GINZBURG–LANDAU VORTICES: WEAK STABILITY AND SCHRÖDINGER EQUATION DYNAMICS

By

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CONTENTS

CHAPTER 1. INTRODUCTORY MATERIAL

1 Introduction

We consider the $\epsilon \to 0$ behavior of the initial value problem for the Ginzburg-Landau Schrödinger equation,

$$GLS_\epsilon \qquad \begin{cases} iu_t^\epsilon - \Delta u^\epsilon + \frac{1}{\epsilon^2}\Big(|u^\epsilon|^2 - 1\Big)u^\epsilon = 0, & u^\epsilon : \mathbb{T}^2 \times [0,T) \longmapsto \mathbb{R}^2, \\ u^\epsilon(x,0) = \phi^\epsilon(x), & x \in \mathbb{T}^2. \end{cases}$$

The quantity

$$(1.1.1)\qquad I^\epsilon[u]=\int_{\mathbb{T}^2} E^\epsilon(u)dx;\ \ E^\epsilon(u)=\frac{1}{2}|Du|^2+\frac{1}{4\epsilon^2}\left(|u|^2-1\right)^2$$

is the Hamiltonian for the evolution GLS_ϵ. We assume that the initial data ϕ^ϵ has a finite number of discrete "vortices", and that the energy of ϕ^ϵ away from the vortices is $O(1)$. We show that these vortex structures are preserved by the evolution and that, under further assumptions on the initial data, their motion can be described. Our main result is that they behave in the limit $\epsilon \to 0$ exactly like classical fluid-dynamical point vortices on the torus $\mathbb{T}^2$. The main results of this paper were announced in [7].

This work is motivated by both mathematical and physical considerations. Mathematically, it is related to recent efforts to study the asymptotic behavior of a number of PDEs associated with the Ginzburg–Landau functional (1.1.1) in the limit $\epsilon \to 0$. Among such PDEs, it is natural to consider the Euler–Lagrange equation, heat equation, and wave equation, as well as the Schrödinger equation:

$$\left.\begin{array}{l} \text{---} \\ k_\epsilon u_t \\ k_\epsilon u_{tt} \\ k_\epsilon i u_t \end{array}\right\} - \Delta u + \frac{1}{\epsilon^2}(|u|^2-1)u = 0.$$

Here k_ϵ denotes a scaling factor, which may be different for different equations.

The limiting behavior of solutions of the Euler–Lagrange equation on a set $U \subset \mathbb{R}^2$ was described in great detail by Bethuel, Brezis and Hélein in [2], with later refinements by Struwe [23] and Lin [16], among others. These works show that, under appropriate assumptions, asymptotics of solutions are completely determined once one knows the location of a number of limiting singular points, or vortices, and, moreover, that the vortex locations are critical points of a renormalized energy which can be computed explicitly. This renormalized energy W is a function on a finite-dimensional space of vortex configurations, that is, $W = W(a), a \in U^m$, for some integer m, which in simple cases is determined by the boundary data.

Remarkably, the same renormalized energy governs the asymptotic behavior of *all* the equations shown above. Limiting behavior of the Ginzburg–Landau heat flow was studied by Lin [17], [18], and Jerrard and Soner [14]. These works demonstrate that vortices evolve on slow time scales by a gradient flow of the renormalized energy.

This paper establishes analogous results for the Schrödinger equation, where the limiting ODE is now a Hamiltonian system. As far as we know, this is the first proof of this result, although numerous formal arguments have appeared in the physics and applied math literature.

Some partial results on the wave equation appear in Lin [15], and a complete description of limiting behavior of solutions of this system is given in Jerrard [13].

Thus, this paper and the others cited above show that, schematically, each of the above PDEs converges as $\epsilon \to 0$ to a finite dimensional problem of the same general type as the original problem:

$$\left.\begin{array}{l} \text{---} \\ \dot{a}(t) \\ \ddot{a}(t) \\ \mathbf{J}\,\dot{a}(t) \end{array}\right\} - D_a W(a(t)) = 0.$$

In the final equation, **J** represents a symplectic matrix, the details of which depend upon the signs of the vortices.

Physically, GLS_ϵ arises in models of superconductivity. The Landau theory of second order phase transitions (see Chapter 8 in [24]) consists of expanding the energy in terms of a parameter which encodes the "order" in the phase and then exploiting energy properties to determine the evolution of the "order parameter". This theory was applied by Ginzburg and Pitaevskii [10] and Pitaevskii [21] to argue that the order parameter describing superfluid helium II evolves according to GLS_ϵ. In this context, I^ϵ is the free energy and u^ϵ is the order parameter "which plays the role of 'the effective wave function' of the superfluid part of the liquid" [10]. The motion of u^ϵ under the GLS_ϵ evolution conserves $I^\epsilon[u^\epsilon]$. If we express $u^\epsilon(x) = \rho(x)e^{i\theta(x)}$, with ρ, θ $\mathbb{R}$-valued, then ρ^2 represents the density of the superfluid and $D\theta$ is the velocity of the superfluid. Gross [11] also derived GLS_ϵ as the Schrödinger equation for a wave function describing a system of interacting bosons. The equation GLS_ϵ is often called the Gross–Pitaevskii equation in the physics literature.

We briefly discuss the contents of this paper. In the rest of Chapter 1 we introduce some notation and background material and state our main results. Some of the background material is well-known from the work of Bethuel et al. [2], and some is new in this context. In particular, a novel feature of our approach is that we identify vortices as points of concentration of the Jacobian $Ju^\epsilon(t) = \det Du^\epsilon(t)$ of solutions of GLS_ϵ. This is physically natural, since the Jacobian more or less corresponds to the vorticity of the superfluid.

Our main results concern not only vortex dynamics, but also some variational results on the renormalized energy, and a detailed characterization of vortex structure. An important consequence of the latter is the topological stability of vortices in weak function spaces.

In Chapter 2 we prove our results on vortex dynamics. The key identity is provided by taking the curl of the equation for conservation of momentum, which may be thought of as writing the Euler equations for the superfluid flow in terms of the vorticity. Through this equation, we are able to control dynamics of vortices by studying limits of spatial gradients Du^ϵ. We also make essential use of results on vortex structure and renormalized energy, that are established in Chapters 3 and 4.

Chapter 3 presents the proofs of the results on vortex structure. We present these results in a general n-dimensional setting, since most of our arguments are quite insensitive to the dimension. We also discuss some extensions. Finally, Chapter 4 contains our results on the renormalized energy.

2 Notation and the quantities $j(u)$, $[Ju]$, μ_u^ϵ

We begin by introducing some basic notation and concepts. We use the summation convention throughout, except where explicitly noted. We employ O and o notation in some of the analysis below. We write, for example, $O_{a,b,c}(1)$ to indicate a quantity is $O(1)$ with respect to the interesting limit $\epsilon \to 0$, with the implicit constant depending only upon the parameters a, b, c. We normally think of solutions u^ϵ of GLS_ϵ as taking values in $\mathbb{R}^2$. In particular, the symbol " $\cdot$ " denotes the scalar product in $\mathbb{R}^2$, *not* multiplication of complex numbers:

$$u \cdot v := u^i v^i; \qquad u = (u^1, u^2), \quad v = (v^1, v^2).$$

We will, however, feel free to use notation such as iu or $e^{i\alpha}u$, etc. These are interpreted in the obvious way.

We define a 2 by 2 matrix $\mathbb{J}$ by

$$\mathbb{J}_{ij} := \begin{cases} 1 & \text{if } i = 1 \text{ and } j = 2, \\ -1 & \text{if } i = 2 \text{ and } j = 1, \\ 0 & \text{if } i = j, \end{cases} \tag{1.2.1}$$

that is,

$$\mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $u, v \in \mathbb{R}^2$ we also use the notation

$$u \times v \ := \ u^1 v^2 - u^2 v^1 \ = \ \mathbb{J}_{ij} u^i v^j,$$

$$\nabla \times u \ := \ \partial_{x_1} u^2 - \partial_{x_2} u^1 \ = \ \mathbb{J}_{ij} \partial_{x_i} u^j.$$

Note that $iu = -\mathbb{J}u$, so that $(iu) \cdot v = u \times v$. Similarly, $(iu) \cdot u = 0$, and $u \cdot v = (iu) \cdot (iv)$.

For a scalar function ϕ, we define $\nabla \times \phi := (\phi_{x_2}, -\phi_{x_1})$, so that $(\nabla \times \phi)^i = \mathbb{J}_{ij}\phi_{x_j}$.

For a sufficiently differentiable $\mathbb{R}^2$-valued function u we define

$$j(u) := (u \times u_{x_1}, u \times u_{x_2}) \in \mathbb{R}^2. \tag{1.2.2}$$

We also write, when convenient, $j(u) = u \times Du$ or $j(u) = (iu) \cdot Du$.

In the physical model for superfluids, if u^ϵ is a solution of GLS_ϵ, then $j(u^\epsilon)$ is interpreted as the current.

If u is written in the form $u = \rho e^{i\theta}$ for $\mathbb{R}$-valued functions ρ and θ, then $j(u) = \rho^2 D\theta$. In particular, if $|u| \equiv 1$, then $j(u)$ is the phase gradient.

We also define the signed Jacobian of u,

$$Ju := \det Du. \tag{1.2.3}$$

These quantites will play a central role in our analysis.

Note that they are related by the identity

$$\nabla \times j(u) \;=\; 2u_{x_1} \times u_{x_2} \;=\; 2Ju.$$

For any function u such that $j(u) \in L^1$, we can use this identity to make sense of the signed Jacobian as a distribution, or as an element of the dual of C^1. We write $[Ju]$ to denote the distributional signed Jacobian of u, defined by

$$\int \phi[Ju] := \frac{1}{2}\int \nabla \times \phi \cdot j(u), \qquad \phi \in C^1_c.$$

In terminology used in elasticity theory, Ju and $[Ju]$ correspond to $\det Du$ and $\operatorname{Det} Du$, respectively.

If a function u is sufficiently smooth, then $[Ju]$ and Ju can be naturally identified with each other. This is certainly true for $u \in H^1(\mathbb{T}^2)$, and it holds more generally whenever $[Ju]$ can be represented by an L^1 function; see [20]. However, it is not true in general. For example, if $u(x) = x/|x|$, then Ju is a function which vanishes a.e., whereas $[Ju] = \delta_0$. We often write $[Ju]$ even when Ju and $[Ju]$ can be identified, to emphasize that we are thinking of the signed Jacobian in the sense of distributions.

We mention a few properties of $j(u)$ and Ju. These are discussed in more detail in Chapter 3, in a more general setting. First, note that by Hölder and Sobolev inequalities, on any bounded two-dimensional set,

$$u \in W^{1,p},\ p \in \left[\frac{4}{3}, 2\right) \qquad \Longrightarrow j(u) \in L^q, \quad \text{for all } 1 \le q \le \frac{2p}{4-p}, \tag{1.2.4}$$

$$(1.2.5) \qquad u \in H^1 \quad \Longrightarrow j(u) \in L^q, \quad \text{for all } 1 \leq q < 2.$$

The following lemma follows from basic facts about weak and strong convergence. As is mentioned in Chapter 3, an appropriate generalization remains valid in higher dimensions.

Lemma 1.2.1 (Weak continuity of Jacobians). *If $u_k \rightharpoonup \bar{u}$ weakly in $W^{1,p}$, then*

$$j(u_k) \rightharpoonup j(\bar{u})$$

weakly in L^q where q is related to p as in (1.2.4) *and* (1.2.5) *above. Also,*

$$[Ju_k] \to [J\bar{u}]$$

in the sense of distributions.

We use $[J\phi^\epsilon]$ as a way of specifying the vortex locations in the initial data ϕ^ϵ. In particular, we always assume that there exist m points $\alpha_1, \dots \alpha_m \in \mathbb{T}^2$ and integers $d_1, \dots, d_m \in \{\pm 1\}$ such that

$$[J\phi^\epsilon] \rightharpoonup \sum \pi d_i \delta_{\alpha_i}.$$

Informally, this specifies that ϕ^ϵ has a vortex of degree d_i near each point α_i.

For a given $u \in H^1(\mathbb{T}^2; \mathbb{R}^2)$, define the measure

$$(1.2.6) \qquad \mu_u^\epsilon(A) = \frac{1}{|\log \epsilon|} \int_A E^\epsilon(u)\, dx; \qquad E^\epsilon(u) = \frac{1}{2}|Du|^2 + \frac{1}{4\epsilon^2}\left(|u|^2 - 1\right)^2,$$

for subsets $A \subset \mathbb{T}^2$. The renormalization factor $1/|\log \epsilon|$ appears naturally upon considering $u = x/|x|$ smoothly cutoff in a ball of radius ϵ centered at $x = 0$. The factor $1/|\log \epsilon|$ will be further examined in the next section.

Sometimes, we write μ_t^ϵ and $[J_t^\epsilon]$ to compactly express $\mu_{u^\epsilon(t)}^\epsilon$ and $[Ju^\epsilon(t)]$, respectively.

In Chapter 3 we work in $\mathbb{R}^n$ with definitions of $j(u)$ and $[Ju]$ that generalize the notions given here.

A geometric norm on measures Next, we introduce some concepts which permit us to say when two measures are close. For an open subset U of a topological space, let $C_0(U)$ be the Banach space of continuous functions on U which vanish on ∂U. Let $\mathcal{M}(U)$ denote the dual of $C_0(U)$, i.e., the space of finite signed Radon measures on U. Similarly, let $\mathcal{M}^1$ denote the dual of $C_0^1(U)$. Each of these spaces is equipped with the appropriate dual norm.

The following fact is well-known and easy to prove.

Lemma 1.2.2. *Suppose that U is a subset of some metric space and $\bar{U}$ is compact. A bounded sequence $\{\mu_n\} \subset \mathcal{M}(U)$ converges to a measure μ in the weak-* topology on $\mathcal{M}(U)$ if and only if*

$$||\mu_n - \mu||_{\mathcal{M}^1(U)} \to 0.$$

We now specialize to the case $U = \mathbb{T}^2$, although versions of the following facts are true in greater generality. However, the fact that the torus has no boundary lets us bypass certain inessential technical issues.

We define a seminorm

$$||\mu||_{\widehat{\mathcal{M}}^1(\mathbb{T}^2)} := \sup\left\{\int \phi d\mu \ : \ ||D\phi||_\infty \le 1, \ \int \phi = 0\right\}.$$

If $\mu(\mathbb{T}^2) = 0$, we can compute $||\mu||_{\mathcal{M}^1}$ by testing μ against functions ϕ such that $\int \phi = 0$. In this case it follows that

$$C||\mu||_{\widehat{\mathcal{M}}^1(\mathbb{T}^2)} \ \le \ ||\mu||_{\mathcal{M}^1(\mathbb{T}^2)} \ \le \ ||\mu||_{\widehat{\mathcal{M}}^1(\mathbb{T}^2)}. \tag{1.2.7}$$

This is a consequence of the fact that

$$||D\phi||_{L^\infty(\mathbb{T}^2)} \le ||\phi||_{C^1(\mathbb{T}^2)} \le C||D\phi||_{L^\infty(\mathbb{T}^2)}$$

whenever $\int \phi = 0$.

If μ has the form

$$\mu = \sum_{i=1}^n \delta_{\xi_i} - \sum_{i=1}^n \delta_{\eta_i}$$

for some points $\xi_1, ..., \xi_n, \eta_1, ..., \eta_n \in \mathbb{T}^2$, not necessarily distinct, then Brezis, Coron and Lieb [5] show that

$$||\mu||_{\widehat{\mathcal{M}}^1} = \min_\pi \sum |\xi_i - \eta_{\pi(i)}|, \tag{1.2.8}$$

where the minimum is taken over all permutations $\pi \in S_n$. For measures of this form we thus have

$$C^{-1} \min_\pi \sum |\xi_i - \eta_{\pi(i)}| \ \le \ ||\mu||_{\mathcal{M}^1} \ \le \ \min_\pi \sum |\xi_i - \eta_{\pi(i)}|. \tag{1.2.9}$$

(It is an easy exercise to verify this directly for μ of the form $\mu = \delta_\eta - \delta_\xi$.)

This demonstrates that the $\mathcal{M}^1$ norm records the geometric distance between the locations of Dirac masses.

The following lemma illustrates the usefulness of the $\mathcal{M}^1$ norm.

Lemma 1.2.3. *Suppose that for every $t \in [0,T]$, μ_t is a measure of the form $\sum_{1=i}^n \delta_{\xi_i(t)}$, for certain points $\xi_i(t), ..., \xi_n(t)$.*

Then $\mu_{(\cdot)}$ is a continuous (resp. Lipschitz) function from $[0,T]$ into $\mathcal{M}^1$ if and only if the points $\{\xi_i(t)\}$ can be labelled in such a way that $\xi_i(\cdot)$ is continuous (resp. Lipschitz) for each i.

Proof. For any s, t, the measure $\mu_t - \mu_s$ has integral zero and so satisfies (1.2.9), so that

$$||\mu_t - \mu_s||_{\mathcal{M}^1(\mathbb{T}^2)} \sim \min_{\pi} \sum |\xi_i(t) - \xi_{\pi(i)}(s)|.$$

The lemma follows immediately. □

Remarks.

1. The $\widehat{\mathcal{M}}^1$ seminorm can be interpreted as the minimum cost in a Monge–Kantorovitch mass transfer problem. Indeed, when $\mu(\mathbb{T}^2) = 0$ as above, μ can be written in the form $\mu = \nu^1 - \nu^2$, where ν^1 and ν^2 are positive, mutually singular measures and $\nu^1(\mathbb{T}^2) = \nu^2(\mathbb{T}^2)$. Then $||\mu||_{\widehat{\mathcal{M}}^1}$ is precisely the minimum cost of "transporting" ν^1 to ν^2, subject to a linear cost functional. See, for example, [9] for a more precise statement and more details. Equivalently, $||\mu||_{\widehat{\mathcal{M}}^1}$ is also known as the distance between ν^1 and ν^2 in the $\mathcal{L}^1$ Wasserstein metric.

2. Any reasonable weak norm on measures would be equally suitable for our purposes. The $\mathcal{M}^1$ norm is a convenient choice, but it is certainly not the only possible choice.

Finally, we note one more property of the $\mathcal{M}^1$ norm.

Lemma 1.2.4. *Suppose $u, v \in W^{1,p}(\mathbb{T}^2; \mathbb{R}^2)$ for some $p \geq 4/3$. Then*

$$||[Ju] - [Jv]||_{\mathcal{M}^1} \leq C||u - v||_{W^{1,p}} \left(||u||_{W^{1,p}} + ||v||_{W^{1,p}}\right).$$

Proof. We present the proof for $p = 4/3$. We use Hölder inequality and

Sobolev inequalities to estimate

$$
\begin{aligned}
2\,||[Ju]-[Jv]||_{\mathcal{M}^1} &= \sup_{||\phi||_{C^1}\le 1}\int \nabla\times\phi\cdot(j(u)-j(v)) \\
&\le \int |j(u)-j(v)| \\
&\le \int |u-v||Du| + |v||Du-Dv|\;dx \\
&\le ||u-v||_{L^4}||Du||_{L^{4/3}} + ||v||_{L^4}||Du-Dv||_{L^{4/3}} \\
&\le C||u-v||_{W^{1,4/3}}(||u||_{W^{1,4/3}}+||v||_{W^{1,4/3}}).
\end{aligned}
$$

□

3 Harmonic maps and renormalized energy

Bethuel, Brezis and Hélein [2] define a renormalized energy which governs the asymptotics of Ginzburg–Landau energy minimizers on bounded subsets of $\mathbb{R}^2$ with prescribed boundary data. Here we reformulate their definition on the torus $\mathbb{T}^2$. Since boundary terms no longer appear, the definition becomes a little simpler.

