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Gibbsian Dynamics and Ergodicity for the Stochastically Forced Navier–Stokes Equation

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Dedicated to Joel L. Lebowitz, on the occasion of his 70th birthday

Abstract: We study stationary measures for the two-dimensional Navier–Stokes equation with periodic boundary condition and random forcing. We prove uniqueness of the stationary measure under the condition that all "determining modes" are forced. The main idea behind the proof is to study the Gibbsian dynamics of the low modes obtained by representing the high modes as functionals of the time-history of the low modes.

1. Introduction and Main Results

We are interested in determining conditions sufficient to insure that the stochastically-forced Navier–Stokes equation (SNS) possesses a unique stationary measure, or equivalently, that the dynamics is ergodic in the phase space. Our main result is that this holds if all the "determining modes" are forced. To prove this, we show that the dynamics of the Navier–Stokes equation can be reduced to the dynamics of the low modes, the so-called determining modes, with memory. This is the stochastic analog of results proved for the deterministic case by Foias et al. [FMRT]. We will work with the periodic boundary condition. But in principle our techniques should also apply for the more physical no-slip boundary condition.

Consider the two-dimensional Navier–Stokes equation with stochastic forcing:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = \dfrac{\partial W(x,t)}{\partial t} \\ \nabla \cdot u = 0 \end{cases} . \tag{1}$$

For simplicity of presentation we will take W to be of the form

$$W(x,t) = \sum_{|k| \le N} \sigma_k w_k(t,\omega) e_k(x) m \tag{2}$$

where the w_k's are standard i.i.d complex valued Wiener process satisfying $w_{-k}(t) = \overline{w}_k(t)$, and $\sigma_k \in \mathbb{C}$, with $|\sigma_k| > 0$ and $\sigma_{-k} = \overline{\sigma}_k$, are the amplitudes of the forcing, $\{e_k(x) = \binom{-ik_2}{ik_1}\frac{e^{ik\cdot x}}{|k|},\ k \in \mathbb{Z}\}$ are the basis in the space of L^2 divergence-free, mean zero vector fields on $\mathbb{T}^2$, the two dimensional torus. Our techniques apply to more general cases when the higher modes are also forced, as long as $|\sigma_k|$ decays sufficiently fast as $|k| \to \infty$ or to forcing which is not diagonal in Fourier space. But we will restrict ourselves to the form in (2) for clarity.

Define $B(u, v) = -P_{div}(u \cdot \nabla)v, \quad \Lambda^2 u = -P_{div}\Delta u$, where P_{div} is the L^2 projection operator onto the space of divergence-free vector fields. Let $\sigma_{\max}^2 = \max\{|\sigma_k|^2 : |k| \le N\}$. $\mathcal{E}_0 = \sum_{|k|\le N} |\sigma_k|^2$ and $\mathcal{E}_1 = \sum_{|k|\le N} |k|^2|\sigma_k|^2$. Writing $u(x) = \sum_k u_k e_k(x)$, we will define $\mathbb{H}^\alpha = \left\{u = (u_k)_{k\in\mathbb{Z}^2},\ u_0 = 0,\ \sum_k |k|^{2\alpha}|u_k|^2 < \infty\right\}$ and $\mathbb{L}^2 = \mathbb{H}^0$.

We will work on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, \theta_t)$. We associate Ω with the canonical space generated by all $d\omega_k(t)$. $\mathcal{F}$ and $\mathcal{F}_t$ are respectively the associated global σ-algebra and filtration generated by $W(t)$. Lastly, θ_t is the shift on Ω defined by $\theta_t d\omega_k(s) = d\omega_k(s + t)$. Notice that θ_t is an ergodic group of measure-preserving transformations with respect to $\mathbb{P}$. Expectations with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$.

Projecting (1) onto $\mathbb{L}^2$, we obtain the the following system of Itô stochastic equation

$$du(x, t) + \nu\Lambda^2 u(x, t)dt = B(u, u)dt + dW(x, t). \tag{3}$$

It can be shown that (3) generates a continuous Markovian stochastic semi-flow on $\mathbb{L}^2$ defined by

$$\varphi^\omega_{s,t} u_0 = u(t, \omega; s, u_0). \tag{4}$$

When $s = 0$, we simply write φ^ω_t (see [Fla94, DPZ96]).

We will take the state space of (3) to be $\mathbb{L}^2$ equipped with the Borel σ-algebra. A measure $\mu(du)$ on $\mathbb{L}^2$ is stationary for the stochastic flow (3) if for all bounded continuous functions F on $\mathbb{L}^2$ and $t > 0$,

$$\int_{\mathbb{L}^2} F(u)\mu(du) = \int_{\mathbb{L}^2} \mathbb{E}F\left(\varphi^\omega_t u\right)\mu(du). \tag{5}$$

Our main result is:

Theorem 1. *There exists some absolute constant $\mathcal{C}$ such that if $N^2 \ge \mathcal{C}\frac{\mathcal{E}_0}{\nu^3}$ then* (3) *has a unique stationary measure on $\mathbb{L}^2$.*

The existence of at least one stationary measure was proved in [Fla94] and [VF88]. The proof proceeds by establishing compactness for a family of empirical measures. The limiting points of these empirical measures are the stationary measures. Uniqueness has been proved under restrictive assumptions when ALL modes are forced. Flandoli and Maslowski [FM95] proved that if the σ_k's decay algebraically, i.e. if the forcing is sufficiently rough spatially, then the system has a unique stationary measure. These results were extended and refined in [Fer97]. In [Mat99], it was proven that if the viscosity was large enough the contraction induced by the Laplacian dominates and the system possesses a trivial random attractor; and hence, a unique stationary measure. We do not address convergence to the stationary measure. This and the coupling construction used to prove convergence are discussed in [Mat00]. Recently Kuksin and Shirikyan [KS] proved uniqueness of stationary measure when the Navier–Stokes equation is perturbed by a bounded degenerate kicked noise. Results similar to ours have also been obtained independently by Bricmont et al. [BKL].

Our main strategy is to reduce the dynamics of the Navier–Stokes equation to the dynamics of a finite dimensional set of low modes with memory. The reduced dynamics is no longer Markovian, but rather Gibbsian (see §2, §4). The finite dimensional Gibbsian dynamics has a non-degenerate noise, and have a unique stationary measure if the memory is short ranged.

Before proceeding further, let us observe that any given stationary measure μ can be extended to a measure on the path space, denoted by μ_p, where p stands for path or past. Consider the example of the path space $C\left((-\infty, 0], \mathbb{L}^2\right)$. Let A be a cylinder set of the type: For some $t_0, t_1, \cdots t_n, t_0 < t_1 < t_2 \cdots t_n \leq 0$,

$$A = \left\{u(s) \in C\left((-\infty, 0], \mathbb{L}^2\right), u(t_i) \in A_i, i = 0, \cdots n\right\}, \tag{6}$$

where the A_i's are Borel sets of $\mathbb{L}^2$. Corresponding to A, let $B \subset \Omega \times \mathbb{L}^2$,

$$B = \{(u, \omega), u \in A_0, \varphi^{\omega}_{t_0,t_i} u \in A_i, i = 1, \cdots n\}. \tag{7}$$

We will define

$$\mu_p(A) = (\mathbb{P} \times \mu)(B), \tag{8}$$

where $(\mathbb{P} \times \mu)$ is the product measure on $\Omega \times \mathbb{L}^2$. Clearly μ_p is consistent on cylinder sets and can be extended to the natural σ-algebra using the Kolmogorov extension theorem. The natural σ-algebra is the one generated by the cylinder sets. The dynamics of the stochastic semi-flow $\{\varphi^{\omega}_t\}$ can be trivially extended to return a function from $C\left((-\infty, t], \mathbb{L}^2\right)$, given an initial function from $C\left((-\infty, 0], \mathbb{L}^2\right)$. One simply flows forward with φ from the initial condition at time 0. To avoid confusion, we will call this map ψ^{ω}_t. Symbolically, if $u(\cdot) \in C\left((-\infty, 0], \mathbb{L}^2\right)$, then $(\psi^{\omega}_t u)(s) = \varphi^{\omega}_s u(0)$ for $s \in [0, t]$ and $(\psi^{\omega}_t u)(s) = u(s)$ for $s \leq 0$.

If we define the shift on trajectories by $(\theta_t v)(s) = v(s + t)$, we can define a dynamics on $C\left((-\infty, 0], \mathbb{L}^2\right)$ by $\theta_t \psi^{\omega}_t$. In other words, $\theta_t \psi^{\omega}_t u$ takes a trajectory u from $C\left((-\infty, 0], \mathbb{L}^2\right)$, extends it t units of time by flowing forward and then shifts the entire resulting trajectory back t units of time so it again lives on $C\left((-\infty, 0], \mathbb{L}^2\right)$.

It is easy to check directly that if μ is invariant then μ_p is invariant in the sense that

$$\int_{C((-\infty,0],\mathbb{L}^2)} F(u) d\mu_p(u) = \mathbb{E} \int_{C((-\infty,0],\mathbb{L}^2)} F(\theta_t \psi^{\omega}_t u) d\mu_p(u) \tag{9}$$

for all bounded functions on $C\left((-\infty, 0], \mathbb{L}^2\right)$, and $t \geq 0$.

Assume that μ and ν are two stationary measures for the stochastic flow (3), and μ_p and ν_p are respectively their induced measure on the path space $C\left((-\infty, 0], \mathbb{L}^2\right)$. It is obvious that $\mu_p = \nu_p$ implies $\mu = \nu$.

2. Reduction to the Gibbsian Dynamics

Define two subspaces of $\mathbb{L}^2$:

$$\mathbb{L}^2_{\ell} = \mathrm{span}\{e_k, |k| \leq N\}, \quad \mathbb{L}^2_h = \mathrm{span}\{e_k, |k| > N\}. \tag{10}$$

We will call $\mathbb{L}^2_{\ell}$ the set of low modes and $\mathbb{L}^2_h$ the set of high modes. Obviously $\mathbb{L}^2 = \mathbb{L}^2_{\ell} \oplus \mathbb{L}^2_h$. Denote by P_{ℓ} and P_h the projections onto the low and high mode spaces.

Since we are concerned with stationary measures of (3), we are interested in (statistically) stationary solutions of (3) that exist for time from $-\infty$ to $+\infty$. We will show in this section that for such solutions, the high modes are completely determined by the past history of the low modes. For this purpose, we write $u(t) = \big(\ell(t), h(t)\big)$ and

$$\begin{aligned} d\ell(t) = &\Big[-\nu\Lambda^2\ell + P_\ell B(\ell,\ell)\Big]dt \\ &+ \Big[P_\ell B(\ell,h) + P_\ell B(h,\ell) + P_\ell B(h,h)\Big]dt + dW(t), \end{aligned} \tag{11}$$

$$\frac{dh(t)}{dt} = \Big[-\nu\Lambda^2 h + P_h B(h,h)\Big] + P_h B(\ell,h) + P_h B(h,\ell) + P_h B(\ell,\ell). \tag{12}$$

Define the set of "nice pasts" $U \subset C\left((-\infty,0],\mathbb{L}^2\right)$ to consist of all $v : (-\infty,0] \to \mathbb{L}^2$ such that:

i) $v(t)$ is in $\mathbb{H}^2$ for all $t \le 0$.
ii) The energy averages correctly. More precisely,

$$\lim_{t\to-\infty} \frac{1}{|t|}\int_t^0 |\Lambda v(s)|^2_{\mathbb{L}^2} ds = \frac{\mathcal{E}_0}{2\nu}.$$

iii) The energy fluctuations are typical. More precisely, there exists a $T = T(v)$ such that

$$|v(t)|^2_{\mathbb{L}^2} \le \mathcal{E}_0 + \max(|t|,T)^{\frac{2}{3}}$$

for $t \le 0$. The following lemma shows that U contains almost all of the trajectories defined on the whole time interval.

