Gibbs/Equilibrium Measures for Functions of Multidimensional Shifts with Countable Alphabets.

Stephen Muir

March 28, 2011

References

[1] Georgii, H-O.: Gibbs Measures and Phase Transitions. DeGruyter, New York (1981)

[2] Israel, Robert B.: Convexity in the Theory of Lattice Gases. Princeton Series in Physics, Princeton Univ. Press, Princeton, N. J. (1979)

[3] Keller, G.: Equilibrium States in Ergodic Theory. London Mathematical Society Student Texts 42. Cambridge University Press, Cambridge (1998)

[4] Muir, S.: Gibbs/Equilibrium measures for functions of multidimensional shifts with countable alphabets. University of North Texas Doctoral Dissertation. (2011)

[5] Ruelle, D.: Thermodynamical formalism: the mathematical structures of classical equilibrium statistical mechanics. Addison-Wesley Pub. Co., Advanced Book Program, Reading, Mass. (1978)

[6] Simon, B.: The Statistical Mechanics of Lattice Gases, Vol. I. Princeton University Press, Princeton (1993)

Consider a classical lattice gas or lattice spin system with configuration space

$$\mathbb{N}^{\mathbb{Z}^d}, \ d \geq 1.$$

Denote a configuration by

$$\omega = (\omega_\lambda)_{\lambda \in \mathbb{Z}^d} \ ,$$

where $\omega_\lambda \in \mathbb{N}$ for each $\lambda \in \mathbb{Z}^d$. For any configuration $\omega \in \mathbb{N}^{\mathbb{Z}^d}$ denote its restriction or projection to a subset $\Lambda \subseteq \mathbb{Z}^d$ by

$$\omega|_\Lambda = \pi_\Lambda(\omega) = (\omega_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda.$$

Consider a classical lattice gas or lattice spin system with configuration space

$$\mathbb{N}^{\mathbb{Z}^d}, \; d \geq 1.$$

Denote a configuration by

$$\omega = (\omega_\lambda)_{\lambda \in \mathbb{Z}^d} \, ,$$

where $\omega_\lambda \in \mathbb{N}$ for each $\lambda \in \mathbb{Z}^d$. For any configuration $\omega \in \mathbb{N}^{\mathbb{Z}^d}$ denote its restriction or projection to a subset $\Lambda \subseteq \mathbb{Z}^d$ by

$$\omega|_\Lambda = \pi_\Lambda(\omega) = (\omega_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda.$$

For any $\Lambda \subseteq \mathbb{Z}^d$ and $A \subseteq \mathbb{N}^\Lambda$ denote the cylinder set

$$[A] = \pi_\Lambda^{-1}(A) = \left\{ \omega \in \mathbb{N}^{\mathbb{Z}^d} : \omega|_\Lambda \in A \right\}.$$

Define the shift (translation) maps $T^{\lambda} : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{N}^{\mathbb{Z}^d}$ coordinatewise by

$$(T^{\lambda}(\omega))_{\lambda'} = \omega_{\lambda+\lambda'}.$$

Define the shift (translation) maps $T^\lambda : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{N}^{\mathbb{Z}^d}$ coordinatewise by

$$(T^\lambda(\omega))_{\lambda'} = \omega_{\lambda+\lambda'}.$$

For any $\Lambda \subseteq \mathbb{Z}^d$, let

$$\begin{aligned} \mathcal{A}_\Lambda &= \mathit{Borel}\left(\mathbb{N}^\Lambda\right) \times \{\emptyset, \mathbb{N}^{\Lambda^C}\} \\ &= \left\{ A \times \mathbb{N}^{\Lambda^C} \ : \ A \in \mathit{Borel}\left(\mathbb{N}^\Lambda\right) \right\} \subset \ \mathit{Borel}\left(\mathbb{N}^{\mathbb{Z}^d}\right). \end{aligned}$$

For a function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ to be $\mathcal{A}_\Lambda$ measurable means that f only depends on the coordinates in Λ.

We now prepare to introduce the definition of a Gibbs Measure for a function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$.
The physical interpretation of f is negative *local energy*, i.e. specific internal energy.

We now prepare to introduce the definition of a Gibbs Measure for a function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$.
The physical interpretation of f is negative *local energy*, i.e. specific internal energy.
Denote the ergodic sums

$$f_\Lambda \equiv \sum_{\lambda \in \Lambda} f \circ T^\lambda$$

for any $\Lambda \subset \mathbb{Z}^d, \; |\Lambda| < \infty$. In particular let

$$f_m \equiv \sum_{\substack{\lambda \in \mathbb{Z}^d \; : \\ |\lambda| < m}} f \circ T^\lambda,$$

where

$$|\lambda| = \max \{ |\lambda_i| \; : \; 1 \leq i \leq d \} .$$

We call a function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ *exp-summable* if

$$\overline{Z}_1(f) = \sum_{a \in \mathbb{N}} e^{\sup f|_{[a]}} < \infty.$$

We call a function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ *exp-summable* if

$$\overline{Z}_1(f) = \sum_{a \in \mathbb{N}} e^{\sup f|_{[a]}} < \infty.$$

If f is exp-summable, then for every finite $\Lambda \subset \mathbb{Z}^d$ the partition function

$$\overline{Z}_\Lambda(f) = \sum_{\alpha \in \mathbb{N}^\Lambda} e^{\sup f_\Lambda|_{[\alpha]}} < \infty.$$