A general reference for everything in this section is Chapter 1 of [2].

Let F solve

$$\Delta F = 2\pi(\delta_0 - 1) \quad \text{on } \mathbb{T}^2. \tag{1.3.1}$$

Evidently

$$\Delta[F(x) - \ln|x|] \equiv -2\pi$$

in an open ball containing the origin, so $F(x) - \ln|x|$ is a C^∞ function in a neighborhood of the origin. We normalize F by imposing the condition

$$\lim_{x\to 0}(F(x) - \ln|x|) = 0. \tag{1.3.2}$$

The canonical harmonic map We define a canonical harmonic map from the punctured torus into S^1. This map is determined up to a phase by the location and degree of its singular points.

Let $\Phi = \Phi(\cdot; a, d)$ solve

$$\Delta\Phi = 2\pi\sum_{i=1}^{n} d_i\delta_{a_i}, \qquad \int \Phi = 0$$

in $\mathbb{T}^2$. Note that

$$\Phi(x;a,d) = \sum_i d_i F(x - a_i), \tag{1.3.3}$$

where F is the fundamental solution defined above.

Note that if u is any function in $W^{1,1}(\mathbb{T}^2;S^1)$ which is smooth away from a finite number of singular points, then, writing $u = e^{i\theta}$,

$$\int j(u)\, dx = \int D\theta\, dx = 2\pi k$$

for some $k = (k_1, k_2) \in \mathbb{Z}^2$, by periodicity. A theorem of Bethuel and Zheng [3] states that such functions are dense in $W^{1,1}(\mathbb{T}^2, S^1)$, so the same fact holds for all functions in this space. Finally, suppose that $\{v^\epsilon\}$ is a sequence of functions that converges weakly in $W^{1,1}(\mathbb{T}^2, \mathbb{R}^2)$ to a limit v satisfying $|v| = 1$ a.e. The weak continuity of j implies that

$$\lim_{\epsilon \to 0} \int j(v^\epsilon)\, dx = 2\pi k \tag{1.3.4}$$

for some $k \in \mathbb{Z}^2$. We will assume $k = 0$ as a normalization condition on the initial data, even if we do not assume that the initial data converges to some weak limit.

With this in mind we state the following proposition, which is essentially proven in [2].

Proposition 1.3.1. *There is a map $H \in C^\infty_{loc}(\mathbb{T}^2 \setminus (a); S^1) \cap W^{1,1}(\mathbb{T}^2; S^1)$ satisfying*

$$\text{div} j(H) = 0, \tag{1.3.5}$$

$$2[JH] = \nabla \times j(H) = 2\pi \sum_{i=1}^m d_i \delta_{a_i} \tag{1.3.6}$$

and

$$\int_{\mathbb{T}^2} j(H) = 0. \tag{1.3.7}$$

The first two equations hold in $\mathcal{D}'(\mathbb{T}^2)$, and they are also true pointwise away from the singluar points (a). Moreover, if $\widetilde{H}$ is any other function satisfying (1.3.5), (1.3.6), (1.3.7) *then $\widetilde{H} = e^{i\alpha} H$ for some $\alpha \in \mathbb{R}$.*

We sketch the proof after first making some comments.

We refer to H as the canonical harmonic map with singularities (a) of degree (d). We have taken the name "canonical harmonic map" from the work of Bethuel, Brezis and Hélein on the Dirichlet problem. In the periodic context it is clearly something of a misnomer, since H is not unique. Nonetheless it seems easiest to use the familiar terminology. The fact that H is a harmonic map into S^1 is expressed in (1.3.5) above, and that it has singularities of degree d_i at points a_i is contained in (1.3.6).

The idea of the proof is as follows: using (1.3.5), (1.3.7), and the definition of Φ, we integrate (1.3.6) to obtain

$$j(H) = -\nabla \times \Phi. \tag{1.3.8}$$

Since H takes its values in S^1, we can make the ansatz

$$H(x) = \exp(i\theta(x)),$$

and (1.3.8) becomes $D\theta = -\nabla \times \Phi$. One can then fix an arbitrary value for θ at some point and use this equation to solve for a (multivalued) θ. One finishes by verifying that $H = \exp(i\theta)$ is well-defined and has the stated properties.

We henceforth assume that we have selected a single H from the one-parameter family of functions satisfying (1.3.8). We will sometimes write $H(x; a, d)$ to indicate the dependence of H on the singularities.

Given a collection of points $a_1, ..., a_m \in \mathbb{T}^2$ and nonzero integers $d_1, ..., d_m$ such that $\sum_i d_i = 0$, define for $\rho > 0$ the set

$$\mathbb{T}^2_\rho := \mathbb{T}^2 \setminus \bigcup_i B_\rho(a_i).$$

We will normally be interested in the case

$$0 < \rho < \tfrac{1}{4} \min_{i \neq j} |a_i - a_j|. \tag{1.3.9}$$

Define also the renormalized energy

$$W(a, d) := -\pi \sum_{i \neq j} d_i d_j F(a_i - a_j). \tag{1.3.10}$$

The renormalized energy W describes to leading order the finite part of the energy associated with a configuration of vortices $(a), (d)$. We next restate this idea in several more precise ways.

The following proposition comes from [2].

Proposition 1.3.2. *Let $H = H(\cdot, a, d)$. For ρ satisfying* (1.3.9),

$$\int_{\mathbb{T}^2_\rho} \frac{1}{2} |DH|^2 dx = m\pi \ln\left(\frac{1}{\rho}\right) + W(a, d) + O(\rho).$$

The proof follows by noting that $|DH|^2 = |D\Phi|^2$, and then integrating by parts to obtain

$$\int_{\mathbb{T}^2_\rho} |DH|^2 dx = -\int_{\bigcup_i \partial B_\rho(a_i)} \Phi \frac{\partial \Phi}{\partial \nu}.$$

The right hand side can be estimated using the explicit representation of Φ (1.3.3).

Following Bethuel, Brezis and Hélein [2], we define

$$I(\epsilon, \rho) = \min \left\{ \int_{B_\rho} E^\epsilon(u) dx \ : \ u \in H^1(B_\rho), u(x) = \frac{x}{|x|} \text{ for } x \in \partial B_\rho \right\}. \tag{1.3.11}$$

It is shown in [2] that

$$I(\epsilon, \rho) = \pi \ln \left(\frac{\rho}{\epsilon}\right) + O(1) \tag{1.3.12}$$

as $\epsilon \to 0$, for ρ fixed; the upper bound can be established easily by constructing an appropriate comparison function.

A construction given in [2] (Lemma VIII.1) can be adapted to show that

Proposition 1.3.3. *Given any distinct $a_1, ...a_m \in \mathbb{T}^2$ and $d_1, ..., d_m \in \{\pm 1\}$, there exists a amily of functions $v^\epsilon \in H^1(\mathbb{T}^2; \mathbb{R}^2)$ such that*

$$[Jv^\epsilon] \rightharpoonup \sum_{i=1}^m \pi d_i \delta_{a_i} \quad \textit{weakly in } \mathcal{M},$$

$$\int j(v^\epsilon) \to 0,$$

and for every $\rho > 0$

$$\int_{\mathbb{T}^2} E^\epsilon(v^\epsilon) dx \le m \left(\pi \ln \left(\frac{1}{\rho}\right) + I(\epsilon, \rho) \right) + W(a, d) + C\rho + o(1)$$

as $\epsilon \to 0$.

On the other hand, we will establish later that

Proposition 1.3.4. *Given any distinct $a_1, ...a_m \in \mathbb{T}^2$ and $d_1, ..., d_m \in \{\pm 1\}$, and any family of functions $v^\epsilon \in H^1(\mathbb{T}^2; \mathbb{R}^2)$ such that*

$$[Jv^\epsilon] \rightharpoonup \sum \pi d_i \delta_{a_i} \quad \textit{weakly in } \mathcal{M}$$

and

$$\int j(v^\epsilon) \to 0,$$

there exists a constant C independent of ϵ, ρ *such that*

$$\int_{\mathbb{T}^2} E^\epsilon(v^\epsilon)\, dx \geq m\left(\pi \ln\left(\frac{1}{\rho}\right) + I(\epsilon,\rho)\right) + W(a,d) - C\rho + o(1)$$

as $\epsilon \to 0$, *for every* $\rho > 0$.

4 Results

We begin with a discussion of our assumptions on the initial data ϕ^ϵ of GLS_ϵ. We then briefly consider an extremely simple example that illustrates some of the analytic issues. Then we state our main results describing the dynamics of vortices of the solution $u^\epsilon(t)$ of GLS_ϵ. We conclude by stating results which describe structural properties of functions satisfying the assumptions on the initial data. The structure results are fundamental to characterizing the vortex dynamics of the Ginzburg–Landau Schrödinger equation. We expect that these structure results will also be useful in other problems involving vortex motion.

Initial data assumptions We impose three conditions on the initial data $\phi^\epsilon : \mathbb{T}^2 \longmapsto \mathbb{C}$. First, we assume

(1.4.1) $$[J\phi^\epsilon] \rightharpoonup \pi \sum_{i=1}^{m} d_i \delta_{\alpha_i} \quad \text{in } \mathcal{M}(\mathbb{T}^2),$$

where $d_i = \pm 1$, with $\sum_{i=1}^m d_i = 0$ and the α_i are distinct points in $\mathbb{T}^2$.

The next assumption is that the energy of ϕ^ϵ is bounded in some appropriate way consistent with (1.4.1). A relatively weak assumption of this sort is that

(1.4.2) $$\int E^\epsilon(\phi^\epsilon) dx \leq m\pi \ln\left(\frac{1}{\epsilon}\right) + \gamma_1$$

for some $\gamma_1 > 0$ and all $\epsilon \in (0, 1]$. We will be able to get more detailed information about asymptotic behavior of solutions under the assumption

(1.4.3) $$\int E^\epsilon(\phi^\epsilon) dx \leq m\left(\pi \ln\left(\frac{1}{\rho}\right) + I(\epsilon,\rho)\right) + W(a,d) + C\rho + o(1)$$

as $\epsilon \to 0$, for every $\rho > 0$.

The third assumption is the normalization condition that ϕ^ϵ has an average phase gradient that converges to 0 as $\epsilon \to 0$,

(1.4.4) $$\lim_{\epsilon \to 0} \int_{\mathbb{T}^2} j(\phi^\epsilon) dx = 0.$$

The rationale for this assumption is discussed following (1.3.4).

We observed in Proposition 1.3.3 that it is possible to construct data ϕ^ϵ satisfying these assumptions.

We will see later that the energy upper bound (1.4.3) forces ϕ^ϵ to converge in $H^1_{loc}(\mathbb{T}^2 \setminus (\alpha))$ to the canonical harmonic map $H = H(\cdot, \alpha, d)$ (modulo a phase). Contrast this with the assumption (1.4.2) which allows ϕ^ϵ to converge in the same sense to $e^{i\psi(\cdot)}H(\cdot, \alpha, d)$ where $\psi \in H^1(\mathbb{T}^2)$ with $\|\psi\|^2_{H^1}$ controlled by γ_1. Thus (1.4.2) is a much weaker assumption than (1.4.3). Stated differently, the assumption (1.4.2) asserts that the energy of ϕ^ϵ is within $O(1)$ of the minimum subject to the constraint (1.4.1). The assumption (1.4.3) strengthens this condition to within $o(1)$ of the minimum.

A simple example Consider the simple situation where $\phi^\epsilon = \rho e^{i\theta}$ for constants $\rho, \theta \in \mathbb{R}$. The solution of GLS_ϵ is easily seen to be

$$u^\epsilon(x,t) = \rho e^{i(\theta + \frac{1}{\epsilon^2}[\rho^2 - 1]t)}$$

and

$$\int_{\mathbb{T}^2} E^\epsilon(\phi^\epsilon)dx = \frac{1}{4\epsilon^2}[\rho^2 - 1]^2.$$

Unless $\rho - 1 = O(\epsilon^2)$, the phase of the solution u^ϵ oscillates rapidly in the t variable for small ϵ. The assumption (1.4.2) forces $|\rho - 1| \leq O(\epsilon)$ and (1.4.3) forces $|\rho - 1| \leq o(\epsilon)$. So, under both energy upper bounds we may have rapid temporal oscillation in the phase of u^ϵ, forcing $u^\epsilon \rightharpoonup 0$ weakly in $L^p(dxdt)$. These observations reveal that most information about u^ϵ is lost upon passing to weak limits. Therefore, to identify vortices in the $\epsilon \to 0$ limit requires a device insensitive to these oscillations.

Note that $j(u) = j(e^{i\beta}u)$ for any constant $\beta \in \mathbb{R}$, which indicates that $j(u)$ is insensitive to temporal oscillations in the phase of u. The insensitivity to phase oscillations leads us to expect $j(u^\epsilon)$ to retain more information under passage to weak limits.

Vortex dynamics Our first result shows that, even under the weaker assumption (1.4.2), vortex paths exist. Moreover, knowing their location, we can determine the average behavior (i.e. weak limits) of the current j^ϵ.

Theorem 1.4.1. *Let u^ϵ be the solution of GLS_ϵ with initial data ϕ^ϵ satisfying* (1.4.1), (1.4.4) *and* (1.4.2). *Then, after passing to a subsequence as $\epsilon \to 0$, there* *exists a* $T > 0$ *(independent of* ϵ*) and Lipschitz paths* $a_i : [0,T) \longmapsto \mathbb{T}^2$, $a_i(0) = \alpha_i$ *such that*

$$\mu^{\epsilon}_{u^\epsilon(t)} \rightharpoonup \pi \sum_{i=1}^{m} \delta_{a_i(t)} \tag{1.4.5}$$

and

$$[Ju^\epsilon(t)] \rightharpoonup \pi \sum_{i=1}^{m} d_i \delta_{a_i(t)} \tag{1.4.6}$$

weakly as measures for all $t \in [0,T)$. *Moreover,*

$$|u^\epsilon(t)|^2 \to 1 \quad \text{in } L^2(dx) \tag{1.4.7}$$

for all $t \in [0,T)$ *and*

$$j(u^\epsilon) \rightharpoonup j(H) \qquad \text{weakly in } L^p(dxdt) \tag{1.4.8}$$

for all $1 \leq p < 2$ *where* $H(\cdot,t) = H(\cdot, a(t), d)$ *is the canonical harmonic map. Finally,* $|a_i(t) - a_j(t)| > 0$ *for all* $t \in [0,T)$, *and*

$$T = \inf\{t > 0 : |a_i(t) - a_j(t)| \to 0 \text{ for some } i \neq j\}. \tag{1.4.9}$$

Under the stronger assumption (1.4.3), the limiting vortex trajectories can be found by solving an ODE, and the weak limits in the above theorem become strong limits. We thus obtain a nearly complete description of the limiting behavior of solutions u^ϵ.

Theorem 1.4.2. *Suppose* ϕ^ϵ *satisfies* (1.4.1), (1.4.4) *and* (1.4.3). *Then for each* i,

$$\begin{cases} \dfrac{d}{dt} a_i = 2 \sum_{j:j\neq i} d_j \nabla \times F(a_i - a_j) = -\dfrac{1}{\pi} d_i \mathbb{J} D_{a_i} W(a,d), \\ \qquad\qquad a_i(0) = \alpha_i. \end{cases} \tag{1.4.10}$$

Also, for every $t \in [0,T)$,

$$\frac{1}{|u^\epsilon|} j(u^\epsilon) \to j(H) \quad \text{strongly in } L^2_{loc}(\mathbb{T}^2 \setminus (a(t))), \tag{1.4.11}$$

and for every $\rho > 0$ *and* $t \in [0,T)$,

$$\lim_{\epsilon \to 0} \min_{\alpha \in [0,2\pi)} \left\| u^\epsilon(\cdot,t) - e^{i\alpha} H(\cdot, a(t), d) \right\|_{H^1(\mathbb{T}^2_\rho)} = 0. \tag{1.4.12}$$

Remarks.

1. Recall that we have arbitrarily fixed one map H from the one-parameter family $e^{i\alpha}H$ solving (1.3.8).

2. The above result is valid globally in time if the trajectories $(a_i(\cdot))$ solving (1.4.10) satisfy $a_i(t) \neq a_j(t)$ for all $i \neq j$ and all $t > 0$. We expect that this condition holds for generic initial data, but not for all data. An example (in a slightly different context) of vortices that collide in finite time is given in Marchioro and Pulvirenti [19].

Vortex structure and topological stability In order to prove the above theorems, we need to carry out a detailed analysis of the local structure of vortices. Perhaps the most important consequence of this analysis is that vortex-like objects in a function $u = u(x,t)$ are locally topologically stable if the evolution $t \mapsto u(\cdot, t)$ is continuous in H^1.

Here we state two results of this sort. The first, local structure theorem, is more basic.

Theorem 1.4.3 (Local Structure). *Suppose that* $\epsilon \leq r \leq 1$, $u \in H^1(B_r; \mathbb{R}^2)$,

$$|| \, [Ju] - \pi d\delta_0 \, ||_{\mathcal{M}^1(B_r)} \leq \frac{\pi}{200} r, \tag{1.4.13}$$

where $d = \pm 1$. *Assume also that*

$$\int_{B_r} E^\epsilon(u) \, dx \leq \pi \ln \left(\frac{r}{\epsilon}\right) + \gamma_1 \tag{1.4.14}$$

for some γ_1. *Then there exists a point* $\xi \in B_{r/2}$ *and a constant* $C_1 = C_1(\gamma_1) > 0$ *such that*

$$\int_{B_\sigma(\xi)} E^\epsilon(u) \, dx \geq \pi \log \left(\frac{\sigma}{\epsilon}\right) - C_1 \tag{1.4.15}$$

for every $\sigma \in [0, r/2]$. *Moreover,*

$$||\mu_u^\epsilon - \pi\delta_\xi||_{\mathcal{M}^1(B_r)} \leq \frac{C_1}{|\ln \epsilon|}, \tag{1.4.16}$$

$$||[Ju] - \pi d\delta_\xi||_{\mathcal{M}^1(B_r)} \leq o_{\gamma_1}(1). \tag{1.4.17}$$

Finally, for any $p \in [1, 2)$, *there exists some* $C_p = C_p(\gamma_1)$ *such that*

$$||Du||_{L^p(B_r)} \leq C_p. \tag{1.4.18}$$

The following result will be more directly useful for our analysis of vortex dynamics. It follows easily from Theorem 1.4.3 above.

