 2.1.

Corollary 2.5. *If G is a GGS-group, then* $\operatorname{Stab}_G(2) \le \gamma_3(G)$.

Proof. Since the defining vector $\mathbf{e}$ of G is different from $(0, \dots, 0)$, it is clear that the rank t of the matrix $C(\mathbf{e}, 0)$ is at least 2. It follows from the previous theorem that $G_2 = G/\operatorname{Stab}_G(2)$ is a p-group of maximal class of order greater than or equal to p^3. Thus $|G_2 : \gamma_3(G_2)| = p^3 = |G : \gamma_3(G)|$, and consequently $\operatorname{Stab}_G(2)$ is contained in $\gamma_3(G)$. □

We have seen in Theorem 2.1 that $G' \le \operatorname{Stab}_G(1)$. Next we want to characterize which elements of $\operatorname{Stab}_G(1)$ belong to G'. This goal will be achieved in Theorem 2.11. If $g \in \operatorname{Stab}_G(1) = \langle b_0, \dots, b_{p-1} \rangle$, then we can write g as a word in $b_0, \dots, b_{p-1}$, i.e. we can write $g = \omega(b_0, \dots, b_{p-1})$, where $\omega = \omega(x_0, \dots, x_{p-1})$ is a group word in the p variables $x_0, \dots, x_{p-1}$.

Definition 2.6. Let ω be a group word in the variables $x_0, \dots, x_{p-1}$, where p is a prime. Then:

(i) The *partial p-weight* of ω with respect to a variable x_i, with $0 \le i \le p-1$, is the sum of the exponents of x_i in the expression for ω, considered as an element of $\mathbb{F}_p$.

(ii) The *total p-weight* of ω is the sum of all of its partial p-weights.

It is not difficult to give examples showing that the representation of an element $g \in \operatorname{Stab}_G(1)$ as a word in $b_0, \dots, b_{p-1}$ is not unique. Our first step towards the proof of Theorem 2.11 will be to see that, however, the partial and total p-weights are the same for all word representations. For this purpose, we need the following lemma.

Lemma 2.7. *Let p be a prime, and let $(a_0, \dots, a_{p-1}) \in \mathbb{F}_p^p$ be a non-zero vector. If $C = C(a_0, \dots, a_{p-1})$, then:*

(i) $\operatorname{rk} C = p - m$, *where m is the multiplicity of 1 as a root of the polynomial $a(X) = a_0 + a_1 X + \dots + a_{p-1} X^{p-1}$. As a consequence, we have* $\operatorname{rk} C < p$ *if and only if* $\sum_{i=0}^{p-1} a_i = 0$.

(ii) *If* $\mathbf{1}$ *represents the column vector of length p with all entries equal to 1, then*

$$\operatorname{rk} C = \operatorname{rk} (C \mid \mathbf{1}).$$

Proof. If we consider the quotient ring $V = \mathbb{F}_p[X]/(X^p - 1)$ as an $\mathbb{F}_p$-vector space, then both

$$\mathcal{B} = \{\overline{1}, \overline{X}, \dots, \overline{X^{p-1}}\}$$

and

$$\mathcal{B}' = \{\overline{1}, \overline{X-1}, \dots, \overline{(X-1)^{p-1}}\}$$

are bases of V. Multiplication by $\overline{a(X)}$ defines a linear map $\varphi : V \to V$, and the matrix of φ with respect to $\mathcal{B}$ is C (we construct the matrix by rows). Thus $\operatorname{rk} C = \operatorname{rk} \varphi$.

On the other hand, we can write $a(X) = (X-1)^m b(X)$, with $b(X) \in \mathbb{F}_p[X]$ and $b(1) \neq 0$. Let $b(X) = b_0 + b_1(X-1) + \cdots + b_{k-1}(X-1)^{k-1}$, where $k = p - m$ and $b_0 \neq 0$. Then the matrix of φ with respect to $\mathcal{B}'$ is the block matrix

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad B = \begin{pmatrix} b_0 & b_1 & \cdots & b_{k-2} & b_{k-1} \\ 0 & b_0 & \cdots & b_{k-3} & b_{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_0 \end{pmatrix},$$

since $\overline{(X-1)^i} = \overline{0}$ in V for all $i \geq p$. Thus $\operatorname{rk} \varphi = k$, and (i) follows.

Let us now prove (ii). We first prove that

$$\operatorname{rk} C = \operatorname{rk} \begin{pmatrix} C \\ 1 \dots 1 \end{pmatrix}. \tag{5}$$

Since C is the matrix of φ with respect to $\mathcal{B}$ constructed by rows, it is clear that (5) is equivalent to $\overline{1 + X + \cdots + X^{p-1}}$ lying in the image of φ. Note that, since we are working with coefficients in $\mathbb{F}_p$, we have

$$1 + X + \cdots + X^{p-1} = (X-1)^{p-1}.$$

Since

$$\varphi(\overline{(X-1)^{k-1}}) = \overline{b_0(X-1)^{p-1}}$$

and $b_0 \neq 0$, it follows that $\overline{(X-1)^{p-1}} \in \operatorname{im} \varphi$, as desired.

Now, since the transpose ${}^tC$ of C is also a circulant matrix, we can apply (5) to ${}^tC$ and get

$$\operatorname{rk} C = \operatorname{rk} {}^tC = \operatorname{rk} \begin{pmatrix} {}^tC \\ 1 \dots 1 \end{pmatrix} = \operatorname{rk} {}^t(C \mid \mathbf{1}) = \operatorname{rk} (C \mid \mathbf{1}).$$

□

Let $g = \omega(b_0, \dots, b_{p-1})$ be an arbitrary element of $\operatorname{Stab}_G(1)$, and suppose that the partial p-weight of ω with respect to x_i is r_i, for $i = 0, \dots, p-1$. It follows from (3) that

$$\psi(g) = (a^{m_1}\omega_1(b_0, \dots, b_{p-1}), \dots, a^{m_p}\omega_p(b_0, \dots, b_{p-1})), \tag{6}$$

where each ω_i is a word of total p-weight r_i (and where r_p is to be understood as r_0), and

$$m_i = (r_0 \; r_1 \; \dots \; r_{p-1}) C^i. \tag{7}$$

Theorem 2.8. *Let G be a GGS-group, and let $g \in \operatorname{Stab}_G(1)$. Then the partial and total p-weights are the same for all representations of g as a word in $b_0, \dots, b_{p-1}$.*

Proof. It suffices to see that, if ω is a word such that $\omega(b_0,\dots,b_{p-1})=1$, then the total p-weight of ω is 0, and the partial p-weight r_i of ω with respect to x_i is equal to 0, for every $i=0,\dots,p-1$. Obviously, the second assertion implies the first one, but the proof will go the other way around.

As in (6), we write

$$\psi(\omega(b_0,\dots,b_{p-1}))=(a^{m_1}\omega_1(b_0,\dots,b_{p-1}),\dots,a^{m_p}\omega_p(b_0,\dots,b_{p-1})). \tag{8}$$

Since this element is equal to 1, it follows that $m_i=0$ for $i=1,\dots,p$. According to (7), this means that

$$(r_0\ r_1\ \dots\ r_{p-1})C=(0\ 0\ \dots\ 0).$$

Now, since $\operatorname{rk} C=\operatorname{rk}(C\mid \mathbf{1})$ by Lemma 2.7, we also have $(r_0\ r_1\ \dots\ r_{p-1})\mathbf{1}=0$, that is,

$$r_0+r_1+\dots+r_{p-1}=0.$$

This proves that the total p-weight of ω is 0.

Now we return to (8). Since $\omega(b_0,\dots,b_{p-1})=1$ by hypothesis, then we also have $\omega_i(b_0,\dots,b_{p-1})=1$ for all $i=1,\dots,p$. Now, since the total p-weight of ω_i is r_i, it follows from the previous paragraph that $r_i=0$. □

The independence of the partial and total p-weights from the word representation allows us to give the following definition.

Definition 2.9. Let G be a GGS-group, and let $g\in \operatorname{Stab}_G(1)$. We define the *partial weight* of g with respect to b_i, and the *total weight* of g, as the corresponding p-weights for any word ω representing g.

We prefer to speak simply about weights instead of p-weights in the case of an element $g\in\operatorname{Stab}_G(1)$, since all elements b_i (with respect to which the weights are considered) have order p. Now the following result is clear.

Theorem 2.10. *Let G be a GGS-group. Then the maps from $\operatorname{Stab}_G(1)$ to $\mathbb{F}_p$ sending every $g\in\operatorname{Stab}_G(1)$ to its partial weight with respect to one of the b_i or to its total weight are well-defined homomorphisms.*

Theorem 2.11. *Let G be a GGS-group. Then the derived subgroup G' consists of all the elements of $\operatorname{Stab}_G(1)$ whose total weight is equal to 0.*

Proof. The map ϑ sending each element of $\operatorname{Stab}_G(1)$ to its total weight is a homomorphism onto the abelian group $\mathbb{F}_p$, and consequently $G'\le\ker\vartheta$. Since $|G:G'|=p^2$ and $|G:\operatorname{Stab}_G(1)|=|\operatorname{Stab}_G(1):\ker\vartheta|=p$, the equality follows. □

Definition 2.12. Let G be a GGS-group. If $g\in\operatorname{Stab}_G(1)$ has partial weight r_i with respect to b_i for $i=0,\dots,p-1$, we say that $(r_0,\dots,r_{p-1})\in\mathbb{F}_p^p$ is the *weight vector* of g.