Lemma 2.1. *Let μ_p be the measure on $\mathbb{C}\left((-\infty,0],\mathbb{L}^2\right)$ induced by a stationary measure μ for* (3). *Then $\mu_p\big(U\big) = 1$.*

Proof of Lemma 2.1. It is proved in [Mat98] or [Fer97] that with probability one, a solution to (3) is in $\mathbb{H}^2$ for all t.

The fact that the last condition is satisfied by a set of full measure is proved in Lemma B.3. All that remains to show is ii).

From Lemma B.2 $|\Lambda v|^2_{\mathbb{L}^2}$ is in $L^1(\mu)$ for any stationary measure μ and $\int |\Lambda v|^2_{\mathbb{L}^2}\, d\mu = \frac{\mathcal{E}_0}{2\nu}$. Since the measure is invariant under shifts back in time and each ergodic component has the same average enstrophy, the ergodic theorem implies that for μ_p–almost every trajectory time average converges to the average of $|\Lambda u|^2_{\mathbb{L}^2}$ against μ. $\square$

Given an arbitrary continuous function of time $\ell(t)$ on $\mathbb{L}^2_\ell$, we can view (12) as a closed equation with some exogenous forcing $\ell(t)$. By $\Phi_{s,t}(\ell, h_0)$, we mean the solution to (12) at time t given the initial condition h_0 at time s and the "forcing" ℓ.

Denote by $\mathcal{P}$ the set of all $\ell \in C\left((-\infty,0],\mathbb{L}^2_\ell\right)$ such that the following two conditions hold. First, $\ell = P_\ell u$ for some $u = (\ell,h) \in U$. Second, $h(t) = \Phi_{s,t}(\ell, h(s))$ for any $s < t \le 0$, where h was the matching high mode so $(\ell,h) \in U$. That is to say $h(t)$ solves (12) with low modes $\ell(t)$ and the total solution (ℓ,h) is in our space of "nice pasts". In light of Lemma 2.1 the set $\mathcal{P}$ is not empty. We now will show that this h is uniquely determined by ℓ.

Lemma 2.2. *There exists an absolute positive constant $\mathcal{C}$ such that if $N^2 > \mathcal{C}\frac{\mathcal{E}_0}{\nu^3}$ then the following holds:*

If there exists two solutions $u_1(t) = \big(\ell(t), h_1(t)\big)$, $u_2(t) = \big(\ell(t), h_2(t)\big)$ corresponding to some (possibly different) realizations of the forcing and such that $u_1, u_2 \in U$, then $u_1 = u_2$, i.e. $h_1 = h_2$.

Furthermore given a solution $u(t) = \big(\ell(t), h(t)\big) \in U$, any $h_0 \in \mathbb{L}^2_h$, and $t \leq 0$, the following limit exists:

$$\lim_{t_0 \to -\infty} \Phi_{t_0,t}(\ell, h_0) = h^*$$

and $h^* = h(t)$.

Proof of Lemma 2.2. We begin with the first clam. Denote by $\rho(t) = h_1(t) - h_2(t)$. From (12) we have

$$\begin{aligned}\frac{d\rho}{dt} &= -\nu\Lambda^2\rho + P_h B(h_1, h_1) - P_h B(h_2, h_2) + P_h B(\ell, \rho) + P_h B(\rho, \ell) \\ &= -\nu\Lambda^2\rho + P_h B(\ell + h_1, \rho) + P_h B(\rho, \ell + h_2) \\ &= -\nu\Lambda^2\rho + P_h B(u_1, \rho) + P_h B(\rho, u_2).\end{aligned} \tag{13}$$

Taking the inner product with ρ, using the fact that $\langle P_h B(u_1, \rho), \rho\rangle_{\mathbb{L}^2} = 0$, gives

$$\frac{1}{2}\frac{d}{dt}|\rho|^2_{L^2} = -\nu|\Lambda\rho|^2_{L^2} + \langle P_h B(\rho, u_2), \rho\rangle_{\mathbb{L}^2}.$$

Since

$$\begin{aligned}|\langle P_h B(\rho, u_2), \rho\rangle_{\mathbb{L}^2}| &\leq \hat{C}\, |\Lambda\rho|_{\mathbb{L}^2}\, |\rho|_{\mathbb{L}^2}\, |\Lambda u_2|_{\mathbb{L}^2} \\ &\leq \frac{\nu}{2}\, |\Lambda\rho|^2_{\mathbb{L}^2} + \frac{\hat{C}^2}{2\nu}\, |\rho|^2_{\mathbb{L}^2}\, |\Lambda u_2|^2_{\mathbb{L}^2}\,,\end{aligned}$$

we get

$$\frac{1}{2}\frac{d}{dt}\, |\rho|^2_{\mathbb{L}^2} \leq -\frac{\nu}{2}\, |\Lambda\rho|^2_{\mathbb{L}^2} + \frac{\hat{C}^2}{2\nu}\, |\Lambda u_2|^2_{\mathbb{L}^2}\, |\rho|^2_{\mathbb{L}^2}\,.$$

Since ρ only contains modes with $|k| > N$, the Poincaré inequality implies

$$\frac{d}{dt}\, |\rho|^2_{\mathbb{L}^2} \leq \left(-\nu N^2 + \frac{\hat{C}^2}{\nu}\, |\Lambda u_2|^2_{\mathbb{L}^2}\right) |\rho|^2_{\mathbb{L}^2}\,.$$

Therefore we have, for $t_0 < t < 0$,

$$|\rho(t)|^2_{\mathbb{L}^2} \leq |\rho(t_0)|^2_{\mathbb{L}^2} \exp\left\{-\nu N^2(t - t_0) + \frac{\hat{C}^2}{\nu}\int_{t_0}^{t} |\Lambda u_2(s)|^2_{\mathbb{L}^2}\, ds\right\}. \tag{14}$$

From the third assumption on functions in U, we know that $\lim \frac{1}{t}\int_{-t}^{0} |\Lambda u_2(s)|^2_{\mathbb{L}^2}\, ds = \frac{\mathcal{E}_0}{2\nu}$. Hence for $t_0 < T_1$, where T_1 depends on t and u_2, we have

$$-\nu N^2(t - t_0) + \frac{\hat{C}^2}{\nu}\int_{t_0}^{t} |\Lambda u_2(s)|^2_{\mathbb{L}^2}\, ds \leq -\frac{\gamma}{2}(t - t_0),$$

where $\gamma = \nu N^2 - \frac{\hat{C}^2 \varepsilon_0}{2\nu^2}$. If we set $\mathcal{C} = \frac{\hat{C}^2}{2}$, then our assumption on N implies $\gamma > 0$.

Now using the last property of paths in U we have for any $t_0 \le T_2$,

$$\begin{aligned}|\rho(t)|^2_{\mathbb{L}^2} \le& |\rho(t_0)|^2_{\mathbb{L}^2} \exp\{-\frac{\gamma}{2}(t-t_0)\} \\ \le& 2[\mathcal{E}_0 + |t_0|^{\frac{2}{3}}] \exp\{-\frac{\gamma}{2}(t-t_0)\} \to 0\end{aligned}$$

as $t_0 \to -\infty$, where T_2 is some finite constant depending on u_1 and u_2. This completes the proof of the first part of Lemma 2.2.

To see the second part, observe that (14) only required control of $\int_{t_0}^t |\Lambda u(s)|^2_{\mathbb{L}^2}\, ds$ for one of the two solutions. If we proceed as before letting the given solution $u(t)$ play the role of u_2 and the solution to (12) starting from h_0 play the role of u_1, the we obtain the estimate

$$|\rho(t)|^2_{\mathbb{L}^2} \le |h(t_0) - h_0|^2_{\mathbb{L}^2} \exp\left\{-\nu N^2(t-t_0) + \frac{\hat{C}^2}{\nu}\int_{t_0}^t |\Lambda u(s)|^2_{\mathbb{L}^2}\, ds\right\}. \qquad (15)$$

Since $u(t) = (\ell(t), h(t)) \in U$, the same reasoning as before shows that $\rho(t)$ goes to zero as $t_0 \to -\infty$. Hence the limit exists and equals $h(t)$. $\square$

In fact the splitting into high and low modes can be accomplished even when all of the modes are forced. One replaces (12) with an Itô stochastic differential equation. This causes little complication as (13) remains a standard PDE. See [Mat98].The ideas in this section are related to the ideas of Lyapunov-Schmidt reduction and those around center and inertial manifolds. See [EFNT94] for a discussion and other references.

From now on we assume that N satisfies

$$N^2 > \mathcal{C}\frac{\mathcal{E}_0}{\nu^3}, \qquad (16)$$

where $\mathcal{C}$ is the constant from Lemma 2.2.

Because of Lemma (2.2), we can define a map Φ_0 which reconstructs the high modes at time zero from a given low mode trajectory stretching from zero back to $-\infty$. Before making this more precise, let us fix some notation. In general, we will use $\ell(t)$ to refer to the value of the low modes at time t and will use L^t to mean the entire trajectory from $-\infty$ to t. Hence $\ell(t) \in \mathbb{L}^2$ and $L^t \in C\big((-\infty, t], \mathbb{L}^2\big)$ and $\ell(s) = L^t(s)$ for $s \le t$. In this notation $h(0) = \Phi_0\big(L^0\big)$, where L^0 is some "low mode past" in $\mathcal{P}$ which is the projection of U to the low modes. By $\Phi_s(L^t, h(0))$ with $s \le t$, we mean the solution to (12) at time s with initial condition $h(0)$ and low mode forcing L^t. Of course $\Phi_s(L^t, h(0))$ only depends on the information in L^t between 0 and s. We can extend the definition of Φ beyond time zero by defining $\Phi_t(L^t) = \Phi_t(L^t, h(0))$, where $h(0) = \Phi_0(L^0)$.

Given the initial low mode past of $L^0 \in \mathcal{P}$, we can solve for the future of ℓ using

$$d\ell(t) = \Big[-\nu\Lambda^2\ell(t) + P_\ell B\big(\ell(t), \ell(t)\big) + G\big(\ell(t), \Phi_t(L^t)\big)\Big]dt + dW(t), \qquad (17)$$

where

$$G\,(\ell, h) = P_\ell B(\ell, h) + P_\ell B(h, \ell) + P_\ell B(h, h). \qquad (18)$$

Thus we have a closed formulation of the dynamics on the low modes given an initial past in $L^0 \in \mathcal{P}$. We write $L^t = \mathsf{S}^\omega_t L^0$. We reiterate that L^t is the entire trajectory from time t back to $-\infty$, whereas $\ell(t)$ is simply the value of the low modes at time t.

Except for the fact the G-term in (17) is history-dependent, (17) has the form of a standard finite dimensional stochastic ODE with non-degenerate forcing, which of course has a unique stationary measure. Our task is reduced to showing that the memory effort in (17) is not strong enough to spoil ergodicity.