Theorem 1.4.4 (Global Structure). *Suppose that* $u \in H^1(\mathbb{T}^2;\mathbb{R}^2)$*, and that there exist points* $x_1,...,x_m \in \mathbb{T}^2$*, integers* $d_1,...,d_m \in \{\pm 1\}$*, and* $\epsilon \leq r := \frac{1}{4}\min_{i\neq j}|x_i - x_j|$ *such that*

$$\| [Ju] - \pi \sum_{i=1}^{m} d_i \delta_{x_i} \|_{\mathcal{M}^1(\mathbb{T}^2)} \leq \frac{\pi}{200} r, \tag{1.4.19}$$

and

$$\int_{\mathbb{T}^2} E^\epsilon(u)\, dx \leq \pi m \log\left(\frac{r}{\epsilon}\right) + \gamma_1 \tag{1.4.20}$$

for some constant γ_1*. Then there exists points* $a_i \in B_{r/2}(x_i)$*,* $i = 1,...,m$ *such that*

$$\|\mu_u^\epsilon - \pi \sum_{i=1}^{m} \delta_{a_i}\|_{\mathcal{M}^1(\mathbb{T}^2)} \leq O_{\gamma_1}\left(\frac{1}{|\log \epsilon|}\right), \tag{1.4.21}$$

$$\|[Ju] - \pi \sum_{i=1}^{m} d_i \delta_{a_i}\|_{\mathcal{M}^1(\mathbb{T}^2)} \leq o_{\gamma_1}(1). \tag{1.4.22}$$

Moreover, for ρ *fixed and* $0 < \rho < r/2$,

$$\int_{\mathbb{T}^2_\rho} E^\epsilon(u) dx \leq O_{\rho,\gamma_1}(1), \tag{1.4.23}$$

$$\|Du\|_{L^2(\mathbb{T}^2_\rho)} \leq O_{\rho,\gamma_1}(1). \tag{1.4.24}$$

Finally, for any $p \in [1,2)$,

$$\|Du\|_{L^p(\mathbb{T}^2)} \leq O_{p,\gamma_1}(1) \tag{1.4.25}$$

and

$$\|j(u)\|_{L^p(\mathbb{T}^2)} \leq O_{p,\gamma_1}(1). \tag{1.4.26}$$

The structure results are proved in Chapter 3. Our results there are stated and proved in a much more general framework, one which applies to Ginzburg–Landau type functionals in arbitrary dimensions and to a Ginzburg–Landau type functional arising in models of superconductivity. Our analysis relies heavily on techniques introduced in [12]; similar ideas appear also in [22].

Results on renormalized energy For the proof of Theorem 1.4.2, we need some results about the renormalized energy. Before stating the result, we note some consequences of the definition of the current $j(u)$, which we recall may be written

$$j(u) = (iu \cdot u_{x_1}, iu \cdot u_{x_2}) = iu \cdot Du.$$

First, since $Du(x) = 0$ for a.e. x such that $u(x) = 0$, we can write

$$\begin{aligned} u_{x_k} &= \left[\frac{iu}{|u|} \cdot u_{x_k}\right] \frac{iu}{|u|} + \left[\frac{u}{|u|} \cdot u_{x_k}\right] \frac{u}{|u|} \\ &= \frac{j^k(u)}{|u|} \frac{iu}{|u|} + |u|_{x_k} \frac{u}{|u|}. \end{aligned} \tag{1.4.27}$$

It follows that

$$|Du|^2 = \frac{|j(u)|^2}{|u|^2} + |\, D|u|\, |^2. \tag{1.4.28}$$

In Chapter 4 we will establish

Theorem 1.4.5. *Suppose that* $u^\epsilon \in H^1$ *is a sequence such that*

$$[Ju^\epsilon] \rightharpoonup \pi \sum_{i=1}^{m} d_i \delta_{a_i}, \tag{1.4.29}$$

weakly as measures, and that there exists some $\gamma_2 \in \mathbb{R}$ *such that*

$$\int E^\epsilon[u^\epsilon] dx \le m \left(\pi \ln\left(\frac{1}{\rho}\right) + I(\epsilon, \rho)\right) + W(a, d) + C\rho + \gamma_2 + o(1) \tag{1.4.30}$$

as $\epsilon \to 0$, *for every* $\rho > 0$. *Then there exists some universal constant* C *such that*

$$\limsup_{\epsilon \to 0} \left\| \frac{1}{|u^\epsilon|} j(u^\epsilon) - j(H) \right\|^2_{L^2(\mathbb{T}^2_\rho)} \le C\gamma_2 \tag{1.4.31}$$

and

$$\limsup_{\epsilon \to 0} \|\, D|u^\epsilon|\, \|^2_{L^2(\mathbb{T}^2_\rho)} \le C\gamma_2 \tag{1.4.32}$$

for every $\rho > 0$. *Here* $H = H(\cdot; a, d)$ *is the canonical harmonic map. Finally,*

$$\limsup_{\epsilon \to 0} \|(|u^\epsilon|^2 - 1)\|^2_{L^2(\mathbb{T}^2_\rho)} \le C\gamma_2 \epsilon^2. \tag{1.4.33}$$

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CHAPTER 2. VORTEX DYNAMICS

In this chapter we prove Theorems 1.4.1 and 1.4.2. The proofs of both of these theorems rely on the Global Structure Theorem 1.4.4. Theorem 1.4.2 also depends upon the result on the renormalized energy Theorem 1.4.5. We begin by deriving evolution equations for certain nonlinear quantities involving u^ϵ assuming that u^ϵ evolves according to GLS_ϵ. In fact, we work in a slightly more general context for these derivations. Then, we present the proofs of Theorems 1.4.1 and 1.4.2.

1 Evolution identities

We first record some identities which hold for sufficiently smooth solutions of a general nonlinear Schrödinger equation of the form

$$iu_t - \Delta u + W'\left(\frac{|u|^2}{2}\right) u = 0. \tag{2.1.1}$$

Most of these are well-known.

First, if u is a smooth solution of (2.1.1), then

$$\begin{aligned} \frac{d}{dt}\frac{|u|^2}{2} &= u \cdot u_t \\ &= (iu)\cdot \Delta u \\ &= \left((iu)\cdot u_{x_j}\right)_{x_j} = \operatorname{div} j(u). \end{aligned} \tag{2.1.2}$$

Let

$$E := \frac{1}{2}|Du|^2 + W\left(\frac{|u|^2}{2}\right).$$

Then

$$\frac{d}{dt}E = Du \cdot Du_t + W'\left(\frac{|u|^2}{2}\right) u \cdot u_t$$
$$= (u_{x_j} \cdot u_t)_{x_j} - \left[\Delta u - W'\left(\frac{|u|^2}{2}\right) u\right] \cdot u_t$$
$$= (u_{x_j} \cdot u_t)_{x_j}, \tag{2.1.3}$$

since $(iu_t) \cdot u_t = 0$. This identity implies that energy is conserved for solutions of (2.1.1). A similar computation yields

$$E_{x_k} = (u_{x_j} \cdot u_{x_k})_{x_j} - (iu_t) \cdot u_{x_k}. \tag{2.1.4}$$

Next, for each $k = 1, 2$, we use (2.1.4) to compute

$$\frac{d}{dt}[(iu) \cdot u_{x_k}] = (iu_t) \cdot u_{x_k} + ((iu) \cdot u_t)_{x_k} - (iu_{x_k}) \cdot u_t$$
$$= 2(iu_t) \cdot u_{x_k} + ((iu) \cdot u_t)_{x_k}$$
$$= 2(u_{x_j} \cdot u_{x_k})_{x_j} + [2E + (iu) \cdot u_t]_{x_k}. \tag{2.1.5}$$

We write this as a vector identity in the form

$$\frac{d}{dt}j(u) = 2 \text{ div } (Du \otimes Du) + D[2E + (iu) \cdot u_t]. \tag{2.1.6}$$

This may be interpreted as expressing the conservation of momentum.

Finally, we take the curl of the above identity, recalling that $Ju = \frac{1}{2}\nabla \times j(u)$. Since the curl of a gradient is zero, we obtain

$$\frac{d}{dt}Ju = \frac{1}{2}\nabla \times j(u)_t$$
$$= \nabla \times \text{div } (Du \otimes Du). \tag{2.1.7}$$

Written out in full, this means that

$$\frac{d}{dt}Ju = \mathbb{J}_{kl}(u_{x_j} \cdot u_{x_k})_{x_j x_l}. \tag{2.1.8}$$

Multiply by a smooth function η and integrate to obtain

$$\int \eta \, [Ju] \,|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_{\mathbb{T}^2} \mathbb{J}_{kl} \eta_{x_j x_l} u_{x_j} \cdot u_{u_k} \, dx \, dt. \tag{2.1.9}$$

In deriving this, we have assumed that u is a smooth solution of (2.1.1). However, for certain choices of W in (2.1.1), in particular the choice making (2.1.1) into GLS_ϵ, the preceding calculations apply equally well to $u(\cdot) \in H^1(\mathbb{T}^2)$.

For $\epsilon > 0$ fixed, GLS_ϵ is a defocussing nonlinear Schrödinger equation. Bourgain has established [4] global wellposedness for GLS_ϵ below H^1. We validate the preceding calculations for $u^\epsilon \in H^1(\mathbb{T}^2)$ as follows using various aspects of Bourgain's result. By continuous dependence on the data, a different solution $\widetilde{u^\epsilon}(t)$ is close to $u^\epsilon(t)$ in $H^1(\mathbb{T}^2)$ provided the corresponding initial data $\widetilde{\phi^\epsilon}$ and ϕ^ϵ are close in $H^1(\mathbb{T}^2)$. Let $\widetilde{\phi^\epsilon}$ be a smooth approximator to ϕ^ϵ. The preceding calculations apply to $\widetilde{u^\epsilon}$ since it remains smooth for all time. The various identities above, in particular (2.1.9), are then validated for $u^\epsilon \in H^1(\mathbb{T}^2)$ by considering a sequence of smooth approximators.

We comment briefly on the usefulness of the identity (2.1.9) in our study of the dynamics of vortices of GLS_ϵ. Suppose we knew that $[Ju(t)] = \pi\delta_{a(t)}$, that $u(t)$ is smooth away from $a(t)$ for all $t \in [t_1, t_2]$, and that u is changing with time in some smooth way. We select a test function η supported in the ball $B_r(a(t_1))$ which is linear in $B_{r/2}(a(t_1))$. Then, the left side of (2.1.9) "feels" the motion of the Dirac mass $a(t_1) \to a(t_2)$ while the right side is controlled by the first derivatives of u inside supp $D^2\eta = B_r(a(t_1)) \setminus B_{r/2}(a(t_1))$, *away from the singularity.*

2 Vortex paths

The idea of the proof of Theorem 1.4.1 is to use the Jacobian evolution identity (2.1.9) to show that vortices move with velocity at most $O(1)$ in in the chosen time scale, at least for short times. It then follows that the hypotheses of the the Global Structure Theorem 1.4.4 are satisfied by $u^\epsilon(t)$, the evolving solution of GLS_ϵ, in a nontrivial time interval $[0, T)$, uniformly for all small ϵ. As a consequence, the t-parametrized measures $[J_t^\epsilon]$ and μ_t^ϵ concentrate to Dirac masses at points $a_i(t) \in \mathbb{T}^2$, and these points follow Lipschitz trajectories in the interval $[0, T)$. We employ some of the notation appearing in the statement of Theorem 1.4.4.

We now establish Theorem 1.4.1.

Proof.

1. Define for $r = \min_{i\neq j} \frac{1}{4}|\alpha_i - \alpha_j|$ with $0 < \epsilon \ll r \leq 1$ the quantity

$$T^\epsilon = \sup\left\{t \geq 0 : \; \|[Ju^\epsilon(s)] - [J\phi^\epsilon]\|_{M^1(\mathbb{T}^2)} \leq \frac{\pi}{400} r \text{ for all } 0 \leq s \leq t\right\}.$$

Recall that $Ju = \det Du = u^1_{x_1}u^2_{x_2} - u^1_{x_2}u^2_{x_1}$. The estimates

$$\|[Ju^\epsilon] - [J\phi^\epsilon]\|_{M^1} \leq \|Ju^\epsilon - J\phi^\epsilon\|_{L^1} \leq C\|u^\epsilon - \phi^\epsilon\|_{H^1}(\|u^\epsilon\|_{H^1} + \|\phi^\epsilon\|_{H^1})$$

and continuity of the flow of $\phi^\epsilon \longmapsto u^\epsilon(t)$ through H^1 guarantee $T^\epsilon > 0$.

2. **Claim.** For s, t satisfying $0 \leq s, t \leq T^\epsilon$ we have

$$\|[Ju^\epsilon(s)] - [Ju^\epsilon(t)]\|_{M^1(\mathbb{T}^2)} \leq c|s-t| + o_{\gamma_1}(1).$$

Proof of Claim. The definition of T^ϵ guarantees for all $t \in [0, T^\epsilon)$ that the hypothesis (1.4.19) of Theorem 1.4.4 holds if ϵ is sufficiently small. Therefore, for each $t \in [0, T^\epsilon)$, we can find points $a_i(t) \in B_{r/2}(\alpha_i)$, $i = 1, \ldots, m$, for which

$$\left\| [Ju^\epsilon(t)] - \pi \sum_{i=1}^{m} d_i \delta_{a_i(t)} \right\|_{M^1(\mathbb{T}^2)} \leq o_{\gamma_1}(1), \tag{2.2.1}$$

by (1.4.22). Of course, the $a_i(t)$ may depend upon ϵ. So we can estimate

$$\|[Ju^\epsilon(s)] - [Ju^\epsilon(t)]\|_{M^1(\mathbb{T}^2)} \leq \left\| \pi \sum_{i=1}^{d} d_i (\delta_{a_i(s)} - \delta_{a_i(t)}) \right\|_{M^1(\mathbb{T}^2)} + o_{\gamma_1}(1).$$

By (1.2.9), we can estimate by

$$\leq \pi \sum_{i=1}^{m} |a_i(s) - a_i(t)| + o_{\gamma_1}(1).$$

The claim will be established once we show for $i = 1, \ldots, m$,

$$|a_i(s) - a_i(t)| \leq c|s-t| + o_{\gamma_1}(1). \tag{2.2.2}$$

3. We prove (2.2.2) by using the identity (2.1.9). Fix i and observe that $a_i(s),\ a_i(t) \in B_{r/2}(\alpha_i)$ for all $s, t \in [0, T^\epsilon)$. There exists an $\eta \in C_c^\infty(B_r(\alpha_i))$ satisfying

$$\eta(x) = \nu \cdot x \quad \text{for } x \in B_{3r/4}(a_i(0)), \quad \nu \in S^1$$

and

$$\pi|a_i(s) - a_i(t)| = \pi d_i \int \eta(\delta_{a_i(s)} - \delta_{a_i(t)}).$$

The conditions on η guarantee that $\operatorname{supp}(D^2\eta) \subset B_r(\alpha_i) \setminus B_{3r/4}(\alpha_i)$. Notice that η depends upon the index i.

Insert the function η described above into (2.1.9). Using (2.2.1) and (2.1.9)

$$\begin{aligned} \pi|a_i(s) - a_i(t)| &= \int_{B_r(\alpha_i)} \eta([Ju^\epsilon(s)] - [Ju^\epsilon(t)])dx + o(1) \\ &= \int_s^t \int_{B_r(\alpha_i)} \eta_{x_j x_l} \mathbb{J}_{jk} u^\epsilon{}_{x_k} \cdot u^\epsilon{}_{x_l} dx\, d\tau + o_{\gamma_1}(1). \end{aligned}$$

The support properties of $\eta_{x_j x_l}$ permit us to replace $B_r(\alpha_i)$ by $B_r(\alpha_i)\backslash B_{3r/4}(\alpha_i)$. Finally, we estimate by

$$\leq |s-t| \; \|D^2\eta\|_{L^\infty} \sup_{\tau\in[s,t]} \|Du^\epsilon(\tau)\|_{L^2(B_r\backslash B_{3r/4})}.$$

The size of $\|D^2\eta\|_{L^\infty}$ depends upon r but is independent of ϵ and (1.4.24) permits us to control the Du term by a constant independent of ϵ, so (2.2.2) follows and the claim is proven. We also note that the claim implies T^ϵ may be taken independently of ϵ, so we denote this quantity by T from now on.

4. The remaining convergence claims follow from the bounds stated in Theorem 1 and passing to subsequences, except for (1.4.7) which follows directly from (1.4.2). We prove (1.4.8). Fix any $p \in [1,2)$. Since the conditions of Theorem 1 hold for every $t \in [0,T)$, we deduce from (1.4.26) that

$$\|j(u^\epsilon)\|_{L^p(\mathbb{T}^2\times[0,T))} \leq O_{p,\gamma_1,T}(1).$$

It follows, upon passing to a subsequence as $\epsilon \to 0$, that

$$j(u^\epsilon) \rightharpoonup \bar{j} \quad \text{weakly in } \; L^p(dxdt)$$

for some $\bar{j}$. We wish to identify $\bar{j}$.

Let $\phi \in C_0^\infty(\mathbb{T}^2 \times [0,T))$. The identity (2.1.2) implies

$$\begin{aligned}\int j(u^\epsilon)\cdot D\phi dxdt &= \int \phi_t \frac{|u^\epsilon|^2}{2} dxdt \\ &= \int \phi_t \frac{1}{2}(|u^\epsilon|^2 - 1) dxdt \\ &\to 0\end{aligned}$$

as $\epsilon \to 0$ for every t, by (1.4.7). Therefore div $\bar{j} = 0$. Moreover, from (1.4.6) we have $\nabla \times \bar{j} = 2[\bar{J}] \otimes dt = 2\pi \sum d_i \delta_{a_i(t)} \otimes dt$ weakly.

Let $H(x,t) = H(x, a(t), d)$. If we define $V = \bar{j} - j(H)$, we have

$$\text{div}\, V = \nabla \times V = 0$$

weakly. Let η^δ be a standard mollifier and set $V^\delta = V * \eta^\delta$. The convolution here is in space and time. The above considerations imply

$$\text{div}\, V^\delta = \nabla \times V^\delta = 0$$

in $\mathbb{T}^2$ for every $t < T$. Since V^δ is smooth, this implies $V^\delta(x,t) = g^\delta(t)$. Letting $\delta \to 0$, we find that V is also constant in x for each fixed t. For any fixed t we have

$$\begin{aligned}\int V(x,t)dx &= \int (\bar{\jmath} - j(H))\,(x,t)dx \\ &= \int \bar{\jmath}(x,t)\, dx \\ &= \lim_{\epsilon\to 0} \int j(u^\epsilon)(x,t)dx \\ &= 0\end{aligned}$$

using (1.4.4).

5. The proof given above may be iterated until the time T given in (1.4.9). □

Remark. A more general Schrödinger evolution equation associated with the ϵ-dependent Hamiltonian $I^\epsilon[u]$ has an ϵ-dependent time scale

$$ik_\epsilon \partial_t u - \Delta u + \frac{1}{\epsilon^2}(|u|^2 - 1).$$

In writing GLS_ϵ, we have implicitly selected the time scale $k_\epsilon = 1$. Theorem 1.4.1 suggests that this is the proper time scale to observe vortex motion. That is, the mobility of the vortices is $O(1)$ in the time scale $k_\epsilon = 1$.

3 Vortex equations of motion

Next, we exploit the renormalized energy result Theorem 1.4.5 to describe the motion of the points $a_i : [0,T) \longmapsto \mathbb{T}^2$ under the more stringent energy upper bound (1.4.3).

The idea is to extract more information out of the identity (2.1.9) than was used in the previous proof. In order to do this, we need sharper control over limits of quadratic terms in Du^ϵ away from the vortices. Informally, we show that if the initial data converges strongly to the canonical harmonic map, then conservation of energy forces the same convergence at later times. This is implemented using a Gronwall's inequality argument and Theorem 1.4.5.

We present the proof of Theorem 1.4.2.

Proof.

1. Let $a_i(t)$, $i = 1, \ldots, m$ denote the paths selected in Theorem 1.4.1, and let ϵ_n be the corresponding subsequence. Let $b_i(t)$ denote the solution of the system

$$\begin{cases} \dfrac{d}{dt} b_i = 2 \sum\limits_{j:j\neq i} d_j \nabla \times F(b_i - b_j), \\ \qquad b_i(0) = \alpha_i. \end{cases} \tag{2.3.1}$$

A calculation shows that

$$D_{a_i}W = -2\pi \sum_{j:j\neq i} d_i d_j DF(a_i - a_j).$$

Therefore, the ODE in (2.3.1) may be reexpressed, as in (1.4.10), in Hamiltonian form showing that the renormalized energy W is conserved. This ODE system has a unique solution on a nontrivial time interval $[0, T')$. Let

$$T_1 = \min(T, T'),$$

where T is as in (1.4.9). Note that T_1 is independent of ϵ. We wish to show for all i that $b_i(t)$ coincides with $a_i(t)$ on the time interval $[0, T_1)$. Observe that this will imply $T_1 = T' = T$.