As we next see, we can analyze the subgroups $\operatorname{Stab}_G(2)$ and $\operatorname{Stab}_G(3)$ by using the weight vector.

Theorem 2.13. *Let G be a GGS-group with defining vector $\mathbf{e}$, and put $C=C(\mathbf{e},0)$. If the weight vector of $g\in\operatorname{Stab}_G(1)$ is $(r_0,\dots,r_{p-1})$, then:*

(i) *We have $g\in\operatorname{Stab}_G(2)$ if and only if $(r_0\ \dots\ r_{p-1})C=(0\ \dots\ 0)$.*
(ii) *If $g\in\operatorname{Stab}_G(3)$, then $(r_0,\dots,r_{p-1})=(0,\dots,0)$.*

Proof. (i) If we write $\psi(g)$ as in (6), then $g \in \operatorname{Stab}_G(2)$ if and only if $m_i = 0$ in $\mathbb{F}_p$ for every $i = 1, \ldots, p$. Now, by (7), this is equivalent to the condition $(r_0 \ \ldots \ r_{p-1})C = (0 \ \ldots \ 0)$.

(ii) Again we use the expression in (6). If $g \in \operatorname{Stab}_G(3)$, then $\omega_i(b_0, \ldots, b_{p-1}) \in \operatorname{Stab}_G(2)$ for all $i = 1, \ldots, p$. As mentioned above, $\omega_i(b_0, \ldots, b_{p-1})$ is an element of total weight r_i. Let $(s_0, \ldots, s_{p-1})$ be the weight vector of this element, so that $r_i = s_0 + \cdots + s_{p-1}$. Then, by (i), we have $(s_0 \ \ldots \ s_{p-1})C = (0 \ \ldots \ 0)$. Since $\operatorname{rk} C = \operatorname{rk}(C \mid \mathbf{1})$ by Lemma 2.7, it follows that $r_i = s_0 + \cdots + s_{p-1} = 0$, as desired. □

One may wonder whether the converse holds in (ii) of the previous theorem, i.e. if the weight vector of an element is $(0, \ldots, 0)$, does it lie in $\operatorname{Stab}_G(3)$? We make things clearer in the following theorem.

Theorem 2.14. *Let G be a GGS-group. Then $\operatorname{Stab}_G(1)'$ consists of all elements of $\operatorname{Stab}_G(1)$ whose weight vector is $(0, \ldots, 0)$. Furthermore, we have $|G : \operatorname{Stab}_G(1)'| = p^{p+1}$.*

Proof. The map ρ which sends every element of $\operatorname{Stab}_G(1)$ to its weight vector is a homomorphism onto $\mathbb{F}_p^p$. Thus $|\operatorname{Stab}_G(1) : \ker \rho| = p^p$. Since $\mathbb{F}_p^p$ is abelian, it follows that $\operatorname{Stab}_G(1)' \leq \ker \rho$. On the other hand, since $\operatorname{Stab}_G(1) = \langle b_0, \ldots, b_{p-1} \rangle$ and every b_i has order p, we have $|\operatorname{Stab}_G(1) : \operatorname{Stab}_G(1)'| \leq p^p$. Hence $\ker \rho = \operatorname{Stab}_G(1)'$ and $|\operatorname{Stab}_G(1) : \operatorname{Stab}_G(1)'| = p^p$. Since $|G : \operatorname{Stab}_G(1)| = p$, we are done. □

In particular, we have $\operatorname{Stab}_G(3) \leq \operatorname{Stab}_G(1)'$. Once we prove Theorem A, it will follow that $|G : \operatorname{Stab}_G(3)| = p^{tp+1-\delta}$, where t is the rank of $C(\mathbf{e}, 0)$ and δ is 1 or 0, according to whether or not $\mathbf{e}$ is symmetric. Since t is always at least 2, we have $|G : \operatorname{Stab}_G(3)| > p^{p+1}$ in every case. Hence $\operatorname{Stab}_G(3)$ is always a proper subgroup of $\operatorname{Stab}_G(1)'$, and the converse of (ii) in Theorem 2.13 never holds.

Next we prove a result which will allow us to reduce, for the calculation of the order of congruence quotients and of the Hausdorff dimension, to the case of GGS-groups with defining vectors of the form $\mathbf{e} = (1, e_2, \ldots, e_{p-1})$. We need the following lemma.

Lemma 2.15. *Let p be a prime, and let $\sigma = (1 \ 2 \ \ldots \ p)$. Assume that $\alpha \in S_p$ satisfies the following two conditions:*

(i) *α normalizes the subgroup $\langle \sigma \rangle$.*
(ii) *$\alpha(p) = p$.*

Then, for every $i = 1, \ldots, p-1$, if $\alpha(i) = j$ we have $\alpha(p-i) = p-j$.

Proof. If we think of S_p as the set of permutations of the field $\mathbb{F}_p$, then σ corresponds to the map $\ell \mapsto \ell + 1$, and the normalizer of $\langle \sigma \rangle$ in S_p corresponds to the affine group over $\mathbb{F}_p$ (see Lemma 14.1.2 of [7]). Thus $\alpha(\ell) = a\ell + b$ for some $a \in \mathbb{F}_p^\times$ and $b \in \mathbb{F}_p$. Since $\alpha(p) = p$, it follows that $b = 0$, and so $\alpha(\ell) = a\ell$ for every $\ell \in \mathbb{F}_p$. Hence α is a linear map and, as a consequence,

$$\alpha(p-i) = \alpha(-i) = -\alpha(i) = -j = p-j.$$

□

We say that an automorphism f of $\mathcal{T}$ has *constant portrait* if f has the same label at all vertices of $\mathcal{T}$. By formula (1) for the labels of a composition, the set of all automorphisms of constant portrait is a subgroup of $\operatorname{Aut} \mathcal{T}$.

Theorem 2.16. *Let G be a GGS-group with defining vector $\mathbf{e} = (e_1, \ldots, e_{p-1})$, and assume that $e_k \neq 0$. Then there exists $f \in \operatorname{Aut} \mathcal{T}$ of constant portrait such that $L = G^f$ is a GGS-group whose defining vector $\mathbf{e}' = (e'_1, \ldots, e'_{p-1})$ satisfies:*

(i) *$\mathbf{e}'$ is a permutation of the vector $\mathbf{e}/e_k$, that is, there exists $\alpha \in S_{p-1}$ such that $e'_i = e_{\alpha(i)}/e_k$ for all $i = 1, \ldots, p-1$.*
(ii) *$\alpha(1) = k$, and so $e'_1 = 1$.*
(iii) *If $\alpha(i) = j$, then $\alpha(p-i) = p-j$. In other words, two values which are placed in symmetric positions of $\mathbf{e}$ are moved (after division by e_k) to symmetric positions of $\mathbf{e}'$. Thus $\mathbf{e}'$ is symmetric if and only if $\mathbf{e}$ is symmetric.*
(iv) *$\operatorname{rk} C(\mathbf{e}, 0) = \operatorname{rk} C(\mathbf{e}', 0)$.*

Furthermore, we have $|G_n| = |L_n|$ for every n, and $\dim_\Gamma \overline{G} = \dim_\Gamma \overline{L}$.

Proof. Observe that there exists a permutation $\beta \in S_p$, in fact only one, that normalizes the subgroup $\langle \sigma \rangle$ and such that $\beta(k) = 1$ and $\beta(p) = p$. Indeed, since $\sigma^\beta = (\beta(1) \ \ldots \beta(p))$ and the positions of 1 and p are already fixed in this last tuple, there is only one way to choose the rest of the images of β if we want to obtain a power of σ. Let r be defined by the condition $\sigma^\beta = \sigma^r$, and set $\alpha = \beta^{-1}$. Note that $\alpha(1) = k$ and that, by Lemma 2.15, if $\alpha(i) = j$, then $\alpha(p-i) = p-j$.