Existence of the solution for memory-dependent stochastic ODEs of the type (17) was considered in the work of Ito et al. [IN].

3. Uniqueness of the Invariant Measure

3.1. Proof of the Main Theorem. Given any "nice low mode past" $L \in \mathcal{P}$, we can reconstruct the "high modes" and hence define a closed dynamics on the paths of the low modes. However, this dynamics is no longer Markovian which will produce difficulties.

Let μ be an ergodic stationary measure on $\mathbb{L}^2$ and μ_p be its extension to the path space $C\big((-\infty, 0], \mathbb{L}^2\big)$. We will also consider the restriction of μ_p to $C\big((-\infty, 0], \mathbb{L}^2_\ell\big)$, still denoted by μ_p. Lemma 2.1 says that $\mu_p(\mathcal{P}) = 1$.

Given any $L^0 \in \mathcal{P}$, let $Q_t(L^0, \cdot)$ be the measure induced on $C\big([0, t], \mathbb{L}^2_\ell\big)$ by the dynamics starting from L^0. In other words, $Q_t(L^0, \cdot)$ is the distribution of $\mathsf{S}^\omega_t L^0$ viewed as a random variable taking values in $C\big([0, t], \mathbb{L}^2_\ell\big)$. Similarly let $Q_\infty(L^0, \cdot)$ be the distribution induced on $C\big([0, \infty), \mathbb{L}^2_\ell\big)$ starting from L^0.

Consider the stochastic process defined by $\theta_t \mathsf{S}^\omega_t L^0$, where L^0 is a random variable on $\mathcal{P}$ distributed according to the invariant measure μ_p. For $t \geq 0$ it is a random process with values in $\mathcal{P}$. This is clear as all of the defining properties of U are asymptotic in t; and hence the addition of a segment of finite length does not destroy them. Since μ_p is invariant with respect to the dynamics, $\theta_t \mathsf{S}^\omega_t L^0$ is a stationary random process. Hence with probability one there exist time averages along trajectories $\theta_t \mathsf{S}^\omega_t L^0$.

Take any bounded measurable functional F from $C\big((-\infty, 0], \mathbb{L}^2_\ell\big) \to \mathbb{R}$ such that $F(L^0)$, $L^0 \in C\big((-\infty, 0], \mathbb{L}^2_\ell\big)$ depends only on a finite range of L^0. Let

$$\bar{F} = \int F(L) d\mu_p(L). \tag{19}$$

Theorem 2. *The SNS equation* (1) *has a unique stationary measure.*

The proof of Theorem 2 is based on the following two lemmas whose proofs will be given later.

Lemma 3.1. *Let L^0_1 and L^0_2 be two initial pasts in $\mathcal{P}$, such that $\ell_1(0) = \ell_2(0)$. Then $Q_\infty(L^0_1, \cdot)$ and $Q_\infty(L^0_2, \cdot)$ are equivalent.*

Recall that $\ell(\tau)$ is the solution of (16) with initial condition L.

Lemma 3.2. *For any past $L \in \mathcal{P}$ and any $t > 0$, the distribution of $\ell(t) \in \mathbb{L}^2_\ell$ conditioned at starting from L at time zero, denoted by $R_t(L, \cdot)$, satisfies the following: there exists a strictly positive function $f_{L,t} \in L^1(\mathbb{L}^2_\ell)$, such that*

$$dR_t(L, \cdot) \geq f_{L,t}(\cdot) dm(\cdot).$$

where $m(\cdot)$ is the Lebesgue measure on $\mathbb{L}^2_\ell$.

For any measure μ on $\mathbb{L}^2$ let $P_\ell\mu$ denote its projection to a measure on the low modes $\mathbb{L}^2_\ell$. Namely, $(P_\ell\mu)(B) = \mu(P_\ell^{-1}(B))$. Then we have the following direct consequence of Lemma 3.2.

Corollary 3.3. *If μ is a stationary measure then $P_\ell\mu$ has a component which is equivalent to the Lebesgue measure.*

Proof of Theorem 2. Assume that there are two different ergodic stationary measures on $\mathbb{L}^2$ called μ_1 and μ_2. They must be mutually singular. Let $\mu_{1,p}$ and $\mu_{p,2}$ be the extensions of these two measures onto the path space $\mathcal{P}$. Let L_i^0 be a random variable on $\mathcal{P}$ distributed as $\mu_{i,p}$. Since $\theta_t \mathsf{S}_t^\omega L_i^0$ is stationary with respect to $\mu_{p,i}$ we can pick a set $\mathcal{P}_i$, of full $\mu_{p,i}$-measure, such that for all $L \in \mathcal{P}_i$ One can find a functional F such as above so that $\bar{F}_1 = \int F(L)d\mu_{p,1}(L) \neq \bar{F}_2 = \int F(L)d\mu_{p,2}(L)$. This assumption will lead to a contradiction. The limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(\theta_t \mathsf{S}_t^\omega L_i^o)dt = \bar{F}_i \tag{20}$$

is well defined for $\mathbb{P}$-almost every ω.

For $\ell \in \mathbb{L}^2_\ell$ define $\mathcal{P}_i(\ell) = \{L \in \mathcal{P}_i \ : \ L(0) = \ell\}$ and let $\mu_{p,i}(\,\cdot\,|\ell)$ be the conditional measure that $L(0) = \ell$. By Fubini's theorem, we know that for $P_\ell\mu_i$-almost every $\ell \in \mathbb{L}^2_\ell$ we have $\mu_{p,i}(\mathcal{P}_i(\ell)\,|\,\ell) = 1$. Hence we can find a set $A_i \subset \mathbb{L}^2_\ell$ such that $\mu_{p,i}(\mathcal{P}_i(\ell)\,|\,\ell) = 1$ for all $\ell \in A_i$ and $P_\ell\mu_i(A_i) = 1$. Define $A = A_1 \cap A_2$. Corollary 3.3 implies that $P_\ell\mu_i(A) > 0$ for $i = 1, 2$. Hence there exists some $\ell^* \in A$.

Since $\ell^* \in A_1 \cap A_2$, we know that $\mu_{p,i}(\mathcal{P}_i(\ell^*)\,|\,\ell^*) = 1$ for $i = 1, 2$. Thus there exist some $L_{*,1} \in \mathcal{P}_1(\ell^*)$ and $L_{*,2} \in \mathcal{P}_2(\ell^*)$. Notice that by construction $L_{*,1}(0) = \ell^* = L_{*,2}(0)$, and hence it follows from Lemma 3.1 that $Q_\infty(L_{*,1},\cdot)$ and $Q_\infty(L_{*,2},\cdot)$ are equivalent. Since $L_{*,i} \in \mathcal{P}_i(\ell^*)$, we know that we can pick $B_i \subset C\big([0,\infty),\mathbb{L}^2\big)$ such that the time average of F converges to $\bar{F}_i$ for all futures in B_i and $Q_\infty(L_{*,i}, B_i) = 1$ for $i = 1, 2$. Since the Q's are equivalent, $Q_\infty(L_{*,1}, B_1 \cap B_2) > 0$ and hence $B_1 \cap B_2$ is non-empty. This in turn implies that $\bar{F}_1 = \bar{F}_2$ which contradicts the assumption that they were not equal. $\square$

3.2. Proofs of the lemmas. We first prove Lemma 3.1. Fix L_1^0 and L_2^0. Most of our construction will depend explicitly on them. With probability one, we can extend each of the initial pasts into the infinite future by $L_i^s = \mathsf{S}_s^\omega L_i^0$ and setting $\ell_i(s) = L_i^t(s)$ for $s \leq t$. We can also reconstruct the entire solution by using Φ_t to obtain the high modes. Set $h_i(s) = \Phi_s(L_i^s)$ and $u_i(s) = \big(\ell_i(s), h_i(s)\big)$. Fix a constant C_0 such that $|u_i(0)|^2_{\mathbb{L}^2} \leq C_0$.

We begin by constructing a set of nice future paths which will contain most trajectories. For any positive K we define

$$A_i(K) = \Big\{ f \in C\big([0,\infty),\mathbb{L}^2_\ell\big) : |v(t)|^2_{\mathbb{L}^2} + 2\nu \int_0^t |\Lambda v(s)|^2_{\mathbb{L}^2}\,ds < C_0 + \mathcal{E}_0 t + K t^{\frac{4}{5}}$$
$$\text{where } v(s) = f(s) + \Phi_s(f, h_i) \Big\}$$

and $A(K) = A_1(K) \cap A_2(K)$.

By Lemma A.5, we know that for any $a \in (0, 1)$ there exists a K such that

$$\mathbb{P}\Big\{\omega : \mathsf{S}_t^\omega L_i^0 \in A_i(K)\Big\} > 1 - \frac{a}{2} \quad \text{for } i = 1, 2,$$

and hence

$$\mathbb{P}\Big\{\omega : \mathsf{S}_t^\omega L_i^0 \in A(K) \quad \text{for } i = 1, 2\Big\} > 1 - a > 0.$$

This is just another way of saying $Q_\infty(L_i^0, A(K)) > 1 - a$.

Lemma 3.4. *Let L_1^0 and L_2^0 be two initial pasts in $\mathcal{P}$ such that $L_1^0(0) = L_2^0(0)$. Let $A(K) \subset C\big([0,\infty), \mathbb{L}_\ell^2\big)$ be as defined above. For any choice of $K > 0$, $Q_\infty(L_1^0, \cdot \cap A(K))$ is equivalent to $Q_\infty(L_2^0, \cdot \cap A(K))$.*

Proof of Lemma 3.1. Since we can choose K so that $A(K)$ has measure arbitrarily close to 1, we have that $Q_\infty(L_1^0, \cdot)$ is equivalent to $Q_\infty(L_2^0, \cdot)$. □

Proof of Lemma 3.4. We intend to use Girsanov's theorem to compare the two induced measures, $Q_\infty(L_1^0, \cdot)$ and $Q_\infty(L_2^0, \cdot)$. However we do not do so directly. To aid in our analysis, we consider the following surrogate processes y which will agree with ℓ on the set $A = A(K)$. As before, we will use $y(t)$ to denote the value of the process at time t and Y^t to be the entire trajectory up to time t.

$$\begin{aligned} dy_i(t) &= \Big[-\nu\Lambda^2 y_i(t) + P_\ell B\big(y_i(t), y_i(t)\big) \\ &\qquad + \Theta_t(Y_i^t) G\big(y_i(t), \Phi_t(Y_i^t, h_i(0))\big)\Big] dt + dW(t) \qquad (21) \\ y_i(0) &= \ell_i(0), \end{aligned}$$

where

$$h_i(0) = \Phi_t(L_i^0),$$

$$\Theta_t(f) = \begin{cases} 1 & \text{if } f \in A|_{[0,t]} \\ 0 & \text{if } f \notin A|_{[0,t]} \end{cases},$$

and $A|_{[0,T]}$ is the low mode paths which agree with a path in A up to time T. Recall that $\Phi_t\big(Y_i^t, h_i(0)\big)$ is the solution to (12) with $\ell = Y$ and $h(0) = h_i(0)$.

Equation (21) is the same as (17) except for the insertion of $\Theta_t(Y_i^t)$. As long as $\Theta_s(Y_i^t) = 1$ for $s \in [0, t]$, then $y_i(s) = \ell_i(s)$ for $s \in [0, t]$.