For $t \in [0, T_1)$, let

$$\zeta(t) = \sum_i |b_i(t) - a_i(t)|.$$

It suffices to prove that, given any $\widetilde{T} < T_1$, we can find some small $\delta(\widetilde{T}) > 0$ and a constant $C = C(\widetilde{T})$ such that

$$\frac{d}{dt}\zeta(t) \leq C\zeta(t) \tag{2.3.2}$$

for a.e. $t \in [0, \widetilde{T}]$ whenever $\zeta(t) \leq \delta$. We will show that (2.3.2) holds at each point where $a_i(\cdot)$ is differentiable for all i; by Rademacher's theorem, this condition is satisfied on a set of full measure.

Fix $\widetilde{T} < T_1$. By (1.4.9), there is some $r = r(\widetilde{T}) > 0$ for which

$$\min_{i\neq j,\ t\leq \widetilde{T}} |a_i(t) - a_j(t)| \geq 4r. \tag{2.3.3}$$

2. We use the fact that b_i solves (2.3.1) and the triangle inequality to estimate

$$\begin{aligned}\frac{d\zeta}{dt} &\leq \sum_i |b_{i,t} - a_{i,t}| \\ &\leq 2\sum_i \Big| \sum_{j:j\neq i} d_j \nabla \times F(b_i - b_j) - \sum_{j:j\neq i} d_j \nabla \times F(a_i - a_j) \Big| \\ &+ \sum_i \Big| a_{i,t} - 2\sum_{j:j\neq i} d_j \nabla \times F(a_i - a_j) \Big| \\ &= Term\ 1 + Term\ 2.\end{aligned}$$

We immediately dispose of $Term$ 1. Fix $s < \widetilde{T}$ and a pair of indices $i \neq j$. Let $h = |(b_i(s) - b_j(s)) - (a_i(s) - a_j(s))|$. Note that by assumption $h \leq \zeta(s) \leq \delta$. By

Taylor's theorem, at the fixed time s, we have

$$|\nabla \times F(b_i - b_j) - \nabla \times F(a_i - a_j)| \le h \max_{\{x:|x-(a_i-a_j)|\le h\}} |D^2 F| \le C\zeta(s).$$

The last inequality follows from (2.3.3) provided $\delta < r$. Therefore, $Term$ 1 satisfies the desired estimate (2.3.2).

3. We turn our attention to $Term$ 2, which is a sum of terms $(Term\ 2)_i$, with $i = 1, \dots, m$. Suppose that each function $a_i(\cdot)$ is differentiable at $s \in [0, \widetilde{T})$. Fix $\eta \in C_0^\infty$ such that $\text{supp}(\eta) \subset B_{r(\widetilde{T})}(a_i(s))$ and $\eta(x) = \nu \cdot x$ in a neighborhood of $a_i(s)$. Here we take $\nu \in S^1$ to satisfy

$$(Term\ 2)_i = d_i \nu \cdot \Big(a_{i,t}(s) - 2\sum_{j:j\ne i} d_j \nabla \times F(a_i(s) - a_j(s))\Big). \tag{2.3.4}$$

Since

$$[Ju^{\epsilon_n}(t)] \rightharpoonup \pi \sum d_i \delta_{a_i(t)} \quad \text{weakly as measures}$$

we can rewrite

$$\begin{aligned} d_i\nu \cdot a_{i,t}(s) &= \lim_{h\to 0} d_i \nu \cdot \frac{1}{h}(a_i(s+h) - a_i(s)) \\ &= \lim_{h\to 0}\lim_{n\to\infty} \frac{1}{\pi h}\int_{\mathbb{T}^2} (\eta[Ju^{\epsilon_n}(s+h)] - \eta[Ju^{\epsilon_n}(s)])\, dx \\ &= \lim_{h\to 0}\lim_{n\to\infty} \frac{1}{\pi h}\int_s^{s+h}\int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} u^{\epsilon_n}_{x_k} \cdot u^{\epsilon_n}_{x_l}\, dx dt. \end{aligned}$$

We used (2.1.9) in the last step.

Let $H(x,t) = H(x, a(t), d)$. We reexpress the remaining term in (2.3.4) using Lemma 2.3.1 which is stated and proven below,

$$\begin{aligned} & d_i\nu \cdot \Big(2\sum_{j:j\ne i} d_j \nabla \times F(a_i(s) - a_j(s))\Big) \\ &= \lim_{h\to 0}\frac{2}{h}\int_s^{s+h} d_i\nu\cdot\Big(\sum_{j:j\ne i} d_j\nabla\times F(a_i(t) - a_j(t))\Big) dt \\ &= \lim_{h\to 0}\frac{1}{\pi h}\int_s^{s+h}\int_{\mathbb{T}^2} \eta_{x_j x_l}\mathbb{J}_{jk} j^k(H) j^l(H)\, dx dt. \end{aligned}$$

Therefore,

$$(Term\ 2)_i = \lim_{h\to 0}\lim_{n\to\infty}\frac{1}{\pi h}\int_s^{s+h}\int_{\mathbb{T}^2}\eta_{x_j x_l}\mathbb{J}_{jk}\left(u^{\epsilon_n}_{x_k}\cdot u^{\epsilon_n}_{x_l} - j^k(H) j^l(H)\right) dx dt.$$

Inside the integral, $H = H(\cdot, a(t), d)$ and $Du^\epsilon = Du^\epsilon(t)$.

On any set where $|u| > 0$, using the decomposition (1.4.27),

$$u^\epsilon_{x_k} \cdot u^\epsilon_{x_l} = \frac{1}{|u^\epsilon|^2} j^k(u^\epsilon) j^l(u^\epsilon) + |u^\epsilon|_{x_k} |u^\epsilon|_{x_l}.$$

We will thus have proved (2.3.2) when we show

(2.3.5)
$$\lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} \int_s^{s+h} \int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} \left(|u^{\epsilon_n}|_{x_k} |u^{\epsilon_n}|_{x_l} \right) dx dt \le C\zeta(s),$$

and

(2.3.6)
$$\lim_{h\to 0} \lim_{n\to\infty} \frac{1}{h} \int_s^{s+h} \int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} \left(\frac{1}{|u^{\epsilon_n}|^2} j^k(u^{\epsilon_n}) j^l(u^{\epsilon_n}) - j^k(H) j^l(H) \right) dx dt \le C\zeta(s).$$

These estimates will follow from the tight upper bound (1.4.3) on the energy and energy conservation.

4. The renormalized energy W is conserved for solutions $b(\cdot)$ of (2.3.1), and $\int_{\mathbb{T}^2} E^\epsilon(u^\epsilon(\cdot,t))(x)dx$ is conserved for solutions u^ϵ of GLS_ϵ. Therefore, for every $t \le \widetilde{T}$ and every $\rho > 0$, the upper bound (1.4.3) gives

$$\begin{aligned} \int_{\mathbb{T}^2} E^\epsilon(u^\epsilon(\cdot,t))(x)dx &= \int_{\mathbb{T}^2} E^\epsilon(\phi^\epsilon(\cdot))(x)dx \\ &\le m\left(\pi \log\left(\frac{1}{\rho}\right) + I(\epsilon,\rho)\right) + W(a(0),d) + C\rho + o(1) \\ &= m\left(\pi \log\left(\frac{1}{\rho}\right) + I(\epsilon,\rho)\right) + W(b(t),d) + C\rho + o(1). \end{aligned}$$

Arguing as in the estimate of $Term$ 1, we see that

$$W(b(t)) - W(a(t)) \le C \sum |b_i(t) - a_i(t)| = C\zeta(t)$$

provided δ is small enough. Therefore

(2.3.7)
$$\int_{\mathbb{T}^2} E^\epsilon(u^\epsilon(\cdot,t))(x)dx \le m\left(\pi \log\frac{1}{\rho} + I(\epsilon,\rho)\right) + W(a(t),d) + C\rho + C\zeta(t) + o(1),$$

as $\epsilon \to 0$ for every $\rho > 0$. We have from Theorem 1.4.1 that

(2.3.8)
$$[Ju^{\epsilon_n}(t)] \rightharpoonup \pi \sum_{i=1}^m d_i \delta_{a_i(t)} \quad \text{weakly as measures.}$$

The conditions (2.3.7), (2.3.8) are precisely the hypotheses of Theorem 1.4.5 with $\gamma_2 = C\zeta(t)$. So, for every $t \in [s, s+h]$,

$$\limsup_{n\to\infty} \|D|u^{\epsilon_n}|(\cdot,t)\|^2_{L^2(\mathbb{T}^2_\rho)} \le C\zeta(t) \tag{2.3.9}$$

and

$$\limsup_{n\to\infty} \left\| \frac{1}{|u^{\epsilon_n}|} j(u^{\epsilon_n}) - j(H) \right\|^2_{L^2(\mathbb{T}^2_\rho)} \le C\zeta(t). \tag{2.3.10}$$

These estimates allow us to prove (2.3.5), (2.3.6).

We quickly estimate (2.3.5). By observing that

$$(2.3.5) \le \lim_{h\to 0} \lim_{n\to\infty} \|D^2\eta\|_{L^\infty(\mathbb{T}^2_\rho)} \frac{C}{h} \int_s^{s+h} \|D|u^{\epsilon_n}|(\cdot,t)\|^2_{L^2(\mathbb{T}^2_\rho)} dt, \tag{2.3.11}$$

and then applying (2.3.9), we obtain the desired upper bound.

5. We now establish (2.3.6). First, we show that

$$\left(\frac{1}{|u^{\epsilon_n}|} j(u^{\epsilon_n}) - j(H)\right) \rightharpoonup 0 \quad \text{weakly in } L^2(\mathbb{T}^2_\rho \times [s, s+h]). \tag{2.3.12}$$

To see this, note that $\frac{1}{|u^{\epsilon_n}|} j(u^{\epsilon_n})$ is uniformly bounded in $L^2(\mathbb{T}^2_\rho \times [s, s+h])$ and hence converges weakly to some limit $\bar{j}$. We know from (1.4.8) that $j(u^{\epsilon_n}) \rightharpoonup j(H)$ weakly in $L^p(dxdt)$ for all $1 \le p < 2$. We also know from (1.4.7) that $|u^{\epsilon_n}|^2 \to 1$ strongly in $L^2(dxdt)$. Thus

$$\begin{aligned} j(H) &= \text{weak } L^1 \lim_{n\to\infty} j(u^{\epsilon_n}) \\ &= \text{weak } L^1 \lim_{n\to\infty} \left(\frac{j(u^{\epsilon_n})}{|u^{\epsilon_n}|} |u^{\epsilon_n}| \right) \\ &= \left(\text{strong } L^2 \lim_{n\to\infty} |u^{\epsilon_n}|\right) \left(\text{weak } L^2 \lim_{n\to\infty} \frac{j(u^{\epsilon_n})}{|u^{\epsilon_n}|}\right) \\ &= \bar{j} \end{aligned}$$

which proves (2.3.12).

For fixed k, l, observe that the quadratic term in (2.3.6) can be reexpressed as

$$\begin{aligned} &\Big(\frac{1}{|u^{\epsilon_n}|} j^k(u^{\epsilon_n}) - j^k(H)\Big)\Big(\frac{1}{|u^{\epsilon_n}|} j^l(u^{\epsilon_n}) - j^l(H)\Big) \\ &\qquad + j^l(H)\Big(\frac{1}{|u^{\epsilon_n}|} j^k(u^{\epsilon_n}) - j^k(H)\Big) + j^k(H)\Big(\frac{1}{|u^{\epsilon_n}|} j^l(u^{\epsilon_n}) - j^l(H)\Big). \end{aligned}$$

Since

$$\left(\frac{1}{|u^{\epsilon_n}|} j^k(u^{\epsilon_n}) - j^k(H)\right) \rightharpoonup 0 \quad \text{weakly in } L^2(\mathbb{T}^2_\rho \times [s, s+h])$$

and $j^l(H)$ does not depend upon n, the second expression contributes nothing as $n \to \infty$. The first expression is controlled using (2.3.10).

6. Since we have appropriately bounded $Term$ 1 and $Term$ 2, we have proven (2.3.2). Gronwall's inequality implies $\zeta = 0$ which gives (1.4.10) of the Theorem. Since $\zeta = 0$, (2.3.10) implies (1.4.11). We conclude by proving (1.4.12). Fix t and $\rho > 0$. Let u^ϵ be a subsequence which converges in $L^2(\mathbb{T}^2_\rho)$ to some limit $\bar{u}$. We may assume, by (1.4.24), that $Du^\epsilon \rightharpoonup D\bar{u}$ weakly in $L^2(\mathbb{T}^2_\rho)$. Also, (2.3.9) and (2.3.10) give

$$\liminf_{\epsilon\to 0} \|Du^\epsilon\|_{L^2(\mathbb{T}^2_\rho)} = \lim_{\epsilon\to 0}\left\| \frac{j(u^\epsilon)}{|u^\epsilon|} \right\|_{L^2(\mathbb{T}^2_\rho)} = \|D\bar{u}\|_{L^2(\mathbb{T}^2_\rho)}.$$

Therefore $Du^\epsilon \to D\bar{u}$ strongly in $L^2(\mathbb{T}^2_\rho)$. Finally, since $j(\bar{u}) = j(H)$, Proposition 1.3.1 implies $\bar{u} = e^{i\alpha}H$ for some $\alpha \in \mathbb{R}$. □

Lemma 2.3.1. *Suppose that* $\eta \in C^2$ *and that*

$$\operatorname{supp}(\eta) \cap \{a_1, ..., a_m\} = \{a_i\}; \quad D^2\eta \equiv 0 \ \text{ in a neighborhood of } a_i.$$

Let $H := H(\cdot; a, d)$ *be the canonical harmonic map. Then*

$$\int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} j^k(H) j^l(H) = d_i D\eta(a_i) \cdot \Big(2\pi \sum_{j:j\neq i} d_j \ \nabla \times F(a_i - a_j)\Big).$$

Remark. This computation remains valid if $d_1, ..., d_m$ assume arbitrary integer values, that is, if we lift the assumption that $d_i = \pm 1$ for all i.

Proof.

1. We reexpress the integral in the lemma. Recall that $j(H) = -\nabla \times \Phi$ where Φ satisfies

$$\Delta\Phi = \sum_{i=1}^{m} 2\pi d_i \delta_{a_i},$$

and, using (1.3.3), we write

$$\Phi(x) = d_i F(x - a_i) + G(x) \tag{2.3.13}$$

where

$$G(x) = \sum_{j:j\neq i} d_j F(x - a_j).$$

Since $j^k(H) = \mathbb{J}_{km}\Phi_{x_m}$, we have

$$\mathbb{J}_{jk} j^k(H) j^l(H) = -\mathbb{J}_{ln}\Phi_{x_n}\Phi_{x_j}.$$

Fix any number ρ so small that $D^2\eta = 0$ on $B_\rho(a_i)$. We have

$$\int_{\mathbb{T}^2} \eta_{x_j x_l} \mathbb{J}_{jk} j^k(H) j^l(H) dx = - \int_{\mathbb{T}^2 \setminus B_\rho(a_i)} \eta_{x_j x_l} \mathbb{J}_{ln} \Phi_{x_n} \Phi_{x_j}$$

$$= \int_{\mathbb{T}^2 \setminus B_\rho} \eta_{x_l} \mathbb{J}_{ln} \Phi_{x_n x_j} \Phi_{x_j} dx + \int_{\partial B_\rho} \eta_{x_l} \mathbb{J}_{ln} \Phi_{x_n} \Phi_{x_j} \nu^j dH^1$$

where $\nu = (\nu^1, \nu^2)$ is the outward unit normal to ∂B_ρ. We recognize $\Phi_{x_n x_j} \Phi_{x_j} = \frac{1}{2}(\Phi_{x_j}\Phi_{x_j})_{x_n}$ and integrate by parts again to find

$$= - \int_{\mathbb{T}^2 \setminus B_\rho} \eta_{x_l x_n} \mathbb{J}_{ln} \frac{1}{2} \Phi_{x_j} \Phi_{x_j} dx - \int_{\partial B_\rho} \eta_{x_l} \mathbb{J}_{ln} \frac{1}{2} \Phi_{x_j} \Phi_{x_j} \nu^n dH^1$$
$$+ \int_{\partial B_\rho} \eta_{x_l} \mathbb{J}_{ln} \Phi_{x_n} \Phi_{x_j} \nu^j dH^1.$$

Since $\eta_{x_l x_n} \mathbb{J}_{ln} = 0$, the integral over $\mathbb{T}^2 \setminus B_\rho$ vanishes and we are left with two boundary integrals I_ρ, II_ρ.

2. We calculate the boundary integrals. We begin with

$$I_\rho = - \int_{\partial B_\rho} \eta_{x_l} \mathbb{J}_{ln} \frac{1}{2} \Phi_{x_j} \Phi_{x_j} \nu^n dH^1.$$

By using (2.3.13), we observe

$$\Phi_{x_j} \Phi_{x_j} = F_{x_j} F_{x_j} + 2 d_i F_{x_j} G_{x_j} + G_{x_j} G_{x_j}. \tag{2.3.14}$$

Since F is even, the first term integrates to zero. We exploit the fact that G_{x_j} is nearly constant on $B_\rho(a_i)$ to calculate the contribution to I_ρ arising from the remaining two terms in (2.3.14). The cross term contributes

(2.3.15)
$$-d_i \eta_{x_l}(a_i) G_{x_j}(a_i) \mathbb{J}_{ln} \int_{\partial B_\rho} F_{x_j} \nu^n dH^1 - d_i \eta_{x_l}(a_i) \mathbb{J}_{ln} \int_{\partial B_\rho} F_{x_j} [G_{x_j}(x) - G_{x_j}(a_i)] \nu^n dH^1.$$

Since $F \sim \log|x - a_i|$, $|F_{x_j}| \sim 1/\rho$ on ∂B_ρ and G is C^1 on B_ρ, the second integral contributes $O(\rho)$. The $G_{x_j} G_{x_j}$ term contributes $O(\rho)$ as well.

Next, we calculate

$$II_\rho = \int_{\partial B_\rho} \eta_{x_l} \mathbb{J}_{ln} \Phi_{x_n} \Phi_{x_j} \nu^j dH^1$$

by expanding using (2.3.13). The $F_{x_n}F_{x_j}$ term again vanishes by symmetry. The $G_{x_n}G_{x_j}$ term contributes $O(\rho)$ and the cross terms remain to be estimated. The first cross term gives

$$d_i\eta_{x_l}(a_i)G_{x_j}(a_i)\mathbb{J}_{ln}\int_{\partial B_\rho} F_{x_n}\nu^j dH^1 + O(\rho). \tag{2.3.16}$$

The second cross term contributes

$$d_i\eta_{x_l}(a_i)G_{x_n}(a_i)\mathbb{J}_{ln}\int_{\partial B_\rho} F_{x_j}\nu^j dH^1 + O(\rho). \tag{2.3.17}$$

3. Since $\int_{\partial B_\rho} F_{x_n}\nu^j dH^1 = \int_{\partial B_\rho} F_{x_j}\nu^n dH^1$, the first terms in (2.3.15) and (2.3.16) cancel and the only remaining contribution is (2.3.17). Finally, observe that

$$\int_{\partial B_\rho} F_{x_j}\nu^j dH^1 = \int_{B_\rho} \Delta F dx = 2\pi - 2\pi^2\rho^2,$$

using (1.3.1). Therefore

$$\int_{\mathbb{T}^2} \eta_{x_j x_l}\mathbb{J}_{jk} j^k(H) j^l(H) dx = 2\pi d_i \eta_{x_l}(a_i)\mathbb{J}_{ln} G_{x_n}(a_i) + O(\rho).$$

Since ρ can be taken arbitrarily small, we have proved the lemma. □

CHAPTER 3. VORTEX STRUCTURE

1 Background on Jacobian and degree

In this chapter we will prove versions of Theorems 1.4.3 and 1.4.4. Because we believe that these sorts of results are extremely useful in questions involving vortex dynamics, we establish them in much greater generality than we require for our analysis of the Ginzburg–Landau Schrödinger equation in Chapter 2.

