GIBBS' PHENOMENON FOR HAUSDORFF MEANS(1)

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1. We define the general Euler means of a sequence

$$s_0, s_1, s_2, \cdots \tag{1.1}$$

by

$$\sigma_n(r) = \sum_{\nu=0}^{n} C_{n,\nu} r^\nu (1-r)^{n-\nu} s_\nu, \qquad n = 0, 1, 2, \cdots; \tag{1.2}$$

here r is a positive number $0 < r \leqq 1$. For $r = 1$

$$\sigma_n(1) = s_n.$$

The transform (1.2) is regular, that is, if $s_n \to s$, then

$$\sigma_n(r) \to s, \qquad \text{as } n \to \infty.$$

We define the Hausdorff means of the sequence (1.1) by

$$h_n = \sum_0^n C_{n,\nu} s_\nu \int_0^1 r^\nu (1-r)^{n-\nu} d\psi(r), \qquad n = 0, 1, 2, \cdots, \tag{1.3}$$

where $\psi(r)$ is of bounded variation in $0 \leqq r \leqq 1$.

The transform (1.3) is regular if and only if

$$\int_0^1 d\psi(r) = \psi(1) - \psi(0) = 1, \tag{1.4}$$

and if

$$\psi(r) \text{ is continuous at } r = 0(^2). \tag{1.5}$$

We may assume that $\psi(0) = 0$; then the conditions (1.4) and (1.5) become

$$\psi(1) = 1, \qquad \psi(+0) = \psi(0) = 0. \tag{1.6}$$

Observe that if $s_n = 1$, $n = 0, 1, 2, \cdots$, then

$$h_n = \int_0^1 \sum C_{n,\nu} r^\nu (1-r)^{n-\nu} d\psi = \int_0^1 d\psi = 1, \qquad n = 0, 1, 2, \cdots.$$

The transform (1.2) is a special case of (1.3), when we choose

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(2) See for example, G. H. Hardy, *Divergent series*, Oxford, 1949, chap. 11.

$$\psi(u) = \begin{cases} 0, & \text{for } 0 \leqq u < r \\ 1, & \text{for } r \leqq u \leqq 1. \end{cases} \tag{1.7}$$

Other special cases are

$$\psi(u) = 1 - (1-u)^p, \qquad Rp > 0, \tag{1.8}$$

$$\psi(u) = \frac{1}{\Gamma(p)} \int_0^u \left(\log \frac{1}{y}\right)^{p-1} dy, \qquad Rp > 0. \tag{1.9}$$

In case (1.8)

$$\int_0^1 r^\nu (1-r)^{n-\nu} d\psi(r) = p \int_0^1 r^\nu (1-r)^{n-\nu+p-1} dr = p \frac{\Gamma(\nu+1)\Gamma(n-\nu+p)}{\Gamma(n+p+1)},$$

$$C_{n,\nu} = \frac{\Gamma(n+1)}{\Gamma(\nu+1)\Gamma(n-\nu+1)},$$

so that (1.3) becomes

$$h_n = \frac{1}{C_{n+p,n}} \sum_0^n C_{n+p-\nu-1,p-1} s_\nu;$$

these are the Cesàro means of order p.

In case (1.9) we get for (1.3)

$$h_n = \frac{1}{\Gamma(p)} \sum_0^n C_{n,\nu} s_\nu \int_0^1 r^\nu (1-r)^{n-\nu} \left(\log \frac{1}{r}\right)^{p-1} dr;$$

these are the Hölder means of order p.

It is known that Cesàro and Hölder means of the same order are equivalent, that is, they sum the same class of series to the same value.

2. The Fourier series

$$\sum_1^\infty \frac{\sin \nu t}{\nu} = \frac{1}{2}(\pi - t), \qquad 0 < t \leqq \pi,$$

converges for all t, and the function has a jump at $t=0$. Hence the convergence is nonuniform at $t=0$; that is, the sequence $s_n(t_n)$, where

$$s_n(t) = \sum_1^n \frac{\sin \nu t}{\nu}, \tag{2.1}$$

and $t_n \to 0$ has several limit points, depending on the manner in which $t_n \to 0$. We have, more precisely,

$$s_n(t_n) \to \int_0^\tau \frac{\sin t}{t} dt, \quad \text{if} \quad nt_n \to \tau, \qquad \tau \geqq 0.$$

The maximal limit point is attained when $\tau=\pi$, in which case

$$s_n(t_n) \to \int^{\pi} \frac{\sin t}{t}\, dt = \frac{\pi}{2} \times 1.17897 \cdots,$$

while

$$(\pi - t)/2 \to \pi/2 \qquad \text{as } t \downarrow 0.$$

This is called Gibbs' phenomenon. It then follows easily that if

$$f(t) \sim \sum_1^{\infty} b_\nu \sin \nu t$$

is the Fourier series of a function of bounded variation, and

$$s_n(t) = \sum_1^{n} b_\nu \sin \nu t,$$

then

$$s_n(t_n) \to \frac{2}{\pi} f(+0) \int_0^{\tau} \frac{\sin t}{t}\, dt, \qquad \text{as } nt_n \to \tau.$$

$(2/\pi)\int_0^{\pi} \frac{\sin t}{t}\, dt = 1.17897 \cdots$ is called the Gibbs ratio for partial sums.