Let $Q_\infty^y(L_1^0, \cdot)$ and $Q_\infty^y(L_2^0, \cdot)$ be the measures induced by Y_1 and Y_2 respectively. If applicable, Girsanov's theorem would imply that these measure are equivalent, that is $Q_\infty^y(L_1^0, \cdot) \sim Q_\infty^y(L_2^0, \cdot)$. For Girsanov's theorem to apply, it is sufficient that the Novikov condition holds. Namely,

$$\mathbb{E}\exp\left\{\frac{1}{2}\int_0^\infty \left|\Sigma^{-1}\Theta_t(Y_1^t) D\Big(y_1(t), \Phi_t\big(Y_1^t, h_1(0)\big), \Phi_t\big(Y_1^t, h_2(0)\big)\Big)\right|^2 dt\right\} < \infty, \qquad (22)$$

where $D(g, f_1, f_2) \stackrel{\text{def}}{=} G(g, f_1) - G(g, f_2)$ and Σ is a diagonal matrix with the σ_k's on its diagonal. Here we have written the condition in terms of the y_1 process. One can also write the condition in terms of the y_2 process; the finiteness of one implies the finiteness of the other.

We will in fact show something much stronger than (22). Since $|\Sigma^{-1}| < \infty$, it would be enough to show that

$$\sup_{\omega} \int_0^{\infty} \left|\Theta_t(Y_1^t) D\left(y_1(t), \Phi_t\big(Y_1^t, h_1(0)\big), \Phi_t\big(Y_1^t, h_2(0)\big)\right)\right|^2 dt < \infty. \tag{23}$$

Putting $h_i(s) = \Phi_s(Y_1^s, h_i(0))$, $u_i(s) = \ell_i(s) + h_i(s)$, $\rho(s) = h_1(s) - h_2(s)$ and using Lemma A.4, we have

$$\left|D\big(\ell_1(s), h_1(s), h_2(s)\big)\right|_{\mathbb{L}^2}^2 \le C' \, |\rho(s)|_{\mathbb{L}^2}^2 \left[|u_1(s)|_{\mathbb{L}^2}^2 + |u_2(s)|_{\mathbb{L}^2}^2\right]. \tag{24}$$

Notice that if $\ell_i \in A|_{[0,T]}$ then for all $t \in [0, T]$,

$$\begin{aligned} |u_i(t)|_{\mathbb{L}^2}^2 &< C_0 + \mathcal{E}_0 t + K t^{\frac{4}{5}}, \\ \int_0^t |\Lambda u_i(s)|_{\mathbb{L}^2}^2 \, ds &< \frac{1}{2\nu}\big(C_0 + \mathcal{E}_0 t + K t^{\frac{4}{5}}\big), \\ |\rho(0)|_{\mathbb{L}^2}^2 &= |u_1(0) - u_2(0)|_{\mathbb{L}^2}^2 \le 2\big(|u_1(0)|_{\mathbb{L}^2}^2 + |u_2(0)|_{\mathbb{L}^2}^2\big) \le 4C_0. \end{aligned}$$

In addition, we can apply the same analysis as in Sect. 2. Starting from (14) and using the above estimates produces

$$\begin{aligned} |\rho(t)|_{\mathbb{L}^2}^2 &\le |\rho(0)|_{\mathbb{L}^2}^2 \exp\left\{-\nu N^2 t + \frac{\hat{C}^2}{\nu} \int_0^t |\Lambda u_2(s)|_{\mathbb{L}^2}^2 \, ds\right\} \\ &\le 4C_0 \exp\left\{-\nu N^2 t + \frac{\hat{C}^2}{2\nu^2}\big(C_0 + \mathcal{E}_0 t + K t^{\frac{4}{5}}\big)\right\}. \end{aligned}$$

Since by assumption $\nu N^2 > \mathcal{C}\frac{\mathcal{E}_0}{\nu^2} = \frac{\hat{C}^2 \mathcal{E}_0}{2\nu^2}$, the second term goes to zero sufficiently fast and hence the estimate on the right-hand side of (24) decays exponentially fast. Thus,

$$\begin{aligned} \sup_{\omega} \int_0^{\infty} &\left|\Theta_t(Y_1) D\big(y_1(t), \Phi_t(Y_1^t, h_1(0)), \Phi_t(Y_1^t, h_2(0))\big)\right|^2 dt \\ &\le \sup_{f \in A} \int_0^{\infty} |D\left(f(r), \Phi_t(f, h_1(0)), \Phi_t(f, h_2(0))\right)|^2 \, dt \\ &< \mathrm{const}(C_0) < \infty, \end{aligned}$$

which implies, $Q_\infty^y(L_1^0, \cdot) \sim Q_\infty^y(L_2^0, \cdot)$. As long as Y_i stays in A, $y_i = \ell_i$. Hence $Q_\infty^y(L_i^0, \cdot \cap A) = Q_\infty(L_i^0, \cdot \cap A)$ and finally $Q_\infty(L_1^0, \cdot \cap A) \sim Q_\infty(L_2^0, \cdot \cap A)$. $\square$

In fact our proof provided more information than stated in Lemma 3.4. It contains some estimates uniform over a class of initial pasts which will be useful in later investigations of the convergence rate. (See [Mat00].) We state the extra information in the following corollary.

Corollary 3.5. *In the setting of the proof of Lemma 3.4, define* $\mathcal{P}' = \{L \in \mathcal{P} : |L(0) + \Phi_0(L)|_{\mathbb{L}^2} < C_0\}$*. Then there exists a constant, depending on* C_0 *and* K*, so that*

$$\sup_{L_1, L_2 \in \mathcal{P}'} \int \left| 1 - \frac{dQ^y_\infty(L_1, g)}{dQ^y_\infty(L_2, g)} \right|^2 dQ^y_\infty(L_2, g) < \text{const}(C_0, K_1) < \infty.$$

We now move to the proof of Lemma 3.2. Fix $L \in \mathcal{P}$. The proof proceeds by comparing the process $\ell(t)$ to the associated Galerkin approximation living on $\mathbb{L}^2_\ell$ which we will denote by $x(t)$. The advantage is that $x(t)$ is a standard non-degenerate diffusion and hence it is Markovian and well understood.

Take $x(t)$ as the solution defined by the following stochastic ODEs:

$$\begin{aligned} dx(t) &= \left[-\nu\Lambda^2 x + P_\ell B(x, x)\right]dt + dW(t), \\ x(0) &= \ell(0). \end{aligned}$$

As in the previous section, we do not compare $x(t)$ directly to $\ell(t)$ but instead to a modified version of $\ell(t)$ which we will denote by $z(t)$. In analogy to before, we will denote the path of this process up to time t by Z^t. Before continuing let us assume without loss of generality that $|\ell(0)|_{\mathbb{L}^2} \le C_0$ and $t \le T$ for some positive C_0 and T. This will give our estimates some uniformity over all initial conditions inside this ball and for times $t \le T$.

The evolution of $z(t)$ is given by

$$\begin{aligned} dz(t) &= \Big[-\nu\Lambda^2 z + P_\ell B(z, z) + \Theta_t(Z^t) G\big(z, \Phi_t\left(Z^t, h_0\right)\big)\Big]dt + dW, \\ z(0) &= \ell(0)\big(= L(0)\big), \end{aligned}$$

where $h_0 = \Phi_0(L)$ and G is defined in (18). As in the last section, $\Theta_t(Z^t)$ is a cut-off function. For any fixed $b_0 > 1$, we define

$$\Theta_s(Z^s) = \begin{cases} 1 & \text{if } \int_0^s |Z^s(r)|^4_{\mathbb{L}^2}\, dr < (b_0 C_0)^4 T \\ 0 & \text{otherwise} \end{cases}.$$

Here b_0 is a fixed constant to be chosen below.

For any $B \subset \mathbb{L}^2_\ell$, define

$$[B] = \left\{v \in C\big([0, t], \mathbb{L}^2_\ell\big) : v(t) \in B\right\}.$$

Then $R_t(L(0), B) = Q_t(L, [B])$.

Letting $Q^x_t(L, \cdot)$ and $Q^z_t(L, \cdot)$ be the two measures induced on $C\big([0, t], \mathbb{L}^2_\ell\big)$ by the dynamics of x and z respectively. Lemma 3.2 will be a consequence of the following two lemmas.

Lemma 3.6. *Fix any* $b_0 > 1$*. (The constant used in defining the* z *process.) Then the following holds: For any* $L \in \mathcal{P}$ *and* $t \ge 0$*,* $Q^x_t(L(0), \cdot)$ *is equivalent to* $Q^z_t(L, \cdot)$*.*

Lemma 3.7. *For any* b_0 *the following holds: For any* $L \in \mathcal{P}$ *and* $t \ge 0$*, there exists a positive function* $g(\cdot)$ *so that* $Q^x_t(L(0), [B] \cap A) \ge \int_B g(y)dm(y)$*, where* $m(\cdot)$ *is the Lebesgue measure.*

We now use these two lemmas to prove Lemma 3.2.

Proof of Lemma 3.2. Observe that by construction as long as the trajectories stay in A, $x(t) = \ell(t)$. Hence using Lemma 3.7, we have

$$R_t(L, B) = Q_t(L, [B]) \geq Q_t(L, [B] \cap A) = Q_t^z(L, [B] \cap A),$$
$$Q_t^x(L(0), [B] \cap A) \geq \int_B g(L(0), y) dm(y),$$

where $g(L(0), y)$ is a positive function in y. Since Lemma 3.6 says that $Q_t^z(\ell, \cdot \cap A)$ is equivalent to $Q_t^x(L(0), \cdot \cap A)$, we know that $R_t(L(0), B)$ is also bounded from below by a positive measure equivalent to the Lebesgue measure. $\square$

We now turn to Lemma 3.6. Our construction gives some measure of uniform control which is useful for estimating the rate the system converges to the stationary measure. (See [Mat00].) We state these more precise estimates in the following corollary.

Corollary 3.8. *Fix a $C_0 > 0$ and define $\mathcal{P}' = \{L \in \mathcal{P} : |L(0) + \Phi_0(L)|_{\mathbb{L}^2} < C_0\}$. Then for any $\alpha \in (0, 1)$ there exists a $b_0 > 0$ (the constant used to define A) so that:*

$$\inf_{t\in[0,T]} \inf_{L\in\mathcal{P}'} \mathbb{P}\left\{\mathsf{S}_t^{\omega} L \in A\right\} > 1 - a,$$
$$\sup_{L\in\mathcal{P}'} \int \left|1 - \frac{dQ_t^z(L, g)}{dQ_t^x(L, g)}\right|^2 dQ_t^x(L, g) < K(C_0, t)$$

for $t \in [0, T]$, where K is a constant depending on C_0 and t such that for each C_0, $K \to 0$ as $t \to 0$.