We start in this section by defining some notation that will be used throughout this chapter, and also quoting some results that we will need. The definitions that we give here reduce to those of Section 2 of Chapter 1 in the case $n = 2$.

Let $\{dx^i\}_{i=1}^n$ be an orthonormal basis for $T^*\mathbb{R}^n$, so that $\{dx^\alpha\}_{\alpha\in I_{k,n}}$ forms an orthonormal basis for $\Lambda^k(T^*\mathbb{R}^n)$, the space of k-covectors on $\mathbb{R}^n$. Here $I_{k,n}$ is the set of all multiindices of the form $\alpha = (\alpha_1, ..., \alpha_k)$ such that $1 \le \alpha_1 < \cdots < \alpha_k \le n$. For such a multiindex, $dx^\alpha := dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$.

We let ω_n denote the volume of the unit ball in $\mathbb{R}^n$.

For vectors $v^1, ..., v^n \in \mathbb{R}^n$, we let $\det(v^1, ..., v^n)$ denote the determinant of the matrix whose columns are the v^is, arranged in the given order.

Suppose that $u \in W^{1,n}(U; \mathbb{R}^n)$ for some $U \subset \mathbb{R}^n$. We define an $n-1$ form

$$j(u) := \sum_{\alpha \in I_{n-1,n}} \det(u, u_{x_{\alpha_1}}, ..., u_{x_{\alpha_{n-1}}}) dx^\alpha. \tag{3.1.1}$$

Note that $\det(u, u_{x_{\alpha_1}}, ..., u_{x_{\alpha_{n-1}}}) \in W^{1,p}$ for every $p \in [1, 1+1/n)$.

For a.e. x, $j(u)(x)$ is well-defined pointwise as an element of $\Lambda^{n-1}(T^*\mathbb{R}^n)$. As such it defines a linear functional on $\Lambda^{n-1}(T\mathbb{R}^n)$. Indeed, given $\tau = \tau^1 \wedge \cdots \wedge \tau^{n-1} \in \Lambda^{n-1}(T\mathbb{R}^n)$, one can check that

$$\langle j(u), \tau \rangle \;=\; \det(u, (\tau^1 \cdot Du), ..., (\tau^{n-1} \cdot Du)) \quad \text{a.e. } x.$$

To see this, note first that the right-hand side of the above identity defines a linear functional on $\Lambda^{n-1}(T\mathbb{R}^n)$, or equivalently an alternating linear functional on $(\mathbb{R}^n)^{n-1}$. It then suffices to verify that this linear functional agrees with (3.1.1) when applied to the standard basis of $\Lambda^{n-1}(T\mathbb{R}^n)$, which is dual to the basis for $\Lambda^{n-1}(T^*\mathbb{R}^n)$.

We also define an n-form

$$\begin{aligned} Ju &:= \frac{1}{n} dj(u) \\ &= \det(u_{x_1}, ..., u_{x_n}) \text{ vol} \\ &= \det Du \text{ vol} \end{aligned} \tag{3.1.2}$$

where $\text{vol} = dx^1 \wedge \cdots \wedge dx^n$ is the standard volume form. We will refer to Ju as the signed Jacobian of u. For $u \in W^{1,n}(U; \mathbb{R}^n)$, we may think of Ju as an L^1 function.

In this context, Stokes' Theorem asserts that for any bounded open set V with smooth boundary,

$$\int_V Ju = \int_{\partial V} \frac{1}{n} \langle j(u), \tau \rangle. \tag{3.1.3}$$

Here τ is the appropriately oriented $(n-1)$ volume element. The trace of a $W^{1,n}$ function belongs to $W^{1,n-1}$, so the right-hand side makes sense.

The Brouwer degree of a function u can be expressed in terms of either $j(u)$ or Ju. Let $u \in W^{1,n}(U; \mathbb{R}^n)$, and suppose that $V \subset U$ and that V is bounded, with smooth boundary. If $\text{ess inf}_{\partial V}|u| > 0$, then the Brouwer degree of u is defined by

$$\deg(u; \partial V) = \int_V \eta(u) Ju \tag{3.1.4}$$

where $\eta \in C^\infty(\mathbb{R}^n)$ satisfies

$$\int \eta = 1, \quad \eta \geq 0, \quad \text{spt}\, \eta \subset B_\rho(0), \quad \rho < \text{ess}\,\inf_{\partial V}|u|.$$

The degree is an integer, and it is independent of the specific choice of η and thus well-defined.

If we write $u = |u|v$, so that $|v| = 1$, then the degree can also be defined by the formula

$$\deg(u; \partial V) = \int_{\partial V} \frac{1}{n\omega_n} \langle j(v), \tau \rangle. \tag{3.1.5}$$

Here $\tau(x)$ is an $(n-1)$ vector of unit length, which represents the appropriately oriented $(n-1)$ tangent plane to ∂V at the point $x \in \partial V$.

A very nice treatment of degree is given by Brezis and Nirenberg [6].

We can also define the signed Jacobian as a distribution (or as an element of the dual of C_0^1), which we write $[Ju]$:

$$\int \eta [Ju] := \frac{1}{n} \int \langle j(u), d^*\eta \rangle.$$

Here d^* is the formal adjoint of the exterior derivative d. This definition makes sense in some spaces which are weaker that $W^{1,n}$, as all it requires is that $j(u)$ be integrable. This condition holds, for example, when $u \in L^\infty \cap W^{1,n-1}$, or $u \in W^{1,p}$ for $p \geq n^2/(n+1)$. Properties of the distributional Jacobian (also called the distributional determinant) have been studied by by S. Müller [20] among other authors.

One such well-known property is the following.

Lemma 3.1.1. (Weak continuity of Jacobians). *If $u_k \to \bar{u}$ weakly in $W^{1,p}(U)$, then*

$$j(u_k) \rightharpoonup j(\bar{u})$$

weakly in L^q where $p \in \left[\frac{n^2}{n+1}, n\right)$ and $1 \leq q \leq \frac{np}{n^2 - p}$. Also,

$$Ju_k \to J\bar{u}$$

in the sense of distributions.

A proof can be found in [8].

We will need the following lemma, which is proved in Alberti, Jerrard and Soner [1].

Lemma 3.1.2. *Suppose that $U \subset \mathbb{R}^n$ and that $u \in W^{1,p}(U;\mathbb{R}^n)$ for $p \geq n-1$. Suppose further that $|u| = 1$ a.e. and that $[Ju]$ is a measure in the sense that*

$$\int \eta[Ju] \leq C\|\eta\|_{C^0}$$

for all $\eta \in C^1$. Then $[Ju]$ has the form

$$[Ju] = \sum_i \omega_n d_i \delta_{\xi_i}$$

for integers d_i and a locally finite collection of points $\xi_i \in U$.

For the reader's convenience we sketch the proof.

The first step is to show that, given *any* $x \in U$, for almost every $r < \mathrm{dist}(x, \partial U)$ there is an integer $k(r)$ such that

$$\int_{B_r} [Ju] = \omega_n k(r).$$

The left-hand side is interpreted as a limit of $\int \eta_n[Ju]$ for a sequence η_n of smooth functions converging pointwise to the characteristic function of the ball, a limit which is independent of the details of the approximating sequence.

This is proved by showing that it holds for "good enough" radii, and verifying that a.e. radius r is good enough.

The next step is to show that in fact the above identity holds for all but finitely many r (with $x \in U$ still fixed.) This follows from the assumption that $[Ju]$ is a measure.

After this it is clear that for every x we can find some $r_0(x) > 0$ such that $\int_{B_r}[Ju]$ is a constant integer multiple of ω_n for all $r < r_0$. The result then follows easily.

Essential degree It is convenient in places to work with an approximation to the degree that enable us to ingore "inessential" components of the set $\{|u| \sim 0\}$. We therefore need to introduce some more definitions. These are not standard.

Assume that $u \in C \cap W^{1,n}(U;\mathbb{R}^n)$ and that $|u| \geq 1/2$ on ∂U.

Let S denote the set on which $|u|$ is small,

$$S := \{x \in U \ : \ |u(x)| \leq 1/2\}. \tag{3.1.6}$$

By assuming that u is continuous, we have avoided any possible subtleties in the definition of S, and we also know as a result that the connected components of S are closed. Each component S_i of S has a well-defined degree given by the definition (3.1.4). The degree is an integer even when ∂S_i is not smooth, as can be seen by approximating S_i by smooth sets.

We may thus define the *essential* part of S,

$$S_E := \bigcup\{\text{components } S_i \text{ of } S : \deg(u;\partial S_i) \neq 0\} \tag{3.1.7}$$

and the *negligible* part of S,

$$\begin{aligned} S_N &:= \bigcup\{\text{components } S_i \text{ of } S : \deg(u;\partial S_i) = 0\} \\ &= S \setminus S_E. \end{aligned} \tag{3.1.8}$$

For any subset $V \subset U$ such that $\partial V \cap S_E = \emptyset$, we use the notation

(3.1.9)

$$\mathrm{dg}(u;\partial V) := \sum \{\deg(u;\partial S_i) \ : \ \text{components } S_i \text{ of } S_E \text{ such that } S_i \subset\subset V\}.$$

If $\partial V \cap S_E \neq \emptyset$ then $\mathrm{dg}(u;\partial V)$ is left undefined.

We will refer to dg as the "essential degree".

The essential degree is a technical device needed to circumvent some difficulties in the covering arguments which are a key part of the proof of Theorem 1.4.3. We emphasize that the distinction between dg and deg can generally be ignored with very little loss of understanding.

Note in particular that

$$\mathrm{dg}(u;\partial V) = \deg(u;\partial V) \quad \text{if } |u| > 1/2 \text{ on } \partial V. \tag{3.1.10}$$

Devices for lower bounds We will need a number of results which are proven in Jerrard [12]. We introduce the n-dimensional analog of the Ginzburg–Landau energy

$$I^\epsilon[u] = \int E^\epsilon(u)dx; \qquad E^\epsilon(u) = \frac{1}{n}|Du|^n + \frac{1}{\epsilon^2}\left(|u|^2 - 1\right)^2.$$

Define

$$\kappa_n = (n-1)^{n/2}\omega_n \tag{3.1.11}$$

and

$$\lambda^\epsilon(r) = \min_{m\in[0,1]} \left[m^n \frac{\kappa_n}{r} + \frac{1}{C^*\epsilon}(1-m)^N\right], \tag{3.1.12}$$

where $C^*, N > 0$ are certain constants that depend only on the dimension n. The quantity λ^ϵ provides us with a useful lower bound of the energy on a sphere as seen in Lemma 3.1.4 below. The first term in the definition of λ^ϵ accounts for the energy associated with the rotation in u while the second term accounts for the stretching in the length of u.

Note that λ^ϵ is nonincreasing.

Further define

$$\Lambda^\epsilon(s) := \int_0^s \lambda^\epsilon(r) \wedge \frac{c_0}{\epsilon} dr, \tag{3.1.13}$$

for some sufficiently small constant c_0, depending on the dimension n.

The first result we quote is established by an interpolation argument. The point is that, since ∂B_r is an $(n-1)$-dimensional surface, $\int_{\partial B_r} |D\rho|^n$ controls the Hölder $1/n$ seminorm of ρ on ∂B_r.

Lemma 3.1.3. *Suppose that $u \in W^{1,n}(U;\mathbb{R}^n)$ and that $B_r \subset U$ with $r \geq \epsilon$. Let $\rho := |u|$ and*

$$\gamma := \int_{\partial B_r} \frac{1}{n}|D\rho|^n + \frac{1}{4\epsilon^2}(\rho^2 - 1)^2 \, dH^{n-1} \in [0, \infty]. \tag{3.1.14}$$

Then

$$||1 - \rho||_{L^\infty(\partial B_r)} \leq (C\epsilon\gamma)^{1/N}$$

for some $C, N > 0$ depending only on the dimension n.

The next lemma contains a basic lower bound relating the energy of a function to its degree. It is convenient to state it in terms of the essential degree dg defined above.

Lemma 3.1.4. *If $u \in C \cap W^{1,n}(U;\mathbb{R}^n)$ and $\mathrm{dg}(u;\partial B_r) \neq 0$ for $B_r \subset U$ with $r \geq \epsilon$, then*

$$\int_{\partial B_r} E^\epsilon \, dH^{n-1} \geq \lambda^\epsilon(r) \wedge \frac{c_0}{\epsilon}.$$

We briefly explain the idea. If $|u| < 1/2$ on ∂B_r, the result follows from Lemma 3.1.3. If not, define $m := \inf_{\partial B_r} |u|$, and write $u = \rho v$, $|v| = 1$. Observing that $|Du|^n \geq |D\rho|^n + \rho^n |Dv|^n$, the result follows from Lemma 3.1.3 and the estimate $\int_{\partial B_r} |Dv|^n \geq \kappa_n / r$, which holds when $\deg(u; \partial B_r) \neq 0$.

We record several useful properties of Λ^ϵ. The ones contained in the next lemma are direct consequences of the definition.

Lemma 3.1.5. *$\Lambda^\epsilon(\cdot)$ is increasing, and moreover*

$$\Lambda^\epsilon(r + s) \leq \Lambda^\epsilon(r) + \Lambda^\epsilon(s) \quad \forall r, s \geq 0; \tag{3.1.15}$$

$$\Lambda^\epsilon(r) \geq \kappa_n \ln\left(\frac{r}{\epsilon}\right) - C(n) \quad \forall r \geq 0. \tag{3.1.16}$$

The next lemma follows by integrating Lemma 3.1.4.

Lemma 3.1.6. *If* $u \in C \cap W^{1,n}(U;\mathbb{R}^n)$, $\epsilon \leq r_0 \leq r_1$, *and* $\mathrm{dg}(u;\partial B_s) \neq 0$ *for all* $s \in [r_0, r_1]$, *then*

$$\int_{B_{r_1}\setminus B_{r_0}} E^\epsilon dx \geq \Lambda^\epsilon(r_1) - \Lambda^\epsilon(r_0). \tag{3.1.17}$$

The final lemma asserts that it is possible to cover the set S_E by balls satisfying a good lower bound, and such that the radius of each ball is at least ϵ. The latter condition is important in our later arguments because of the condition $r_0 \geq \epsilon$ in Lemma 3.1.6.

Lemma 3.1.7. *Suppose that* $u \in C \cap W^{1,n}(U;\mathcal{R}^n)$ *and that* $|u| \geq 1/2$ *on* ∂U. *Then there is a collection of closed, pairwise disjoint balls* $\{B_i\}_{i=1}^k$ *with radii* r_i *such that*

$$S_E \subset \bigcup_{i=1}^k B_i, \tag{3.1.18}$$

$$r_i \geq \epsilon \quad \forall i, \tag{3.1.19}$$

$$B_i \cap S_E \neq \emptyset \quad \textit{for each } i, \tag{3.1.20}$$

$$\int_{B_i\cap U} E^\epsilon dx \;\geq\; \frac{c_0}{\epsilon} r_i \;\geq\; \Lambda^\epsilon(r_i). \tag{3.1.21}$$

The idea of the proof is as follows: Around each component S_i of S_E, place a small ball of radius $r_i = \max(\mathrm{diam}\ S_i, \epsilon)$. Consider one of these balls. If $r_i > \epsilon$, then (3.1.21) holds as a result of Lemma 3.1.3. If $r_i = \epsilon$, (3.1.21) holds because

$$\int_{S_i} |Du|^n\, dx \geq C^{-1}\int_{S_i} |Ju|\, dx \geq C^{-1}\left|\int_{S_i} Ju dx\right| \geq C^{-1}|\mathrm{deg}(u;\partial S_i)|. \tag{3.1.22}$$

If two or more balls intersect, this can be controlled by combining them into larger balls and using the Besicovitch Covering Theorem to control the overlap.

The lower bound (3.1.22) is useless if S_i has degree zero, which makes it impossible, in general, to cover S_N with balls satisfying the stated conditions. It is this fact that forces us to introduce the essential degree dg.

2. Concentration of energy

In this section we prove the following result:

Theorem 3.2.1. *Suppose that* $\epsilon \le r \le 1$, $u \in W^{1,n}(B_r;\mathbb{R}^n)$,

$$|| \, [Ju] - \omega_n d\delta_0 \, ||_{\mathcal{M}^1(B_r)} \le \gamma_0 r, \tag{3.2.1}$$

where $d = \pm 1$ *and* $\gamma_0 = \gamma_0(n)$ *is a constant which will be fixed below. Assume also that*

$$\int_{B_r} E^\epsilon(u)\, dx \le \kappa_n \log\left(\frac{r}{\epsilon}\right) + \gamma_1. \tag{3.2.2}$$

Then there exists a point $\xi \in B_{r/2}$ *and a constant* $C_1 > 0$ *such that*

$$\int_{B_\sigma(\xi)} E^\epsilon(u)\, dx \ge \kappa_n \log\left(\frac{\sigma}{\epsilon}\right) - C_1 \tag{3.2.3}$$

for every $\sigma \in [0, r/2]$. *Moreover,*

$$||\mu_u^\epsilon - \kappa_n \delta_\xi||_{\mathcal{M}^1(B_r)} \le \frac{C_1}{|\ln \epsilon|}, \tag{3.2.4}$$

and for any $p \in [1, n)$, *there exists some* C_p *such that*

$$||Du||_{L^p(B_r)} \le C_p. \tag{3.2.5}$$

Remarks.

1. In the case $n = 2$, we can take $\gamma_0 = \pi/200$.
2. The constants C_1 and C_p above depend only on the dimension n and the constant γ_1 in the assumed upper bound (3.2.2). In particular, they are valid for all u as above, uniformly for $\epsilon \in (0, 1]$. Note, however, that it suffices to prove the theorem only for $\epsilon \le \epsilon_0 = \epsilon_0(C, n)$.
3. Theorem 3.2.1 immediately implies several other estimates. Suppose that u satisfies the hypotheses of the theorem. From (3.2.2) we have

   $$|||u|^2 - 1||_{L^2(B_r)} \le C\epsilon(\ln(r/\epsilon) + 1). \tag{3.2.6}$$

   Interpolation inequalities and (3.2.5) then imply that for any $p < \infty$ we can find a constant C depending on p, n and γ_1, such that

   $$||u||_{L^p(B_r)} \le C. \tag{3.2.7}$$

This bound and (3.2.5) imply that for any $p \in [1, \frac{n}{n-1})$ there is a constant C such that

$$||j(u)||_{L^p(B_r)} \leq C. \tag{3.2.8}$$

Recall that j is defined in (1.2.2).

