Gifted Ninth Graders' Notions of Proof: Investigating Parallels in Approaches of Mathematically Gifted Students and Professional Mathematicians

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High school students normally encounter the study and use of formal proof in the context of Euclidean geometry. Professional mathematicians typically use an informal trial-and-error approach to a problem, guided by intuition, to arrive at the truth of an idea. Formal proof is pursued only after mathematicians are intuitively convinced about the truth of an idea. Is the use of intuition to arrive at the plausibility of a mathematical truth unique to the professional mathematician? How do mathematically gifted students form the truth of an idea? In this study, 4 mathematically gifted freshmen with no prior exposure to proof nor high school geometry were given the task of establishing the truth or falsity of a nonroutine geometry problem, sometimes referred to as "circumscribing a triangle" problem. This problem asks whether it is true that for every triangle there is a circle that passes through each of the vertices. This paper describes and interprets the processes used by the mathematically gifted students to establish truth and compares these processes to those used by professional mathematicians. All 4 students were able to think flexibly, as evidenced in their ability to reverse the direction of a mental process and arrive at the correct conclusion. This paper further validates the use of Krutetskiian constructs of flexibility and reversibility of mental processes in gifted education as characteristics of the mathematically gifted student.

Introduction

The *Principles and Standards for School Mathematics* (2000), published by the National Council of Teachers of Mathematics (NCTM), envisions classrooms in which students "make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures" (p. 3). The NCTM envisions students in all grade levels approaching mathematics in a manner akin to professional mathematicians. For instance, at the early elementary level, teachers are

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encouraged to create learning experiences that allow students to develop pattern-recognition and classification skills and to encourage students to justify their answers via use of empirical evidence and short chains of deductive reasoning grounded in previously accepted facts. As students progress to middle grades, they are expected to have frequent experiences with formulating generalizations and conjectures, evaluating conjectures, and constructing mathematical arguments. Finally, in high school, students are expected to become adept at working formally with definitions, axioms, and theorems and be able to write proofs.

The recommendations outlined by the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) are generic and meant to apply to all students. However, there is a substantial body of research in gifted education that indicates that mathematically gifted students are different from their peer groups in many ways. For instance, mathematically gifted students differ from their peers in their capacity for learning at a faster pace (Chang, 1985; Heid, 1983) and in their desire to understand the conceptual ideas (Johnson, 1983; Sheffield, 1999). Further, they differ from their peers in their ability to abstract and generalize (Greenes, 1981; Kanevsky, 1990; Krutetskii, 1976; Shapiro, 1965; Sriraman, 2002, 2003a), information-processing abilities and data management (Greenes; Yakimanskaya, 1970), flexibility and reversibility of operations (Krutetskii), and tenacity and decision-making abilities in problem-solving situations (Frensch & Sternberg, 1992; Sriraman, 2003a). Instructional studies at Stanford University (Suppes & Binford, 1965; Goldberg & Suppes, 1972) showed that, with instruction, talented students could master inference principles such as *modus ponens*, *modus tollens*, and hypothetical syllogism, which are all precursors to proof, as early as the fifth grade!

This led me to hypothesize that mathematically gifted students may have an intuitive notion of proof and its role in mathematics, even if they have never had any prior instruction on proof. In other words, do mathematically gifted students have a natural capacity to approach proof in a manner akin to mathematicians? Usually, mathematicians first form a personal belief about the truth of an idea and use that as a guide for more formal analytic methods of establishing truth. For example, a mathematician may intuitively arrive at the result of a theorem, but realize that deduction is needed to establish truth publicly (Fischbein, 1980; Kline, 1976; Polya, 1954). Thus, intuition convinces the mathematician about the truth of an idea, while serving to organize the direction of more formal methods (Fischbein), namely, the construction of a proof to publicly establish the validity of the finding (Bell, 1976; Manin, 1977; Mason, Burton, & Stacey, 1992). This leads to the following questions:

1. How do mathematically gifted students arrive at an intuition of truth?
2. How do mathematically gifted students convince themselves and others about their intuition of truth?
3. Do the approaches used by gifted students parallel those used by professional mathematicians? If so, what are the parallels?

Literature Review

Epp (1990) stated that the kind of thinking done by mathematicians in their own work is "distinctly different from the elegant deductive reasoning found in mathematics texts" (p. 257). This statement puts into perspective the challenges faced by a student when expected to construct deductive arguments upon first encountering geometry in high school. When one talks to mathematicians about mathematical discovery, they acknowledge making illogical steps in arguments, wandering around in circles (Lampert, 1990), trying guesses (Davis & Hersh, 1981; Poincaré, 1948), and looking at analogous examples (Fawcett, 1938; Polya, 1954) for help; and, yet, the end result does not give the student this insight into the hidden struggle beneath the crisp, dry proof.