Proof of Lemma 3.6 and Corollary 3.8. Girsanov's theorem would imply the result if the Novikov condition

$$\mathbb{E} \exp \left\{ \frac{1}{2} \int_0^t |\Theta_s(Z^s)|^2 \left|G\big(z(s), \Phi_s(Z^s, h_0)\big)\right|_{\mathbb{L}^2}^2 ds \right\} < \infty$$

holds. As in the proof of Lemma 3.4, we will prove the stronger condition

$$\sup_{z(\cdot)\in A} \int_0^t \left|G\big(z(s), \Phi_s(Z^s, h_0)\big)\right|_{\mathbb{L}^2}^2 ds < \infty.$$

Using Lemma A.4, we obtain the following estimate on G:

$$\left|G\big(z(s), \Phi_s(Z^s, h_0)\big)\right|_{\mathbb{L}^2}^2 \leq C'\big[|z(s)|_{\mathbb{L}^2}^2 \, |h(s)|_{\mathbb{L}^2}^2 + |h(s)|_{\mathbb{L}^2}^4\big],$$

where $h(s) = \Phi_s(Z^s, h_0)\big)$. By Lemma C.1 we know that if z is in A then $\sup_{s\in[0,t]} |h(t)|_{\mathbb{L}^2}$ is less than some C_1, where C_1 depends on $|h_0|_{\mathbb{L}^2}$ and the b_0, C_0 and T used to define A. Hence for any $z \in A$, we have

$$\begin{aligned}
\int_0^t \left|G\big(z(s), \Phi_s(Z^s, h_0)\big)\right|_{\mathbb{L}^2}^2 ds &\leq C' \int_0^t \big[|z(s)|_{\mathbb{L}^2}^2 \, |h(s)|_{\mathbb{L}^2}^2 + |h(s)|_{\mathbb{L}^2}^4\big] ds \\
&\leq C' \left(\int_0^t |z(s)|_{\mathbb{L}^2}^4 \, ds\right)^{\frac{1}{2}} \left(\int_0^t |h(s)|_{\mathbb{L}^2}^4 \, ds\right)^{\frac{1}{2}} + C' C_1^4 t \\
&\leq C'(b_0 C_0)^2 T^{\frac{1}{2}} C_1^2 t^{\frac{1}{2}} + C' C_1^4 t.
\end{aligned}$$

Hence Novikov's condition holds and the lemma is proven. □

Proof of Lemma 3.7. The basic idea is as follows. Some of the paths which satisfy the condition defining A can be described by requiring that some norm of the paths be less than some fixed $f_k^*(t)$ at time t. Such a condition has the advantage that it corresponds to fixing a zero boundary condition along the boundary of some region for the associated Fokker-Planck equation. Since the diffusion is nondegenerate this process has a positive density on the interior of this region. By carefully picking f_k^* we can have the region contain sets arbitrarily far away from the origin. We now make this precise.

Fix a $L \in \mathcal{P}$, and a $t > 0$. For $k = 0, 1, 2, \dots$ define the disk D_k by

$$D_k = \left\{ f \in \mathbb{L}^2_\ell : |f|^4_{\mathbb{L}^2} \in [2^k, 2^{k+1}) \right\}$$

and let $\bar{D}_k$ be the closure of D_k. We will construct $g(\,\cdot\,) = \sum g_k(\,\cdot\,)\mathbf{1}_{D_k}$, where g_k is strictly positive on $\bar{D}_k$ and zero outside of $\bar{D}_k$.

Let f_k^* be a non-decreasing, positive, real-vaued C^∞ function f_k^* such that $f_k^*(s) = (C_0^4 + \alpha_k)^{\frac{1}{4}}$ for $s \in [0, (1-\alpha_k)t - \varepsilon]$ and $f_k^*(s) = (100 \cdot 2^{k+1})^{\frac{1}{4}}$ for $s \in [(1-\alpha_k)t, t]$ and linearly interpolates in $[(1-\alpha_k)t - \varepsilon, (1-\alpha_k)t]$. α_k is some number in $(0, 1)$ chosen so that $\int_0^t (f_k^*(r))^4 dr < (b_0 C_0)^4 T$. This is possible as long as $b_0 > 1$ and $t \le T$.

Now define the subset H_k of $C\big([0, t], \mathbb{L}^2_\ell\big)$ by

$$H_k = \left\{ f \in C\big([0, t], \mathbb{L}^2_\ell\big) : \sup_{s \in [0,t]} |f(s)|_{\mathbb{L}^2} \le f_k^*(s) \right\}.$$

By the choice of f_k^* it is clear that $H_k \subset A$, where A is the same set used in the definition of z.

Now consider the process $x_k'(t)$ which follows the same equation as $x(t)$ except that it is killed whenever the trajectory leaves H_k. Another way of saying this is $x_k'(t)$ is the process $x(t)$ conditioned on staying in H_k. The transition density of this process $g_k'(s, \ell(0), y)$ is the solution to the Kolmogorov equation with the same generator as x but with zero boundary conditions along the boundary of H_k. Since the generator is elliptic, we know that $g_k'(t, \ell(0), y)$ is strictly positive everywhere in the interior of H_k. Since the trace of H_k at time t strictly contains D_k, we know that $g_k'(t, \ell(0), y)$ is strictly positive for $y \in \bar{D}_k$. Also by construction it is clear that $Q_t^x(\ell(0), H_k) > 0$ for all k. Let $a_k = Q_t^x(\ell(0), H_k)$ and set $g_k(\,\cdot\,) = a_k g_k'(t, \ell(0), \,\cdot\,)\mathbf{1}_{D_k}(\,\cdot\,)$.

All that remains is to verify that this choice of g_k constructs a g with the desired minorization property since it is clearly everywhere positive. Without loss of generality it is enough to show it for a B contained in some arbitrary D_k. Then

$$\begin{aligned} Q_t^x(\ell(0), [B] \cap A) &\ge Q_t^x(\ell(0), [B] \cap H_k) \ge \mathbb{P}_{\ell(0)}\{x \in [B] \,\&\, x \in H_k\} \\ &\ge \mathbb{P}_{\ell(0)}\{x \in [B] \,\big|\, x \in H_k\} \mathbb{P}_{\ell(0)}\{x \in H_k\} \\ &\ge a_k \int_B g_k'(t, \ell(0), y) dm(y) = \int_B g_k(y) dm(y). \end{aligned}$$

□

4. Stationary Measures and Thermodynamical Formalism

In this section we make a few general heuristic remarks about the methodology behind our approach.

The starting point of our construction is rewriting the original Navier–Stokes equation with random forcing as a finite-dimensional system of ordinary stochastic differential equations whose drift coefficients depends on the whole past:

$$d\ell = [-\nu\Lambda^2\ell + P_\ell B(\ell,\ell) + G(\ell,\Phi_t(L^t))]dt + dW. \tag{25}$$

From (25)

$$dW = d\ell - [-\nu\Lambda^2\ell + P_\ell B(\ell,\ell) + G(\ell,\Phi_t(L^t))]dt. \tag{26}$$

The measure corresponding to all $dw_k(t)$, $k \in \mathcal{Z}_\nu$, $-\infty < t < \infty$ can be symbolically written as

$$\int \exp\left\{-\frac{1}{2}\sum_{k\in\mathcal{Z}_\nu}\frac{1}{|\sigma_k|^2}\int_{-\infty}^{\infty}\left|\frac{dw_k(t)}{dt}\right|^2 dt\right\} \quad \prod_k dw_k(t).$$

Here $\mathcal{Z}_\nu$ is the set of modes that are forced. The substitution of the expression for dw_k from (26) gives

$$\exp\left\{\int_{-\infty}^{\infty}\mathcal{L}_1(\ell(t))dt + \int_{-\infty}^{\infty}\mathcal{L}_2(\ell(t))dt - \frac{1}{2}\sum_{k\in\mathcal{Z}_\nu}\frac{1}{|\sigma_k|^2}\int_{-\infty}^{\infty}\left|\frac{d\ell_k(t)}{dt}\right|^2 dt\right\} \quad \prod_k d\ell_k(t),$$

where

$$\mathcal{L}_1(\ell(t)) = -\frac{1}{2}\big(-\nu\Lambda^2\ell + P_\ell B(\ell,\ell) + G(\ell,\Phi_t(L^t))\big)^2,$$

$$\int_{-\infty}^{\infty}\mathcal{L}_2(\ell(t))dt = \int_{-\infty}^{\infty}\sum_{k\in\mathcal{Z}_\nu}\frac{1}{|\sigma_k|^2}\big(-\nu\Lambda^2\ell + P_\ell B(\ell,\ell) + G(\ell,\Phi_t(L^t))\big)_k d\ell_k(t).$$

The factor $\exp\left\{-\frac{1}{2}\sum_{k\in\mathcal{Z}_\nu}\frac{1}{|\sigma_k|^2}\int_{-\infty}^{\infty}\left|\frac{d\ell_k(t)}{dt}\right|^2 dt\right\} \quad \prod_k d\ell_k(t)$ can be considered as the differential of a "free measure" which in our case is a finite-dimensional white noise.

The "Lagrangians" $\mathcal{L}_1$, $\mathcal{L}_2$ describe the non-local interaction of $\ell(t)$ with the past. The whole expression shows that the stationary measure for the SNS system is actually a Gibbs state constructed with the help of Lagrangians $\mathcal{L}_1$, $\mathcal{L}_2$.

The estimations of the growth of $\mathcal{L}_1$, $\mathcal{L}_2$ as a function of the growth of $|\ell_k(s)|_{\mathbb{L}^2}$, $s \to -\infty$ show the class of realizations for which the conditional distributions can be defined. Therefore we have a weaker form of the Gibbs state. R. L. Dobrushin in his last papers and talks stressed the importance of this class of probability distributions. Since we are dealing all the time with probability distributions, the free energy of our Gibbs state is zero. It would be interesting to develop a general theory of existence and uniqueness of Gibbs states for general Lagrangians $\mathcal{L}_1$, $\mathcal{L}_2$ so that our result becomes a particular case of a more general statement.

5. Conclusion

When analyzing the ergodic properties of an infinite dimensional stochastic process, one of the most delicate aspects is often finding the correct topology in which to work. One of the principle advantages of the approach presented in this paper is that it evades this difficulty. We trade an infinite dimensional diffusion process for a finite dimensional Itô process with memory.

We have tried to present the simplest case of our theory, so that the exposition would be unencumbered. In fact the proofs contained in this work have proved a more general theorem than originally stated. Consider forcing defined by

$$W(x,t) = \sum_{k\in\mathcal{Z}} \sigma_k w_k(t,\omega) e_k(x),$$

where $\mathcal{Z}$ is some finite subset of $\mathbb{Z}^2$ such that $(0,0) \notin \mathcal{Z}$ and $k \in \mathcal{Z}$ if and only if $\sigma_k > 0$. If we define

$$\mathbb{L}^2_\ell = \mathrm{span}\{e_k, k \in \mathcal{Z}\}, \quad \mathbb{L}^2_h = \mathrm{span}\{e_k, k \notin \mathcal{Z}\}$$

and

$$N_- = \sup\bigl\{N : k \in \mathcal{Z} \text{ for all } k \text{ with } 0 < |k| \le N\bigr\}.$$

With these definitions all of the previous lemmas and theorems hold with the role of N replaced by N_-. In particular, if $N_-^2 > \mathcal{C}\frac{\mathcal{E}_0}{\nu^3}$ the system has a unique invariant measure.

This formulation emphasizes the nature of our principle assumption. By requiring that all of the low modes are forced, we are essentially requiring that the reduced Gibbsian dynamics are elliptic in nature. Some steps towards dealing with a hypo-elliptic setting have been made. In [EMatt], finite dimensional truncations of the two dimensional SNS equation were studied and shown to be ergodic under minimal assumptions. In [EM], a reaction diffusion equation was studied under degenerate forcing.