Now we define an automorphism f of $\mathcal{T}$ by choosing the labels at all vertices of $\mathcal{T}$ equal to β. We claim that $L = G^f$ satisfies the properties of the statement of the theorem. We have

$$(g^f)_{(v)} = \beta^{-1} g_{(f^{-1}(v))} \beta$$

for every $g \in G$ and every vertex v of the tree. It readily follows that $a^f = a^r$. We now consider $c = b^f$. Let S be the set of all vertices of the form $p.\overset{n}{\ldots}.pi$, where $n \geq 0$ and $1 \leq i \leq p-1$. If $v \in S$, then we have $f(v) = p.\overset{n}{\ldots}.p\beta(i)$, and consequently $f^{-1}(v) = p.\overset{n}{\ldots}.p\alpha(i)$. Thus

$$c_{(v)} = \beta^{-1} b_{(p \ldots p\alpha(i))} \beta = (\sigma^{e_{\alpha(i)}})^\beta = \sigma^{re_{\alpha(i)}}$$

in this case. On the other hand, if $v \notin S$, then also $f^{-1}(v) \notin S$, and so we have $b_{(f^{-1}(v))} = 1$ and $c_{(v)} = 1$. Thus c is the automorphism given by the recursive relation

$$\psi(c) = (a^{re_{\alpha(1)}}, \ldots, a^{re_{\alpha(p-1)}}, c).$$

Now, let ℓ be the inverse of $re_{\alpha(1)}$ modulo p, and put $b' = c^\ell$. Then $L = \langle a, b' \rangle$, where b' is the automorphism defined by

$$\psi(b') = (a^{e'_1}, \ldots, a^{e'_{p-1}}, b'),$$

i.e. L is the GGS-group with defining vector $\mathbf{e}'$. This proves (i), (ii), and (iii).

Let us now check (iv). If $C = C(\mathbf{e}, 0)$, $C' = C(\mathbf{e}', 0)$ and we define $e_p = 0$, then

$$c'_{ij} = e_{\alpha(j-i+1)}/e_k = e_{\alpha(j)-\alpha(i)+\alpha(1)}/e_k = c_{\alpha(i)-\alpha(1)+1, \alpha(j)}/e_k,$$

since we know that α is a homomorphism by the proof of Lemma 2.15. (Here, all indices are taken modulo p between 1 and p.) By observing that the maps $i \mapsto \alpha(i) - \alpha(1) + 1$ and $j \mapsto \alpha(j)$ are permutations of $\mathbb{F}_p$, we conclude that $\operatorname{rk} C = \operatorname{rk} C'$.

Finally, note that, since G and L are conjugate, we clearly have $|G_n| = |L_n|$, and then by (2), also $\dim_\Gamma \overline{G} = \dim_\Gamma \overline{L}$. □

We want to stress the fact that the automorphism f conjugating G to L in the previous theorem has constant portrait. This has nice consequences, such as the following one.

Proposition 2.17. *Let J and K be two subgroups of $\operatorname{Aut}\mathcal{T}$, where J is contained in $\operatorname{Stab}(1)$. If $f \in \operatorname{Aut}\mathcal{T}$ has constant portrait, then we have*

$$K \times \overset{p}{\cdots} \times K \subseteq \psi(J)$$

if and only if

$$K^f \times \overset{p}{\cdots} \times K^f \subseteq \psi(J^f).$$

Proof. Since f^{-1} is also an automorphism of constant portrait, it suffices to prove the 'only if' part. Let β be the permutation appearing at all labels of f. Then we can write $f = ch$, where c is the rooted automorphism corresponding to β and $h \in \operatorname{Stab}(1)$ is such that $\psi(h) = (f, \dots, f)$.

Let us now consider an arbitrary tuple $(k_1, \dots, k_p)$, with $k_i \in K$ for every $i = 1, \dots, p$. By hypothesis, there exists $j \in J$ such that $\psi(j) = (k_1, \dots, k_p)$. Then $\psi(j^c) = (k_{\beta^{-1}(1)}, \dots, k_{\beta^{-1}(p)})$, and consequently

$$\psi(j^f) = \psi(j^c)^{\psi(h)} = (k_{\beta^{-1}(1)}, \dots, k_{\beta^{-1}(p)})^{(f,\dots,f)} = (k^f_{\beta^{-1}(1)}, \dots, k^f_{\beta^{-1}(p)}).$$

Clearly, this implies that $K^f \times \cdots \times K^f \subseteq \psi(J^f)$. □

The previous proposition will be useful when we want to find a branch structure in a GGS-group. The same can be said about the following result.

Proposition 2.18. *Let G be a GGS-group, and let L and N be two normal subgroups of G. If $L = \langle X \rangle^G$ for a subset X of G, and $(x, 1, \dots, 1) \in \psi(N)$ for every $x \in X$, then*

$$L \times \overset{p}{\cdots} \times L \subseteq \psi(N).$$

Proof. By Proposition 2.2, if $g \in G$ there exists $h \in \operatorname{Stab}_G(1)$ such that the first component of $\psi(h)$ is g. Since $(x, 1, \dots, 1) \in \psi(N)$ and N is normal in G, it follows that $(x^g, 1, \dots, 1) \in \psi(N)$ for every $x \in X$ and $g \in G$. Hence

$$L \times \{1\} \times \overset{p-1}{\cdots} \times \{1\} \subseteq \psi(N),$$

since $L = \langle x^g \mid x \in X,\ g \in G \rangle$.

Now, if $\psi(n) = (\ell_1, \ell_2, \dots, \ell_p)$, then $\psi(n^a) = (\ell_p, \ell_1, \dots, \ell_{p-1})$. As a consequence,

$$\{1\} \times \cdots \times \{1\} \times L \times \{1\} \times \cdots \times \{1\} \subseteq \psi(N),$$

where L may appear at any position. The result follows. □

3. GGS-groups with non-constant defining vector

In this section we prove Theorems A and B in the case that the defining vector $\mathbf{e}$ of the GGS-group G is not constant. As it turns out, the key is to prove that G has a certain branch structure. We begin by recalling the concepts that we will need about branching in $\operatorname{Aut}\mathcal{T}$.

Definition 3.1. Let G be a self-similar spherically transitive group of automorphisms of a regular tree, and let K be a non-trivial subgroup of $\operatorname{Stab}_G(1)$. We say that G is *weakly regular branch* over K if

$$K \times \cdots \times K \subseteq \psi(K).$$

If furthermore K has finite index in G, we say that G is *regular branch* over K.

It is well known (and an immediate consequence of Proposition 2.2) that every GGS-group G is self-similar and spherically transitive. We next see that, if $\mathbf{e}$ is not constant, then G is regular branch over $\gamma_3(G)$.

Lemma 3.2. *Let G be a GGS-group with non-constant defining vector. Then*

$$\psi(\gamma_3(\mathrm{Stab}_G(1))) = \gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G).$$

In particular,

$$\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G) \subseteq \psi(\gamma_3(G)),$$

and G is a regular branch group over $\gamma_3(G)$.

Proof. Since $\psi(\mathrm{Stab}_G(1))$ is contained in $G \times \overset{p}{\cdots} \times G$, it clearly suffices to prove the inclusion $\supseteq$. By Theorem 2.16 and Proposition 2.17, we may assume that $\mathbf{e} = (1, e_2, \ldots, e_{p-1})$. If $e_{p-1} = 0$, then

$$\psi(b) = (a, \ldots, a^{e_{p-2}}, 1, b),$$

and consequently

$$\psi([b_0, b_1, b_0]) = ([a, b, a], 1, \ldots, 1)$$

and

$$\psi([b_0, b_1, b_1]) = ([a, b, b], 1, \ldots, 1).$$

Since $G = \langle a, b \rangle$, it follows that $\gamma_3(G) = \langle [a, b, a], [a, b, b] \rangle^G$, and then by Proposition 2.18, we have $\gamma_3(G) \times \cdots \times \gamma_3(G) \subseteq \psi(\gamma_3(\mathrm{Stab}_G(1)))$. Thus we may assume that $e_{p-1} \neq 0$.

Now we consider the following two cases separately:

(i) There exists $k \in \{2, \ldots, p-2\}$ such that (e_{k-1}, e_k) and (e_k, e_{k+1}) are not proportional.

(ii) (e_{k-1}, e_k) and (e_k, e_{k+1}) are proportional for all $k = 2, \ldots, p-2$.

Observe that if $p = 3$, then case (ii) vacuously holds.

(i) Let us put

$$g_k = b_{p-k+1}^{e_k} b_{p-k}^{-e_{k-1}}$$

for $2 \leq k \leq p-2$, so that

$$\psi(g_k) = (a^{e_k^2 - e_{k-1}e_{k+1}}, \ldots, 1).$$

(The intermediate values represented by the dots are not necessarily 1 in this case.) Since (e_{k-1}, e_k) and (e_k, e_{k+1}) are not proportional, we have $e_k^2 - e_{k-1}e_{k+1} \neq 0$. Hence there is a power g of g_k such that

$$\psi(g) = (a, \ldots, 1).$$

On the other hand, since

$$\psi(b_1 b_{p-1}^{-e_{p-1}}) = (ba^{-e_2 e_{p-1}}, \ldots, 1),$$

with the help of g we can get an element $h \in \mathrm{Stab}_G(1)$ such that

$$\psi(h) = (b, \ldots, 1).$$

Consequently,

$$\psi([b_0, b_1, g]) = ([a, b, a], 1, \dots, 1)$$

and

$$\psi([b_0, b_1, h]) = ([a, b, b], 1, \dots, 1),$$

and the result follows as before from Proposition 2.18.