Chazan (1993) examined high school geometry students' justification for their views of empirical evidence and mathematical proof and reported his findings from in-depth interviews with 17 high school students from geometry classes that employed empirical evidence. The focus of Chazan's analysis was on students' reasons for viewing empirical evidence as proof and mathematical proof simply as evidence. In the first part of the interview, students were asked to compare and contrast arguments based on the measurement of examples and deductive proof. The second part of the interview focused on the textbook deductive proof and sought to clarify if interviewees believed that a deductive proof proves the conclusion true for all objects satisfying the given. They were also asked to draw counterexamples if they could. Chazan concluded that students had a good reason to believe that *evidence is proof* in the realm of triangles because there was sufficient evidence to support the claim. These students expressed skepticism of the ability of a deductive proof to guarantee that no counterexamples exist.

The van Hiele model (1986) of geometric thought emerged from the doctoral works of Dina van Hiele-Geldof and Pierre van Hiele in the Netherlands. The model consists of five levels of understanding, which can be labeled as visualization, induction, induction with informal deduction, formal deduction, and, finally, proof. These labels describe the characteristics of the thinking at each stage. The first level is characterized by students' recognizing figures by their global appearance or seeing geometric figures as a visual whole. Students at the second level (analysis) are able to list properties of geometric figures; the properties of the geometric figures become a vehicle for identification and description. In the third level, students begin to relate and integrate the properties into necessary and sufficient sets for geometric shapes. In the fourth level, students develop sequences of statements to deduce one statement from another; formal deductive proof appears for the first time at this level. In the fifth level, students are able to analyze and compare different deductive systems. The van Hiele levels of geometric thinking are sequential and discrete, rather than continuous, and the structure of geometric knowledge is unique for each level and a function of age. Van Hiele believed that instruction plays the biggest role in students moving from one level of geometric thinking to the next higher level. He also claimed that, without instruction, students may remain indefinitely at a given level. I do not agree with van Hiele's claim that the levels are discrete and a function of age. This claim may hold true for nongifted students, but they certainly do not apply to mathematically gifted students, as will be argued in the next paragraph. Moreover, the van Hiele model does not take a holistic view of mathematical ability and is strictly confined to the realm of geometry.

There were numerous experiments conducted in the former Soviet Union from the 1950s to the 1970s (Ivanitsyna, 1970; Krutetskii, 1976; Menchinskaya, 1959; Shapiro, 1965; Yakimanskaya, 1970) with mathematically "capable" students that demonstrated that gifted students have a repertoire of abilities that cannot be pigeonholed into discrete levels within a narrow subdomain of mathematics, such as Euclidean geometry. Instead, these researchers characterized the mathematical abilities of gifted children holistically, comprised of analytic, geometric, and harmonic components, and argued that gifted children usually have a preference for one component over the others. The analytic type has a mathematically abstract cast of mind, the geometric type has a mathematically pictorial cast of mind, whereas a harmonic type is a combination of analytic and geometric types. For instance, given the same problem, one gifted child might pursue an analytic approach, whereas another would pursue a geometric approach.

Strunz (1962) gave a different classification of styles of mathematical giftedness and suggested the empirical type and the conceptual type. In this classification, the empirical type would have a preference for applied situations, immediately observable relations, and induction, whereas the conceptual type would have a preference for theoretical situations and deduction.

Krutetskii (1976) observed that one of the attributes of mathematically gifted students is the ability to switch from a direct to a reverse train of thought (reversibility), which they performed with relative ease. The mathematical context in which this reversibility was observed was in transitions from usual proof to proof via contradiction (reductio ad absurdum) or when moving from a theorem to its converse.

The researchers cited in these paragraphs have acknowledged the use of intuitive ability in gifted children. To my knowledge, there are no studies that have looked at how gifted students use their intuition in mathematics. There are, however, a limited number of studies with mathematicians who have tried to increase our understanding of how mathematicians use intuition (Fischbein, 1980; Kline, 1976; Sriraman, 2003b).

Kline (1976) found that a group of mathematicians said they began with an informal trial-and-error approach guided by intuition. It is this process that helped them convince themselves about the truth of a mathematical idea. After the initial conviction, formal methods were pursued:

> The logical approach to any branch of mathematics is usually a sophisticated, artificial reconstruction of discoveries that are refashioned many times and then forced into a deductive system. The proofs are no longer natural or guided by intuition. Hence one does not really understand them through logical presentation. (p. 451)

Fischbein (1980) believed that intuition is an essential component of all levels of argument and referred to the use of intuition as anticipatory: "While trying to solve a problem one suddenly has the feeling that one has grasped the solution even before one can offer an explicit, complete justification for that solution" (p. 10).