We call a function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ *exp-summable* if

$$\overline{Z}_1(f) = \sum_{a \in \mathbb{N}} e^{\sup f|_{[a]}} < \infty.$$

If f is exp-summable, then for every finite $\Lambda \subset \mathbb{Z}^d$ the partition function

$$\overline{Z}_\Lambda(f) = \sum_{\alpha \in \mathbb{N}^\Lambda} e^{\sup f_\Lambda|_{[\alpha]}} < \infty.$$

We call a function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ *regular* if

$$\sum_{n=1}^{\infty} n^{d-1} \delta_n(f) \; < \; \infty,$$

wherein appears the modulus of continuity

$$\delta_n(f) = \sup \{ |f(x) - f(y)| \; : \; x_\lambda = y_\lambda \; \forall \; |\lambda| < n \} .$$

DLR equations for a function

A *Gibbs measure* for a regular, exp-summable function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ is a Borel probability measure μ on $\mathbb{N}^{\mathbb{Z}^d}$ for which

$$\mu\left([\alpha]|\mathcal{A}_{\Lambda^c}\right)(\beta) = \lim_{m\to\infty} \frac{\exp f_m(\alpha, \beta|_{\Lambda^c})}{\sum_{\alpha' \in \mathbb{N}^\Lambda} \exp f_m(\alpha', \beta|_{\Lambda^c})}$$

holds for every $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, every $\alpha \in \mathbb{N}^\Lambda$ and μ-almost-every $\beta \in \mathbb{N}^{\mathbb{Z}^d}$.

Local Permutation Characterization of Gibbs Measures for a Function

For $\Lambda \subset \mathbb{Z}^d, \ |\Lambda| < \infty$ let

$$\mathfrak{S}_\Lambda = \left\{ \tau = (\tau_\lambda)_{\lambda \in \mathbb{Z}^d} \ : \ \forall \ \lambda \in \mathbb{Z}^d \ \tau_\lambda \in \mathfrak{S}_\mathbb{N}, \ \forall \ \lambda \in \Lambda^C \ \tau_\lambda = \mathrm{Id} \right\}.$$

Local Permutation Characterization of Gibbs Measures for a Function

For $\Lambda \subset \mathbb{Z}^d, \ |\Lambda| < \infty$ let

$$\mathfrak{S}_\Lambda = \left\{ \tau = (\tau_\lambda)_{\lambda \in \mathbb{Z}^d} \ : \ \forall \ \lambda \in \mathbb{Z}^d \ \tau_\lambda \in \mathfrak{S}_\mathbb{N}, \ \forall \ \lambda \in \Lambda^C \ \tau_\lambda = \mathrm{Id} \right\}.$$

For any finite sub-alphabet $F \subset \mathbb{N}$, let

$$\mathfrak{S}_{\Lambda,F} = \left\{ \tau \in \mathfrak{S}_\Lambda \ : \ \forall \ \lambda \in \Lambda \ \tau_\lambda|_{F^C} = \mathrm{Id} \right\}.$$

Local Permutation Characterization of Gibbs Measures for a Function

For $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$ let

$$\mathfrak{S}_\Lambda = \left\{ \tau = (\tau_\lambda)_{\lambda \in \mathbb{Z}^d} \ : \ \forall \ \lambda \in \mathbb{Z}^d \ \tau_\lambda \in \mathfrak{S}_\mathbb{N}, \ \forall \ \lambda \in \Lambda^C \ \tau_\lambda = \mathrm{Id} \right\}.$$

For any finite sub-alphabet $F \subset \mathbb{N}$, let

$$\mathfrak{S}_{\Lambda,F} = \{ \tau \in \mathfrak{S}_\Lambda \ : \ \forall \ \lambda \in \Lambda \ \tau_\lambda|_{F^C} = \mathrm{Id} \}.$$

If f is regular, Λ, F are finite subsets of $\mathbb{Z}^d, \mathbb{N}$, respectively, and $\tau \in \mathfrak{S}_{\Lambda,F}$, then the *infinite volume energy loss*

$$f_\tau \ \equiv \ \lim_{n \to \infty} (f_n \circ \tau^{-1} - f_n)$$

converges absolutely and is uniformly continuous, and the family $\{f_\tau : \tau \in \mathfrak{S}_{\Lambda,F}\}$ is equicontinuous.