Consider now the h_n means of the sequence (2.1). The question arises: When does the sequence $\{h_n(t)\}$ present a Gibbs' phenomenon? This will depend on the choice of the function $\psi(t)$. In the case of the general Euler transform (1.2) our result is (see [5])(³)

$$h_n(t_n) \to \int_0^{r\tau} \frac{\sin t}{t}\, dt, \qquad \text{as } nt_n \to \tau.$$

Thus the maximal limit point is the same as in the case $r=1$ of the partial sums, although Euler's summation method is quite powerful.

For Cesàro means of order p Cramér [1] proved the existence of a constant c, $0<c<1$, such that Gibbs' phenomenon occurs for $p<c$, and does not occur for $p \geqq c$. The Gibbs ratio is obtained by evaluating the expression

$$G = \max_{\tau>0} \frac{2}{\pi} \int_0^{\tau} \left(1 - \frac{u}{\tau}\right)^p \frac{\sin u}{u}\, du = \max \frac{2}{\pi} \int_0^1 (1-u)^p \frac{\sin \tau u}{u}\, du.$$

Cooke [2] has shown that

$$0.40 < c < 0.48.$$

(³) Numbers in brackets refer to the references cited at the end of the paper.

Earlier Gronwall gave the approximate value

$$c = 0.4395.$$

3. In this section we shall prove:

THEOREM 1a. *If* $nt_n \to \tau < \infty$, *then, as* $n \to \infty$

$$h_n(t_n) \to \int_0^1 \int_0^\tau \frac{\sin ry}{y} dy d\psi(r).$$

Here $h_n(t)$ is the Hausdorff transform of the sequence (2.1). Clearly

$$s_n(t) = -\frac{t}{2} + \int_0^t \frac{\sin (n + 1/2)x}{2 \sin x/2} dx,$$

so that

$$\sum_0^n C_{n,\nu} r^\nu (1 - r)^{n-\nu} s_\nu = -\frac{t}{2} + \frac{J}{2} \int_0^t \frac{1}{\sin x/2} (1 - r + re^{ix})^n e^{ix/2} dx,$$

and, in view of (1.3),

$$h_n(t) = -\frac{t}{2} \int_0^1 d\psi(r) + \frac{J}{2} \int_0^1 \int_0^t \frac{1}{\sin x/2} (1 - r + re^{ix})^n e^{ix/2} dx d\psi(r).$$

Let $1 - r + re^{ix} = \rho e^{i\alpha}$, where ρ and α depend on r and x; if $r = 0$, then $\rho = 1$, and $\alpha = 0$; if $r = 1$, then $\rho = 1$, $\alpha = x$. In general

(3.1) $$\rho^2 = (1 - r + r \cos x)^2 + r^2 \sin^2 x = 1 - 2r(1 - r)(1 - \cos x),$$

(3.2) $$\rho \cos \alpha = 1 - r + r \cos x, \qquad \rho \sin \alpha = r \sin x.$$

Now

$$\begin{aligned} h_n(t) &= -\frac{t}{2} + \frac{J}{2} \int_0^1 \int_0^t \frac{\rho^n}{\sin x/2} e^{i(n\alpha + x/2)} dx d\psi(r) \\ &= -\frac{t}{2} + \frac{1}{2} \int_0^1 \int_0^t \rho^n \frac{\sin (n\alpha + x/2)}{\sin x/2} dx d\psi(r) \\ &= -\frac{t}{2} + \frac{1}{2} \int_0^1 \int_0^t \rho^n \sin n\alpha \cot x/2 dx d\psi(r) \\ &\qquad + \frac{1}{2} \int_0^1 \int_0^t \rho^n \cos n\alpha dx d\psi(r) \\ &= -\frac{t}{2} + A_1(t) + A_2(t), \quad \text{say.} \end{aligned}$$

Here, as $0 < \rho \leqq 1$,

$$|A_2(t)| < \frac{t}{2}\int_0^1 |d\psi(r)| = O(t), \qquad t \to 0,$$

so that

$$h_n(t) = A_1(t) + O(t), \qquad t \to 0.$$

Furthermore, from $0<y-\sin y<y^3$,

$$\left|\cot y - \frac{1}{y}\right| = \left|\cos y \frac{y - \sin y}{y \sin y} - \frac{1 - \cos y}{y}\right| \leqq \frac{\pi}{2} y + \frac{y}{2},$$

hence

$$\begin{aligned} A_1(t) &= \frac{1}{2}\int_0^1\int_0^t \rho^n \sin n\alpha \cot x/2 dx d\psi(r) \\ &= \int_0^1\int_0^t \rho^n \frac{\sin n\alpha}{x} dx d\psi(r) \\ &\quad + O\left(\int_0^1\int_0^t \rho^n x |\sin n\alpha d\psi(r)| dx\right). \end{aligned}$$