To determine qualitative characteristics of creative behavior, I interviewed 5 mathematicians (Sriraman, 2003b). In that study, the mathematicians were questioned about how they formed an intuition of the truth of a proposition. All of the mathematicians in that study mentioned that the last thing they looked at was a formal proof. They went about forming an intuition about truth by consciously trying to construct examples and counter-examples (Sriraman). In other words, they worked a problem both ways, constructing examples to verify truth, as well as looking for counterexamples that would establish its falsity, thus using a back-and-forth approach of conscious guessing (Bell, 1976; Lampert, 1990; Polya, 1954; Usiskin, 1987). The "ideal mathematician," a fictional entity constructed by Davis and Hersh (1981), when asked by a student of philosophy, "What is a mathematical proof?" replied with lots of examples, such as "the fundamental theorem of this, the fundamental theorem of that, etc." (Hersh, 1993, p. 389). When probed by the philosophy student, the ideal mathematician finally succumbed and confided that "Formal logic is rarely employed in proving theorems, that the real truth of the matter is that proof is just a convincing argument, as judged by competent judges" (p. 389). This leads us once again to the questions posed earlier about mathematically gifted students:

1. How do mathematically gifted students arrive at an intuition of truth?
2. How do mathematically gifted students convince themselves and others about their intuition of truth ?
3. Do the approaches used by gifted students parallel those used by professional mathematicians? If so, what are the parallels?

Methodology

The Participants

Since one of the goals of this study was to determine whether mathematically gifted students have intuitive notions about proof, it was important to select students who had no prior instruction on proof. The participants selected for this study were 4 freshmen in a large, rural midwestern high school, enrolled in various sections of Integrated Mathematics I (an NSF-funded curriculum aligned to NCTM standards and developed at Western Michigan University). I was a full-time mathematics teacher and gifted coordinator of this high school. The 4 students had been enrolled previously in the same K–8 school district, one of the feeder schools of the high school, and had been identified as mathematically gifted by the K–8 district guidelines based on test scores on the Stanford Achievement Test (95 percentile) and teacher nominations. I was provided with

Table 1

Testing Profiles of the Four Students

	SAT (Grade 1) Math raw score[1] (out of 90 items)	SAT (Grade 8) Math raw score[2] (out of 82 items)	Percentile rank (national)
Jill	90	82	99
Yuri	89	80	99
Kevin	89	81	99
Sarah	88	80	99

[1]The math portion of the 8th edition consisted of 90 items subdivided into items that measured number concepts (34), computation (26), and applications (30).
[2]The math portion of the 9th edition consisted of 82 items subdivided into items that measured problem solving (52) and procedures (30).

this information by the K–8 district. Table 1 gives the achievement profiles of the 4 students and shows that they were in the top one percentile on the Stanford Achievement Test.

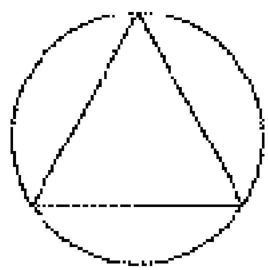
In addition, at the end of their first semester in the high school, their respective ninth-grade math teachers identified and recommended the 4 students for the high school gifted program. The achievement profiles in mathematics, along with the nomination by math teachers in the K–8 district, as well as the high school, evidenced the mathematical giftedness of these 4 students.

The 4 students were not enrolled in any of the math courses taught by me. They were invited to participate in this study through a letter stating that the gifted coordinator (author) was interested in studying the mathematical thinking of gifted students. The 4 students consented to participate in this study and completed a survey about their previous K–8 coursework in mathematics and answered specific questions about their familiarity with geometry and proof. The surveys indicated that their previous math course was Algebra, with some enrichment. All 4 students indicated studying classification of geometric figures based on properties in grades 4 and 6. One of the students mentioned an interest in geometric constructions, but reported that no instruction was given in school. The mathematics curricula in grades 7 and 8 did not include any instruction on Euclidean geometry or proof. The only content that related to geometry and proof, respectively, were (a) a small unit on the use of formulas to determine surface areas and volumes of geometric shapes and (b) an enrichment unit on establishing identities in ratio and proportion.

The Problem

The problem selected for this experiment is sometimes referred to as the "circumscribing a triangle problem." A perusal of commonly used textbooks in high schools indicated that this problem is normally encountered as an enrichment problem toward the end of the school year in geometry. This problem is also found in analytic geometry books because it can be solved using analytic, algebraic, or both tools.

The problem states as follows: Consider the triangle below. The circle passes through each vertex of this triangle:



1. Is it true that for **every** triangle there is a circle that passes through each of the vertices?
2. If yes, why? If no, how would you go about finding out?

This problem was deemed suitable for an extended investigation because of the following reasons:

1. The problem was simply stated and easy to understand, and the 4 students had not encountered such a problem before. Therefore, they were confronted with a novel task.
2. The problem presented visual information on the basis of which false inferences could be made.
3. The problem could be approached in a variety of ways—algebraically, analytically, empirically, via geometric construction and logically—thereby allowing various styles of solutions to manifest.
4. The problem was stated generally, although a particular case was presented in the figure.

Data-Collection Procedures

The clinical interview technique attributed to and pioneered by Piaget (1975) to study the thinking processes of the students was followed. Each student was individually interviewed after school.

The interviews were task based, centered on the aforementioned problem and open ended with the purpose of getting students to verbalize their thought processes while solving the given problem. Each of the four interviews lasted approximately 1 hour. The students were probed at length and asked to "think aloud." The following questions were asked:

1. How would you go about convincing someone who thinks that the statement is (the opposite of what the student has said)?
2. How would one find the center, radius, or both of the circle circumscribing a triangle?
3. If the student based inferences on the given figure, they were asked "Why?"
4. What constitutes a proof in mathematics?