New Gibbs Measure Characterization[4]:
For a regular, exp-summable local energy function f, a Borel probability measure μ on $\mathbb{N}^{\mathbb{Z}^d}$ is a Gibbs measure for f if and only if for every finite subset Λ of $\mathbb{Z}^d$, every finite subalphabet F of $\mathbb{N}$, and every $\tau \in \mathfrak{S}_{\Lambda,F}$, the image measure $\mu \circ \tau^{-1}$ is absolutely continuous w.r.t. μ and has Radon-Nikodym derivative

$$\frac{d\mu \circ \tau^{-1}}{d\mu} = e^{f_\tau}.$$

New Gibbs Measure Characterization[4]:
For a regular, exp-summable local energy function f, a Borel probability measure μ on $\mathbb{N}^{\mathbb{Z}^d}$ is a Gibbs measure for f if and only if for every finite subset Λ of $\mathbb{Z}^d$, every finite subalphabet F of $\mathbb{N}$, and every $\tau \in \mathfrak{S}_{\Lambda,F}$, the image measure $\mu \circ \tau^{-1}$ is absolutely continuous w.r.t. μ and has Radon-Nikodym derivative

$$\frac{d\mu \circ \tau^{-1}}{d\mu} = e^{f_\tau}.$$

Existence Theorem[4]:
If f is a regular, exp-summable function of $\mathbb{N}^{\mathbb{Z}^d}$ then the set of Gibbs measures for f is a nonempty, convex, and weakly compact set in which the shift invariant Gibbs measures form a nonempty, convex, weakly compact subset.

Entropy

Entropy

Let $\mathcal{M}_{+,1}$ be the set of Borel probability measures on $\mathbb{N}^{\mathbb{Z}^d}$, and $\mathcal{M}_{+,1,T}$ be the set of shift (translation) invariant Borel probability measures.

Entropy

Let $\mathcal{M}_{+,1}$ be the set of Borel probability measures on $\mathbb{N}^{\mathbb{Z}^d}$, and $\mathcal{M}_{+,1,T}$ be the set of shift (translation) invariant Borel probability measures. For $\mu \in \mathcal{M}_{+,1,T}$, define the entropy

$$h(\mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{\alpha\in\mathbb{N}^{\Lambda_n}} -\mu([\alpha]) \log(\mu([\alpha])) \in [0,\infty].$$

Entropy

Let $\mathcal{M}_{+,1}$ be the set of Borel probability measures on $\mathbb{N}^{\mathbb{Z}^d}$, and $\mathcal{M}_{+,1,T}$ be the set of shift (translation) invariant Borel probability measures. For $\mu \in \mathcal{M}_{+,1,T}$, define the entropy

$$h(\mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{\alpha \in \mathbb{N}^{\Lambda_n}} -\mu([\alpha]) \log(\mu([\alpha])) \in [0, \infty].$$

Fact[4]: If f is a regular, exp-summable function of $\mathbb{N}^{\mathbb{Z}^d}$ that also satisfies

$$\sum_{a \in \mathbb{N}} \sup f|_{[a]} e^{\sup f|_{[a]}} > -\infty \qquad (1)$$

then every shift invariant Gibbs measure for f has finite entropy. Conversely, if the sum diverges to $-\infty$ then every shift invariant Gibbs measure for f has infinite entropy.

Equilibrium Measures

"Pressure" Theorem and Variational Principle [4]: If $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ is uniformly continuous and exp-summable with $\delta_1(f) < \infty$, then

$$\begin{aligned} \mathbb{R} \ni p(f) &\equiv \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \overline{Z}_\Lambda(f) \\ &= \sup \left\{ \int f d\mu + h(\mu) : \mu \in \mathcal{M}_{+,1,T} \text{ and } \int f d\mu > -\infty \right\}. \end{aligned}$$

Equilibrium Measures

"Pressure" Theorem and Variational Principle [4]: If $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ is uniformly continuous and exp-summable with $\delta_1(f) < \infty$, then

$$\begin{aligned} \mathbb{R} \ni p(f) &\equiv \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \overline{Z}_\Lambda(f) \\ &= \sup \left\{ \int f d\mu + h(\mu) : \mu \in \mathcal{M}_{+,1,T} \text{ and } \int f d\mu > -\infty \right\}. \end{aligned}$$

An equilibrium measure for a regular, exp-summable function f is any $\mu \in \mathcal{M}_{+,1,T}$ which maximizes the "negative free energy", i.e. for which

$$\int f \, d\mu + h(\mu) = p(f).$$

Equilibrium Measures

"Pressure" Theorem and Variational Principle [4]: If $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ is uniformly continuous and exp-summable with $\delta_1(f) < \infty$, then

$$\begin{aligned} \mathbb{R} \ni p(f) &\equiv \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \overline{Z}_\Lambda(f) \\ &= \sup \left\{ \int f d\mu + h(\mu) : \mu \in \mathcal{M}_{+,1,T} \text{ and } \int f d\mu > -\infty \right\}. \end{aligned}$$

An equilibrium measure for a regular, exp-summable function f is any $\mu \in \mathcal{M}_{+,1,T}$ which maximizes the "negative free energy", i.e. for which

$$\int f \, d\mu + h(\mu) = p(f).$$

Theorem (physical significance of Gibbs measures)[5]:
If $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ is a regular, exp-summable local energy function and $\mu \in \mathcal{M}_{+,1,T}$ with finite entropy, then μ is a Gibbs measure for f if and only if it is an equilibrium measure for f. If f additionally satisfies criterion 1 then the equilibrium measures for f are precisely the shift invariant Gibbs measures for f.