From (3.1), $\rho^2 \leqq 1$, so that

$$\begin{aligned} A_1(t) &= \int_0^1\int_0^t \rho^n \frac{\sin n\alpha}{x} dx d\psi(r) + O\left(t^2 \int_0^1 |d\psi(r)|\right) \\ &= \int_0^1\int_0^t \rho^n \frac{\sin n\alpha}{x} dx d\psi(r) + O(t^2) \\ &= B(t) + O(t^2), \quad \text{say,} \end{aligned}$$

and

$$h_n(t) = \int_0^1\int_0^t \rho^n \frac{\sin n\alpha}{x} dx d\psi(r) + O(t).$$

We now assume that $t=t_n \to 0$, and $nt_n \to \tau < \infty$. For $0<\rho<1$

$$1 - \rho^n = (1-\rho)\sum_0^{n-1} \rho^\nu < n(1-\rho) < n(1-\rho^2),$$

hence

$$1 - \rho^n = \lambda n(1-\rho^2), \qquad 0 < \lambda < 1. \tag{3.3}$$

It follows that

$$h_n(t) = \int_0^1\int_0^t \frac{\sin n\alpha}{x} dx d\psi(r) + n\int_0^1\int_0^t \lambda(1-\rho^2)\frac{\sin n\alpha}{x} dx d\psi + O(t);$$

from(3.1)

$$1-\rho^2 = 4r(1-r)\sin^2\frac{x}{2} < x^2,$$

so that

$$h_n(t) = \int_0^1\int_0^t \frac{\sin n\alpha}{x}\,dxd\psi(r) + O\left\{n\int_0^1\int_0^t xdx\,|\,d\psi(r)\,|\right\} + O(t).$$

Here the second term is $O(nt^2)=O(t)$, hence

$$h_n(t) = \int_0^1\int_0^t \frac{\sin n\alpha}{x}\,dxd\psi(r) + O(t).$$

From (3.2) for small values of x

$$\tan\alpha = \frac{r\sin x}{1-r(1-\cos x)} = z, \quad \text{say},$$

and

$$\alpha = \arctan z = \sum_1^\infty (-1)^{\nu-1}\frac{z^{2\nu-1}}{2\nu-1} = rx + O(rx^3). \tag{3.4}$$

The formula

$$\sin a - \sin b = 2\cos\frac{a+b}{2}\sin\frac{a-b}{2}$$

now yields

$$\sin n\alpha - \sin nrx = 2\cos n\,\frac{\alpha+rx}{2}\sin O(nrx^3).$$

Thus

$$\begin{aligned} h_n(t) &= \int_0^1\int_0^t \frac{\sin nrx}{x}\,dxd\psi(r) + O\left\{\int_0^1\int_0^t nx^2dx\,|\,d\psi(r)\,|\right\} + O(t) \\ &= \int_0^1\int_0^{nt} \frac{\sin ry}{y}\,dyd\psi + O(nt^3) + O(t) \\ &= \int_0^1\int_0^{nt} \frac{\sin ry}{y}\,dyd\psi + O(t); \end{aligned}$$

this proves Theorem 1.

4. To discuss the case $nt_n \to \infty$ we need some sharper estimates in place of (3.3) and (3.4). The result is:

THEOREM 1b. *If* $nt_n \to \infty$, *then*

$$\lim_{r\to\infty} h_n(t_n) = \pi/2.$$

Here $h_n(t)$ *is the Hausdorff transform of the sequence* (2.1).

We first estimate for small positive ϵ the difference

$$e^{-n\epsilon} - (1-\epsilon)^n.$$

We have

$$e^{-\epsilon} > 1 - \epsilon, \qquad 0 < \epsilon < 1,$$

hence

$$e^{-n\epsilon} > (1-\epsilon)^n, \quad \text{or} \quad 1 > (1-\epsilon)^n e^{n\epsilon}.$$

Furthermore

$$0 < 1 - \left\{(1-\epsilon)e^{\epsilon}\right\}^n < n\left\{1 - (1-\epsilon)e^{\epsilon}\right\},$$

and

$$0 < 1 - (1-\epsilon)e^{\epsilon} < \epsilon^2 \qquad \text{for } 0 < \epsilon \leqq 1/2.$$

It follows that

$$0 < e^{-n\epsilon} - (1-\epsilon)^n < n\epsilon^2 e^{-n\epsilon},$$

and, if we replace ϵ by $1-\rho^2 = 2r(1-r)(1-\cos x) \leqq 1/2$

$$0 < e^{-n(1-\rho^2)} - \rho^{2n} < n(1-\rho)^2 e^{-n(1-\rho^2)},$$

or

$$\left\{e^{-n(1-\rho^2)/2} - \rho^n\right\}\left\{e^{-n(1-\rho^2)/2} + \rho^n\right\} < n(1-\rho^2)^2 e^{-n(1-\rho^2)}.$$