Our arguments can be easily extended to the case where the forcing of the k^{th} mode has the form $f_k + \sigma_k dw_k(t)$, f_k is a constant, $f_k = 0$ and $\sigma_k = 0$ for $k \notin \mathcal{Z}$ or the case when the forcing is not diagonal in Fourier space.

Our approach can also be extended in several other different directions. We can consider the case when the high modes are also forced. As long as the forcing of the high modes decays sufficiently fast, our argument still applies with almost no change. The Wiener process in the forcing can be replaced by other diffusion processes such as the Ornstein-Uhlenbeck process. Dissipative PDEs such as the Cahn-Hilliard equation and the Ginzburg-Landau equations can also be studied using the same method. Finally, exponential convergence of empirical distributions to the stationary distribution can be proved.

A. Energy Estimates

In this Appendix, we prove a number of estimates controlling the evolution of the energy and enstrophy. Estimates for higher Sobolev norms are also possible, see [Mat98] for examples. In all cases, they are analogous to the standard results in the deterministic setting. Here we do not limit ourselves to forcing with only finitely many active modes. We will characterize the forcing in terms of the $\mathcal{E}_l$ defined by $\mathcal{E}_l \stackrel{\text{def}}{=} \sum |k|^{2l}|\sigma_k|^2$. We begin with the basic energy and enstrophy estimates in the stochastic setting.

Lemma A.1. *For any $p > 1$, we have*

$$\begin{aligned}
&\mathbb{E}\,|u(t)|^{2p}_{\mathbb{L}^2} + 2p\nu \int_0^t \mathbb{E}\,|\Lambda u(s)|^2_{\mathbb{L}^2}\,|u(s)|^{2(p-1)}_{\mathbb{L}^2}\,ds \\
&\qquad \le \mathbb{E}\,|u(0)|^{2p}_{\mathbb{L}^2} + C_0 \int_0^t \mathbb{E}\,|u(s)|^{2(p-1)}_{\mathbb{L}^2}\,ds, \\
&\mathbb{E}\,|\Lambda u(t)|^{2p}_{\mathbb{L}^2} + 2p\nu \int_0^t \mathbb{E}\left|\Lambda^2 u(s)\right|^2_{\mathbb{L}^2}|\Lambda u(s)|^{2(p-1)}_{\mathbb{L}^2}\,ds \\
&\qquad \le \mathbb{E}\,|\Lambda u(0)|^{2p}_{\mathbb{L}^2} + C_1 \int_0^t \mathbb{E}\,|\Lambda u(s)|^{2(p-1)}_{\mathbb{L}^2}\,ds.
\end{aligned}$$

Here $C_i = p\mathcal{E}_i + 2p(p-1)\sigma^2_{\max}$ and $\sigma^2_{\max} = \sup|\sigma_k|^2$. In the case $p = 1$, we have the equalities

$$\mathbb{E}\,|u(t)|^2_{\mathbb{L}^2} + 2\nu \int_0^t \mathbb{E}\,|\Lambda u(s)|^2_{\mathbb{L}^2} = \mathbb{E}\,|u(0)|^2_{\mathbb{L}^2} + \mathcal{E}_0 t, \tag{27}$$

$$\mathbb{E}\,|\Lambda u(t)|^2_{\mathbb{L}^2} + 2\nu \int_0^t \mathbb{E}\left|\Lambda^2 u(s)\right|^2_{\mathbb{L}^2} = \mathbb{E}\,|\Lambda u(0)|^2_{\mathbb{L}^2} + \mathcal{E}_1 t. \tag{28}$$

Proof. We begin by fixing a positive integer M and considering the Galerkin approximation defined by $u^{(M)}(t) = \sum_{|k|\le M} u^{(M)}_k(t)e_k$. $u^{(M)}(t)$ satisfies an equation of exactly the same form as the full solution except the nonlinearity has been projected to those terms of order less than or equal to M. We will also need $\mathcal{E}^M_l \stackrel{\text{def}}{=} \sum\limits_{|k|\le M} |k|^{2l}|\sigma_k|^2$. Our estimates will be independent of the order of approximation M. For simplicity, we will sometimes neglect the superscript M.

Applying Itô's formula to the map $\{u_k\} \mapsto \left(\sum |u_k|^2\right)^p$ produces,

$$\begin{aligned}
d\,|u(t)|^{2p}_{\mathbb{L}^2} &= 2p\,|u(t)|^{2(p-1)}_{\mathbb{L}^2}\left[-\nu\,|\Lambda u(t)|^2_{\mathbb{L}^2}\,dt + \langle u(t), dW\rangle_{\mathbb{L}^2}\right] \\
&\quad + 2p(p-1)\,|u(t)|^{2(p-2)}_{\mathbb{L}^2}\left(\sum_k |u_k(t)|^2|\sigma_k|^2\right)dt + p\,|u(t)|^{2(p-1)}_{\mathbb{L}^2}\,\mathcal{E}^M_0\,dt
\end{aligned} \tag{29}$$

for the energy moments and

$$\begin{aligned}
d\,|\Lambda u(t)|^{2p}_{\mathbb{L}^2} &= 2p\,|\Lambda u(t)|^{2(p-1)}_{\mathbb{L}^2}\left[-\nu\left|\Lambda^2 u(t)\right|^2_{\mathbb{L}^2}dt + \langle \Lambda^2 u(t), dW\rangle_{\mathbb{L}^2}\right] \\
&\quad + 2p(p-1)\,|\Lambda u(t)|^{2(p-2)}_{\mathbb{L}^2}\left(\sum_k |k|^2|\sigma_k|^2|u_k(t)|^2\right)dt \\
&\quad + p\,|\Lambda u(t)|^{2(p-1)}_{\mathbb{L}^2}\,\mathcal{E}^M_1\,dt
\end{aligned} \tag{30}$$

for the enstrophy moments.

Here $\langle \Lambda^\alpha u(t), dW(t)\rangle_{\mathbb{L}^2}$ is shorthand for $\sum |k|^\alpha u_k(t)\sigma_k dw_k(t)$. In the first, we have used the fact that $\langle B(u,u), u\rangle_{\mathbb{L}^2} = 0$ and in the second the fact that $\langle B(u,u), \Lambda^2 u\rangle_{\mathbb{L}^2} = 0$. Since, on the torus, the structure of the energy and the enstrophy equations are the same we will continue giving all of the details for analysis of the enstrophy equation.

The analysis for the energy equation proceeds analogously, see [Mat99, Mat98]. For a fixed $H > 0$, we introduce the stopping time

$$T = \inf\left\{t \geq 0 : \left|\Lambda^2 u(t)\right|^2_{\mathbb{L}^2} \geq H^2\right\}.$$

Denoting by M_t the local martingale term in (30) , we define the stopped martingale M_t^T by

$$M_t^T = \int_0^t 2p\,|\Lambda u(s\wedge T)|^{2(p-1)}_{\mathbb{L}^2}\,\langle \Lambda^2 u(s\wedge T), dW(s)\rangle_{\mathbb{L}^2}.$$

M_t^T has the advantage that its quadratic variation, denoted by $[M^T, M^T]_t$, is clearly finite.

$$\begin{aligned}
[M^T, M^T]_t &\leq 2p\sigma^2_{\max}\int_0^t \left|\Lambda^2 u(s\wedge T)\right|^{2p}_{\mathbb{L}^2} ds \\
&\leq 2p\sigma^2_{\max}\int_0^t \left|\Lambda^2 u(s\wedge T)\right|^{2p}_{\mathbb{L}^2} ds \leq 2p\sigma^2_{\max} H^{2p} t < \infty.
\end{aligned}$$

Because $\mathbb{E}[M^T, M^T]_t < \infty$ we know that $\mathbb{E}M_t^T = 0$. And because $t\wedge T$ is a bounded stopping time the Optional Stopping Time Lemma says that $\mathbb{E}M^T_{t\wedge T} = 0$. Since $M_{t\wedge T} = M^T_{t\wedge T}$, we have

$$\mathbb{E}\,|\Lambda u(t\wedge T)|^2_{\mathbb{L}^2} + 2\nu\mathbb{E}\int_0^{t\wedge T}\left|\Lambda^2 u(s)\right|^2_{\mathbb{L}^2} ds = \mathbb{E}\,|\Lambda u(0)|^2_{\mathbb{L}^2} + \mathcal{E}_1^M\mathbb{E}(t\wedge T),$$

and when $p > 1$,

$$\begin{aligned}
&\mathbb{E}\,|\Lambda u(t\wedge T)|^{2p}_{\mathbb{L}^2} + 2p\nu\mathbb{E}\int_0^{t\wedge T}|\Lambda u(t)|^{2(p-1)}_{\mathbb{L}^2}\left|\Lambda^2 u(s)\right|^2_{\mathbb{L}^2} ds \\
&\quad = \mathbb{E}\,|\Lambda u(0)|^{2p}_{\mathbb{L}^2} + \mathbb{E}\int_0^{t\wedge T} 2p(p-1)\,|\Lambda u(s)|^{2(p-2)}_{\mathbb{L}^2}\left(\sum_k |k|^2|\sigma_k|^2|u_k(s)|^2\right) \\
&\qquad + p\,|\Lambda u(s)|^{2(p-1)}_{\mathbb{L}^2}\,\mathcal{E}_1^M ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}\,|\Lambda u(t\wedge T)|^{2p}_{\mathbb{L}^2} + 2p\nu\mathbb{E}\int_0^{t\wedge T}|\Lambda u(t)|^{2(p-1)}_{\mathbb{L}^2}\left|\Lambda^2 u(s)\right|^2_{\mathbb{L}^2} ds \\
&\qquad \leq \mathbb{E}\,|\Lambda u(0)|^{2p}_{\mathbb{L}^2} + \left[2p(p-1)\sigma^2_{\max} + p\mathcal{E}_1^M\right]\mathbb{E}\int_0^{t\wedge T}|\Lambda u(s)|^{2(p-1)}_{\mathbb{L}^2} ds.
\end{aligned}$$

Since $u(t)$ is continuous in time, $T\to\infty$ as $H\to\infty$ and hence $T\wedge t\to t$. Thus we obtain

$$\mathbb{E}\,|\Lambda u(t)|^2_{\mathbb{L}^2} + 2\nu\mathbb{E}\int_0^t\left|\Lambda^2 u(s)\right|^2_{\mathbb{L}^2} ds = \mathbb{E}\,|\Lambda u(0)|^2_{\mathbb{L}^2} + \mathcal{E}_1^M t,$$

$$\begin{aligned}\mathbb{E}\,|\Lambda u(t)|_{\mathbb{L}^2}^{2p} &+ 2p\nu\mathbb{E}\int_0^t |\Lambda u(t)|_{\mathbb{L}^2}^{2(p-1)} \left|\Lambda^2 u(s)\right|_{\mathbb{L}^2}^2 ds \\ &\le \mathbb{E}\,|\Lambda u(0)|_{\mathbb{L}^2}^{2p} \left[2p(p-1)\sigma_{\max}^2 + p\mathcal{E}_1^M\right]\mathbb{E}\int_0^t |\Lambda u(s)|_{\mathbb{L}^2}^{2(p-1)} ds.\end{aligned}$$

Recall that we have been calculating with an M^{th} order Galerkin approximation. For the $p = 1$ equation, the right hand side converges to the desired right hand side. With this bound on $\mathbb{E}\,|\Lambda u(t)|_{\mathbb{L}^2}^2$ in hand we can take the $M \to \infty$ limit of the $p = 2$ equation. Analogously, once we have taken the limit in the p^{th} equation we have the dominating bound needed to take the limit in the $p + 1$ equation. $\square$

In our setting, the Poincaré inequality reads $|\Lambda f|_{\mathbb{L}^2}^2 > |f|_{\mathbb{L}^2}^2$ and $\left|\Lambda^2 f\right|_{\mathbb{L}^2}^2 > |\Lambda f|_{\mathbb{L}^2}^2$. This allows us to close the above inequalities. After applying Gronwall's inequality, we obtain the following estimates which are uniform in time.