The students were asked to explain their reasoning in great detail. The interviews were audiotaped, transcribed verbatim, and rechecked for errors. The students were provided with a copy of the interview transcript and asked to make clarifications they thought were necessary. The objective of doing this was to not misconstrue what the students said, to have a complete and accurate interview transcript, and to ensure compatibility between what students said and what they had meant to say. The 4 students were satisfied with the clearness of the transcripts and did not make any clarifications or corrections. In addition, I recorded my impressions about the interview immediately after each interview. The data consisted of interview artifacts (student work), interview transcripts, and my notes.

Data Coding and Data Analysis

The transcribed data was coded and analyzed using techniques from grounded theory (Glaser & Strauss, 1977). Coding began by reading the interview transcripts line by line and spontaneously memoing words that described the mental processes employed by the 4 students. The goal of coding was to delineate the processes and build categories (Corbin & Strauss, 1998). I purposefully looked for actions that corresponded to a process, noting their evolution as students responded to the problem. The constant comparative method was applied to compare the actions of the 4 students and to isolate the similarities of their thought processes as found in the data. The following categories emerged as a result of data coding and analysis.

The category of visualization emerged when students repeatedly verbalized the visual information provided, indicating that the circumscribed triangle was equilateral. There were 108 memos of words and phrases, such as "It looks equilateral," "The angles and sides look equal," "It looks like a perfect triangle," and so forth. The category of intuition emerged as a result of 137 memos of phrases like "It just seems right . . . I don't know why," "It just seems obvious," "I'm sure there is a way," and so forth. In other words, these memos pointed to assertions of self-evidence. There were 212 memos for words indicating measurement and use of concrete examples, which led to the creation of the category of empiricism. Finally, there were 82 memos for phrases hinting at a reversal of the process of looking at the problem. Such phrases as "How can I fit points inside," "What if I started with the circle," and so forth led to the category of reversibility. The four categories were defined.

Definitions

Visualization. This is the process by which a student makes inferences by transforming or inspecting pictures (Hershkowitz, 1989). Although visualization is an internal mental phenomenon, it can be externally observed through a student's transformatory actions on a given figure.

Empiricism. This refers to the repeated use of examples that provide conforming (nonconforming) evidence in order to support the truth of an idea. Empiricism also involves the use of specific measurements to make inferences (Chazan, 1993; Polya, 1954; Strunz, 1962).

Intuition. Intuition is the affective mood associated with having grasped the solution while trying to solve a problem before one can offer an explicit and complete justification for that solution (Fischbein, 1980; Kline, 1976). It involves the use of reasoning that is not formal, the use of everyday terms, and invoking empirical examples for purposes of justification (Poincaré, 1948; Polya, 1954).

Reversibility. This is the process (or ability) to switch from a direct chain of thought to a reverse chain. It is the ability to reverse a mental operation (Krutetskii, 1976). This includes the ability to solve (or think about) the same problem in several different ways.

Validity

I used the strategy of intersubjectivity (Rubin & Babbie, 1997) by having a colleague analyze the data from the interviews using the coding technique developed. The colleague coded and analyzed 36 random slices of interview data and came to the same conclusions I did. For the slices of data coded independently by the colleague, there was an agreement of 93% for processes indicating visualization, 91% for empiricism, 92% for intuition, and 96% for reversibility. This lends validity to the findings of the research effort.

Results

The results of the study are first presented under the categories that emerged as a result of data coding and analysis. The categories that emerged as processes used to construct a "proof" were visualization, intuition, empiricism, and reversibility. I have presented the students' pathways to "proof" in the form of tables summarizing the patterns in each category. This is followed by an extensive interpretation and commentary on the observed patterns and their isomorphisms to mathematical techniques used by professional mathematicians. Finally, I have established validity of the findings by using "triangulation by theory" (Kelly & Lesh, 2000), which is the application of various explanations from the literature to the data at hand and the selection of the most plausible ones to explain the research results.

All 4 students came to the conclusion that the statement was true for every triangle by an inductive process of trial and error. The process of proving the statement began with the intuition that the statement was true only for equilateral triangles (based on the visual information). The students then ascertained this truth for equilateral triangles by intuitively constructing the center and formulating counterexamples to validate their conjecture that the statement was false in general. They finally determined the truth of the statement by dramatically reversing their thinking. An overall picture of this process is found in Figure 1.