Finally, the Jacobian Ju satisfies

$$||Ju||_{L^1(B_r \setminus B_\sigma(\xi))} \leq C \tag{3.2.9}$$

as well as

$$||[Ju]||_{\mathcal{M}^1(B_r)} \leq C, \quad ||[Ju]||_{W^{-1,p}(B_r)} \leq C \tag{3.2.10}$$

for any $p > n$, where the constant of course depends on p. Recall that $W^{-1,p}(U)$ is by definition the dual space of $W_0^{1,q}(U)$, where $1/p + 1/q = 1$. The last estimate thus follows from (3.2.8) and the fact that $Ju = \frac{1}{n} dj(u)$.

4. We also immediately see from the above theorem that if $\epsilon \leq r \leq 1$, $u \in W^{1,n}(B_r; \mathbb{R}^n)$, and

$$|| \, [Ju] - \omega_n d\delta_0 \, ||_{\mathcal{M}^1(B_r)} \leq \gamma_0 r,$$

for $d = \pm 1$, then

$$\int_{B_r} E^\epsilon dx \geq \kappa_n \ln\left(\frac{r}{\epsilon}\right) - C. \tag{3.2.11}$$

Before giving the proof, we sketch the main ideas:

Step 1: We first show that if $|| \, [Ju] - \omega_n d\delta_0 \, ||_{\mathcal{M}^1(B_r)}$ is small and the energy is not too large, then the set of radii $s \leq r$ satisfying

$$\text{dg}(u; \partial B_s) = d \tag{3.2.12}$$

has large measure. This is carried out in Lemmas 3.2.1 and 3.2.2. The key point in the latter lemma is to choose an appropriate test function ϕ in the definition of the $|| \cdot ||_{\mathcal{M}^1}$ norm.

Step 2: By a covering argument we find a collection of balls which cover S_E and satisfy, for example,

$$\int_B E^\epsilon \, dx \geq \Lambda^\epsilon(\rho)\,; \qquad \rho = \text{ radius of } B; \tag{3.2.13}$$

and

$$\mathrm{dg}(u;\partial B)=0 \quad \text{if } B\cap\partial B_r=\emptyset. \tag{3.2.14}$$

(See Figure 1.)

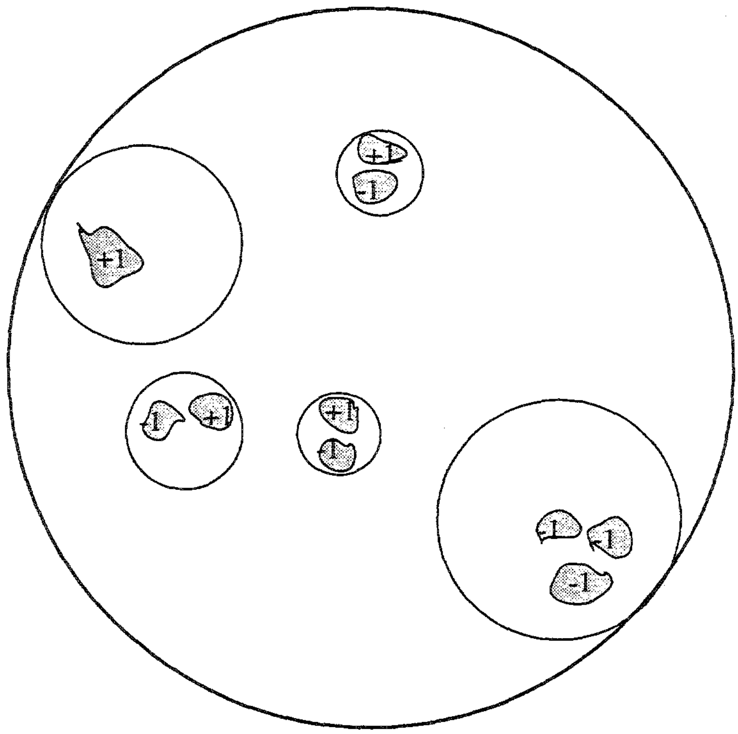

Figure 1. Any ball B in the collection covering S_E not hitting ∂B_r has $\mathrm{dg}(u;\partial B)=0$.

These covering arguments are presented in Section 3 of this chapter. They include various refinements which play a crucial role in the arguments outlined in Step 5 below.

Step 3: Condition (3.2.14) implies that every radius satisfying (3.2.12) must intersect one of the balls from Step 2. (See Figure 2.) Thus Step 1 gives a lower estimate on the sum of the radii of the balls from Step 2.

Step 4: In general $\Lambda^\epsilon(r)+\Lambda^\epsilon(s)$ is considerably larger than $\Lambda^\epsilon(r+s)$, which implies that a collection of many small balls has much more energy than one large ball, where all balls are assumed to satisfy (3.2.13). Using this fact and the assumed upper bound (3.2.2), we show that the collection of balls found above must contain at least one large ball, say $B_{r_1}(x_1)$, with $r_1\geq r/8$.

Step 5: We now focus on the single large ball $B_{r_1}(x_1)$ found above, and we define the "good radii" to be those $s \in [0, r_1]$ such that $\mathrm{dg}(u; \partial B_s) \neq 0$. We know that a lower energy bound holds on these radii. All other radii are said to be "bad radii". Reasoning similar to that of Step 4 shows that, if the set of bad radii is large, then the total energy of the ball $B_{r_1}(x_1)$ must also be large, and this possibility is ruled out by (3.2.2). We thus find that the set of bad radii has measure at most $C\epsilon$.

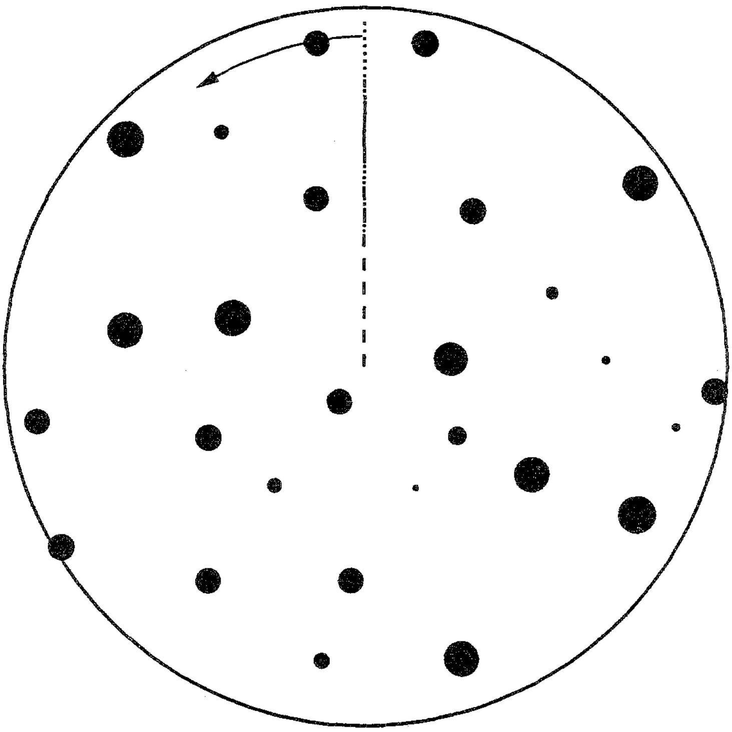

Figure 2. The set of radii, represented by the dashed vertical line, sweeps out circles which must intersect the balls in the cover.

Step 6: At this point all the conclusions of Theorem 3.2.1 follow quite easily.

Step 7: There are some assertions in Theorem 1.4.3 which are not included in Theorem 3.2.1. These other points are proved in Section 4 of this chapter by a compactness argument.

We now present the proofs.

Lemma 3.2.1. *There is some number $\alpha \in (0,1)$, depending only on the dimension n, such that if $r \geq \epsilon$ and $\epsilon > 0$ is sufficiently small, then either*

$$C\epsilon^{-\alpha} \leq \int_{\partial B_r} E^{\epsilon} dH^{n-1} \tag{3.2.15}$$

or

$$\left|\deg(u;\partial B_r) - \frac{1}{\omega_n}\int_V Ju\right| \le C\epsilon^\alpha \left(\int_{\partial B_r} E^\epsilon dH^{n-1} + H^{n-1}(\partial B_r)\right). \tag{3.2.16}$$

Remark. It follows from (3.1.5) that, if $V \subset U$ with ∂V smooth, and $u \in W^{1,n}(\partial V, \partial B_1)$, then

$$\frac{1}{\omega_n}\int_V Ju = \deg(u;\partial V).$$

This lemma asserts that this remains approximately true if we relax the constraint that $|u| = 1$ on ∂V, but instead merely require that that E^ϵ is not too large on ∂V. For convenience it is proved when V is a ball, but in fact it is more generally true.

Proof.

1. As above we let $v = u/|u|$ and $\rho = |u|$ and we define $m := \operatorname{ess\,inf}_{\partial B_r} \rho$. Using (3.1.5) we compute

$$\begin{aligned}
\omega_n \deg(u;\partial B_r) - \int_{B_r} Ju &= \int_{\partial B_r} \langle j(v), \tau\rangle - \int_{\partial B_r} \langle j(u), \tau\rangle \\
&= \int_{\partial B_r} (1-\rho^n)\langle j(v), \tau\rangle \\
&\le ||1-\rho^n||_{L^\infty(\partial B_r)} \int_{\partial B_r} |j(v)| dH^{n-1} \\
&\le C||1-\rho^n||_{L^\infty(\partial B_r)} \int_{\partial B_r} |Dv|^{n-1} dH^{n-1} \\
&\le C||1-\rho^n||_{L^\infty(\partial B_r)} \frac{1}{m^{n-1}} \int_{\partial B_r} (\rho|Dv|)^{n-1} dH^{n-1} \\
&\le C||1-\rho^n||_{L^\infty(\partial B_r)} \frac{1}{m^{n-1}} \int_{\partial B_r} (E^\epsilon + 1)\, dH^{n-1}.
\end{aligned}$$

In the last step above we have used Young's inequality.

2. Define γ as in (3.1.14). From Lemma 3.1.3 we deduce that

$$\frac{1}{m^{n-1}} \le \frac{1}{[1-(C\epsilon\gamma)^{1/N}]^{n-1}}.$$

Also

$$||\rho^n - 1||_{L^\infty(\partial B_r)} \le \min\{(C\epsilon\gamma)^{1/N}, (C\epsilon\gamma)^{n/N}\}.$$

Let $\alpha := n/(N+n)$. If (3.2.15) does not hold, then $\gamma < C\epsilon^{-\alpha}$ and

$$||1-\rho^n||_{L^\infty(\partial B_r)} \frac{1}{m^{n-1}} \le C\epsilon^{n(1-\alpha)/N} = C\epsilon^{-\alpha},$$

if ϵ is sufficiently small. The conclusion follows directly from this and Step 1. □

In order to prove Theorem 3.2.1, it will be convenient to work with continuous functions which satisfy

$$\mathcal{L}^1(S_d(u)) \geq 3r/4 \tag{3.2.17}$$

where

$$S_d(u) := \{s \in [0,r] \ : \ \mathrm{dg}(u;\partial B_s) = d\}. \tag{3.2.18}$$

Recall that dg is defined in (3.1.9), and that the definition requires that u be continuous.

The next lemma shows that smooth approximators to u satisfy (3.2.17).

Lemma 3.2.2. *Let u satisfy the hypotheses of Theorem* 3.2.1, *and let $u^\delta := \eta^\delta * u$, where η^δ is a standard mollifier. If $\epsilon > 0$ is sufficiently small, then*

$$\liminf_{\delta\to 0} \mathcal{L}^1(S_d(u^\delta)) \geq 3r/4.$$

Proof.

1. If δ is sufficiently small, then

$$\|\, [Ju^\delta] - \omega_n d\delta_0 \,\|_{\mathcal{M}^1(B_r)} \leq 2\gamma_0 r. \tag{3.2.19}$$

We will show that if γ_0 is chosen to be sufficiently small (for example, $\gamma_0 := \omega_n/200$ would suffice), then (3.2.17) is satisfied whenever (3.2.19) holds.

We may suppose without any loss of generality that $d = +1$. We also omit the superscripts δ, and assume that u is a continuous function satisfying (3.2.19).

First suppose that $s \in (\epsilon, r)$ satisfies

$$\int_{\partial B_s} E^\epsilon(u)dH^{n-1} < C\epsilon^{-\alpha/2}, \tag{3.2.20}$$

where α is the constant from Lemma 3.2.1; and

$$\left|1 - \frac{1}{\omega_n}\int_{B_s} Ju\right| < \frac{1}{4}. \tag{3.2.21}$$

We claim that these two conditions imply that $s \in S_d(u)$. To see this, note that (3.2.20) and Lemma 3.2.1 imply that (3.2.16) holds, i.e., that

$$\begin{aligned}\left|\deg(u;\partial B_s) - \frac{1}{\omega_n}\int_{B_s} Ju\right| &\leq C\epsilon^\alpha \left(\int_{\partial B_s} E^\epsilon dH^{n-1} + H^{n-1}(\partial B_s)\right)\\ &\leq C\epsilon^{\alpha/2},\end{aligned}$$

using (3.2.20) again. Since $\deg(u;\partial B_s)$ is an integer, the above inequality and (3.2.21) imply that $\deg(u;\partial B_s) = 1$.

Finally, (3.2.20) and Lemma 3.1.3 imply that $\min_{\partial B_s}|u| \geq 3/4$. By (3.1.10) this implies that $\mathrm{dg}(u;\partial B_s) = \deg(u;\partial B_s) = 1$, which is our claim.

2. Define

$$\mathcal{B}_1 := \{s \in (\epsilon, r) \ : \ (3.2.20) \text{ does not hold}\},$$

$$\mathcal{B}_2 := \{s \in (\epsilon, r) \ : \ (3.2.21) \text{ does not hold}\}.$$

From Step 1 we see that it suffices to show that $\mathcal{L}^1(\mathcal{B}_1 \cup \mathcal{B}_2) \leq \frac{1}{4}r - \epsilon$.

From (3.2.2) it is easy to see that $\mathcal{L}^1(\mathcal{B}_1) \leq \epsilon^{\alpha/4}$. Thus we only need to show that $\mathcal{L}^1(\mathcal{B}_2) \leq r/8$.

To do this, write $\mathcal{B}_2 = \mathcal{B}_2^+ \cup \mathcal{B}_2^-$, where

$$\mathcal{B}_2^+ = \{s \in (\epsilon, r) \ : \ \frac{1}{\omega_n}\int_{B_s} Ju \geq \tfrac{5}{4}\} \quad \text{and} \quad \mathcal{B}_2^- = \{s \in (\epsilon, r) \ : \ \frac{1}{\omega_n}\int_{B_s} Ju \leq \tfrac{3}{4}\}.$$

Then

$$\begin{aligned} 5/4 &\leq \frac{1}{\mathcal{L}^1(B_2^+)}\int_{s\in\mathcal{B}_2^+}\frac{1}{\omega_n}\int_{B_s} Ju\ dx\ ds \\ &= \frac{1}{\mathcal{L}^1(B_2^+)}\frac{1}{\omega_n}\int_{B_r}\int_{s\in\mathcal{B}_2^+}\chi_{|x|\leq s} Ju\ ds\ dx \\ &= \frac{1}{\omega_n}\int_{B_r}\psi(x) Ju\ ds\ dx \end{aligned} \tag{3.2.22}$$

for

$$\psi(x) := \frac{\mathcal{L}^1(\{s \in B_2^+ \ : \ s \geq |x|\})}{\mathcal{L}^1(B_2^+)}.$$

By (3.2.19) and the definition of the $\mathcal{M}^1$ norm,

$$\left|\omega_n\psi(0) - \int_{B_r} J(u)\psi\right| \leq 2\gamma_0 r||\psi||_{W^{1,\infty}}. \tag{3.2.23}$$

However, we easily check that $\psi(0) = 1 = ||\psi||_\infty$ and $||D\psi||_\infty = (\mathcal{L}^1(B_2^+))^{-1}$. By substituting these into (3.2.23) and combining with (3.2.22) we obtain

$$\frac{\omega_n}{4} \ \leq \ 2r\gamma_0||\psi||_{W^{1,\infty}} \ \leq \ \frac{2r\gamma_0}{\omega_n}\Big(1 + \frac{1}{\mathcal{L}^1(B_2^+)}\Big).$$

Rearranging, this becomes

$$\mathcal{L}^1(B_2^+) \leq \frac{8r\gamma_0}{\omega_n^2 - 8r\gamma_0} \leq \frac{8\gamma_0}{\omega_n^2 - 8\gamma_0}r,$$

using our assumption that $r \le 1$. Fixing γ_0 sufficiently small, we obtain

$$\mathcal{L}^1(B_2^+) \le \frac{r}{16}.$$

The same argument shows that $\mathcal{L}^1(B_2^-) \le r/16$, so we are finished. □

Remark. In fact we have shown that if $u \in C \cap W^{1,n}(B_r;\mathbb{R}^n)$ with $r \le 1$, if (3.2.2) holds, and

$$||[Ju] - \omega_n d\delta_0||_{\mathcal{M}^1(B_r)} \le h, \tag{3.2.24}$$

then for any $0 < r_1 < r_2 \le r$,

$$\mathcal{L}^1(S_d(u) \cap [r_1, r_2]) \ge r_2 - r_1 - O(h) - o(1) \tag{3.2.25}$$

as $\epsilon \to 0$.

The other chief technical ingredient in the proof of Theorem 3.2.1 is the following lemma, the proof of which is deferred until the next section.

We first introduce some notation. Suppose a function $u \in C \cap W^{1,n}(U;\mathbb{R}^n)$ and $\epsilon > 0$ are given. Let $x \in U$ and $r > 0$. We say that r is a *good radius* about x if $r \ge \epsilon$ and $\mathrm{dg}(u;\partial B_r) \ne 0$. By Lemma 3.1.4, if r is a good radius about x, then

$$\int_{\partial B_r(x)} E^\epsilon dH^{n-1} \ge \lambda^\epsilon(s) \wedge \frac{c_0}{\epsilon}. \tag{3.2.26}$$

If r is not a *good radius*, then it is said to be a *bad radius*.

We define

$$\beta(x,r) = \mathcal{L}^1\left(\{s \in (0,r] \ : \ s \text{ is a bad radius about } x\}\right). \tag{3.2.27}$$

Lemma 3.2.3. *Let U be any bounded subset of $\mathbb{R}^n$, and suppose that $u \in C \cap W^{1,n}(U;\mathbb{R}^n)$ and that $|u| \ge \frac{1}{2}$ on ∂U.*

Then we can find a collection of balls $\{B_i = B_{r_i}(x_i)\}_{i=1}^k$ with pairwise disjoint interiors, such that

$$S_E \subset \bigcup_i B_i, \qquad \text{and } B_i \cap S_E \ne \emptyset \quad \forall i; \tag{3.2.28}$$

$$\mathrm{dg}\,(u;\partial B_i) = 0 \qquad \text{for all } i \text{ such that } B_i \subset U; \tag{3.2.29}$$

$$\int_{B_i \cap U} E^\epsilon dx \ge \Lambda^\epsilon(r_i) + \frac{1}{24}\Lambda^\epsilon((\beta(x_i, r_i) - C_1\epsilon)^+) \tag{3.2.30}$$

$$\ge \Lambda^\epsilon(r_i) \tag{3.2.31}$$

for all $i = 1, ..., k$. Here C_1 is a constant depending only on the dimension n.

The difference between this lemma and the earlier Lemma 3.1.7 is that the balls found earlier are very small, in general of radius $\sim \epsilon$, whereas the balls found here are in some sense as large as possible. This is the meaning of condition (3.2.29), which asserts that every ball of nonzero degree hits the boundary. This lemma also gives control over the size of the sets of "bad radii".

We now present the proof of Theorem 3.2.1.

Proof.