(ii) Since $e_1 = 1$, it follows that $e_i = e_2^{i-1}$ for every $i = 1, \dots, p-1$. (Note that this is valid all the same if $p = 3$.) Hence $\mathbf{e} = (1, m, m^2, \dots, m^{p-2})$ with $m \neq 1$, because $\mathbf{e}$ is not constant. Since $e_{p-1} \neq 0$, we also have $m \neq 0$, and consequently $m^{p-1} = 1$. Then

$$\psi(b_0 b_1^{-m}) = (ab^{-m}, 1, \dots, 1, ba^{-1})$$

and

$$\psi(b_1 b_2^{-m}) = (ba^{-1}, ab^{-m}, 1, \dots, 1).$$

Hence

$$\psi([b_0, b_1, b_1 b_2^{-m}]) = ([a, b, ba^{-1}], 1, \dots, 1)$$

and

$$\psi([b_2^m, b_1, b_0 b_1^{-m}]) = ([a, b, ab^{-m}], 1, \dots, 1).$$

Now, since $G' = \langle [a, b] \rangle^G$ and $\langle ab^{-m}, ba^{-1} \rangle = \langle b^{1-m}, ba^{-1} \rangle$ is the whole of G (at this point, it is essential that $m \neq 1$), it follows that

$$\gamma_3(G) = \langle [a, b, ab^{-m}], [a, b, ba^{-1}] \rangle^G.$$

Thus the result is again a consequence of Proposition 2.18. □

As a consequence of the previous lemma, we can show that, for $\mathbf{e}$ non-constant and $n \geq 3$, there is a close relation between $\mathrm{Stab}_G(n)$ and $\mathrm{Stab}_G(n-1)$ in a GGS-group G.

Lemma 3.3. *Let G be a GGS-group with non-constant defining vector $\mathbf{e}$. Then, for every $n \geq 3$ we have*

$$\psi(\mathrm{Stab}_G(n)) = \mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(n-1)$$

and

$$\psi_{n+1}(\mathrm{Stab}_{G_{n+1}}(n)) = \mathrm{Stab}_{G_n}(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_{G_n}(n-1).$$

Proof. Clearly, it suffices to prove the first equality. By using Corollary 2.5 and Lemma 3.2, we have

$$\mathrm{Stab}_G(2) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(2) \subseteq \gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G) = \psi(\gamma_3(\mathrm{Stab}_G(1))).$$

Thus $\mathrm{Stab}_G(n-1) \times \cdots \times \mathrm{Stab}_G(n-1)$ is contained in the image of $\mathrm{Stab}_G(1)$ under ψ for all $n \geq 3$, and the result follows. □

If the vector $\mathbf{e}$ is non-symmetric, we can improve Lemma 3.2 as follows.

Lemma 3.4. *Let G be a GGS-group with non-symmetric defining vector. Then*

$$\psi(\mathrm{Stab}_G(1)') = G' \times \overset{p}{\cdots} \times G'. \tag{9}$$

In particular,

$$G' \times \overset{p}{\cdots} \times G' \subseteq \psi(G'),$$

and G is a regular branch group over G'.

Proof. Observe that we only need to care about the inclusion $\supseteq$. By Theorem 2.16 and Proposition 2.17, we may assume that $e_1 = 1$ and $e_{p-1} \neq 1$, since $\mathbf{e}$ is non-symmetric. Let us write m for e_{p-1}.

By using (3), we get

$$\begin{aligned}
\psi([b_0, b_1]) &= ([a,b], 1, \dots, 1, [b, a^m]) \\
&\equiv ([a,b], 1, \dots, 1, [a,b]^{-m}) \pmod{\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G)}, \\
\psi([b_{p-1}, b_0]^m) &= (1, \dots, 1, [b, a^m]^m, [a,b]^m) \\
&\equiv (1, \dots, 1, [a,b]^{-m^2}, [a,b]^m) \pmod{\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G)}, \\
&\vdots \\
\psi([b_1, b_2]^{m^{p-1}}) &= ([b, a^m]^{m^{p-1}}, [a,b]^{m^{p-1}}, 1, \dots, 1) \\
&\equiv ([a,b]^{-m^p}, [a,b]^{m^{p-1}}, 1, \dots, 1) \pmod{\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G)}.
\end{aligned}$$

Since $m^p = m$ (recall that $m \in \mathbb{F}_p$), if we multiply together all the expressions above, we obtain that

$$\psi([b_0, b_1][b_{p-1}, b_0]^m \dots [b_1, b_2]^{m^{p-1}}) \equiv ([a,b]^{1-m}, 1, \dots, 1) \pmod{\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G)}.$$

If we use the inclusion

$$\gamma_3(G) \times \overset{p}{\cdots} \times \gamma_3(G) \subseteq \psi(\mathrm{Stab}_G(1)'),$$

which is a consequence of Lemma 3.2, we get

$$([a,b]^{1-m}, 1, \dots, 1) \in \psi(\mathrm{Stab}_G(1)').$$

Now, since $G = \langle a, b \rangle$ and $m \neq 1$, it follows that G' is the normal closure of $[a,b]^{1-m}$. By Proposition 2.18, we conclude that $G' \times \cdots \times G' \subseteq \psi(\mathrm{Stab}_G(1)')$. $\square$

If $\mathbf{e}$ is symmetric non-constant, then equality (9) does not hold, but we are able to measure how far $G' \times \overset{p}{\cdots} \times G'$ is from $\psi(\mathrm{Stab}_G(1)')$.

Lemma 3.5. *Let G be a GGS-group with symmetric non-constant defining vector. Then*

$$|G' \times \overset{p}{\cdots} \times G' : \psi(\mathrm{Stab}_G(1)')| = p.$$

Proof. Since $\mathrm{Stab}_G(1) = \langle b_0, b_1, \dots, b_{p-1} \rangle$, it follows that

$$\mathrm{Stab}_G(1)' = \langle [b_i, b_j]^h \mid 0 \leq i, j \leq p-1,\ h \in \mathrm{Stab}_G(1) \rangle. \tag{10}$$

Let $\overline{\psi}$ be the map from $\mathrm{Stab}_G(1)'$ to $G'/\gamma_3(G) \times \overset{p}{\cdots} \times G'/\gamma_3(G)$ which is obtained by first applying ψ and then reducing every component modulo $\gamma_3(G)$. Observe that $G'/\gamma_3(G) \times \overset{p}{\cdots} \times G'/\gamma_3(G)$ can be seen as a vector space of dimension p over $\mathbb{F}_p$, since $|G' : \gamma_3(G)| = p$. Since we may assume that $e_1 = 1$, and since $e_{p-1} = e_1$, we have

$$\psi([b_i, b_{i+1}]) = (1, \dots, 1, [b,a], [a,b], 1, \dots, 1), \quad \text{for } i = 1, \dots, p-1,$$

where $[b,a]$ appears at the ith position. Now, $G'/\gamma_3(G)$ is generated by the image of $[b,a]$, and so it readily follows that the dimension of $\overline{\psi}(\mathrm{Stab}_G(1)')$ is at least $p-1$.

Hence

$$|G' \times \cdots \times G' : \psi(\mathrm{Stab}_G(1)')(\gamma_3(G) \times \cdots \times \gamma_3(G))| = 1 \text{ or } p.$$

Since $\gamma_3(G) \times \cdots \times \gamma_3(G) \le \psi(\mathrm{Stab}_G(1)')$ by Lemma 3.2, we get

$$|G' \times \cdots \times G' : \psi(\mathrm{Stab}_G(1)')| = 1 \text{ or } p.$$

Thus it suffices to see that $([a,b], 1, \ldots, 1) \notin \psi(\mathrm{Stab}_G(1)')$ in order to conclude that $|G' \times \cdots \times G' : \psi(\mathrm{Stab}_G(1)')| = p$, as desired.

Let $\lambda : \mathrm{Stab}_G(1) \longrightarrow \mathbb{F}_p$ be the homomorphism given by

$$g \longmapsto \sum_{i=0}^{p-1} i r_i,$$

where $(r_0, \ldots, r_{p-1})$ is the weight vector of g. If $g \in \mathrm{Stab}_G(1)$, then the weight vector of g^b is also $(r_0, \ldots, r_{p-1})$, and the weight vector of g^a is $(r_{p-1}, r_0, \ldots, r_{p-2})$. Hence $\lambda(g^b) = \lambda(g)$, and if $g \in G'$, then furthermore

$$\lambda(g^a) = \sum_{i=0}^{p-1} i r_{i-1} = \sum_{i=0}^{p-1} r_{i-1} + \sum_{i=0}^{p-1} (i-1) r_{i-1} = \lambda(g),$$

since $r_0 + \cdots + r_{p-1} = 0$ by Theorem 2.11. It follows that $\lambda(g^h) = \lambda(g)$ for every $g \in G'$ and $h \in G$.