Corollary A.2.

$$\mathbb{E}\,|u(t)|_{\mathbb{L}^2}^2 \le e^{-2\nu t}\mathbb{E}\,|u(0)|_{\mathbb{L}^2}^2 + \frac{\mathcal{E}_0}{2\nu}\left(1 - e^{-2\nu t}\right),$$
$$\mathbb{E}\,|\Lambda u(t)|_{\mathbb{L}^2}^2 \le e^{-2\nu t}\mathbb{E}\,|\Lambda u(0)|_{\mathbb{L}^2}^2 + \frac{\mathcal{E}_1}{2\nu}\left(1 - e^{-2\nu t}\right).$$

For any $p > 1$,

$$\mathbb{E}\,|u(t)|_{\mathbb{L}^2}^{2p} \le e^{-2\nu t}\mathbb{E}\,|u(0)|_{\mathbb{L}^2}^{2p} + C_0\int_0^t e^{-2\nu(t-s)}\mathbb{E}\,|u(s)|_{\mathbb{L}^2}^{2(p-1)} ds,$$
$$\mathbb{E}\,|\Lambda u(t)|_{\mathbb{L}^2}^{2p} \le e^{-2\nu t}\mathbb{E}\,|\Lambda u(0)|_{\mathbb{L}^2}^{2p} + C_1\int_0^t e^{-2\nu(t-s)}\mathbb{E}\,|\Lambda u(s)|_{\mathbb{L}^2}^{2(p-1)} ds.$$

We use standard estimates in the tri-linear term $\langle B(u, v), w\rangle_{\mathbb{L}^2}$ specialized to our two dimensional setting. Its proof can be found in [CF88] for example.

Lemma A.3. *Let α, β, γ be positive real numbers such that $\alpha + \beta + \gamma \ge 1$ and $(\alpha, \beta, \gamma) \ne (0, 0, 1)$, or $(0, 1, 0)$, or $(1, 0, 0)$,*

$$|\langle B(u, v), w\rangle_{\mathbb{L}^2}| \le C\left|\Lambda^\alpha u\right|_{\mathbb{L}^2}\left|\Lambda^{\beta+1} v\right|_{\mathbb{L}^2}\left|\Lambda^\gamma w\right|_{\mathbb{L}^2}.$$

Using this lemma we prove the following estimate specialized to the two dimensional setting with periodic boundary conditions.

Lemma A.4. *Let $\{e_k, k \in \mathbb{Z}^2\}$ be a basis for $\mathbb{L}^2$. Consider a splitting of $\mathbb{L}^2 = \mathbb{L}_\ell^2 + \mathbb{L}_h^2$. Let N^+ be in $\sup\{|k| : \exists\, e_k$ with $e_k \in \mathbb{L}_\ell^2\}$ and P_ℓ be the projector onto $\mathbb{L}_\ell^2$. If $u, v \in \mathbb{L}^2$ then*

$$|P_\ell B(u, v)| \le C(N^+)^3\,|u|_{\mathbb{L}^2}\,|v|_{\mathbb{L}^2}.$$

Proof of Lemma A.4. In the periodic setting, P_ℓ, P_{div}, and $(-\Delta)^s$ all are simply Fourier multipliers and hence commute with one other. Recall that $B(u,v) = P_{div}(u\cdot\nabla)v$ and hence,

$$\begin{aligned}|P_\ell B(u,v)| &= \sup_{\substack{w\in\mathbb{L}^2\\|w|=1}} |\langle P_\ell B(u,v), w\rangle_{\mathbb{L}^2}| = \sup_{\substack{w\in\mathbb{L}^2\\|w|=1}} |\langle B(u,v), P_\ell w\rangle_{\mathbb{L}^2}|\\ &= \sup_{\substack{w\in\mathbb{L}^2\\|w|=1}} |\langle B(u,P_\ell w), v\rangle_{\mathbb{L}^2}| \le C\,|u|_{\mathbb{L}^2}\,|v|_{\mathbb{L}^2} \sup_{\substack{w\in\mathbb{L}^2\\|w|=1}} \left|\Lambda^3 P_\ell w\right|_{\mathbb{L}^2}\\ &\le C(N^+)^3\,|u|_{\mathbb{L}^2}\,|v|_{\mathbb{L}^2} \sup_{\substack{w\in\mathbb{L}^2\\|w|=1}} |w|_{\mathbb{L}^2} \le C(N^+)^3\,|u|_{\mathbb{L}^2}\,|v|_{\mathbb{L}^2}\,. \qquad \square\end{aligned}$$

Lemma A.5. *Fix any $\delta > \frac{1}{2}$, $a \in (0,1)$ and $C_1 > 0$. Let $u(t) = \varphi_t^\omega u_0$. There exists a $K_1 > 0$ such that whenever $|u_0|^2_{\mathbb{L}^2} < C_0$,*

$$\mathbb{P}\left\{|u(t)|^2_{\mathbb{L}^2} + 2\nu\int_0^t |\Lambda u(s)|^2_{\mathbb{L}^2}\,ds \le C_0 + \mathcal{E}_0 t + K_1(t+1)^\delta \text{ for all } t\ge 0\right\} \ge 1-a.$$

Proof of Lemma A.5. The energy equation reads

$$|u(t)|^2_{\mathbb{L}^2} + 2\nu\int_0^t |\Lambda u(s)|^2_{\mathbb{L}^2}\,ds = |u_0|^2_{\mathbb{L}^2} + \mathcal{E}_0 t + \int_0^t \langle u(s), dW(s)\rangle_{\mathbb{L}^2}.$$

Since $|u_0|^2_{\mathbb{L}^2} < C_0$, all we need to show is that

$$\mathbb{P}\left\{M_t \le K_1(t+1)^\delta \text{ for } t\ge 0\right\} \ge 1-a$$

for K_1 large enough, where $M_t = \int_0^t \langle u(s), dW(s)\rangle_{\mathbb{L}^2}$. The quadratic variation $[M,M]_t$ can be calculated and one sees that

$$[M,M]_t \le \sigma^2_{\max}\int_0^t |u(s)|^2_{\mathbb{L}^2}\,,$$

and hence

$$([M,M]_t)^p \le \sigma^{2p}_{\max}\left(\int_0^t |u(s)|^2_{\mathbb{L}^2}\right)^p \le \sigma^{2p}_{\max} t^{p-1}\int_0^t |u(s)|^{2p}_{\mathbb{L}^2}\,ds.$$

From Corollary A.2, we know that if $|u(0)|^2_{\mathbb{L}^2} < C_0$, then there exists a constant $C_p(C_0)$ so that $\mathbb{E}\,|u(t)|^{2p}_{\mathbb{L}^2} \le C_p$ for all $t\ge 0$ and $p\ge 1$.

Now define the events

$$A_k = \left\{\sup_{s\in[0,k]} |M_s| > K_1 k^\delta\right\}.$$

By the Doob–Kolmogorov martingale inequality we have

$$\mathbb{P}\{A_k\} \leq \frac{\mathbb{E}([M,M]_t)^p}{K_1^{2p} k^{2p\delta}} \leq \frac{\sigma_{\max}^{2p} C_p}{K_1^{2p}} \frac{k^p}{k^{2p\delta}}.$$

Lastly observe that

$$\mathbb{P}\{M_t \leq K_1(t+1)^\delta\} \geq 1 - \mathbb{P}\Big\{\bigcup_k A_k\Big\} \geq 1 - \sum_k \mathbb{P}\{A_k\}.$$

By the previous estimate on $\mathbb{P}\{A_k\}$, for any $\delta > \frac{1}{2}$ we see that the sum is finite for p sufficiently large. Specifically, we need $\delta > \frac{1}{2}(1 + \frac{1}{p})$. Lastly, the sum can be made as small as we want by increasing K_1. □

B. Properties of Stationary Measures

We now establish a number of properties, derived from the dynamics, which any stationary measure must possess.

Lemma B.1. *For any stationary measure all energy moments are finite. In fact for any $p \geq 1$ there exist a constant $C_p < \infty$ such that*

$$\int_{\mathbb{L}^2} |u|_{\mathbb{L}^2}^{2p} \, d\mu(u) < C_p$$

for all stationary measures μ. In particular $C_1 = \frac{\mathcal{E}_0}{2\nu}$.

Proof. We will consider the case when $p = 1$. The other cases follow by the same method. For any $\epsilon > 0$ there exists a b_ϵ such that $\mu\{u \in \mathbb{L}^2 : |u|_{\mathbb{L}^2}^2 \leq b_\epsilon\} > 1 - \epsilon$. Let B_ϵ denote $\{u \in \mathbb{L}^2 : |u|_{\mathbb{L}^2}^2 \leq b_\epsilon\}$. For any $H > 0$ and $t > 0$, we have

$$\begin{aligned}
\int_{\mathbb{L}^2} \left(|u|_{\mathbb{L}^2}^2 \wedge H\right) d\mu(u) &= \int_{\mathbb{L}^2} \mathbb{E}\left(\left|\varphi_{0,t}^\omega u\right|_{\mathbb{L}^2}^2 \wedge H\right) d\mu(u) \\
&\leq H\epsilon + \int_{B_\epsilon} \mathbb{E}\left(\left|\varphi_{0,t}^\omega u\right|_{\mathbb{L}^2}^2 \wedge H\right) d\mu(u) \\
&\leq H\epsilon + \int_{B_\epsilon} \mathbb{E}\left(\left|\varphi_{0,t}^\omega u\right|_{\mathbb{L}^2}^2\right) d\mu(u).
\end{aligned}$$

Applying the first bound in Corollary A.2 gives

$$\int_{\mathbb{L}^2} \left(|u|_{\mathbb{L}^2}^2 \wedge H\right) d\mu(u) \leq H\epsilon + \frac{\mathcal{E}_0}{2\nu} + e^{-2\nu t}\left(b_\epsilon - \frac{\mathcal{E}_0}{2\nu}\right).$$

Taking the limit as $t \to \infty$ and then observing that ϵ was arbitrary, we obtain

$$\int_{\mathbb{L}^2} \left(|u|_{\mathbb{L}^2}^2 \wedge H\right) d\mu(u) = \int_U \left(|u|_{\mathbb{L}^2}^2 \wedge H\right) d\mu(u) \leq \frac{\mathcal{E}_0}{2\nu}.$$

Taking $H \to \infty$ gives that the energy of any stationary measure is bounded by $\frac{\mathcal{E}_0}{2\nu}$. The argument for higher moments of the energy is the same □

Lemma B.2. *For any stationary measure* μ*,*

$$\int_{\mathbb{L}^2} |\Lambda u|^2_{\mathbb{L}^2}\, d\mu(u) = \frac{\mathcal{E}_0}{2\nu}.$$

In addition if the forcing is such that $\mathcal{E}_1 < \infty$ *then*

$$\int_{\mathbb{L}^2} \left|\Lambda^2 u\right|^2_{\mathbb{L}^2} d\mu(u) = \frac{\mathcal{E}_1}{2\nu} \quad \textit{and} \quad \int_{\mathbb{L}^2} |\Lambda u|^{2p}_{\mathbb{L}^2}\, d\mu(u) < C_1(p) < \infty$$

for all $p \geq 1$.