Thus

$$e^{-n(1-\rho^2)/2} - \rho^n < n(1-\rho^2)^2 e^{-n(1-\rho^2)/2}$$

or

$$e^{-n(1-\rho^2)/2} - \rho^n = \beta n(1-\rho^2)^2 e^{-n(1-\rho^2)/2}, \qquad 0 < \beta < 1.$$

We now find

$$B(t) = \int_0^1 \int_0^t \rho^n \frac{\sin n\alpha}{x}\, dx d\psi = \int_0^1 \int_0^t e^{-n(1-\rho^2)/2} \frac{\sin n\alpha}{x}\, dx d\psi(r)$$

$$+ n\int_0^1 \int_0^t \beta(1-\rho^2)^2 e^{-n(1-\rho^2)/2} \frac{\sin n\alpha}{x}\, dx d\psi(r)$$

$$= C_1(t) + nC_2(t), \quad \text{say}.$$

Next

$$|C_2(t)| < \int_0^1 \int_0^t 4r^2(1-r)^2(1-\cos x)^2 e^{-nr(1-r)(1-\cos x)} \frac{dx}{x} |d\psi(r)|$$
$$= O\int_0^1 \int_0^t r^2(1-r)^2 x^3 \exp\left(-\frac{n}{2\pi^2} r(1-r)x^2\right) dx\, |d\psi(r)|.$$

Now

$$\int_0^t x^3 \exp\left(-\frac{n}{2\pi^2} r(1-r)x^2\right) dx = \frac{1}{2}\int_0^t x^2 \exp\left(-\frac{n}{2\pi^2} r(1-r)x^2\right) dx^2$$
$$= \frac{1}{2}\int_0^{t^2} y \exp\left(-\frac{n}{2\pi^2} r(1-r)y)\right) dy$$
$$< \frac{t^2}{2}\int_0^{t^2} \exp\left(-\frac{n}{2\pi^2} r(1-r)y\right) dy$$
$$= \frac{t^2}{2}\left[-\exp\left(\frac{-n/2\pi^2 r(1-r)y}{nr(1-r)} 2\pi^2\right)\right]$$
$$= \frac{t^2\pi^2}{nr(1-r)}\left[1-\exp\left(-\frac{n}{2\pi^2} r(1-r)t^2\right)\right]$$

so that

$$nC_2(t) = O\int_0^1 t^2 r(1-r)\,|d\psi(r)| = O(t).$$

Hence,

$$B(t) = C_1(t) + O(t),$$
$$h_n(t) = C_1(t) + O(t).$$

Next

$$\exp(-a\sin^2 x/2) > \exp\left(-\frac{a}{4}x^2\right), \qquad a = 2nr(1-r),$$

and

$$\exp(-a\sin^2 x/2) - \exp\left(-\frac{a}{4}x^2\right)$$
$$= \exp\left(-\frac{a}{4}x^2\right)\left\{\exp a\left(\frac{1}{4}x^2 - \sin^2 x/2\right)\right\}.$$

Furthermore

$$0 < x^2/4 - \sin^2 x/2 = (x/2 - \sin x/2)(x/2 + \sin x/2) < 2x\frac{x^3}{8} = \frac{x^4}{4},$$

hence

$$a(x^2/4 - \sin^2 x/2) < nr(1-r)\frac{x^4}{2},$$

and

$$\begin{aligned}\exp(a(x^2/4 - \sin^2 x/2)) - 1 &< \exp(nr(1-r)x^4/2) - 1 \\ &= \exp(nr(1-r)x^4/2)\{1 - \exp(-nr(1-r)x^4/2\},\end{aligned}$$

so that

$$\begin{aligned}\exp(a(x^2/4 - \sin^2 x/2)) - 1 &< nr(1-r)\frac{x^4}{2}\exp(nr(1-r)x^4/2) \\ &= O(nr(1-r)x^4).\end{aligned}$$

It follows that

$$\exp(-a\sin^2 x/2) - \exp(-\frac{a}{4}x^2) = \exp\left(-\frac{a}{4}x^2\right)O(nr(1-r)x^4),$$

$$\begin{aligned}\int_0^t \exp(-2nr(1-r)\sin^2 x/2)\frac{\sin n\alpha}{x}dx \\ &= \int_0^t \exp\left(-\frac{n}{2}r(1-r)x^2\right)\frac{\sin n\alpha}{x}dx \\ &\quad + O\left\{nr(1-r)\int_0^t \exp\left(-\frac{n}{2}r(1-r)x^2\right)x^3dx\right\} \\ &= \int_0^t \exp\left(-\frac{n}{2}r(1-r)x^2\right)\frac{\sin n\alpha}{x}dx + O(t^2),\end{aligned}$$

so that

$$h_n(t) = \int_0^1\int_0^t \exp\left(-\frac{n}{2}r(1-r)x^2\right)\frac{\sin n\alpha}{x}dxd\psi(r) + O(t).$$