Now we define $\Lambda : G' \times \cdots \times G' \longrightarrow \mathbb{F}_p$ by means of

$$\Lambda(g_1, \ldots, g_p) = \lambda(g_1) + \cdots + \lambda(g_p).$$

By the preceding paragraph, we have

$$\Lambda(g^h) = \Lambda(g), \quad \text{for all } g \in G' \times \cdots \times G' \text{ and } h \in G \times \cdots \times G.$$

Hence $\ker \Lambda$ is a normal subgroup of $G \times \cdots \times G$.

For every $1 \le i < j \le p$, we have

$$\begin{aligned}\psi([b_i, b_j]) &= (1, \ldots, 1, [b, a^{e_{i-j}}], 1, \ldots, 1, [a^{e_{j-i}}, b], 1, \ldots, 1) \\ &= (1, \ldots, 1, b_0^{-1} b_{e_{i-j}}, 1, \ldots, 1, b_{e_{j-i}}^{-1} b_0, 1, \ldots, 1),\end{aligned}$$

where the non-trivial components are at positions i and j. Since $\mathbf{e}$ is symmetric, we have $e_{i-j} = e_{j-i}$, and consequently

$$\Lambda(\psi([b_i, b_j])) = e_{i-j} - e_{j-i} = 0.$$

Hence $\psi([b_i, b_j]) \in \ker \Lambda$, and since $\ker \Lambda$ is a normal subgroup of $G \times \cdots \times G$, it follows from (10) that $\psi(\mathrm{Stab}_G(1)') \le \ker \Lambda$. Since

$$\Lambda([a,b], 1, \ldots, 1) = \Lambda(b_1^{-1} b_0, 1, \ldots, 1) = -1,$$

we deduce that $([a,b], 1, \ldots, 1) \notin \psi(\mathrm{Stab}_G(1)')$, which completes the proof. □

Now we can proceed to calculate the order of G_n for every $n \ge 1$, and the Hausdorff dimension of $\overline{G}$ in Γ, provided that the defining vector $\mathbf{e}$ is not constant. We will use the following result of Šunić (see [18, Proposition 6]).

Theorem 3.6. *Let G be an infinite self-similar subgroup of Γ, and assume that, for some $m \ge 1$, we have*

$$\psi(\mathrm{Stab}_G(n)) = \mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(n-1)$$

for every $n > m$. *If* $|G_m| = p^{r/(p-1)}$ *and* $|G \times \overset{p}{\cdots} \times G : \psi(\mathrm{Stab}_G(1))| = p^s$, *then*

$$\log_p |G_n| = \frac{r-s+1}{p-1}\, p^{n-m} + \frac{s-1}{p-1}$$

for every $n \geq m$, *and the Hausdorff dimension of* $\overline{G}$ *in* Γ *is* $(r-s+1)/p^m$.

We first deal with the case when $\mathbf{e}$ is not symmetric, and then we consider GGS-groups with $\mathbf{e}$ symmetric but not constant.

Theorem 3.7. *Let* G *be a GGS-group with non-symmetric defining vector* $\mathbf{e}$. *Then*

$$\log_p |G_n| = tp^{n-2} + 1, \quad \textit{for every } n \geq 2,$$

where t *is the rank of* $C(\mathbf{e}, 0)$, *and*

$$\dim_\Gamma \overline{G} = \frac{(p-1)t}{p^2}.$$

Proof. We apply Theorem 3.6. Let m, r, and s be as in the statement of that theorem. By Lemma 3.3, we can take $m = 2$. On the other hand, by Theorem 2.4, we have $r = (t+1)(p-1)$. Finally, observe that

$$\begin{aligned} |G \times \overset{p}{\cdots} \times G : \psi(\mathrm{Stab}_G(1))| &= \frac{|G \times \overset{p}{\cdots} \times G : \psi(\mathrm{Stab}_G(1)')|}{|\psi(\mathrm{Stab}_G(1)) : \psi(\mathrm{Stab}_G(1)')|} \\ &= \frac{|G \times \overset{p}{\cdots} \times G : G' \times \overset{p}{\cdots} \times G'|}{|\mathrm{Stab}_G(1) : \mathrm{Stab}_G(1)'|} = \frac{p^{2p}}{p^p} = p^p, \end{aligned}$$

by using (9) and Theorem 2.14. Consequently $s = p$, and the result follows. □

We can similarly prove the following theorem, by using Lemma 3.5 instead of (9).

Theorem 3.8. *Let* G *be a GGS-group with non-constant symmetric defining vector* $\mathbf{e}$. *Then*

$$\log_p |G_n| = tp^{n-2} + 1 - \frac{p^{n-2}-1}{p-1}, \quad \textit{for every } n \geq 2,$$

where t *is the rank of* $C(\mathbf{e}, 0)$, *and*

$$\dim_\Gamma \overline{G} = \frac{(p-1)t-1}{p^2}.$$

4. GGS-groups with constant defining vector

In this section, we deal with the case where the defining vector is constant, say $\mathbf{e} = (e, \ldots, e)$, where $e \in \mathbb{F}_p^\times$. Let m be the inverse of e in $\mathbb{F}_p^\times$, and $b^* = b^m$. Then $G = \langle a, b^* \rangle$, and $\psi(b^*) = (a, \ldots, a, b^*)$. For this reason, we may assume in the remainder of this section that $\mathbf{e} = (1, \ldots, 1)$.

We begin by defining a sequence of elements of G that will be fundamental in the sequel. We put $y_0 = ba^{-1}$ and, more generally, $y_i = y_0^{a^i}$ for every integer i. Thus $y_i^{a^j} = y_{i+j}$ for all $i, j \in \mathbb{Z}$. Also,

$$y_i^b = y_i^{aa^{-1}b} = y_{i+1}^{y_1}. \tag{11}$$

Observe that $y_i = y_j$ if $i \equiv j \pmod p$, so that the set $\{y_0, \ldots, y_{p-1}\}$ already contains all the y_i. In the following lemma, we collect some important properties of the elements y_i. We adopt the following convention: given a vector v of length

p and an integer i, not lying in the range $\{1,\dots,p\}$, the ith position of v is to be understood as the jth position, where $j\in\{1,\dots,p\}$ and $i\equiv j \pmod p$.

Lemma 4.1. *Let G be a GGS-group with constant defining vector. Then:*

(i) $y_{p-1}y_{p-2}\dots y_1y_0=1$.

(ii) *If z_i is the tuple of length p having y_2 at position $i-2$, y_1^{-1} at position $i-1$, and 1 elsewhere, then*

$$\psi([y_i,y_j])=z_iz_j^{-1},\quad \textit{for every } i \textit{ and } j. \tag{12}$$

(iii) *We have*

$$[y_i,y_j]=[y_i,y_{i-1}][y_{i-1},y_{i-2}]\dots[y_{j+1},y_j],\quad \textit{for every } i>j. \tag{13}$$

Proof. (i) We have

$$\begin{aligned}y_{p-1}y_{p-2}\dots y_1y_0 &= a^{-(p-1)}ba^{p-2}\cdot a^{-(p-2)}ba^{p-3}\dots a^{-1}b\cdot ba^{-1}\\ &= a^{-(p-1)}b^pa^{-1}=1.\end{aligned}$$

(ii) Clearly, it is enough to see the result for $i>j$. On the other hand, since both sequences $\{y_i\}$ and $\{z_i\}$ are periodic of period p, we may assume that i and j lie in the set $\{3,\dots,p+2\}$. If $r=j-3$ and $k=i-r$, then

$$[y_i,y_j]=[y_k^{a^r},y_3^{a^r}]=[y_k,y_3]^{a^r},$$

and so $\psi([y_i,y_j])$ is the result of applying to $\psi([y_k,y_3])$ the permutation which moves every element r positions to the right. It readily follows that it suffices to prove (12) for $[y_k,y_3]$ with $4\le k\le p+2$.

Since $y_i=a^{-i}ba^{i-1}=a^{-1}b_{i-1}$ for every i, we have

$$[y_k,y_3]=b_{k-1}^{-1}ab_2^{-1}b_{k-1}a^{-1}b_2=b_{k-1}^{-1}b_1^{-1}b_{k-2}b_2=(b_1^{-1}b_{k-2})^{b_{k-1}}(b_{k-1}^{-1}b_2). \tag{14}$$

Now, it follows from (3) that

$$\begin{aligned}\psi((b_1^{-1}b_{k-2})^{b_{k-1}}) &= (y_1^{-1},1,\overset{k-4}{\dots},1,y_1,1,\dots,1)^{(a,\overset{k-2}{\dots},a,b,a,\dots,a)}\\ &= \begin{cases}(y_2^{-1},1,\overset{k-4}{\dots},1,y_2,1,\dots,1), & \text{if } 4\le k\le p+1,\\ (y_1^{-1}y_2^{-1}y_1,1,\dots,1,y_2), & \text{if } k=p+2.\end{cases}\end{aligned}$$

Here, we have used the fact that $y_1^b=y_2^{y_1}$ by (11). Similarly,

$$\psi(b_{k-1}^{-1}b_2)=\begin{cases}(1,y_1,1,\overset{k-4}{\dots},1,y_1^{-1},1,\dots,1), & \text{if } 4\le k\le p+1,\\ (y_1^{-1},y_1,1,\dots,1), & \text{if } k=p+2.\end{cases}$$

By taking these values to (14), we obtain that $\psi([y_k,y_3])=z_kz_3^{-1}$, as desired.