Let "E.S." stand for equilibrium state, "G.S." for Gibbs state, and "C.E.S." for constructive equilibrium state, to be defined shortly.

Let "E.S." stand for equilibrium state, "G.S." for Gibbs state, and "C.E.S." for constructive equilibrium state, to be defined shortly. The strategy of proof for the "physical significance" theorem is as follows:

$$\begin{aligned} \text{E.S.}(f) &\subseteq \overline{co}\,(\text{C.E.S.}^x(f)) \cap \{h < \infty\} \\ &\subseteq \text{G.S.}^I(f) \cap \{h < \infty\} \\ &\subseteq \text{E.S.}(f). \end{aligned}$$

Let “E.S.” stand for equilibrium state, “G.S.” for Gibbs state, and “C.E.S.” for constructive equilibrium state, to be defined shortly. The strategy of proof for the “physical significance” theorem is as follows:

$$\begin{aligned} \text{E.S.}(f) &\subseteq \overline{co}\,(\text{C.E.S.}^x(f)) \cap \{h < \infty\} \\ &\subseteq \text{G.S.}^I(f) \cap \{h < \infty\} \\ &\subseteq \text{E.S.}(f). \end{aligned}$$

Note- a more direct proof using relative entropy can be adapted from [1], though the technique is originally attributed to Chris Preston.

Finite Volume Gibbs Ensembles.

Fix a configuration $x \in \mathbb{N}^{\mathbb{Z}^d}$ and a finite $\Lambda \subset \mathbb{Z}^d$. The space $\mathbb{N}^\Lambda \times \{x|_{\Lambda^c}\}$ is of countable cardinality and supports the probability measure

$$\pi\left(\omega, x|_{\Lambda^c}\right) = \frac{e^{f_\Lambda\left(\omega, x|_{\Lambda^c}\right)}}{\sum_{\omega' \in \mathbb{N}^\Lambda} e^{f_\Lambda\left(\omega', x|_{\Lambda^c}\right)}}.$$

Finite Volume Gibbs Ensembles.

Fix a configuration $x \in \mathbb{N}^{\mathbb{Z}^d}$ and a finite $\Lambda \subset \mathbb{Z}^d$. The space $\mathbb{N}^\Lambda \times \{x|_{\Lambda^c}\}$ is of countable cardinality and supports the probability measure

$$\pi\left(\omega, x|_{\Lambda^c}\right) = \frac{e^{f_\Lambda\left(\omega, x|_{\Lambda^c}\right)}}{\sum_{\omega' \in \mathbb{N}^\Lambda} e^{f_\Lambda\left(\omega', x|_{\Lambda^c}\right)}}.$$

It is an old, classical result (proved by Jensen's inequality or Lagrange multipliers) that this measure π is an equilibrium measure, in the variational sense, for the energy function $(f_\Lambda)|_{\mathbb{N}^\Lambda \times \{x|_{\Lambda^c}\}}$.

Finite Volume Gibbs Ensembles.

Fix a configuration $x \in \mathbb{N}^{\mathbb{Z}^d}$ and a finite $\Lambda \subset \mathbb{Z}^d$. The space $\mathbb{N}^\Lambda \times \{x|_{\Lambda^c}\}$ is of countable cardinality and supports the probability measure

$$\pi\left(\omega, x|_{\Lambda^c}\right) = \frac{e^{f_\Lambda\left(\omega, x|_{\Lambda^c}\right)}}{\sum\limits_{\omega' \in \mathbb{N}^\Lambda} e^{f_\Lambda\left(\omega', x|_{\Lambda^c}\right)}}.$$

It is an old, classical result (proved by Jensen's inequality or Lagrange multipliers) that this measure π is an equilibrium measure, in the variational sense, for the energy function $(f_\Lambda)|_{\mathbb{N}^\Lambda \times \{x|_{\Lambda^c}\}}$. This motivates us to study the purely atomic measures

$$\pi_\Lambda^x = \sum_{\omega \in \mathbb{N}^\Lambda} \frac{e^{f_n\left(\omega, x|_{\Lambda^c}\right)}}{\sum_{\omega' \in \mathbb{N}^\Lambda} e^{f_n\left(\omega', x|_{\Lambda C}\right)}} \delta_{\omega, x|_{\Lambda^c}},$$

on the space $\mathbb{N}^{\mathbb{Z}^d}$, wherein δ_x is the Dirac delta measure supported entirely on the configuration x.