We next prove the lemma:

LEMMA. *For small values of x and for $0 \leqq r \leqq 1$*

$$\alpha = \alpha(r, x) = rx + O(r(1-r)x^3).$$

From (3.4) for $r = 1$, $\alpha = x$,

$$x = \sum_1^\infty (-1)^{\nu-1} \frac{1}{2\nu - 1}\left(\frac{\sin x}{\cos x}\right)^{2\nu-1},$$

$$rx - \alpha = r\sum_1^\infty (-1)^{\nu-1} \frac{(\sin x)^{2\nu-1}}{2\nu - 1}\left\{\frac{1}{(\cos x)^{2\nu-1}} - \frac{r^{2\nu-2}}{(1 - r + r\cos x)^{2\nu-1}}\right\}$$

$$= r\sum_1^\infty (-1)^{\nu-1} \frac{(\sin x)^{2\nu-1}}{2\nu - 1} \frac{(1 - r + r\cos x)^{2\nu-1} - r^{2\nu-2}(\cos x)^{2\nu-1}}{(\cos x)^{2\nu-1}(1 - r + r\cos x)^{2\nu-1}}.$$

Here the first term is

$$r \sin x \frac{1 - r + r\cos x - \cos x}{\cos x(1 - r + r\cos x)} = r(1 - r)\sin x \frac{1 - \cos x}{\cos x(1 - r + r\cos x)},$$

which is positive and less than $r(1-r)\sin x(1-\cos x)/\cos^2 x$. Let

$$(1 - r + r\cos x)^{2\nu-1} - r^{2\nu-2}(\cos x)^{2\nu-1} = g_\nu(r),$$

then $g_\nu(1) = 0$, and the mean value theorem yields

$$0 \leqq g_\nu(r) \leqq 2(1 - r)(2\nu - 1), \qquad \nu \geqq 2;\ 0 \leqq r \leqq 1.$$

It now follows that

$$|rx - \alpha| < r(1 - r)\frac{x^3}{\cos^2 x} + 2r(1 - r)\sum_2^\infty \frac{x^{2\nu-1}}{(\cos x)^{4\nu-2}},$$

which proves our lemma. Thus

$$\alpha = rx + \delta r(1 - r)x^3, \qquad \text{where } |\delta| \leqq \delta_0 \text{ a constant.}$$

Now the formula

$$\sin a - \sin b = 2\cos\frac{a + b}{2}\sin\frac{a - b}{2}$$

yields

$$\sin n\alpha - \sin nrx = 2\cos n\left(rx + \frac{1}{2}\delta r(1 - r)x^3\right)\sin n\frac{\delta}{2}r(1 - r)x^3,$$

so that

$$\int_0^t \exp\left(-\frac{n}{2}r(1 - r)x^2\right)\frac{\sin n\alpha}{x}dx = \int_0^t \exp\left(-\frac{n}{2}r(1 - r)x^2\right)\frac{\sin nrx}{x}dx$$

$$+ O\left\{\int_0^t \exp\left(-\frac{n}{2}r(1 - r)x^2\right)nr(1 - r)x^2dx\right\}.$$

The last term is

$$O\left\{nr(1-r)\int_0^{t^2}\exp\left(-\frac{n}{2}r(1-r)y\right)dy\right\} = O(t),$$

hence

$$(4.1)\qquad h_n(t) = \int_0^1\int_0^t \exp\left(-\frac{n}{2}r(1-r)x^2\right)\frac{\sin nrx}{x}\,dx + O(t).$$

Let

$$(4.2)\qquad \int_0^t \exp\left(-\frac{n}{2}r(1-r)x^2\right)\frac{\sin nrx}{x}\,dx = q_n(r, t),$$

$$(4.3)\qquad q_n(0, t) = 0,\qquad q_n(1, t) = \int_0^t \frac{\sin nx}{x}\,dx = \int_0^{nt}\frac{\sin y}{y}\,dy.$$

For $0<r<1$

$$q_n(r, t) = \int_0^{nrt}\exp\left(-\frac{1-r}{2rn}y^2\right)\frac{\sin y}{y}\,dy.$$

Let

$$\int_y^\infty \frac{\sin x}{x}\,dx = \mathrm{Si}\,(y),$$

then

$$\begin{aligned} q_n(r, t) &= -\int_0^{nrt}\exp\left(\frac{1-r}{2rn}y^2\right)d\,\mathrm{Si}\,(y) = \pi/2 \\ (4.4)\qquad &\quad - \mathrm{Si}\,(rnt)\exp\left(-\frac{1-r}{2}rnt^2\right) \\ &\quad - \frac{1-r}{rn}\int_0^{rnt} y\,\mathrm{Si}\,(y)\exp\left(-\frac{1-r}{2rn}y^2\right)dy. \end{aligned}$$