(iii) This follows immediately from (ii), since

$$\begin{aligned}\psi([y_i,y_j]) &= (z_iz_{i-1}^{-1})(z_{i-1}z_{i-2}^{-1})\dots(z_{j+1}z_j^{-1})\\ &= \psi([y_i,y_{i-1}])\psi([y_{i-1},y_{i-2}])\dots\psi([y_{j+1},y_j])\\ &= \psi([y_i,y_{i-1}][y_{i-1},y_{i-2}]\dots[y_{j+1},y_j]).\end{aligned}$$

□

Next we introduce a maximal subgroup K of G that will play a key role in the determination of the order of G_n in the case that $\mathbf{e}$ is constant.

Lemma 4.2. *Let G be a GGS-group with constant defining vector, and let $K = \langle ba^{-1}\rangle^G$. Then:*

(i) *$G' \leq K$ and $|G:K| = p$.*

(ii) *$K = \langle y_0, y_1, \dots, y_{p-1}\rangle$ and $K' = \langle [y_1, y_0]\rangle^G$.*

(iii) *$K' \times \overset{p}{\cdots} \times K' \subseteq \psi(K') \subseteq \psi(G') \subseteq K \times \overset{p}{\cdots} \times K$. In particular, G is a weakly regular branch group over K'.*

(iv) *If $L = \psi^{-1}(K' \times \overset{p}{\cdots} \times K')$ (which, by (iii), is contained in K'), then the conjugates $[y_{i+1}, y_i]^{b^j}$, where $0 \leq i, j \leq p-1$, generate K' modulo L.*

Proof. (i) Since $[a, ba^{-1}] = [a, b]^{a^{-1}} \in K$ and K is normal in G, it follows that G' is contained in K. Then $|G:K| = |G/G' : K/G'| = p$.

(ii) Let us first prove that $K = \langle y_0, y_1, \dots, y_{p-1}\rangle$. For this purpose, it suffices to see that $N = \langle y_0, y_1, \dots, y_{p-1}\rangle$ is a normal subgroup of G. This is clear, since $y_i^a = y_{i+1}$ and $y_i^b = y_{i+1}^{y_1}$ for every i.

It follows that

$$K' = \langle [y_i, y_j] \mid 0 \leq j < i \leq p-1\rangle^K = \langle [y_i, y_j] \mid 0 \leq j < i \leq p-1\rangle^G,$$

where the second equality holds because K' is normal in G. By (13), every commutator $[y_i, y_j]$ with $0 \leq j < i \leq p-1$ can be expressed in terms of the $[y_k, y_{k-1}]$ with $k = 1, \dots, p-1$. Since $[y_k, y_{k-1}] = [y_1, y_0]^{a^{k-1}}$, we conclude that $K' = \langle [y_1, y_0]\rangle^G$.

(iii) Let us first prove the inclusion $\psi(G') \subseteq K \times \overset{p}{\cdots} \times K$. We have

$$\begin{aligned}\psi([b,a]) = \psi(b^{-1}b^a) &= (a^{-1}, a^{-1}, \dots, a^{-1}, b^{-1})(b, a, \dots, a, a) \\ &= (a^{-1}b, 1, \dots, 1, b^{-1}a) \in K \times \overset{p}{\cdots} \times K.\end{aligned}$$

Now, since K is normal in G, it readily follows that

$$\psi([b,a]^g) \in K \times \overset{p}{\cdots} \times K, \quad \text{for every } g \in G.$$

This proves the desired inclusion.

Now we focus on proving that $K' \times \overset{p}{\cdots} \times K' \subseteq \psi(K')$. By Proposition 2.18 and (ii), it suffices to see that

$$([y_1, y_0], 1, \dots, 1) \in \psi(K').$$

We consider the cases $p \geq 5$ and $p = 3$ separately.

Suppose first that $p \geq 5$. By using (12), we have

$$\psi([y_1, y_2]) = (y_1, 1, \dots, 1, y_2, y_1^{-1}y_2^{-1})$$

and

$$\psi([y_3, y_4]) = (y_2, y_1^{-1}y_2^{-1}, y_1, 1, \dots, 1).$$

If $k = [[y_3, y_4], [y_1, y_2]]$, it follows that

$$\psi(k) = ([y_2, y_1], 1, \dots, 1),$$

since $p \geq 5$. Hence

$$([y_1, y_0], 1, \dots, 1) = \psi(k^{b^{-1}}) \in \psi(K'),$$

as desired.

Assume now that $p = 3$. We have

$$\psi([y_1, y_0]) = (y_1 y_0, y_0^{-1}, y_1^{-1}),$$

since $y_2y_1y_0 = 1$, by (i) of Lemma 4.1. Hence

$$\begin{aligned}\psi([y_0, y_1]^b) &= (y_0^{-1}y_1^{-1}, y_0, y_1)^{(a,a,b)} = (y_1^{-1}y_2^{-1}, y_1, y_1^b)\\ &= ((y_2y_1)^{-1}, y_1, y_2^{y_1}) = (y_0, y_1, (y_0^{-1}y_1^{-1})^{y_1})\\ &= (y_0, y_1, y_1^{-1}y_0^{-1}),\end{aligned}$$

and

$$([y_1, y_0], 1, 1) = \psi([y_0, y_1]^{ba}[y_1, y_0]) \in \psi(K'),$$

which completes the proof.

(iv) Let us consider an arbitrary element $g \in G$, and let us write $g = ha^ib^j$, for some $i, j \in \mathbb{Z}$, $h \in G'$. Then

$$[y_1, y_0]^g = ([y_1, y_0][y_1, y_0, h])^{a^ib^j} \equiv [y_1, y_0]^{a^ib^j} = [y_{i+1}, y_i]^{b^j} \pmod{L},$$

since $\psi([y_1, y_0, h]) \in \psi(G'') \subseteq K' \times \overset{p}{\cdots} \times K'$ by (iii). Now, since the conjugates $[y_1, y_0]^g$ generate K' by (ii), the result follows. □

In the following results, we consider the action of an element of G by conjugation as an endomorphism of K/K', which allows us to multiply several conjugates of an element of K, modulo K', by adding the elements by which we are conjugating. This gives a meaning to expressions like $g^{1+a+\cdots+a^{p-1}} \in K'$ for an element $g \in K$.

Lemma 4.3. *Let G be a GGS-group with constant defining vector, and let $K = \langle ba^{-1}\rangle^G$. If $g \in K$, then*

$$g^{1+a+\cdots+a^{p-1}} \in K'.$$

Proof. The map R sending $g \in K$ to $g^{1+a+\cdots+a^{p-1}}K'$ is a well-defined homomorphism from K to K/K', and we want to see that R is the trivial homomorphism. Since $K = \langle y_0, \dots, y_{p-1}\rangle$ by (ii) of Lemma 4.2, it suffices to check that $y_i \in \ker R$ for every i. Now,

$$R(y_i) = y_iy_{i+1}\dots y_{p-1}y_0\dots y_{i-1}K' = y_{p-1}y_{p-2}\dots y_1y_0K' = K'$$

by (i) of Lemma 4.1, and we are done. □

Lemma 4.4. *Let G be a GGS-group with constant defining vector, and let $K = \langle ba^{-1}\rangle^G$. If $g \in K'$ and we write $\psi(g) = (g_1, \dots, g_p)$, then:*

(i) $g_pg_{p-1}\dots g_1 \in K'$.

(ii) $\prod_{i=1}^{p-1} g_i^{a+a^2+\cdots+a^i} \in K'$.

Similarly, if

$$g \in K' \operatorname{Stab}_G(n)$$

for some $n \geq 1$, then $g_pg_{p-1}\dots g_1$ and $\prod_{i=1}^{p-1} g_i^{a+a^2+\cdots+a^i}$ lie in $K' \operatorname{Stab}_G(n-1)$.

Proof. We first deal with the case that $g \in K'$. Let us consider the following two maps:

$$\begin{array}{rccc} P & : & K \times \overset{p}{\cdots} \times K & \longrightarrow & K/K' \\ & & (g_1, \dots, g_p) & \longmapsto & g_p\dots g_1K', \end{array}$$

and

$$\begin{array}{rccc} Q & : & K \times \overset{p}{\cdots} \times K & \longrightarrow & K/K' \\ & & (g_1, \dots, g_p) & \longmapsto & \prod_{i=1}^{p-1} g_i^{a+a^2+\cdots+a^i}K'. \end{array}$$

Clearly, P and Q are homomorphisms. By (iii) of Lemma 4.2, $\psi(K')$ is contained in the domain of P and Q, and our goal is to prove that it is actually in the kernels of these maps. Since the image of $K' \times \overset{p}{\cdots} \times K'$ is trivial, it suffices to see that $\psi(g) \in \ker P$ and $\psi(g) \in \ker Q$ for every g in a system of generators of K' modulo L, where $L = \psi^{-1}(K' \times \overset{p}{\cdots} \times K')$. By (iv) of Lemma 4.2, the conjugates $[y_{i+1}, y_i]^{b^j}$, for $i, j = 0, \ldots, p-1$, constitute such a set of generators.