“Constructive” Equilibrium Measures.

The measure π_Λ^x is called a finite volume Gibbs ensemble (within Λ, with boundary condition x) for f.

"Constructive" Equilibrium Measures.

The measure π_Λ^x is called a finite volume Gibbs ensemble (within Λ, with boundary condition x) for f.The broad idea for constructive equilibria is to look for weak limits as $\Lambda \to \mathbb{Z}^d$ of finite volume Gibbs ensembles within Λ.

"Constructive" Equilibrium Measures.

The measure π_Λ^x is called a finite volume Gibbs ensemble (within Λ, with boundary condition x) for f.The broad idea for constructive equilibria is to look for weak limits as $\Lambda \to \mathbb{Z}^d$ of finite volume Gibbs ensembles within Λ.
However there are subtleties, because we want both

1. every equilibrium state (in the sense of extremal free energy) should be in the closed convex hull of the constructive equilibrium states,
   and
2. every constructive equilibrium state should be a translation invariant Gibbs state.

"Constructive" Equilibrium Measures.

The measure π_Λ^x is called a finite volume Gibbs ensemble (within Λ, with boundary condition x) for f.The broad idea for constructive equilibria is to look for weak limits as $\Lambda \to \mathbb{Z}^d$ of finite volume Gibbs ensembles within Λ.
However there are subtleties, because we want both

1. every equilibrium state (in the sense of extremal free energy) should be in the closed convex hull of the constructive equilibrium states,
   and
2. every constructive equilibrium state should be a translation invariant Gibbs state.

It does not hurt to broaden the definition of constructive equilibria to facilitate the proof of (1), as long as (2) can still be proved.

To make the constructive equilibria shift invariant we will have to do spatial averaging of the finite volume Gibbs ensembles, denoted

$$A_{\Lambda'}\pi_\Lambda^x = |\Lambda'|^{-1} \sum_{\lambda' \in \Lambda'} \pi_\Lambda^x \circ T^{-\lambda'}.$$

To make the constructive equilibria shift invariant we will have to do spatial averaging of the finite volume Gibbs ensembles, denoted

$$A_{\Lambda'}\pi^x_\Lambda = |\Lambda'|^{-1} \sum_{\lambda' \in \Lambda'} \pi^x_\Lambda \circ T^{-\lambda'}.$$

To ensure that the constructive equilibria capture all the true equilibrium states we have to allow perturbations of the energy function at each finite volume stage.

To make the constructive equilibria shift invariant we will have to do spatial averaging of the finite volume Gibbs ensembles, denoted

$$A_{\Lambda'}\pi^x_\Lambda = |\Lambda'|^{-1} \sum_{\lambda' \in \Lambda'} \pi^x_\Lambda \circ T^{-\lambda'}.$$

To ensure that the constructive equilibria capture all the true equilibrium states we have to allow perturbations of the energy function at each finite volume stage. Thus we reach the definition that $\mu \in \mathcal{M}_{+,1}$ is a constructive equilibrium state (with boundary configuration x) for the regular, exp-summable function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ if there is some sequence $\{g^n\}_{n=1}^\infty \subset BC\left(\mathbb{N}^{\mathbb{Z}^d}\right)$ for which $(Reg(g^n), \|g^n\|_\infty) \to 0$ as $n \to \infty$ and μ is a weak limit point of the sequence $A_n\tilde{\pi}^x_n$, wherein

$$\tilde{\pi}^x_n = \sum_{\omega \in \mathbb{N}^{\Lambda_n}} \delta_{\left(\omega, x|_{\Lambda^c_n}\right)} \frac{e^{(f+g^n)_n(\omega, x|_{\Lambda^c_n})}}{\sum_{\omega' \in \mathbb{N}^{\Lambda_n}} e^{(f+g^n)_n(\omega', x|_{\Lambda^c_n})}}.$$

Theorems Regarding Constructive Equilibria

For short, let $\text{C.E.S.}^{x}(f)$ denote the set of constructive equilibrium states for f with boundary configuration x.