For fixed $t>0$, $q_n(r, t)$ is a continuous function of r in $0\leqq r\leqq 1$. From (4.3)

$$q_n(0, t_n)\to 0,\qquad q_n(1, t_n)\to \pi/2,\qquad \text{as } n\to\infty.$$

From (4.4) for $\epsilon\leqq r\leqq 1$, $\omega>0$

$$\begin{aligned} &| q_n(r, t_n) - \pi/2 | \\ &< |\mathrm{Si}\,(rnt_n)| + \frac{1-r}{rn}\max_{y>0}|\mathrm{Si}\,(y)|\int_0^\omega y\exp\left(-\frac{1-r}{2rn}y^2\right)dy \\ &\quad + \frac{1-r}{rn}\max_{y\geqq\omega}|\mathrm{Si}\,(y)|\int_\omega^\infty y\exp\left(-\frac{1-r}{2rn}y^2\right)dy \end{aligned}$$

$$< |\operatorname{Si}(rnt_n)| + \max_{y>0} |\operatorname{Si}(y)| \left[1 - \exp\left(-\frac{1-r}{2rn}\omega^2\right)\right]$$
$$+ \max_{y \geqq \omega} |\operatorname{Si}(y)|.$$

Let now $\omega = \omega_n \to \infty$ so that $\omega_n^2 = o(n)$; it then follows that $q_n(r, t_n) \to \pi/2$, uniformly in $\epsilon \leqq r \leqq 1$, and boundedly in $0 \leqq r \leqq 1$. Hence

$$\int_\epsilon^1 q_n(r, t_n) d\psi(r) \to \frac{\pi}{2} \int_\epsilon^1 d\psi(r), \qquad \text{as } n \to \infty;$$

letting $\epsilon \downarrow 0$, we get, in view of bounded convergence,

$$\int_{+0}^1 q_n(r, t_n) d\psi(r) \to \frac{\pi}{2} \int_{+0}^1 d\psi(r) = \frac{\pi}{2} \{\psi(1) - \psi(+0)\} = \frac{\pi}{2}.$$

Theorem 1b now follows from (4.1) and (4.2).

5. Combining Theorems 1a and 1b, we have the result:

THEOREM 2. *If $nt_n \to \tau \leqq \infty$, then*

$$h_n(t_n) \to \int_0^1 \int_0^\tau \frac{\sin ry}{y} dy d\psi(r), \qquad \textit{as } n \to \infty.$$

For $\tau < \infty$ we find

$$\int_0^1 \int_0^\tau \frac{\sin ry}{y} dy d\psi(r) = \int_0^\tau \frac{\sin y}{y} dy - \int_0^1 \psi(r) \frac{\sin r\tau}{r} dr$$
$$= \int_0^1 \{1 - \psi(r)\} \frac{\sin r\tau}{r} dr.$$

Thus the Gibbs ratio is

$$\frac{2}{\pi} \max_{\tau>0} \int_0^1 \{1 - \psi(r)\} \frac{\sin r\tau}{r} dr = \frac{2}{\pi} \max_{\tau>0} \int_0^\tau \{1 - \psi(y/\tau)\} \frac{\sin y}{y} dy.$$

In view of Theorem 2 we get the following result:

THEOREM 3. *For the Hausdorff means $h_n(t)$*

$$\limsup_{n\to\infty, t\to 0} h_n(t) = \max_{\tau>0} \int_0^1 \{1 - \psi(r)\} \frac{\sin \tau r}{r} dr;$$

if this maximum is attained for $\tau = \tau'$, then

$$\limsup_{n\to\infty, t\to 0} h_n(t) = \lim_{nt_n \to \tau'} h_n(t_n).$$

For Euler means it follows by (1.7) that the Gibbs ratio is

$$\frac{2}{\pi}\max_{\tau>0}\int_0^{\tau}\frac{\sin \tau u}{u}\,du = \frac{2}{\pi}\max_{\tau>0}\int_0^{\tau}\frac{\sin y}{y}\,dy.$$

For Cesàro means the Gibbs ratio is by (1.8)

$$G = \max_{\tau>0}\frac{2}{\pi}\int_0^1 (1-r)^p\,\frac{\sin \tau r}{r}\,dr.$$

For Hölder means the Gibbs ratio is by (1.9)

$$G = \max_{\tau>0}\frac{2}{\pi}\int_0^1\left\{1-\frac{1}{\Gamma(p)}\int_0^r\left(\log\frac{1}{x}\right)^{p-1}dx\right\}\frac{\sin \tau r}{r}\,dr.$$