1. By Lemma 3.2.2 and an approximation argument, it suffices to prove the theorem for u which is continuous and satisfies (3.2.17).

We want to use Lemma 3.2.3, for which we need

$$|u| \geq 1/2 \quad \text{on } \partial B_r. \tag{3.2.32}$$

If this condition is not satisfied, we may replace B_r by $B_{\hat{r}}$, where

$$\hat{r} := \max\{s < r \ : \ |u| \geq 1/2 \text{ on } \partial B_s\}.$$

Note that $\mathrm{dg}\,(u; \partial B_s)$ is undefined for all $s > \hat{r}$, so (3.2.17) remains valid on $B_{\hat{r}}$:

$$\mathcal{L}^1(S_d(u)) \geq 3r/4,$$

where $S_d(u)$ is now redefined as $\{s \in [0, \hat{r}] : \mathrm{dg}(u; \partial B_s) = d\}$. So we may assume without loss of generality that (3.2.32) holds.

2. Let $\{B_i\}_{i=1}^M$ be the collection of balls found in Lemma 3.2.3.

We claim first that

$$S_d(u) \subset \bigcup_{i=1}^{M} \{s : \partial B_s \cap B_i \neq \emptyset\}.$$

Indeed, fix $s \in S_d(u)$, so that $\mathrm{dg}\,(u; \partial B_s) = d$. We must show that $\partial B_s \cap B_i \neq \emptyset$ for some i. Suppose, to the contrary, that

$$\partial B_s \cap B_i = \emptyset$$

for all $i \in \{1, \ldots, M\}$. Then (3.2.28), (3.1.9), and (3.2.29) imply that

$$\mathrm{dg}(u; \partial B_s) = \sum_{B_i \subset B_s} \mathrm{dg}(u^\epsilon; \partial B_i) = 0,$$

which is impossible.

3. We now have, using (3.2.17),

$$
\begin{aligned}
3r/4 &\le \mathcal{L}^1(S_d(u)) \\
&\le \mathcal{L}^1\left(\bigcup_i \{s \in (0,r) : \partial B_s \cap B_i \neq \emptyset\}\right) \\
&\le \sum_i \mathcal{L}^1(\{s \in (0,r) \ : \ \partial B_s \cap B_i \neq \emptyset\}) \\
&\le \sum_i 2r_i. \qquad (3.2.33)
\end{aligned}
$$

Let $r_{\max} := \max_j\{r_j\} := r_1$, say. We claim that $r_{\max} \ge r/8$. If not, we can find some subset $I := \{i_1, ..., i_j\} \subset \{1, ..., m\}$ such that

$$\sum_{i \in I} r_i \in \left[\frac{r}{8}, \frac{r}{4}\right].$$

This follows from (3.2.33), which with our choice of I also implies that

$$\sum_{i \notin I} r_i \ge \frac{r}{8}.$$

Then, using (3.2.31) and the subadditivity of Λ^ϵ,

$$
\begin{aligned}
\int_{B_r} E^\epsilon dx &\ge \sum_i \Lambda^\epsilon(r_i) \\
&\ge \Lambda^\epsilon\left(\sum_{i\in I} r_i\right) + \Lambda^\epsilon\left(\sum_{i\notin I} r_i\right) \\
&\ge 2\Lambda^\epsilon(r/8) \\
&\ge 2\kappa_n \ln\left(\frac{r}{8\epsilon}\right) - C.
\end{aligned}
$$

In view of (3.2.2) and (3.1.16), this is impossible for small ϵ

We will take ξ to be x_1, the center of the big ball $B_{r_1}(x_1)$. At this stage we do not know that $\xi \in B_{r/2}$; in the final step of the proof we will show that this can be arranged to hold.

4. We next show that $\beta(x_1, r_1)$ is small. We note first that by (3.2.30),

$$\kappa_n \ln\left(\frac{r}{\epsilon}\right) + C \ge \Lambda^\epsilon(r_1) + \frac{1}{24}\Lambda^\epsilon\left((\beta(r_1, s_1) - \epsilon C_1)^+\right).$$

Since $r_1 \ge r/8$, this implies that

$$
\begin{aligned}
C &\ge \Lambda^\epsilon\left((\beta(r_1, s_1) - \epsilon C_1)^+\right) \\
&\ge \ln\left(\left(\frac{\beta(x_1, r_1)}{\epsilon} - C_1\right)^+\right) - C.
\end{aligned}
$$

Thus

$$\beta(x_1, r_1) \le C\epsilon \tag{3.2.34}$$

for some constant C which depends on γ_1 but is independent of ϵ.

5. The rest is now fairly straightforward. First, fix any $0 \le \sigma < \tau \le r_1$, and let

$$\mathcal{G} := \{s \in [\sigma, \tau] \mid s \text{ is a good radius about } x_1\}.$$

Then

$$\begin{aligned}
\int_{B_\tau(x_1)\setminus B_\sigma(x_1)} E^\epsilon dx &= \int_\sigma^\tau \int_{\partial B_s} E^\epsilon \, dH^{n-1} \, ds \\
&\ge \int_{\mathcal{G}} \int_{\partial B_s} E^\epsilon \, dH^{n-1} \, ds \\
&\ge \int_{\mathcal{G}} \lambda^\epsilon(s) \wedge \frac{c_0}{\epsilon} \, ds \quad \text{using (3.2.26)} \\
&\ge \int_{\tau - \mathcal{L}^1(\mathcal{G})}^{\tau} \lambda^\epsilon(s) \wedge \frac{c_0}{\epsilon} \, ds \quad \text{since } \lambda(\cdot) \text{ is decreasing.}
\end{aligned}$$

Also, from (3.2.34) we easily deduce that

$$\tau - \mathcal{L}^1(\mathcal{G}) \;=\; \sigma + \beta(x_1, r_1) \;\le\; \sigma + C\epsilon.$$

These together imply that

$$\int_{B_\tau(x_1)\setminus B_\sigma(x_1)} E^\epsilon dx \ge \Lambda^\epsilon(\tau) - \Lambda^\epsilon(\sigma) - C. \tag{3.2.35}$$

In particular, taking $\sigma = 0$ and remembering (3.1.16), we obtain (3.2.3).

6. For any $\epsilon \le \sigma \le r_1/2$, we have

$$\begin{aligned}
\int_{B_{2\sigma}(x_1)\setminus B_\sigma(x_1)} E^\epsilon dx &\le \int_{B_r} E^\epsilon - \int_{B_{r_1}(x_1)\setminus B_{2\sigma}(x_1)} E^\epsilon - \int_{B_\sigma(x_1)} E^\epsilon \\
&\le C
\end{aligned} \tag{3.2.36}$$

independent of ϵ, using (3.2.2), (3.2.35), various properties of Λ^ϵ, and the fact that $r_1 \ge r/8$.

We use this to verify (3.2.4). To do this, fix any $\phi \in C_0^1(B_r)$ such that $\|\phi\|_{C^1} \le 1$. We then have

$$\left| \int \phi d\mu_u^\epsilon - \int \phi \kappa_n \delta_{x_1} \right| \le I_1 + I_2,$$

where

$$I_1 := \frac{1}{|\ln \epsilon|} \int_{B_r} |\phi(x) - \phi(x_1)| \, E^\epsilon(x) \, dx \;\text{ and }\; I_2 := |\phi(x_1)| \left| \kappa_n - \frac{1}{|\ln \epsilon|} \int_{B_r} E^\epsilon dx \right|.$$

We easily estimate from (3.2.2) and (3.2.3) that $I_2 \le C|\ln\epsilon|^{-1}$, and

$$|\ln\epsilon|\, I_1 \le \int_{B_r} |x - x_1| E^\epsilon(x)dx \\ \le O(1) + \sum_{i=0}^{K-1} r_1 2^{-i} \int_{B_{\sigma_i}(x_1)\setminus B_{\sigma_{i+1}}(x_1)} E^\epsilon.$$

Here $\sigma_i := 2^{-i} r_1$, and K is chosen so that $\epsilon < \sigma_K \le 2\epsilon$. The $O(1)$ error terms come from integrating over $B_{\sigma_K}(x_1)$ and $B_r \setminus B_{r_1}(x_1)$. Using (3.2.36) we see that the right-hand side is bounded independent of ϵ.

The previous few inequalities thus show that

$$\left| \int \phi d\mu_u^\epsilon - \int \phi\kappa_n\delta_{x_1} \right| \le \frac{C}{|\ln\epsilon|}$$

for any ϕ as above, which by definition is (3.2.4).

7. It remains to prove (3.2.5). To do this, write $\int_{B_r} |Du|^p dx$ as a sum of integrals over annuli, as in Step 6 above. The stated estimate then follows by applying Hölder's inequality on each annulus and using (3.2.36). A similar argument with more details included can be found in Struwe [23].

8. Finally, by taking γ_0 smaller, we can assume that $S_d(u) \cap [\epsilon, r/2] \ge 3r/8$. Then the above arguments apply to the ball $B_{r/2}$, so we can find a point ξ having the desired properties and such that $\xi \in B_{r/2}$ as desired. □

3. Covering arguments

In this section, we prove Lemma 3.2.3. We start by giving a relatively easy covering argument which contains most of the main ideas.

Lemma 3.3.1 (First covering argument). *Suppose that*

$$u \in C \cap W^{1,n}(U; \mathcal{R}^n)$$

and that $|u| \ge 1/2$ *on* ∂U, *where* U *is an open bounded subset of* $\mathbb{R}^n$.

Then we can find a collection of balls $\{B_i = B_{r_i}(x_i)\}_{i=1}^k$ *with pairwise disjoint interiors, such that*

(3.3.1) $$\int_{B_i \cap U} E^\epsilon dx \ge \Lambda^\epsilon(r_i), \quad \forall i = 1, \dots, k;$$

(3.3.2) $$S_E \subset \bigcup_i B_i, \qquad \text{and } B_i \cap S_E \ne \emptyset \quad \forall i;$$

(3.3.3) $$\mathrm{dg}(u;\partial B_i) = 0 \quad \textit{for all } i \textit{ such that } B_i \subset U.$$

Proof. Let $\{B_i\}_{i=1}^M$ be the collection of balls given by Lemma 3.1.7. These satisfy (3.3.1) and (3.3.2) by construction. If (3.3.3) holds, there is nothing to prove. We therefore assume that it does not hold. Recall also that by construction, the balls $\{B_i\}$ are pairwise disjoint.

We will successively modify the balls in such a way that after each step we obtain a new collection satisfying (3.3.1) and (3.3.2), and such that (3.3.3) is eventually satisfied. After each modification we relabel the balls, so that each succesive collection is called $\{B_i\}$. This makes the notation less burdensome and should not cause any confusion.

1. Since (3.3.3) is not satisfied, we may find a ball, say $B_1 := B_{r_1}(x_1)$, such that

$$B_1 \subset U \quad \text{and} \quad \mathrm{dg}(u;\partial B_1) \neq 0.$$

Let

(3.3.4) $$\tilde{r}_1 := \inf\{\rho \geq r_1 : B_\rho(x_1) \cap (\partial U \cup (\bigcup_{i\geq 2} B_i)) \neq \emptyset\}.$$

Our choice of B_1 implies that $\tilde{r}_1 > r_1$. For any $\rho \in (r_1, \tilde{r}_1)$, we see from (3.3.2) that

$$B_\rho(x_1) \cap S_E = B_{r_1}(x_1) \cap S_E$$

and hence that $\mathrm{dg}(u;\partial B_\rho) = \mathrm{dg}(u;\partial B_{r_1}) \neq 0$. So we may use (3.1.17) and the fact that B_1 satisfies (3.3.1) to estimate

$$\begin{aligned}\int_{B_{\tilde{r}_1}(x_1)} E^\epsilon dx &= \int_{B_{\tilde{r}_1}(x_1)\setminus B_{r_1}(x_1)} E^\epsilon dx + \int_{B_{r_1}(x_1)} E^\epsilon dx \\ &\geq [\Lambda^\epsilon(\tilde{r}_1) - \Lambda^\epsilon(r_1)] + \Lambda^\epsilon(r_1) = \Lambda^\epsilon(\tilde{r}_1).\end{aligned}$$

2. Relabel $B_1 = (B_1)_{\text{new}} := B_{\tilde{r}_1}(x_1)$ and $r_1 = (r_1)_{\text{new}} := (\tilde{r}_1)_{\text{old}}$. We now have a new collection of balls satisfying (3.3.1) and (3.3.2). They may not be pairwise disjoint, but nonetheless their interiors are pairwise disjoint, as a result of (3.3.4).

If the balls are not pairwise disjoint, select two balls B_i, B_j that intersect and replace them by a single larger ball B' of radius $r' = r_i + r_j$ such that $B_i \cup B_j \subset B'$. Then

$$\begin{aligned}\int_{B'\cap U} E^\epsilon dx &\geq \int_{B_i\cap U} E^\epsilon dx + \int_{B_j\cap U} E^\epsilon dx \\ &\geq \Lambda^\epsilon(r_i) + \Lambda^\epsilon(r_j) \;\geq\; \Lambda^\epsilon(r').\end{aligned}$$

We have used the subadditivity (3.1.15) of Λ^ϵ.

If B' intersects some other ball, say B_k, combine as before into a larger ball B'' containing $B' \cup B_k$ and with radius $r'' \leq r' + r_k$. The same calculation then shows that

$$\int_{B''\cap U} E^\epsilon dx \;\geq\; \Lambda^\epsilon(r_i) + \Lambda^\epsilon(r_j) + \Lambda^\epsilon(r_k) \;\geq\; \Lambda^\epsilon(r'').$$

Observe that this is true even if $B' \cap B_k$ has nonempty interior, since we estimate the energy in the new ball B'' only using balls from the earlier collection $\{B_i\}$, which have pairwise disjoint interiors.

We can thus continue to combine balls until we achieve a pairwise disjoint collection satisfying (3.3.1) and (3.3.2).

3. If the balls in this new collection satisfy (3.3.3), we are finished. If not, we are in exactly the situation of the beginning of Step 1, except that there are now fewer balls and their radii are larger. We may thus iterate the argument as long as (3.3.3) does not hold. The process must eventually terminate, as the number of balls is finite and decreases with each iteration, and when it terminates the construction is complete. □

A more careful version of the above argument will establish Lemma 3.2.3. We will use the fact that the estimate $\Lambda^\epsilon(r) + \Lambda^\epsilon(s) \geq \Lambda^\epsilon(r+s)$ can be improved when either r or s is not too small.

Lemma 3.3.2. *There exists some $C_1 = C_1(n) > 1$ such that if $\epsilon \leq r_0 \leq r_1$ and $r_1 \geq \epsilon C_1$, then*

$$\Lambda^\epsilon(r_0) + \Lambda^\epsilon(r_1) \geq \Lambda^\epsilon(r_1 + 2r_0) + \tfrac{1}{24}\Lambda^\epsilon(2r_0).$$

Proof.

1. We first find some constant C_0 such that

$$\lambda^\epsilon(s) - \lambda^\epsilon(2s) - \lambda^\epsilon(3s) \geq \tfrac{1}{12}\lambda^\epsilon(s) \tag{3.3.5}$$

whenever $s \geq \epsilon C_0$.

Define

$$f(m,s) := m^n \frac{\kappa_n}{s} + \frac{(1-m)^N}{C^*\epsilon},$$

where C^*, N are in the definition (3.1.12) of λ^ϵ, so that $\lambda^\epsilon(s) = \min_{m\in[0,1]} f(m,s)$. Fix s and find $\bar{m} \in (0,1)$ such that $\lambda^\epsilon(s) \;=\; f(\bar{m}, s)$. Then $\frac{d}{dm} f(\bar{m}, s) = 0$, which implies that

$$\frac{(1-\bar{m})^N}{C^*\epsilon} = (1-\bar{m})\bar{m}^{n-1}\frac{n}{N}\frac{\kappa_n}{s}.$$

Also, it is clear that $\lambda^\epsilon(s) \le f(1,s) = \kappa_n/s$, so

$$(1-\bar{m}) \le \left(\frac{C^*\kappa_n\epsilon}{s}\right)^{1/N}.$$

Combining these, we obtain

$$\frac{(1-\bar{m})^N}{C^*\epsilon} \le C\left(\frac{\epsilon}{s}\right)^{1/N}\frac{\kappa_n}{s}$$

for some C. In particular, the previous two equations imply that, if $s \ge \epsilon C_0$ for some sufficiently large C_0, then

$$\frac{(1-\bar{m})^N}{C^*\epsilon} \le \frac{1}{24}\bar{m}^n\frac{\kappa_n}{s}.$$

When this holds we estimate

$$\begin{aligned}\lambda^\epsilon(s) - \lambda^\epsilon(2s) - \lambda^\epsilon(3s) &\ge f(\bar{m},s) - f(\bar{m},2s) - f(\bar{m},3s)\\ &= \bar{m}^n\frac{\kappa_n}{6s} - \frac{(1-\bar{m})^N}{C^*\epsilon}\\ &\ge \bar{m}^n\frac{\kappa_n}{12s} + \frac{(1-\bar{m})^N}{C^*\epsilon} \ \ge\ \frac{1}{12}\lambda^\epsilon(s).\end{aligned}$$

2. Next, we will select a constant $C_1 > C_0$ such that, if $r_1 \ge C_1\epsilon$ and $\epsilon \le s \le r_1$, then

$$\lambda^\epsilon(s) - \lambda^\epsilon(r_1+s) - \lambda^\epsilon(r_1+2s) \ge \tfrac{1}{12}\lambda^\epsilon(s). \tag{3.3.6}$$

If $\epsilon C_0 \le s \le r_1$, this follows from (3.3.5) and the fact that λ^ϵ is nonincreasing. So we assume that $\epsilon \le s \le C_0\epsilon < C_1\epsilon \le r_1$.

As before select $\bar{m} = \bar{m}(s)$ such that $\lambda^\epsilon(s) = f(\bar{m},s)$. One easily sees that

$$\min_{s\ge\epsilon}\bar{m}(s) > 0. \tag{3.3.7}$$

Then

$$\begin{aligned}\lambda^\epsilon(s) - \lambda^\epsilon(r_1+s) - \lambda^\epsilon(r_1+2s) &\ge f(\bar{m},s) - f(1,r_1+s) - f(1,r_1+2s)\\ &= \left(\bar{m}^n - \frac{s}{r_1+s} - \frac{s}{r_2+s}\right)\frac{\kappa_n}{s} + \frac{(1-\bar{m})^N}{C^*\epsilon}\\ &\ge \left(\bar{m}^n - 2\frac{C_0}{C_1}\right)\frac{\kappa_n}{s} + \frac{(1-\bar{m})^N}{C^*\epsilon}.\end{aligned}$$

In view of (3.3.7), we can easily choose C_1 large enough that (3.3.6) holds.

3. By taking C_1 still larger, if necessary, we may assume that

$$\lambda^\epsilon(s) \le \frac{c_0}{4\epsilon} \tag{3.3.8}$$

whenever $s \geq C_1\epsilon$, where c_0 is the constant in the definition (3.1.13) of Λ^ϵ.