Theorems Regarding Constructive Equilibria

For short, let C.E.S.$^x(f)$ denote the set of constructive equilibrium states for f with boundary configuration x.If f is regular and exp-summable and $x \in \mathbb{N}^{\mathbb{Z}^d}$ has only finitely many different letters, then [4]

1. C.E.S.$^x(f)$ is nonempty.

Theorems Regarding Constructive Equilibria

For short, let $\text{C.E.S.}^x(f)$ denote the set of constructive equilibrium states for f with boundary configuration x.If f is regular and exp-summable and $x \in \mathbb{N}^{\mathbb{Z}^d}$ has only finitely many different letters, then [4]

1. $\text{C.E.S.}^x(f)$ is nonempty.
2. $\text{C.E.S.}^x(f) \subseteq \text{G.S.}^I(f)$

Theorems Regarding Constructive Equilibria

For short, let C.E.S.$^x(f)$ denote the set of constructive equilibrium states for f with boundary configuration x.If f is regular and exp-summable and $x \in \mathbb{N}^{\mathbb{Z}^d}$ has only finitely many different letters, then [4]

1. C.E.S.$^x(f)$ is nonempty.
2. C.E.S.$^x(f) \subseteq$ G.S.$^I(f)$
3. $\overline{co}\,$(C.E.S.$^x(f)) \supseteq$ E.S.(f).
4. the extreme equilibrium states i.e. the ones ergodic under the shift action are all constructive equilibrium states.
   (note - for there to be any equilibrium states f must be non-pathologically exp-summable, in which case E.S.=G.S.^{I} and therefore weak compactness of E.S. follows from that of G.S.^{I})

Interactions

In all of the references shown but [3], the energy due to interaction between lattice sites is described as follows:
For each finite $\Lambda \subset \mathbb{Z}^d$, define a bounded $\mathcal{A}_\Lambda$ measurable function $\Phi_\Lambda : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ to represent the potential energy due to the joint interaction of all the particles/spins located at lattice sites within Λ.

Interactions

In all of the references shown but [3], the energy due to interaction between lattice sites is described as follows:

For each finite $\Lambda \subset \mathbb{Z}^d$, define a bounded $\mathcal{A}_\Lambda$ measurable function $\Phi_\Lambda : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ to represent the potential energy due to the joint interaction of all the particles/spins located at lattice sites within Λ.

Assume the interaction is *translation invariant*, i.e. for every $\Lambda \in \mathcal{P}_F(\mathbb{Z}^d)$ and $\lambda \in \mathbb{Z}^d$

$$\Phi_{\Lambda+\lambda} = \Phi_\Lambda \circ T^\lambda.$$

Also assume the interaction Φ is *admissible* by some apriori measure p on $\mathbb{N}$, Also assume the interaction Φ is *admissible* by some apriori measure p on $\mathbb{N}$,i.e. for every $\Lambda \subset \mathbb{Z}^d,\ |\Lambda| < \infty$, the Hamiltonian series

$$H_\Lambda \equiv \sum_{\substack{\Lambda' \subset \mathbb{Z}^d\ : \\ \Lambda' \cap \Lambda \neq \emptyset \text{ and } |\Lambda'| < \infty}} \Phi_{\Lambda'}\ ,$$

converges pointwise on $\mathbb{N}^{\mathbb{Z}^d}$ and the "partition function"

$$Z_\Lambda^\beta = \sum_{\alpha \in \mathbb{N}^\Lambda} e^{-H_\Lambda(\alpha,\beta)} p^\Lambda(\alpha),$$

converges for every $\beta \in \mathbb{N}^{\Lambda^c}$.

Gibbs Measure for an Interaction

Under these assumptions a *Gibbs measure for* Φ is defined as any Borel probability measure μ on $\mathbb{N}^{\mathbb{Z}^d}$ with the conditional expectations

$$\mu\left([\alpha]|\mathcal{A}_{\Lambda^c}\right)(\beta) = \frac{e^{-H_\Lambda(\alpha,\beta|_{\Lambda^c})} p^\Lambda(\alpha)}{Z_\Lambda^\beta},$$

for every finite $\Lambda \subset \mathbb{Z}^d$, every $\alpha \in \mathbb{N}^\Lambda$ and μ-a.e. $\beta \in \mathbb{N}^{\mathbb{Z}^d}$.

Gibbs Measure for an Interaction

Under these assumptions a *Gibbs measure for* Φ is defined as any Borel probability measure μ on $\mathbb{N}^{\mathbb{Z}^d}$ with the conditional expectations

$$\mu\left([\alpha] | \mathcal{A}_{\Lambda^c}\right)(\beta) = \frac{e^{-H_\Lambda(\alpha, \beta|_{\Lambda^c})} p^\Lambda(\alpha)}{Z_\Lambda^\beta},$$

for every finite $\Lambda \subset \mathbb{Z}^d$, every $\alpha \in \mathbb{N}^\Lambda$ and μ-a.e. $\beta \in \mathbb{N}^{\mathbb{Z}^d}$. These are called the DLR equations for the interaction Φ (with apriori measure p).