Visualization

Visualization played an important role in the process of establishing the truth of the statement. All 4 students insisted that the given triangle was equilateral because it "looked" like one. Although the statement clearly asked if it was possible to circumscribe a circle over *every* triangle, the students couldn't ignore the visual image. This led them to conjecture that the given statement applied to

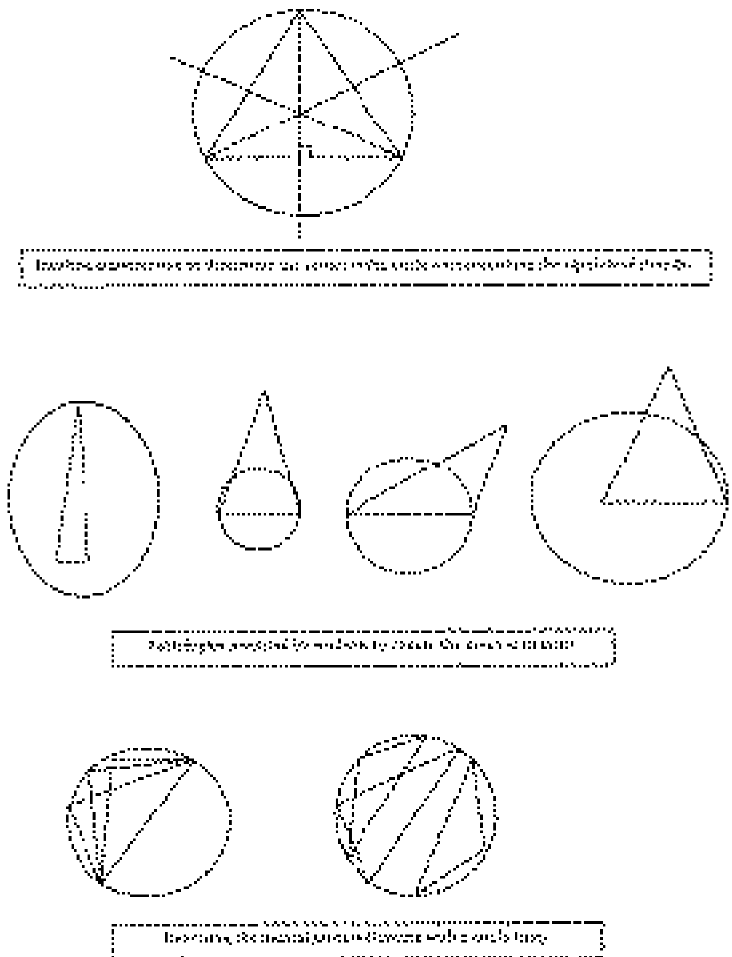


Figure 1. Samples of interview artifacts.

only equilateral triangles or "special" triangles. Table 2 provides examples of student verbalizations of their conjectures based on the visual information.

Intuition

The 4 students followed their initial intuition that the statement applied only to equilateral triangles. Three of the 4 were also able to determine the correct construction to locate the center of the cir-

Table 2

Seeing Is Believing

Category	Examples of process	Student
Visualization	It would only work for equilateral triangles (*pointing to the figure that looks like an equilateral triangle*).	Jill
	Right now you have an equilateral triangle. At least it looks like one.	Yuri
	It's working over here because it is an equilateral triangle.	Kevin
	This looks like an equilateral triangle with equal distances.	Sarah

cle circumscribing an equilateral triangle. This was remarkable because they had never been taught this construction before. However, their intuition guided them into discovering the construction. It is important to note that students did not have a compass and straightedge available and all constructions were done freehand (see Figure 1). Table 3 provides glimpses of student intuition used to construct the center of the circle.

Empiricism

Students were probed as to whether the construction they had discovered applied only to equilateral triangles. This led them to construct counterexamples (see Figure 1) to substantiate their intuition that the statement applied only to equilateral triangles and was false in general. Table 4 provides glimpses of this empirical process of constructing counterexamples used by the 4 students.

Reversibility

At this stage of the interview, each of the 4 students was almost convinced that the statement was false in general. It is noteworthy that they weren't willing to commit to saying the statement was false, in spite of the counterexamples they had constructed. The students wanted to try a different approach, which was evidence of their flexibility in

Table 3

The Center of the Circle and the Equilateral Triangle Coincide . . . I'm Certain

Category	Examples of process	Student
Intuition	Draw the perpendiculars that pass through midpoints and then, when you have the center, you can take the distance to one of the vertices as the radius and join them. . . . *It just seems right. I don't know why.*	Jill
	I draw an altitude to each side of the triangle. . . . Where they intersect . . . *it just seems obvious that this will give the center.*	Yuri
	I know there is some way to do it. *It's obvious that there is some way to find it.*	Kevin
	I'll draw the equilateral triangle, the perpendicular . . . and another perpendicular and the point where they cross would be the center.	Sarah

thinking, a trait of mathematically gifted students (Krutetskii, 1976). Table 5 shows similarities in how students dramatically reversed their thinking by starting with an arbitrary circle first instead of the triangle. By reversing their train of thought, they were able to convince themselves that the statement was, indeed, true.

Interpretations and Isomorphisms

In the preceding tables and figure, similarities in the student pathways to their proof were reconstructed. The thinking processes of the 4 students show remarkable isomorphisms to those of professional mathematicians, as will be discussed in this section. Mathematics is often viewed as an activity of creating relationships, some of which are based on visual imagery (Casey, 1978; Presmeg, 1986). The image presented to the 4 students immediate-

Table 4

Look at All These Weird Triangles!