Proof. Using Eq. (27), we have that for any initial condition $u_0 \in \mathbb{L}^2$,

$$\mathbb{E}\left|\varphi_{0,t}u_0\right|^2_{\mathbb{L}^2} + 2\nu \int_0^t \mathbb{E}\left|\Lambda\varphi_{0,s}u_0\right|^2_{\mathbb{L}^2} ds = |u_0|^2_{\mathbb{L}^2} + \mathcal{E}_0 t.$$

Here we have switched the time integral and the expectation by the Fubini–Tonelli theorem because the integrand is non-negative. We know from Lemma B.1 that any stationary measure has finite energy moments. Hence averaging with respect to the stationary measure gives

$$\begin{aligned}\int_{\mathbb{L}^2} \mathbb{E}\left|\varphi_{0,t}u_0\right|^2_{\mathbb{L}^2} d\mu(u_0) + 2\nu \int_{\mathbb{L}^2}\int_0^t \mathbb{E}\left|\Lambda\varphi_{0,s}u_0\right|^2_{\mathbb{L}^2} ds\, d\mu(u_0)\\ = \int_{\mathbb{L}^2} |u_0|^2_{\mathbb{L}^2}\, d\mu(u_0) + \mathcal{E}_0 t.\end{aligned}$$

Because μ was a stationary measure, we have that

$$\int_{\mathbb{L}^2} \mathbb{E}\left|\varphi_{0,t}u_0\right|^2_{\mathbb{L}^2} d\mu(u_0) = \int_{\mathbb{L}^2} |u_0|^2_{\mathbb{L}^2}\, d\mu(u_0)$$

and

$$\int_{\mathbb{L}^2}\int_0^t \mathbb{E}\left|\Lambda\varphi_{0,s}u_0\right|^2_{\mathbb{L}^2} ds = t\int_{\mathbb{L}^2} |\Lambda u_0|^2_{\mathbb{L}^2}\, d\mu(u_0).$$

Hence $2\nu \int_{\mathbb{L}^2} |\Lambda u_0|^2_{\mathbb{L}^2}\, d\mu(u_0) = \mathcal{E}_0$, concluding the proof of the first claim.

We now turn to the enstrophy moments. By the first part of this lemma, we know that there exist a $U \subset \mathbb{H}^1$ such that $\mu(U) = 1$. We now can proceed just as in Lemma B.1 to prove that all of the enstrophy moments are finite.

To find the expected value of the $\mathbb{H}^2$ norm we use Eq. (28). Then we proceed exactly as we did to obtain the expected value of the enstrophy (the $\mathbb{H}^1$ norm). □

Lemma B.3. *Let* μ_p *be the measure induced on* $C\big((-\infty,0],\mathbb{L}^2_\ell\big)$ *by any given stationary measure* μ. *Fix any* $K_0 > 0$ *and* $\delta > \frac{1}{2}$. *Then for* μ_p*-almost every trajectory in* $C\big((-\infty,0],\mathbb{L}^2_\ell\big)$, $v(s)$, *there exists a constant* T *such that for* $s \leq 0$,

$$|v(s)|^2_{\mathbb{L}^2} \leq \mathcal{E}_0 + K_0 \min(T, |s|)^\delta.$$

Proof. The basic energy estimate, derived from (29), reads:

$$|v(t)|^2_{\mathbb{L}^2} = |v(t_0)|^2_{\mathbb{L}^2} + \mathcal{E}_0(t-t_0) - 2\nu\int_{t_0}^t |\Lambda v(s)|^2_{\mathbb{L}^2}\,ds + \int_{t_0}^t \langle v(s), dW(s)\rangle_{\mathbb{L}^2},$$

for any $t_0 < t \leq 0$. There is no problem writing the integration against the Wiener path in the above integral. Our stochastic PDE had pathwise defined solutions. Therefore if we know the initial condition $v(t_0)$ and the trajectory of $v(s)$ for $s \in [t_0, t]$ the increments of the Wiener process on the interval $[t_0, t]$ are uniquely defined.

For any $k \geq 1$, the above estimate implies

$$\sup_{s\in[-k,-k+1]} |v(s)|^2_{\mathbb{L}^2} \leq |v(-k)|^2_{\mathbb{L}^2} + \mathcal{E}_0 + \sup_{s\in[-k,-k+1]} F_k(s),$$

where $F_k(s) = -2\nu\int_{-k}^s |\Lambda v(r)|^2_{\mathbb{L}^2}\,dr + M_k(s)$ and $M_k(s) = \int_{-k}^s \langle v(r), dW(r)\rangle_{\mathbb{L}^2}$.

Now define

$$A_k = \left\{ v(s) : \sup_{s\in[-k,-k+1]} |v(s)|^2_{\mathbb{L}^2} \leq \mathcal{E}_0 + K_0|k-1|^\delta \right\}$$

and $U_T = \cap_{k>T} A_k$. Since the U_T are an increasing collection of sets it will be sufficient to prove that the $\lim_{T\to\infty} \mu_p(U_T) = 1$. This is the same as showing that $\lim_{T\to\infty} \mu_p(U_T^c) = 0$. Now since $\mu_p(U_T^c) \leq \sum_{k>T} \mu_p(A_k^c)$, we need only to show that $\sum_{k>0} \mu_p(A_k^c) < \infty$:

$$\begin{aligned}\mu_p(A_k^c) \leq \mu_p &\left\{ v(s) : |v(-k)|^2_{\mathbb{L}^2} \geq \frac{K_0}{2}|k-1|^\delta \right\}\\ &+ \mu_p\left\{ v(s) : \sup_{s\in[-k,-k+1]} F_k(s) \geq \frac{K_0}{2}|k-1|^\delta \right\},\end{aligned}$$

The first term is the most straightforward. Lemma B.2 implies that the second moment of the energy is uniformly bounded by some constant C_2. Hence Chebyshev's inequality produces

$$\mu_p\left\{ v(s) : |v(-k)|^2_{\mathbb{L}^2} \geq \frac{K_0}{2}|k-1|^\delta \right\} \leq \frac{4}{K_0^2|k-1|^{2\delta}}\mathbb{E}\,|v(-k)|^4_{\mathbb{L}^2} \leq \frac{4C}{K_0^2|k-1|^{2\delta}}$$

which is summable as long as $\delta > \frac{1}{2}$.

The second term proceeds in the same way but with Chebyshev's inequality replaced by the exponential martingale estimate. The exponential martingale inequality controls the size of a martingale minus something proportional to its quadratic variation (see [RY94, Mao97] for example). The details are given in the following.

The key observation is that we can control $F_k(s)$ by controlling $M_k(s) - \alpha[M_k, M_k](s)$, where $[M_k, M_k](s)$ is the quadratic variation of the martingale $M_k(s)$ and α is a constant we will choose presently. First notice that with probability one,

$$\begin{aligned}[M_k, M_k](s) &= \int_{-k}^s \sum_l |\sigma_l|^2 |v_l(r)|^2 dr \leq \sigma^2_{\max}\int_{-k}^s |v(r)|^2_{\mathbb{L}^2}\,dr\\ &\leq \sigma^2_{\max}\int_{-k}^s |\Lambda v(r)|^2_{\mathbb{L}^2}\,dr\end{aligned}$$

and hence

$$F_k(s) \leq M_k(s) - \frac{2\nu}{\sigma_{\max}^2}[M_k, M_k](s)$$

almost surely. In this setting, the exponential martingale inequality states that for positive α and β,

$$\mathbb{P}\left\{\sup_{s\in[-k,0]} M_k(s) - \frac{\alpha}{2}[M_k, M_k](s) > \beta\right\} \leq e^{-\alpha\beta}.$$

Taking $\alpha = \frac{4\nu}{\sigma_{\max}^2}$ we find

$$\mu_p\left\{v(s) : \sup_{s\in[-k,-k+1]} F_k(s) \geq \frac{K_0}{2}|k-1|^\delta\right\} \leq \exp\left(-\frac{2\nu K_0}{\sigma_{\max}^2}|k-1|^\delta\right).$$

Since this is summable for any $\delta > 0$, the proof is complete. □

C. Control of High Modes by Low Modes

Lemma C.1. *If $h(t)$ is the solution to* (12) *with some low mode forcing $\ell \in C\big([0,t], \mathbb{L}_\ell^2\big)$, then* $\sup_{s\in[0,t]} |h(s)|_{\mathbb{L}^2}$ *is bounded by a constant depending on $|h(0)|_{\mathbb{L}^2}$ and $\int_0^t |\ell|_{\mathbb{L}^2}^4\, ds$.*

Proof. Taking the inner product of (12) with h produces

$$\frac{1}{2}\frac{d}{dt}|h(t)|_{\mathbb{L}^2}^2 = -\nu\,|\Lambda h|_{\mathbb{L}^2}^2 + \langle P_h B(h,\ell), h\rangle_{\mathbb{L}^2} + \langle P_h B(\ell,\ell), h\rangle_{\mathbb{L}^2}$$

because $\langle P_h B(\ell,h), h\rangle_{\mathbb{L}^2} = \langle P_h B(h,h), h\rangle_{\mathbb{L}^2} = 0$. Next using Lemma A.3 produces,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|h(t)|_{\mathbb{L}^2}^2 &\leq -\nu\,|\Lambda h|_{\mathbb{L}^2}^2 + C\,|\Lambda h|_{\mathbb{L}^2}\,|h|_{\mathbb{L}^2}\,|\Lambda\ell|_{\mathbb{L}^2} + C\,|\Lambda h|_{\mathbb{L}^2}\,|\Lambda\ell|_{\mathbb{L}^2}^2\\ &\leq \frac{C}{2\nu}\,|h|_{\mathbb{L}^2}^2\,|\Lambda\ell|_{\mathbb{L}^2}^2 + \frac{C}{2\nu}\,|\Lambda\ell|_{\mathbb{L}^2}^4\end{aligned}$$

Since $\ell \in \mathbb{L}_\ell^2$ we have $|\Lambda\ell|_{\mathbb{L}^2} \leq (N^+)\,|\ell|_{\mathbb{L}^2}$, where $N^+ = \sup\{|k| : \exists\, e_k \text{ with } e_k \in \mathbb{L}_\ell^2\}$, and hence after applying Gronwall's Lemma we have

$$\begin{aligned}|h(t)|_{\mathbb{L}^2}^2 \leq C_1\,|h(0)|_{\mathbb{L}^2}^2 &\exp\left(a_1\int_0^t |\ell|_{\mathbb{L}^2}^2\,ds\right)\\ &+ C_2\left(\int_0^t |\ell|_{\mathbb{L}^2}^4\,ds\right)\exp\left(a_1\int_0^t |\ell|_{\mathbb{L}^2}^2\,ds\right).\end{aligned}$$

Since by Hölder inequality,

$$\int_0^t |\ell|_{\mathbb{L}^2}^2\,ds \leq t\int_0^t |\ell|_{\mathbb{L}^2}^4\,ds,$$

the proof is complete. □

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