The Local Energy Function Associated to an Interaction

Given an interaction for a lattice model, Ruelle [5] proposed two different formulas for its local energy:

$$A_\Phi = \sum_{\substack{\Lambda \subset \mathbb{Z}^d \ : \\ \Lambda \ni 0 \text{ and } |\Lambda| < \infty}} \frac{1}{|\Lambda|} \Phi_\Lambda \ ,$$

and

$$\hat{A}_\Phi = \sum_{\substack{\Lambda \subset \mathbb{Z}^d \ : \\ \text{lex.ord.}(0 \in \Lambda) = \left\lceil \frac{|\Lambda|}{2} \right\rceil}} \Phi_\Lambda .$$

The Local Energy Function Associated to an Interaction

Given an interaction for a lattice model, Ruelle [5] proposed two different formulas for its local energy:

$$A_\Phi = \sum_{\substack{\Lambda \subset \mathbb{Z}^d \; : \\ \Lambda \ni 0 \text{ and } |\Lambda| < \infty}} \frac{1}{|\Lambda|} \Phi_\Lambda \; ,$$

and

$$\hat{A}_\Phi = \sum_{\substack{\Lambda \subset \mathbb{Z}^d \; : \\ \text{lex.ord.}(0 \in \Lambda) = \left\lceil \frac{|\Lambda|}{2} \right\rceil}} \Phi_\Lambda .$$

The connection between the DLR equations for an interaction and the DLR equations for a function is this [4]: With suitable convergence assumptions, taking either $f = -A_\Phi$ or $f = -\hat{A}_\Phi$ in the local energy DLR equations for the function f yields the DLR equations for the interaction Φ.

Georgii [1] treats equilibrium measures as follows:

Georgii [1] treats equilibrium measures as follows: Let p be an apriori probability measure on $\mathbb{N}$, and let Φ be an interaction strongly summable in the sense

$$\sum_{\Lambda \ni 0, |\Lambda| < \infty} \|\Phi_\Lambda\|_\infty < \infty.$$

Georgii [1] treats equilibrium measures as follows: Let p be an apriori probability measure on $\mathbb{N}$, and let Φ be an interaction strongly summable in the sense

$$\sum_{\Lambda \ni 0, |\Lambda| < \infty} \|\Phi_\Lambda\|_\infty < \infty.$$

Under these assumptions, an equilibrium measure for Φ is defined as any $\mu \in \mathcal{M}_{+,1,T}$ for which

$$\int A_\Phi d\mu - h(\mu | p^{\mathbb{Z}^d}) = \inf_{\nu \in \mathcal{M}_{+,1,T}} \int A_\Phi d\nu - h(\nu | p^{\mathbb{Z}^d}),$$

and the equilibrium measures for Φ are proven to coincide with the shift invariant Gibbs measures for Φ.

Georgii [1] treats equilibrium measures as follows: Let p be an apriori probability measure on $\mathbb{N}$, and let Φ be an interaction strongly summable in the sense

$$\sum_{\Lambda \ni 0, |\Lambda| < \infty} \|\Phi_\Lambda\|_\infty < \infty.$$

Under these assumptions, an equilibrium measure for Φ is defined as any $\mu \in \mathcal{M}_{+,1,T}$ for which

$$\int A_\Phi d\mu - h(\mu | p^{\mathbb{Z}^d}) = \inf_{\nu \in \mathcal{M}_{+,1,T}} \int A_\Phi d\nu - h(\nu | p^{\mathbb{Z}^d}),$$

and the equilibrium measures for Φ are proven to coincide with the shift invariant Gibbs measures for Φ.
Note that $\int A_\Phi d\mu = \int \hat{A}_\Phi d\mu$ for any $\mu \in \mathcal{M}_{+,1,T}$, so we could use either A or $\hat{A}$ in the definition.

Advantage of local energy function approach: example

Let $\mathcal{S}$ be the "Small" Banach space of translation invariant interactions with finite norm

$$\|\Phi\|_S = \sum_{0 \in \Lambda \in \mathcal{P}_F(\mathbb{Z}^d)} \|\Phi_\Lambda\|_\infty < \infty.$$

These are the interactions treated by Georgii's variational theorem.

Advantage of local energy function approach: example

Let $\mathcal{S}$ be the "Small" Banach space of translation invariant interactions with finite norm

$$\|\Phi\|_S = \sum_{0 \in \Lambda \in \mathcal{P}_F(\mathbb{Z}^d)} \|\Phi_\Lambda\|_\infty < \infty.$$

These are the interactions treated by Georgii's variational theorem. We will produce an interaction that is not in this space but whose local energy is never the less regular and therefore subject to our theorems.

Example

Consider $d \geq 2$ with an interaction Ψ which is nonzero only on cubes and takes the form

$$\Psi_{\Lambda_m}(x) = m^{-(1+\epsilon)} \prod_{n=1}^{m} \psi_{m,n}\left(x|_{\Lambda_n \setminus \Lambda_{n-1}}\right),$$

for $m \geq 2$.