Category	Examples of process	Student
Empiricism	If you have a triangle, say, like this (*draws a thin scalene triangle)*, you couldn't get a circle to pass through this triangle. It would be more like an ellipse.	Jill
	You can't always use the altitudes to determine the center. Let me draw another triangle.	Yuri
	Now, if you take a really different type of triangle, then it won't work. Here is a triangle and it is not working.	Kevin
	Yeah, I tried fitting a circle around these other triangles; it didn't work.	Sarah

ly kindled the formation of a preliminary conjecture: The given triangle was equilateral. This in turn led to the question of determining the center of the circumscribed circle, which resulted in the students discovering the relationship that the center of the circle coincided with the point of intersection of the three perpendicular bisectors. It is important to understand that it is impossible to find out directly how children (and hence how mathematicians) create images; however, the manner in which they use images can be studied from their actions on a given problem (Inhelder & Piaget, 1971). We, as adults, have often seen children do some strange things when faced with a mathematical task. One often comes across reactions from teachers where they view the child's intuitive actions as bizarre (Kamii & DeClark, 1985). This kind of reaction reflects the teacher's inability to imagine things from the child's perspective. Someone reading an abstract proof is in a similar situation because of being unable to imagine the proof from the creative mathematician's perspective, as well as being unaware of the images used by the mathematician in its creation. The 4 students were able to isolate attributes (equal sides and equal angles) that they deemed critical (Hershkowitz, 1989) in order to form the initial conjecture about the truth of the given statement. In other

Table 5

Let's Start With the Circle First!

Category	Examples of process	Student
Reversibility	Wait a minute . . . I suppose it is true. You can draw a circle and always fit some triangle inside it. *(Draws an example) You can fit it as long as it is inside the circle.*	Jill
	I've found a new construction: What is stopping me from fixing the base points elsewhere? *I can fix the two points elsewhere* (drawing chords) *and then choose the third point. Yes, the statement is true.*	Yuri
	Let me try something else. I'll draw a really queer triangle, and I'm going to make it look like this (draws an obtuse scalene triangle). Would this work? *But if you pick these 3 points on the circle, it seems to work . . . Yeah!* You can always pick the points on the circle and then draw the triangle.	Kevin
	I'm trying to think here (tearing the paper in frustration). *What if I follow the circle and pick points?* (Pause) Yeah. I'm trying to visually look at the triangles and I . . . guess the fact that no matter what kind of triangle I draw, *if I can draw a circle first, then I can draw any triangle in it* (trying more examples). *I must draw the circle first. Yes, it's true, it's a true statement.*	Sarah

words, the figure served as a visual reference point to kindle the mathematical process of proving.

Intuition is the guide that mathematicians use to convince themselves about the validity of a proposition (Burton, 1999;

Fischbein, 1980; Kline, 1976; Sriraman, 2003b). For the 4 students, the process of proving began with the intuition that the statement was true for equilateral triangles. This can be interpreted as the intuitive action of specializing the given statement for equilateral triangles. Mathematical thinking is often characterized by four processes: *specializing*, *conjecturing*, *generalizing*, and *convincing* (Burton, 1984, 1999). Once the students had specialized and conjectured that the statement was true for equilateral triangles, they were asked if this implied that the statement was true for all triangles. This led to a quasiempirical (Ernest, 1991; Lerman, 1983) approach to proof in which the students tried to construct mathematical pathologies (Figure 1) or mathematical monsters (Lakatos, 1976) in the form of triangles that would disobey the given proposition. This quasiempirical process again shows remarkable similarities to a view of mathematical thinking introduced by the eminent mathematical philosopher Imre Lakatos, in which mathematics is presented as a model of possibilities subject to conjecture, proof, and refutation. In other words, mathematics is not viewed as an absolute, immutable body of knowledge, but is, instead, subjected to the scientific process of constantly revising and refining preliminary hypotheses. In this view of mathematics, no theorem or proof is perfect because there is always the possibility of a better revision. The quasiempirical process of constructing pathologies by the 4 students is a common trait among mathematicians when working on problems. Pathologies serve the purpose of revisiting the problem and refining the hypotheses or assumptions.

The quasiempirical process employed by the 4 students led to the revised conjecture that the statement was, perhaps, false in general. However, at this juncture, they were still unwilling to commit to saying that the statement was false, in spite of the counterexamples they had constructed. A common trait among professional mathematicians is for them to work on a problem for a prolonged period of time, and, if no breakthrough occurs, they often stand back and "sleep on it." In other words, they let the problem incubate and hope that an insight or breakthrough will eventually occur. This is the Gestalt view of mathematical thinking (Hadamard, 1945; Poincaré, 1948; Sriraman, 2003b; Wallas, 1926; Wertheimer, 1945). Mathematicians often characterize this as the stage where the "problem talks to you." I contend that this occurred in a microcosmic way with the 4 students. After having spent close to an hour on the problem, they put their pencils aside and mulled on the problem in silence for a few minutes. Remarkably enough, the insight to reverse their thinking dawned upon them (Table 5). This process has many interpretations. Insightful mathematical thinking and creativity can be viewed as a process of nonalgorithmic decision making (Ervynck, 1991; Poincaré, 1948). Decisions that have to be made by mathematicians may be of a widely divergent nature and always involve a *crucial* choice (Birkhoff, 1969; Ervynck; Poincaré). In an age where the use of the computing power of machines to gain insights into results is slowly becoming a valid approach, it is interesting that mathematicians view a crucial aspect of their craft as nonalgorithmic decision making. This was the most intense and frustrating stage for the 4 students where conceptual activity (Ervynck, 1991) occurred and manifested in an illumination (Wallas, 1926; Wertheimer, 1945), namely, the decision or choice to reverse the structure of the problem. It is common among mathematicians to work on the problem one day and then on its converse the next day or, simultaneously, work the problem both ways to gain an insight. This process of reversibility is viewed as an aspect of flexibility in thinking, a trait of mathematically gifted students (Krutetskii, 1976), and it connects well with the back-and-forth approach employed by mathematicians when tackling a problem.