Thus if $p=1$, in either case

$$G = \max\frac{2}{\pi}\int_0^1 (1-r)\,\frac{\sin \tau r}{r}\,dr$$

$$= \max\frac{2}{\pi}\int_0^{\tau}\left(1-\frac{r}{\tau}\right)\frac{\sin r}{r}\,dr.$$

Now

$$\int_0^{\tau}\frac{\sin r}{r}\,dr = \int_0^{\tau}\frac{1}{r}\,d(1-\cos r) = \frac{1-\cos r}{r}\Big]_0^{\tau} + \int_0^{\tau}\frac{1-\cos r}{r^2}\,dr$$

$$= \frac{1-\cos\tau}{\tau} + \int_0^{\tau}\frac{2\sin^2 r/2}{r^2}\,dr$$

$$= \frac{1-\cos\tau}{\tau} + \int_0^{\tau/2}\frac{2\sin^2 y}{4y^2}\,2dy$$

$$= \frac{1-\cos\tau}{\tau} + \int_0^{\tau/2}\left(\frac{\sin y}{y}\right)^2 dy;$$

thus

$$G = \frac{2}{\pi}\max\left\{\frac{1-\cos\tau}{\tau} + \int_0^{\tau/2}\frac{\sin y}{y}\,dy - \frac{1}{\tau}\int_0^{\tau}\sin r\,dr\right\}$$

$$= \frac{2}{\pi}\max_{\tau>0}\int_0^{\tau}\left(\frac{\sin y}{y}\right)^2 dy$$

$$= 1.$$

The same is true for $p>1$ from elementary properties of the Cesàro and Hölder means.

To discuss the case $0<p<1$, observe that in general $\max\int_0^1\{1-\psi(u)\}$

$(\sin \tau u/u)du$ is reached when

$$\int_0^1 \{1 - \psi(u)\} \cos \tau u \, du = 0.$$

Now, if $\psi(u)$ is differentiable,

$$\begin{aligned}\int_0^1 \{1 - \psi(u)\} \cos \tau u \, du &= \int_0^1 \{1 - \psi(u)\} d_u \frac{(\sin \tau u)}{\tau} \\ &= \frac{1 - \psi(u)}{\tau} \sin \tau u \Big]_0^1 + \frac{1}{\tau}\int_0^1 \sin \tau u \cdot \psi'(u) du \\ &= \frac{1}{\tau}\int_0^1 \psi'(u) \sin \tau u \, du = 0,\end{aligned}$$

if

$$\int_0^1 \psi'(u) \sin \tau u \, du = 0. \tag{5.1}$$

For Hölder means by (1.9)

$$1 - \psi(u) = \frac{1}{\Gamma(p)}\int_u^1 \left(\log \frac{1}{y}\right)^{p-1} dy,$$

thus (5.1) becomes

$$\int_0^1 \left(\log \frac{1}{u}\right)^{p-1} \sin \tau u \, du = 0.$$

Let

$$\frac{1}{\Gamma(p)}\int_0^1 \int_u^1 \left(\log \frac{1}{y}\right)^{p-1} dy \frac{\sin \tau u}{u} du = g_p(\tau) = g(\tau), \tag{5.2}$$

then there is no Gibbs' phenomenon if and only if

$$g(\tau) \leqq \pi/2 \qquad \text{for all } \tau > 0.$$

We have

$$\begin{aligned} g'(\tau) &= \int_0^1 \{1 - \psi(u)\} \cos \tau u \, du \\ &= \frac{1}{\tau}\int_0^1 \psi'(u) \sin \tau u \, du. \end{aligned} \tag{5.3}$$

Here

$$
(5.4)\qquad
\begin{aligned}
\psi(u) &= \frac{1}{\Gamma(p)}\int_0^u \left(\log\frac{1}{y}\right)^{p-1} dy,\\
\psi'(u) &= \frac{1}{\Gamma(p)}\left(\log\frac{1}{u}\right)^{p-1},\\
\psi''(u) &= \frac{1-p}{\Gamma(p)}\,\frac{1}{u}\left(\log\frac{1}{u}\right)^{p-2} > 0, \qquad 0 < p < 1.
\end{aligned}
$$

Pólya [4, pp. 374–375] proved that if $f(t)>0$ and $f'(t)>0$, then

$$V(z) = \int_0^1 f(t)\sin zt\,dt$$

has only real and simple zeros; there is no zero in $0<z\leqq\pi$, and each of the intervals $\nu\pi<z<(\nu+1)\pi$, $\nu=1, 2, 3, \cdots$, contains exactly one zero. Pólya also proved that

$$(5.5)\qquad \operatorname{sgn} V(\nu\pi) = (-1)^{\nu-1}, \qquad \nu = 1, 2, 3, \cdots,$$

and

$$(5.6)\qquad \frac{V(z)}{z\sin z} = \frac{V'(0)}{z} + \sum_{\nu=1}^{\infty}(-1)^{\nu}\frac{V(\nu\pi)}{\nu\pi}\left(\frac{1}{z-\nu\pi}+\frac{1}{z+\nu\pi}\right).$$