Example

Consider $d \geq 2$ with an interaction Ψ which is nonzero only on cubes and takes the form

$$\Psi_{\Lambda_m}(x) = m^{-(1+\epsilon)} \prod_{n=1}^{m} \psi_{m,n}\left(x|_{\Lambda_n \setminus \Lambda_{n-1}}\right),$$

for $m \geq 2$. Let

$$\psi_{m,n} = f_{m,n}\left(\prod_{\lambda \in \Lambda_n \setminus \Lambda_{n-1}} x_\lambda\right),$$

where

$$f_{m,n}(t) = a_{m,n}^{-1} + t^{-1}(a_{m,n} - a_{m,n}^{-1})$$

and the constants

$$a_{m,n} = \sqrt{\frac{(n-1)^{d-1}(mn^{d-1}-1)}{n^{d-1}(m(n-1)^{d-1}-1)}}.$$

We have

$$\mathrm{Range}(\psi_{m,n}) = (a_{m,n}^{-1}, a_{m,n}],$$

a small interval around 1 with the upper bound achieved when the configuration is identically the number 1 and the lower bound approached as the product of the spins in $\Lambda_n \setminus \Lambda_{n-1}$ tends to infinity.

We have

$$\text{Range}(\psi_{m,n}) = (a_{m,n}^{-1}, a_{m,n}],$$

a small interval around 1 with the upper bound achieved when the configuration is identically the number 1 and the lower bound approached as the product of the spins in $\Lambda_n \setminus \Lambda_{n-1}$ tends to infinity.
From this it follows that

$$\delta_n(\Psi_{\Lambda_m}) = m^{-(2+\epsilon)} n^{-(d-1)}$$

and

$$\|\Psi_{\Lambda_m}\|_\infty = m^{-(2+\epsilon)}.$$

From the supnorms one can check $\Psi \notin \mathcal{S}$, so Georgii's variational theorem does not apply.

We have

$$\mathrm{Range}(\psi_{m,n}) = (a_{m,n}^{-1}, a_{m,n}],$$

a small interval around 1 with the upper bound achieved when the configuration is identically the number 1 and the lower bound approached as the product of the spins in $\Lambda_n \setminus \Lambda_{n-1}$ tends to infinity.
From this it follows that

$$\delta_n(\Psi_{\Lambda_m}) = m^{-(2+\epsilon)} n^{-(d-1)}$$

and

$$\|\Psi_{\Lambda_m}\|_\infty = m^{-(2+\epsilon)}.$$

From the supnorms one can check $\Psi \notin \mathcal{S}$, so Georgii's variational theorem does not apply.
On the other hand, with the oscillations one can check that $\hat{A}_\Psi$ *is* regular (and bounded).

Let p be any finite apriori measure on the alphabet $\mathbb{N}$.

Let p be any finite apriori measure on the alphabet $\mathbb{N}$.
Then $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ by

$$f(\omega) = -\hat{A}_\Psi(\omega) + \log p(\omega|_0)$$

is regular and exp-summable, so that our theorems apply.

Let p be any finite apriori measure on the alphabet $\mathbb{N}$.
Then $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ by

$$f(\omega) = -\hat{A}_{\Psi}(\omega) + \log p(\omega|_0)$$

is regular and exp-summable, so that our theorems apply.
Moreover,

Let p be any finite apriori measure on the alphabet $\mathbb{N}$.
Then $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ by

$$f(\omega) = -\hat{A}_\Psi(\omega) + \log p(\omega|_0)$$

is regular and exp-summable, so that our theorems apply.
Moreover,

1. The DLR equations for f are equal to the DLR equations for Ψ with the apriori measure p, so that f and Ψ define the same Gibbs measures.

Let p be any finite apriori measure on the alphabet $\mathbb{N}$.
Then $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ by

$$f(\omega) = -\hat{A}_\Psi(\omega) + \log p(\omega|_0)$$

is regular and exp-summable, so that our theorems apply.
Moreover,

1. The DLR equations for f are equal to the DLR equations for Ψ with the apriori measure p, so that f and Ψ define the same Gibbs measures.
2. $$\int f \; d\mu + h(\mu) = -\left(\int \hat{A}_\Psi \; d\mu - h(\mu | p^{\mathbb{Z}^d}) \right),$$

   so that Ψ and f define the same "free energy" functional on $\mathcal{M}_{+,1,T}$ and hence have the same equilibrium measures.

However we haven't proved that $\hat{A}_\Psi$ couldn't arise from some other interaction $\Psi' \in \mathcal{S}$.

However we haven't proved that $\hat{A}_\Psi$ couldn't arise from some other interaction $\Psi' \in \mathcal{S}$.
Open Question: Describe exactly the image of the small space of interactions under the maps A and $\hat{A}$.

Acknowledgments

1. Without the consistent support of my phd advisor, Mariusz Urbański, I would never have gotten started on this project, much less finished it.
2. The gracious suggestions of H.O. Georgii were invaluable in reconciling my project with the "state of the art" presented in his book [1].
3. Gerhard Keller's book [3] provided the inspiration for a function-based approach to Gibbs measures in higher dimensions, and his responses to my questions and early drafts led to substantial improvements.
4. Thanks to my committee for spending the time and effort to read critically, and to all of you for listening!