After the students had convinced themselves that the statement was true for all triangles, they were asked how they would convince others about this truth. In other words, students were probed about the methods they would employ to establish the truth publicly. It is remarkable that all 4 students intuitively knew that one counterexample was sufficient to establish falsity; however, establishing truth involved more work and would require "substantial" evidence. The students relied on the use of empirical evidence to explain truth and were convinced that numerous visual examples were enough to convince others about the truth of the statement. In other words, a proof for them was explaining and convincing (Bell, 1976; Kline, 1976). This is a very natural view of proof, even among professional mathematicians. "The formal logic view of proof is a fascinating topic of study for logic . . . but it is not a truthful picture of real-life mathematical proof" (Hersh, 1993, p. 391). The views expressed by the 4 students about the role of proof are sophisticated for ninth graders and, again, show remarkable isomorphisms to that held by professional mathematicians and some philosophers of mathematics, as illustrated by the following quotes:

> I look for examples that will support it and those that won't support it. I have to be looking for examples that would disprove the statement, otherwise I would be wasting a lot of time doing work for nothing. . . . Proof is written explanation or

> examples explaining, based on previous things that I believe to be true. (Yuri)

This quote is astoundingly isomorphic to the view of proof expressed by one professional mathematician, a brilliant analyst:

> First I get an idea that something along a certain line should be true and then I start to prove it, and in that proof I run into some difficulties and then I say, can I construct an example from those difficulties? If in constructing the example I run into difficulties . . . then can those difficulties be put into this proof you know, so I do this back and forth, usually at some gut level I have the belief that something along that line is true. Not always is that intuition correct but it is correct often enough and . . . I am able to prove something that I suspect is true. (quoted from Sriraman, 2003b)

This next quote shows startling similarities to Lakatos' (1976) thesis of mathematics being an ever-evolving process of conjecture, proof, and refutation.

> You can always find one case where it doesn't work. So to prove something is true, even like in science you can have a theory that it will work, but you can never definitely be sure that it will always work. . . . You never prove something is true. You can take a bunch of different types of cases and see if it works and then it is generally accepted that it's true . . . unless someone comes around and proves it's false. There are things that are believed to be true for 200 years and someone will come around with a particular case where it's false. (Kevin)

Informal, quasi-empirical mathematics does not grow through a monotonous increase in the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations (Lakatos, 1976).

Finally, the use of visual arguments by the students to convince others about the truth of a statement has historically found expression in Indian mathematics. The following quotes illustrate this isomorphism:

> I guess you would start out like this visual, get your arguments down, and then put it into words. I remember a lot of times starting out visually and just seeing and working with that and being able to put it into a proof[1]. . . . (Laughing) I just don't see

> the point sometimes. Like, if I know it's true, why do I have to go through 16 steps in proving it? It's much more effective in getting the point across with visual examples. (Sarah)

> In general the mathematics of Europe was influenced by Greek mathematics while Indian mathematics, despite influences from Greece and Arabia establish a unique tradition. There was no conflict between . . . visual demonstration and numerical calculation and . . . proof by deduction. (Almeida, 2003, p. 7)

> Our inherited notion is that rigorous proof is not carved in marble. People will modify that notion, will allow machine computation, numerical evidence, probabilistic algorithms, if they find it advantageous to do so. Then, we are misleading our pupils, if in the classroom we treat rigorous proof as shibboleth." (Hersh, 1993, p. 395, as quoted by Almeida, 2003, p. 7)

Triangulation by Theory and Implications

In this study, 4 mathematically gifted ninth-grade students with no formal exposure to proof or Euclidean geometry were given the task of establishing the truth or falsity of a statement. The strategies used by the students to construct a "proof" were documented, coded, and analyzed. It was found that these students relied on visualization (Hershkowitz, 1989; Yakimanskaya, 1970), empiricism involving the use of examples and counterexamples or conscious guessing (Bell, 1976; Lampert, 1990; Polya, 1954; Sriraman, 2003b), and reversibility (Krutetskii, 1976; Shapiro, 1965) to arrive at the truth. This entire process was guided by their strong intuition as evidenced by their ability to formulate conjectures and devise constructions to validate their initial conjecture about equilateral triangles. It is noteworthy that, although the 4 students were faced with nonconforming evidence in the form of "weird" triangles that seemed uncircumscribable, they were unwilling to commit to stating the statement was false. It is easier to say that something is false, based on a poorly constructed counterexample, as is evidenced in high school geometry (Senk, 1985; Usiskin, 1987), whereas to state something is true in mathematics involves the conviction that the statement holds for a potentially infinite number of cases. The 4 gifted students were aware of this distinction, whereas most high school students in geometry think otherwise and believe statements to be true just for a particular figure (Mason, 1996; Senk).