This result is applicable to $f(t)=\psi'(t)$ if $0<p<1$. Thus, all zeros of $g'(t)$ are real and simple, and each of the intervals $\nu\pi<\tau<(\nu+1)\pi$ contains exactly one zero. Furthermore

$$\frac{\tau g'(\tau)}{\tau\sin\tau} = \frac{1}{\tau}\int_0^1 u\psi'(u)du + \sum_{\nu=1}^{\infty}(-1)^{\nu}g'(\nu\pi)\left(\frac{1}{\tau-\nu\pi}+\frac{1}{\tau+\nu\pi}\right).$$

Now

$$\int_0^1 u\psi'(u)du = \frac{1}{\Gamma(p)}\int_0^1 u\left(\log\frac{1}{u}\right)^{p-1}du = \frac{1}{\Gamma(p)}\int_0^{\infty} x^{p-1}e^{-2x}dx = 2^{-p},$$

so that, using (5.5) and (5.6)

$$g'(t) = 2^{-p}\frac{\sin\tau}{\tau} - 2\sum_{\nu=1}^{\infty}\left|g'(\nu\pi)\right|\frac{\tau\sin\tau}{\tau^2-\nu^2\pi^2}.$$

It follows that

$$\int_0^t g'(\tau)dt = g(t) = 2^{-p}\int_0^t \frac{\sin\tau}{\tau}d\tau - 2\sum_{\nu=1}^{\infty}\left|g'(\nu\pi)\right|\int_0^t \frac{\tau\sin\tau}{\tau^2-\nu^2\pi^2}d\tau.$$

We write

$$\max_{\tau>0} g_p(t) = m(p);$$

it follows from the definition of Gibbs' phenomenon and from elementary properties of the Hölder means that $m(p)$ decreases to $\pi/2$ as p increases, thus there is a constant γ, $0<\gamma\leqq 1$, so that the Hölder means present a Gibbs' phenomenon for $p<\gamma$, but not for $p\geqq\gamma$. Clearly $m(p)=\pi/2$ for $p\geqq\gamma$.

Denote the positive zeros of $g'(\tau)$ by $\tau_1, \tau_2, \cdots$, then

$$\pi < \tau_1 < 2\pi < \tau_2 < 3\pi < \cdots .$$

$g(\tau)$ has a maximum at $\tau=\tau_1$, and a minimum at $\tau=\tau_2$. Decomposing the integration in (5.2) into subintervals in which $\sin \tau u$ is either positive or negative, one can show that for $p=0.6$ and $\tau>3\pi$

$$\left| g_p(\tau) \right| < \pi/2. \tag{5.7}$$

From (5.3) and (5.4)

$$\begin{aligned}\Gamma(p)\tau g'(\tau) &= \int_0^1 \left(\log\frac{1}{u}\right)^{p-1} \sum_1^\infty (-1)^{\nu-1} \frac{\tau^{2\nu-1}u^{2\nu-1}}{(2\nu-1)!}\,du \\ &= \frac{\Gamma(p)}{2^p}\sum_1^\infty (-1)^{\nu-1}\frac{\tau^{2\nu-1}}{(2\nu-1)!}\,\frac{1}{\nu^p},\end{aligned}$$

hence

$$g(\tau) = \frac{1}{2^p}\sum_1^\infty (-1)^{\nu-1}\frac{\tau^{2\nu-1}}{(2\nu-1)!(2\nu-1)}\,\frac{1}{\nu^p}\,. \tag{5.8}$$

Using this formula, further computation shows that (5.7) holds also for $p=0.6$ and $0<\tau<3\pi$. It follows that $\gamma<0.6$, and (5.7) holds for $p>0.6$ and $\tau>0$. We can also employ (5.8) to find a lower bound for γ. The computations were carried out by Fanny Gordon under the direction of Dr. Gertrude Blanch. It was found that for

$$p = 0.5800, \qquad \tau = 4.25318, \qquad 2^p\tau g'(\tau) = -0.0000329,$$

and

$$g(\tau) = 1.571899\cdots > \pi/2 = 1.570796\cdots .$$

Thus, $\gamma>0.5800$. Further calculation shows that $0.58<\gamma<0.585$, and for $p=0.5826\cdots$, we have $\tau_1=4.2598\cdots$, and $g(\tau_1)=1.5708\cdots$.

It is noticeable that γ is essentially larger than the corresponding constant for Cesàro means: $c=0.4395\cdots$.

For the discussion of $g(\tau)$ and $g'(\tau)$ we may also employ asymptotic estimates similar to those of Gronwall.

In closing we mention two notes on the Gibbs' phenomenon by B. Kuttner in J. London Math. Soc. vol. 20 (1945) pp. 136–139, and vol. 22 (1947) pp. 295–298. Here a different type of transforms is considered; we shall give more details in a forthcoming paper.

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