Let $c \in \Gamma$ be defined by means of $\psi(c) = (a, a, \ldots, a)$. We claim that

$$g^b \equiv g^c \pmod{L}, \quad \text{for every } g \in K'. \tag{15}$$

Indeed, we have $\psi(b) = \psi(c)(1, \ldots, 1, a^{-1}b)$, and so

$$\begin{aligned}\psi(g^b) = \psi(g^c)^{(1,\ldots,1,a^{-1}b)} &= \psi(g^c)[\psi(g^c), (1, \ldots, 1, a^{-1}b)] \\ &\equiv \psi(g^c) \pmod{K' \times \overset{p}{\cdots} \times K'},\end{aligned}$$

since $\psi(g^c) \in K \times \overset{p}{\cdots} \times K$ and $a^{-1}b \in K$.

As a consequence of (15), it suffices to see that $\psi([y_{i+1}, y_i]^{c^j})$ lies in both $\ker P$ and $\ker Q$. Since

$$P(\psi([y_{i+1}, y_i]^{c^j})) = P(\psi([y_{i+1}, y_i]))^{a^j}$$

and

$$Q(\psi([y_{i+1}, y_i]^{c^j})) = Q(\psi([y_{i+1}, y_i]))^{a^j},$$

we have reduced ourselves to proving that $\psi([y_{i+1}, y_i])$ is in the kernel of P and Q for every i. According to (12), we have $\psi([y_{i+1}, y_i]) = z_{i+1}z_i^{-1}$, with z_i as defined in Lemma 4.1. Now, one can easily check that

$$P(z_i) = y_1^{-1}y_2K' \quad \text{and} \quad Q(z_i) = y_2^{-1}K' \quad \text{for every } i,$$

where in the case of Q and $i = 1$ we need to use the fact that

$$y_2^{a+a^2+\cdots+a^{p-1}} \equiv y_2^{-1} \pmod{K'},$$

by Lemma 4.3. It readily follows that $\psi([y_{i+1}, y_i])$ lies in both $\ker P$ and $\ker Q$, as desired.

Assume now that $g \in K' \operatorname{Stab}_G(n)$, and let us write $g = fh$, with $f \in K'$ and $h \in \operatorname{Stab}_G(n)$. Put $\psi(f) = (f_1, \ldots, f_p)$ and $\psi(h) = (h_1, \ldots, h_p)$. Since $h_1, \ldots, h_p \in \operatorname{Stab}_G(n-1)$, which is a normal subgroup of G, we have

$$g_p \ldots g_1 = f_p h_p \ldots f_1 h_1 = f_p \ldots f_1 h^*,$$

for some $h^* \in \operatorname{Stab}_G(n-1)$. Since $f \in K'$, we already know that $f_p \ldots f_1 \in K'$, and so we conclude that $g_p \ldots g_1 \in K' \operatorname{Stab}_G(n-1)$, as desired. The second assertion can be proved in a similar way. □

Theorem 4.5. *Let G be a GGS-group with constant defining vector, and let $K = \langle ba^{-1} \rangle^G$ and $L = \psi^{-1}(K' \times \overset{p}{\cdots} \times K')$. Then the following isomorphisms hold:*

$$K'/L \cong K/K' \times \overset{p-2}{\cdots} \times K/K',$$

and

$$K' \operatorname{Stab}_G(n)/L \operatorname{Stab}_G(n) \cong K/K' \operatorname{Stab}_G(n-1) \times \overset{p-2}{\cdots} \times K/K' \operatorname{Stab}_G(n-1),$$

for every $n \geq 3$.

Proof. Let π be the map given by

$$\begin{array}{ccc} K \times \overset{p}{\cdots} \times K & \longrightarrow & K/K' \times \overset{p-2}{\cdots} \times K/K' \\ (g_1, \ldots, g_p) & \longmapsto & (g_1K', \ldots, g_{p-2}K'), \end{array}$$

and let R be the composition of $\psi : K' \longrightarrow K \times \overset{p}{\cdots} \times K$ with π. If we see that R is surjective, and that $\ker R = L$, then the first isomorphism of the statement follows.

Let $g \in K'$ be an element lying in $\ker R$. If $\psi(g) = (g_1, \ldots, g_p)$, then we have $g_1, \ldots, g_{p-2} \in K'$. By (ii) of Lemma 4.4, it follows that

$$g_{p-1}^{a+\cdots+a^{p-1}} \in K',$$

and by applying Lemma 4.3, we get $g_{p-1} \in K'$. Now, (i) of Lemma 4.4 immediately yields that also $g_p \in K'$. This proves that $\ker R = L$.

Now we prove that

$$K/K' \times \{\overline{1}\} \times \cdots \times \{\overline{1}\} \subseteq R(K'). \tag{16}$$

Then, by arguing as in the proof of Proposition 2.18, it follows that R is surjective. By (12), we have

$$\psi([y_1, y_2]) = (y_1, 1, \ldots, 1, h_{p-1}, h_p)$$

for some elements $h_{p-1}, h_p \in K$. Hence

$$\psi([y_1, y_2]^{b^{i-1}}) = (y_i, 1, \ldots, 1, h_{p-1}^*, h_p^*)$$

for every i, and we are done, since $K = \langle y_0, \ldots, y_{p-1} \rangle$.

The second isomorphism can be proved in a similar way. Observe that the condition $n \geq 3$ guarantees that $\mathrm{Stab}_G(n-1) \leq G' \leq K$, so that it makes sense to write $K/K' \,\mathrm{Stab}_G(n-1)$. This time consider the homomorphism

$$\begin{array}{cccc} \pi_n & : & K \times \overset{p}{\cdots} \times K & \longrightarrow & K/K' \,\mathrm{Stab}_G(n-1) \times \overset{p-2}{\cdots} \times K/K' \,\mathrm{Stab}_G(n-1) \\ & & (g_1, \ldots, g_p) & \longmapsto & (g_1K' \,\mathrm{Stab}_G(n-1), \ldots, g_{p-2}K' \,\mathrm{Stab}_G(n-1)), \end{array}$$

and let R_n be the composition of $\psi : K' \longrightarrow K \times \overset{p}{\cdots} \times K$ with π_n. Observe that the surjectiveness of R already implies that R_n is surjective. Let us prove that $\ker R_n = L \,\mathrm{Stab}_G(n) \cap K'$. The same proof as above, but using the last part of Lemma 4.4, shows that

$$\begin{aligned} \psi(\ker R_n) &= (K' \,\mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times K' \,\mathrm{Stab}_G(n-1)) \cap \psi(K') \\ &= (K' \times \overset{p}{\cdots} \times K')(\mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(n-1)) \cap \psi(K'). \end{aligned}$$

Since $K' \times \overset{p}{\cdots} \times K' \subseteq \psi(K')$, we can apply Dedekind's Law to get

$$\psi(\ker R_n) = (K' \times \overset{p}{\cdots} \times K')\big((\mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(n-1)) \cap \psi(K')\big).$$

Now, since $n \geq 3$, we have

$$\begin{aligned} (\mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times \mathrm{Stab}_G(n-1)) \cap \psi(K') &= \psi(\mathrm{Stab}_G(n)) \cap \psi(K') \\ &= \psi(\mathrm{Stab}_G(n) \cap K'), \end{aligned}$$

and it follows that

$$\begin{aligned} \psi(\ker R_n) = (K' \times \overset{p}{\cdots} \times K')\psi(\mathrm{Stab}_G(n) \cap K') &= \psi(L)\psi(\mathrm{Stab}_G(n) \cap K') \\ &= \psi(L(\mathrm{Stab}_G(n) \cap K')). \end{aligned}$$

Hence

$$\ker R_n = L(\mathrm{Stab}_G(n) \cap K') = L\,\mathrm{Stab}_G(n) \cap K',$$

as claimed.

Now, we can readily obtain the desired isomorphism:

$$\begin{aligned} K'\,\mathrm{Stab}_G(n)/L\,\mathrm{Stab}_G(n) &\cong K'/(L\,\mathrm{Stab}_G(n) \cap K') = K'/\ker R_n \\ &\cong R_n(K') = K/K'\,\mathrm{Stab}_G(n-1) \times \overset{p-2}{\cdots} \times K/K'\,\mathrm{Stab}_G(n-1). \end{aligned}$$

□

Theorem 4.6. *Let G be a GGS-group with constant defining vector, and let $K = \langle ba^{-1}\rangle^G$. Then, for every $n \geq 2$, the quotient $G/K'\,\mathrm{Stab}_G(n)$ is a p-group of maximal class of order p^{n+1}.*

Proof. For simplicity, let us write $T_n = K'\,\mathrm{Stab}_G(n)$, $Q_n = G/T_n$ and $A_n = K/T_n$ (take into account that $\mathrm{Stab}_G(2) \leq G' \leq K$). Since $|Q_n : Q_n'| = |G : G'| = p^2$ and A_n is an abelian maximal subgroup of Q_n, it follows from Lemma 2.3 that Q_n is a p-group of maximal class. As a consequence, if we want to prove that $|Q_n| = p^{n+1}$, it suffices to see that the nilpotency class of Q_n is n.