Mathematicians have an awareness of the generality of a statement by distinguishing between *looking through* and *looking at*.

Looking through is analogous to generalizing through the particular, whereas looking at is analogous to specializing (identifying) a particular case in general (Dubinsky, 1991; Mason, 1996). A simplistic example that comes to mind and is also given by Mason is in a high school geometry setting when a teacher draws a (particular) triangle on the board and says that the sum of the angles of a triangle is 180 degrees. Most often what is stressed is the empirical fact, namely 180 degrees.

> The generality of the statement is hidden in the indefinite article a. A student looking through this statement sees the general in the particular and recognizes that the essence of the statement is the invariance of the angle sum in the domain of all possible triangles. Looking through entails recognizing the attribute of invariance in an implied domain of generality. (Mason, 1996, p. 65)

The 4 gifted students in this study were able to look through the statement posed in the problem and recognize the attribute of invariance, a quality of professional mathematicians. The gifted students were also aware of the differences between convincing themselves and convincing others. This was clear when they said that convincing the class entailed organizing the evidence and constructing an argument in a coherent way (Hersh, 1993; Hoyles, 1997; Mason et al., 1992). They demonstrated flexibility in thinking about the problem differently. This manifested in the remarkable way they reversed their strategy (Krutetskii, 1976) to conclude that the statement was indeed true.

In terms of Strunz's (1962) classification of styles of mathematical giftedness, the 4 gifted students showed a preference for immediately observable relations and induction, but were conceptually aware that proving a statement entailed all possible cases. They could reason in the abstract and would be classified as amalgams of the "empirical" and "conceptual" types. If one utilized the holistic classification of the Soviet researchers, the 4 students' mathematical giftedness was of the harmonic type, a combination of the analytic and geometric types. They were able to use their pictorial representations to induce the truth of the statement analytically (Ivanitsyna, 1970; Krutetskii, 1976; Menchinskaya, 1959; Shapiro, 1965; Yakimanskaya, 1970).

Finally, the gifted students exhibited great tenacity and perseverance (Burton, 1984; Diezmann & Watters, 2003; Sriraman, 2003a) and stuck with the problem until they were absolutely convinced about their conclusion. Their approach to proof in this study was very different from the logical approach to proof found in most textbooks and very similar to those used by professional mathematicians. The processes used by the gifted students to prove the truth of the statement showed remarkable isomorphisms to those employed by professional mathematicians as discussed in the previous section.

The logical approach is an artificial reconstruction of discoveries that are being forced into a deductive system, and, in this process, the intuition that guided the discovery process gets lost. The implication here is that many teachers use the logical approach to proof in the classroom and thereby subdue gifted students' intuition and natural ways of thinking about the problem. These 4 students would eventually encounter the study of geometry from an intuitive standpoint in the second year of the research-based and standards-aligned integrated mathematics course in which geometry is introduced from an inductive and intuitive standpoint in the context of transformations. In this sequence, the necessity for formal proof is gradually introduced. However gifted students enrolled in a traditional sequence of mathematics courses encounter the study of Euclidean geometry and deductive proof, which robs them of using their natural instincts to establish truth like mathematicians do. The implication for gifted education is the need to develop mathematics curricula that create opportunities for gifted students to develop their intuition about proof and make use of challenging and worthwhile mathematical tasks.

Limitations

The students in this study were freshmen enrolled in various sections of integrated math in a rural high school. Demographically speaking, they were all White, with middle-class socioeconomic backgrounds. They all had the same K–8 educational background. All 4 students had very high academic aspirations and intended to take Integrated Math 4 and AP Calculus concurrently in their senior year. The 4 students had a positive disposition toward mathematics and had enjoyed a high degree of success in all previous mathematical endeavors. These students had not been exposed to constructing mathematical proofs, nor had they formally studied geometry. The results of this study are attributable to the unique characteristics of the population, the particular problem chosen, and the interview design. The processes used by the mathematically gifted students to construct a proof and their intuitive notions of proof showed remarkable similarities to those of professional mathematicians. In order to generalize these findings to mathematically gifted students who are entering high school with similar middle school backgrounds, more research is needed at the early high school level. It is certainly feasible to replicate this experiment with similar types of open-ended problems that call for establishing the truth or falsity of mathematical statements.

I conjecture that mathematically gifted students have the natural intuitive dispositions of mathematicians. It would be a worthwhile endeavor for the gifted education community to have a deeper understanding of these dispositions in order to develop high school curricula and pedagogies that nurture these natural talents.

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Endnote

1. Sarah is referring to exercises in ratio and proportion that give a sequence of steps to establish basic identities.