We need an auxiliary result. Let $\{x_i\}_{i\geq 1}$ be a sequence of elements of G such that $\{x_1, x_2\} = \{a, b\}$ and $x_i \in \{a, b\}$ for every $i \geq 3$. We claim that, for every $i \geq 2$, the section $\gamma_i(Q_n)/\gamma_{i+1}(Q_n)$ is generated by the image of the commutator $[x_1, x_2, \ldots, x_i]$. We argue by induction on i. If $i = 2$, then we have to show that the image of $[a, b]$ generates $\gamma_2(Q_n)/\gamma_3(Q_n)$. This follows immediately from (i) in Lemma 2.3, since $[a, b] = [a, a^{-1}b]$, where $bT_n \in Q_n \smallsetminus A_n$ and $a^{-1}bT_n = (ba^{-1}T_n)^a \in A_n \smallsetminus \gamma_2(Q_n)$. Now, if we assume that the result holds for $i - 1$, we get it for i by using (ii) of Lemma 2.3.

Let us now prove that the class of Q_n is n, by induction on n. Assume first that $n = 2$. We have

$$\psi([b, a]) = (a^{-1}b, 1, \ldots, 1, b^{-1}a)$$

and

$$\psi([b, a, b]) = ([a^{-1}b, a], 1, \ldots, 1, [b^{-1}a, b]) = ([b, a], 1, \ldots, 1, [a, b]),$$

so that $[b, a, b] \in \mathrm{Stab}_G(2)$. It follows that the image of $[b, a, b]$ in Q_2 is trivial. By the previous paragraph, we necessarily have $\gamma_3(Q_2) = \gamma_4(Q_2)$. Hence $\gamma_3(Q_2) = 1$, and the class of Q_2 is at most 2. If Q_2 is of class 1, then $[b, a] \in K'\,\mathrm{Stab}_G(2)$ and, by Lemma 4.4, $a^{-1}b \in K'\,\mathrm{Stab}_G(1)$. Hence $a^{-1} \in \mathrm{Stab}_G(1)$, which is a contradiction. Thus Q_2 is of class 2.

Now we assume the result for $n - 1$, and we prove it for n. We have

$$\psi([b, a, b, \overset{n-1}{\ldots}, b]) = ([b, a, \overset{n-1}{\ldots}, a], 1, \ldots, 1, [a, b, \overset{n-1}{\ldots}, b]),$$

and

$$[b, a, \overset{n-1}{\ldots}, a], [a, b, \overset{n-1}{\ldots}, b] \in K'\,\mathrm{Stab}_G(n-1),$$

since Q_{n-1} has class $n - 1$ by the induction hypothesis. Thus

$$\psi([b, a, b, \overset{n-1}{\ldots}, b]) \in K'\,\mathrm{Stab}_G(n-1) \times \overset{p}{\cdots} \times K'\,\mathrm{Stab}_G(n-1). \tag{17}$$

Now,

$$\begin{aligned}
&(K' \operatorname{Stab}_G(n-1) \times \overset{p}{\cdots} \times K' \operatorname{Stab}_G(n-1)) \cap \psi(G) \\
&\qquad = (K' \times \overset{p}{\cdots} \times K')(\operatorname{Stab}_G(n-1) \times \overset{p}{\cdots} \times \operatorname{Stab}_G(n-1)) \cap \psi(G) \\
&\qquad \subseteq \psi(K')(\operatorname{Stab}_G(n-1) \times \overset{p}{\cdots} \times \operatorname{Stab}_G(n-1)) \cap \psi(G) \\
&\qquad = \psi(K')(\operatorname{Stab}_G(n-1) \times \overset{p}{\cdots} \times \operatorname{Stab}_G(n-1) \cap \psi(G)) \\
&\qquad = \psi(K')\psi(\operatorname{Stab}_G(n)) = \psi(K' \operatorname{Stab}_G(n)).
\end{aligned}$$

It follows that $[b, a, b, \overset{n-1}{\ldots}, b] \in K' \operatorname{Stab}_G(n)$, and so this commutator becomes trivial in Q_n. Since the image of this commutator generates the quotient $\gamma_{n+1}(Q_n)/\gamma_{n+2}(Q_n)$, we have $\gamma_{n+1}(Q_n) = 1$. Hence the class of Q_n is at most n.

If Q_n has class strictly less than n, then since the image of $[b, a, b, \overset{n-2}{\ldots}, b]$ generates $\gamma_n(Q_n)/\gamma_{n+1}(Q_n)$, it follows that

$$[b, a, b, \overset{n-2}{\ldots}, b] \in K' \operatorname{Stab}_G(n).$$

Since

$$\psi([b, a, b, \overset{n-2}{\ldots}, b]) = ([b, a, \overset{n-2}{\ldots}, a], 1, \ldots, 1, [a, b, \overset{n-2}{\ldots}, b]),$$

it follows from Lemma 4.4 that

$$[b, a, \overset{n-2}{\ldots}, a] \in K' \operatorname{Stab}_G(n-1).$$

This is a contradiction, since Q_{n-1} is of class $n-1$, and $\gamma_{n-1}(Q_{n-1})/\gamma_n(Q_{n-1})$ is generated by the image of $[b, a, \overset{n-2}{\ldots}, a]$. Thus we conclude that the nilpotency class of Q_n is n, which completes the proof of the theorem. □

Theorem 4.7. *Let G be a GGS-group with constant defining vector. Then*

$$\log_p |G_n| = p^{n-1} + 1 - \frac{p^{n-2} - 1}{p-1} - \frac{p^{n-2} - (n-2)p + n - 3}{(p-1)^2},$$

for every $n \geq 2$, and

$$\dim_\Gamma \overline{G} = \frac{p-2}{p-1}.$$

Proof. As on previous occasions, the formula for the Hausdorff dimension of $\overline{G}$ is immediate once we obtain $\log_p |G_n|$. For that purpose, we argue by induction on n. If $n = 2$, then by Theorem 2.4, we have $\log_p |G_2| = t + 1$, where t is the rank of the matrix $C = C(1, \overset{p-1}{\ldots}, 1, 0)$. By Lemma 2.7, $p - t$ is the multiplicity of 1 as a root in $\mathbb{F}_p$ of the polynomial $X^{p-2} + \cdots + X + 1$. Thus $t = p$ and $\log_p |G_2| = p + 1$, as desired.

Assume now that $n \geq 3$. Let $K = \langle ba^{-1} \rangle^G$, and $L = \psi^{-1}(K' \times \overset{p}{\cdots} \times K')$. Then we have the following decomposition of the order of G_n:

$$|G_n| = |G : K' \operatorname{Stab}_G(n)||K' \operatorname{Stab}_G(n) : L \operatorname{Stab}_G(n)||L \operatorname{Stab}_G(n) : \operatorname{Stab}_G(n)|. \tag{18}$$

By Theorem 4.6, we know that $|G : K' \operatorname{Stab}_G(n)| = p^{n+1}$. On the other hand, since

$$K' \operatorname{Stab}_G(n)/L \operatorname{Stab}_G(n) \cong K/K' \operatorname{Stab}_G(n-1) \times \overset{p-2}{\cdots} \times K/K' \operatorname{Stab}_G(n-1)$$

by Theorem 4.5, and since $|K/K' \operatorname{Stab}_G(n-1)| = p^{n-1}$ (again by Theorem 4.6), it follows that

$$|K' \operatorname{Stab}_G(n) : L \operatorname{Stab}_G(n)| = p^{(n-1)(p-2)}.$$

Finally,

$$\begin{aligned}|L\operatorname{Stab}_G(n):\operatorname{Stab}_G(n)| &= |L:\operatorname{Stab}_L(n)| = |\psi(L):\psi(\operatorname{Stab}_L(n))| \\ &= |K'\times\overset{p}{\cdots}\times K' : \operatorname{Stab}_{K'}(n-1)\times\overset{p}{\cdots}\times\operatorname{Stab}_{K'}(n-1)| \\ &= |K':\operatorname{Stab}_{K'}(n-1)|^p = |K'\operatorname{Stab}_G(n-1):\operatorname{Stab}_G(n-1)|^p \\ &= |G/\operatorname{Stab}_G(n-1)|^p/|G/K'\operatorname{Stab}_G(n-1)|^p \\ &= |G_{n-1}|^p p^{-np}.\end{aligned}$$

Now, from (18) we get

$$\begin{aligned}\log_p|G_n| = p\log_p|G_{n-1}| &+ n + 1 + (n-1)(p-2) - np \\ &= p\log_p|G_{n-1}| - n - p + 3,\end{aligned}$$

and the result follows by applying the induction hypothesis to G_{n-1}. □

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