Suppose now that $C_1\epsilon \leq r_1$ and that $\epsilon \leq r_0 \leq r_1$. Then using (3.3.6) and (3.3.8) and the fact that λ^ϵ is nonincreasing, we find

$$\begin{aligned}
&\Lambda^\epsilon(r_0) + \Lambda^\epsilon(r_1) - \Lambda^\epsilon(r_1 + 2r_0) \\
&= \int_0^{r_0} \left[\lambda^\epsilon(s) \wedge \frac{c_0}{\epsilon}\right] ds - \int_{r_1}^{r_1+2r_0} \lambda^\epsilon(s)\, ds \\
&= \int_0^{r_0} \left[\lambda^\epsilon(s) \wedge \frac{c_0}{\epsilon}\right] - \lambda^\epsilon(r_1 + s) - \lambda^\epsilon(r_1 + r_0 + s)\ ds \\
&\geq \int_0^{r_0} [\lambda^\epsilon(s) - \lambda^\epsilon(r_1 + s) - \lambda^\epsilon(r_1 + 2s)] \wedge \left[\frac{c_0}{\epsilon} - 2\lambda^\epsilon(r_1)\right] ds \\
&\geq \int_0^{r_0} \frac{1}{12} \left[\lambda^\epsilon(s) \wedge \frac{c_0}{\epsilon}\right]\ ds \\
&= \frac{1}{12}\Lambda^\epsilon(r_0) \ \geq\ \frac{1}{24}\Lambda^\epsilon(2r_0).
\end{aligned}$$

The final inequality follows from the subadditivity of Λ^ϵ. □

We are now ready to establish Lemma 3.2.3.

Proof.

1. We follow the same strategy as in the proof of the Lemma 3.3.1. That is, we start with a collection of pairwise disjoint balls $\{B_i = B_{r_i}(x_i)\}_{i=1}^k$ satisfying

$$S_E \subset \bigcup_i B_i, \qquad \text{and } B_i \cap S_E \neq \emptyset \quad \forall i; \tag{3.3.9}$$

and

$$\int_{B_i \cap U} E^\epsilon dx \geq \Lambda^\epsilon(r_i) + \frac{1}{24}\Lambda^\epsilon\left((\beta(x_i, r_i) - C_1\epsilon)^+\right) \tag{3.3.10}$$

for all $i = 1, ..., k$.

Recall that $\beta(x, r)$ is defined in (3.2.27).

We will successively modify the balls to obtain new collections again satisfying (3.3.9) and (3.3.10), and in such a way that (3.3.3) is eventually satisfied. As before, we relabel the collections as we proceed.

As in the earlier proof, we start with the collection of balls provided by Lemma 3.1.7. It is immediate from (3.1.18) that (3.3.9) holds.

We claim that (3.3.10) is also verified. For this, we need to select C_1 appropriately. Indeed, it is not hard to see from the definition of Λ^ϵ that if C_1 is sufficiently large then

$$\frac{c_0}{\epsilon} r \geq 2\Lambda^\epsilon(r) \quad \text{whenever } r \geq C_1\epsilon.$$

This implies that

$$\frac{c_0}{\epsilon}r \geq \Lambda^\epsilon(r) + \frac{1}{24}\Lambda^\epsilon\left((r - C_1\epsilon)^+\right)$$

for all $r \geq 0$. By taking C_1 large enough, we may assume that this is true, and that the conclusions of Lemma 3.3.2 hold whenever $r_1 \geq \frac{1}{2}\epsilon C_1$ and $\epsilon \leq r_0 \leq r_1$.

Now from (3.1.21) we know that

$$\int_{B_i} E^\epsilon dx \geq \frac{c_0}{\epsilon} r_i.$$

The choice of C_1 above then implies that (3.3.10) holds, since it is clear from the definition (3.2.27) of β that $\beta(x,r) \leq r$ for all x, r.

2. If (3.3.3) holds, we are finished, so we assume that it does not hold. We may therefore find a ball, say $B_1 := B_{r_1}(x_1)$, such that

$$B_1 \subset U \quad \text{and} \quad \mathrm{dg}(u; \partial B_1) \neq 0.$$

We now expand this ball exactly as in Step 2 of the proof of Lemma 3.3.1. We must verify that the resulting ball, say $B_{\tilde{r}_1}(x_1)$, satisfies (3.3.10).

Recall that $\tilde{r}_1$ is chosen so that $\mathrm{dg}(u; \partial B_\rho(x_1)) = \mathrm{dg}(u; \partial B_{r_1}(x_1)) \neq 0$ for all $\rho \in [r_1, \tilde{r}_1)$. Thus all the radii $\rho \in [r_1, \tilde{r}_1)$ are by definition good radii, and so

$$\beta(x_1, \tilde{r}_1) = \beta(x_1, r_1).$$

As before we use (3.1.17) and the fact that B_1 satisfies (3.3.10) to estimate

$$\begin{aligned}\int_{B_{\tilde{r}_1}(x_1)} E^\epsilon dx &= \int_{B_{\tilde{r}_1}(x_1)\setminus B_{r_1}(x_1)} E^\epsilon dx + \int_{B_{r_1}(x_1)} E^\epsilon dx \\ &\geq [\Lambda^\epsilon(\tilde{r}_1) - \Lambda^\epsilon(r_1)] + \Lambda^\epsilon(r_1) + \frac{1}{24}\Lambda^\epsilon\left((\beta(x_1, r_1) - C_1\epsilon)^+\right) \\ &= \Lambda^\epsilon(\tilde{r}_1) + \frac{1}{24}\Lambda^\epsilon\left((\beta(x_1, \tilde{r}_1) - C_1\epsilon)^+\right).\end{aligned}$$

3. Relabel $B_1 = (B_1)_{\text{new}} := B_{\tilde{r}_1}(x_1)$ and $r_1 = (r_1)_{\text{new}} := (\tilde{r}_1)_{\text{old}}$. We now have a new collection of balls satisfying (3.3.1) and (3.3.2). They may not be pairwise disjoint, but nonetheless their interiors are pairwise disjoint, as a result of (3.3.4).

If the balls are not pairwise disjoint,we select two balls B_i, B_j that intersect. For the sake of concreteness, we assume that $r_i \leq r_j$, and we consider two different cases.

Case 1: $r_j \leq \frac{1}{2}\epsilon C_1$.

In this case we combine the two balls to form a ball B' with radius $r' = r_i + r_j$ exactly as in Step 3 of the proof of Lemma 3.3.1. We need to verify that (3.3.10) holds for the resulting ball B'. This is clear, because in the case we are considering, $r' - \epsilon C_1 \leq 0$, so $\Lambda^\epsilon((r' - \epsilon C_1)^+) = 0$ and the desired result follows by subadditivity as in the earlier proof.

Case 2: $r_j \geq \frac{1}{2}\epsilon C_1$.

This assumption implies that we are in a regime where Lemma 3.3.2 is applicable. In this case we define a new ball B' with with center x_j and radius $r' := r_j + 2r_i$. Note that $B_i \cup B_j \subset B'$. It is clear from the definition (3.2.27) of β that

$$\beta(x_j, r') = \beta(x_j, r_j + 2r_i) \leq \beta(x_j, r_j) + 2r_i.$$

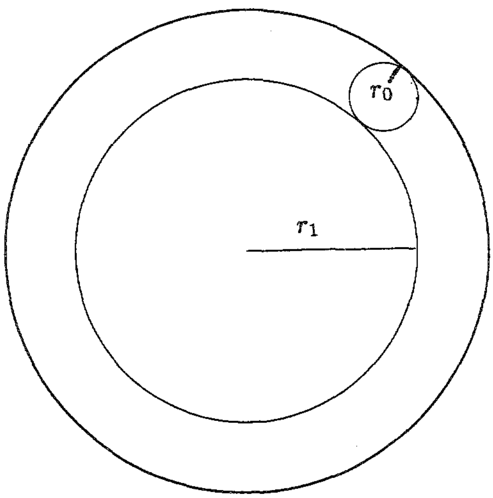

Figure 3. $\Lambda^\epsilon(r_1) + \Lambda^\epsilon(r_0) \geq \Lambda^\epsilon(r_1 + 2r_0) + \frac{1}{24}\Lambda^\epsilon(2r_0)$ = lower bound for large ball + estimate of additional "bad radii".

We may thus use Lemma 3.3.2 and the fact that B_i and B_j satsify (3.3.10) to estimate (see Figure 3)

$$\begin{aligned}\int_{B'} E^\epsilon dx &\geq \Lambda^\epsilon(r_i) + \Lambda^\epsilon(r_j) + \frac{1}{24}\Lambda^\epsilon\left((\beta(x_j, r_j) - \epsilon C_1)^+\right) \\ &\geq \Lambda^\epsilon(r_j + 2r_i) + \frac{1}{24}\left[\Lambda^\epsilon(2r_i) + \Lambda^\epsilon\left((\beta(x_j, r_j) - \epsilon C_1)^+\right)\right] \\ &\geq \Lambda^\epsilon(r') + \frac{1}{24}\Lambda^\epsilon\left((\beta(x_j, r') - \epsilon C_1)^+\right).\end{aligned}$$

So in either case, (3.3.10) is satisfied.

As in the proof of Lemma 3.3.1, we can continue to combine balls as necessary until we achieve a collection of balls which is pairwise disjoint.

4. By alternately expanding and combining balls, we eventually arrive at a collection which also satisfies (3.3.3), and at this point the proof of the lemma is finished. □

4. Concentration of Jacobian and global structure

In this section we complete our proof of the local structure theorem, and we prove the global structure theorem. The remaining statements to be proven concern the properties of the Jacobian measure $[Ju]$. The main point in both proofs is that $[Ju]$ vanishes away from the vortices.

Recall that Theorem 3.2.1 applies to a function $u \in W^{1,n}(B_r; \mathcal{R}^n)$ satisfying

$$\| \, [Ju] - \omega_n d\delta_0 \, \|_{\mathcal{M}^1(B_r)} \leq \gamma_0 r, \tag{3.4.1}$$

where $d = \pm 1$ and $\gamma_0 = \gamma_0(n)$ is some constant which could in principle be computed explicitly, and

$$\int_{B_r} E^\epsilon(u)\, dx \leq \kappa_n \ln\left(\frac{r}{\epsilon}\right) + \gamma_1 \tag{3.4.2}$$

for $\epsilon \leq r$.

We will prove

Theorem 3.4.1. *If u is a function satisfying* (3.4.1) *and* (3.4.2) *then*

$$\left\| \frac{d}{\omega_n}[Ju] - \frac{1}{\kappa_n}\mu_u^\epsilon \right\|_{\mathcal{M}^1(B_r)} \leq o_{\gamma_1}(1), \tag{3.4.3}$$

whenever u is a function satisfying (3.4.1) *and* (3.4.2).

Remark. Theorems 3.2.1 and 3.4.1 together make precise the statement that

$$\frac{d}{\omega_n}[Ju] \sim \frac{1}{\kappa_n}\mu_u^\epsilon \sim \delta_\xi. \tag{3.4.4}$$

In particular, (3.4.3) and (3.2.4) imply that under the hypotheses of Theorem 3.4.1, there exists some $\xi \in B_{r/2}$ such that

$$\left\| \frac{d}{\omega_n}[Ju] - \delta_\xi \right\|_{\mathcal{M}^1(B_r)} \leq o_{\gamma_1}(1). \tag{3.4.5}$$

Proof.

1. Suppose, toward a contradiction, that u^{ϵ_n}, $\epsilon_n \to 0$, is a sequence satisfying (3.4.1), (3.4.2), and

$$\liminf_{\epsilon \to 0} \left\| \frac{d}{\omega_n}[Ju^\epsilon] - \frac{1}{\kappa_n}\mu_{u^\epsilon}^\epsilon \right\|_{\mathcal{M}^1(B_r)} := \alpha > 0. \tag{3.4.6}$$

(We will omit the subscripts and write ϵ for ϵ_n.)

From Theorem 3.2.1 and the remarks which follow, we know that $\{u^\epsilon\}$ and related functions are (weakly) precompact in a variety of senses. In particular, after passing to a subsequence (still denoted u^ϵ), we may assume that there is some point $\xi \in B_{r/2}$ such that

$$\mu^\epsilon_{u^\epsilon} \rightharpoonup \kappa_n \delta_\xi \quad \text{weak-* in } \mathcal{M};$$

$$u^\epsilon \rightharpoonup \bar{u} \quad \text{weak-* in } W^{1,n}_{\text{loc}}(B_r \setminus \{\xi\}) \cap W^{1,p}(B_r), \quad \text{for every } p \in [1, n).$$

It is clear from (3.2.6) that $|\bar{u}| = 1$ a.e. We further have from (3.2.8)

$$j(u^\epsilon) \rightharpoonup \bar{j} \quad \text{weak-* in } L^p(B_r) \quad \text{for every } p \in \left[1, \frac{n}{n-1}\right),$$

and from (3.2.9) and (3.2.10)

$$[Ju^\epsilon] \rightharpoonup [\bar{J}] \quad \text{weak-* in } \mathcal{M}_{\text{loc}}(B_r \setminus \{\xi\}) \cap W^{-1,p}(B_r), \quad p \text{ as above},$$

where (by the weak continuity of Jacobians) $[\bar{J}] = [J(\bar{u})] = \frac{1}{n} d\bar{j}(\bar{u})$.

We will eventually show that $[\bar{J}] = d\omega_n \delta_\xi$, which will lead to a contradiction of (3.4.6). Note that at this stage we do not yet know that $[\bar{J}]$ is a measure.

2. Let $U \subset B_r \setminus \{\xi\}$ be any open set.

Since $|\bar{u}| = 1$ a.e. $x \in U$, we have $J\bar{u} = 0$ a.e. $x \in U$. Indeed, it is clear that this holds if $\bar{u}$ is smooth, since $D\bar{u}(x)$ then has rank at most $n-1$ for every x. A result of Bethuel and Zheng [3] shows that $C^\infty(U; S^{n-1})$ is dense in $W^{1,n}(U; S^{n-1})$, so the claim follows by an approximation argument for arbitrary $\bar{u} \in W^{1,n}(U; S^{n-1})$.

Since $\bar{u} \in W^{1,n}_{\text{loc}}$ away from ξ, it follows that $[\bar{J}] = 0$ in U and hence that the support of $[\bar{J}]$ as a distribution is contained in $\{\xi\}$.

3. Fix any $n < q < \infty$ and $1/p + 1/q = 1$. The embedding $C^1_0 \subset W^{1,q}_0$ implies by duality that

$$[\bar{J}] \in W^{-1,p} = [W^{1,q}]^* \subset [C^1_0]^*.$$

Since $\text{supp}[\bar{J}] \subset \{\xi\}$, for any $\phi \in C^1_0$ we have the representation

$$\int \phi[\bar{J}] = a_0 \phi(\xi) + \sum_{i=1}^{n} a_i \phi_{x_i}(\xi)$$

for certain constants $a_0, ..., a_n$. If $a_i \neq 0$ for any $i \geq 1$, this would not extend continuously to $W^{1,q}$. We therefore deduce that $[\bar{J}]$ is a measure of the form $[\bar{J}] = a_0 \delta_\xi$.

Once we know this, we immediately deduce from Lemma 3.1.2 and (3.4.1) that $[\bar{J}] = \omega_n d\delta_\xi$.

4. Our above arguments have established that

$$\frac{d}{wn}[Ju^\epsilon] - \mu^\epsilon_{u^\epsilon} \rightharpoonup 0$$

weak-* in $\mathcal{M}$. Lemma 1.2.2 then implies that

$$\frac{d}{\omega_n}[Ju^\epsilon] - \frac{1}{\kappa_n}\mu^\epsilon_{u^\epsilon}$$

converges to zero in the $\mathcal{M}^1$ norm, in contradiction to (3.4.6). □

Remarks.

1. Once it is known that $[Ju] = a_0\delta_\xi$, one could give a direct argument to prove that $a_0 = \omega_n d$, which does not rely on Lemma 3.1.2. One such argument would use (3.2.17) and a construction similar to that of Lemma 3.2.2 to produce a uniformly Lipschitz sequence of functions $\phi^k \in C^1_0$ such that $\phi^k(\xi) = 1$ for every k and

$$\int \phi^k[Ju^{\epsilon_k}] \to \omega_n d$$

as $k \to \infty$, implying the result.

2. It is evident that the role of $\mathcal{M}^1$ here is essentially to provide us with a convenient way of making the statement that the weak-* $\mathcal{M}$ convergence of

$$\frac{d}{wn}[Ju^\epsilon] - \frac{1}{\kappa_n}\mu^\epsilon_{u^\epsilon}$$

to zero is uniform for all u^ϵ satisfying (3.4.2) with a given constant γ_1.

We now prove the Global Structure Theorem 1.4.4. We first restate the theorem, in the general n-dimensional setting:

Theorem 3.4.2 (Global Structure). *Suppose that* $u \in W^{1,n}(\mathbb{T}^n; \mathcal{R}^n)$, *and that there exist points* $a_1, ..., a_m \in \mathbb{T}^n$, *integers* $d_1, ..., d_m \in \{\pm 1\}$, *and* $\epsilon \leq r := \frac{1}{4}\min_{i \neq j}|a_i - a_j|$ *such that*

$$\| \, [Ju] - \pi \sum_{i=1}^{m} d_i \delta_{a_i} \, \|_{\mathcal{M}^1(B_r)} \leq \gamma_0 r, \tag{3.4.7}$$

where $\gamma_0(n)$ *is the constant from Theorem* 3.2.1; *and*

$$\int_{\mathbf{T}^2} E^\epsilon(u)\, dx \leq \kappa_n m \ln\left(\frac{r}{\epsilon}\right) + \gamma_1 \tag{3.4.8}$$

for some γ_1. Then there exists points $\tilde{a}_i \in B_{r/2}(a_i)$, $i = 1, ..., m$ and a constant $C_1 = C_1(\gamma_1) > 0$ such that

$$||\mu_u^\epsilon - \kappa_n \sum \delta_{\tilde{a}_i}||_{\mathcal{M}^1} \leq \frac{C_1}{|\ln \epsilon|}, \tag{3.4.9}$$

$$||[Ju] - \kappa_n \sum d_i \delta_{\tilde{a}_i}||_{\mathcal{M}^1} \leq o_{\gamma_1}(1). \tag{3.4.10}$$

Moreover,

$$\int_{\mathbb{T}^n \setminus \bigcup_i B_\sigma(\tilde{a}_i)} E^\epsilon dx \leq C(\sigma, \gamma_1). \tag{3.4.11}$$

Finally, there exists constants C_p and C'_p, depending only on γ_1, such that

$$||Du||_{L^p(\mathbb{T}^n)} \leq C_p \quad \textit{for } p \in [1, n), \tag{3.4.12}$$

and

$$||j(u)||_{L^p(\mathbb{T}^n)} \leq C'_p \quad \textit{for } p \in \left[1, \frac{n}{n-1}\right). \tag{3.4.13}$$

Proof. First note that (3.4.7) implies that

$$\int_{B_r(a_i)} E^\epsilon dx \geq \kappa_n \ln\left(\frac{r}{\epsilon}\right) - C$$

for each i, by (3.2.11). Together with (3.4.8), this forces

$$\int_{B_r(a_i)} E^\epsilon dx \leq \kappa_n \ln\left(\frac{r}{\epsilon}\right) + C$$

for each i. In particular, the hypotheses of Theorem 3.2.1 are satisfied on each ball $B_r(a_i)$, $i = 1, ..., m$. The conclusions of Theorem 3.4.2 all follow easily from Theorem 3.2.1 and the remarks that follow, with the exception of (3.4.10).

This last claim follows by exactly the compactness argument used to prove Theorem 3.4.1. Indeed, if (3.4.10) is false, then we can find a sequence u^ϵ satisfying (3.4.7) and (3.4.8), but with

$$\left|\left|[Ju^\epsilon] - \kappa_n \sum d_i \delta_{\tilde{a}_i}\right|\right|_{\mathcal{M}^1}$$

bounded away from zero. Arguing as in the proof of Theorem 3.4.1, we can extract a subsequence such that $[Ju^\epsilon]$ converges weakly to a limit, which is a collection of point masses with weights $d_i \kappa_n$, where d_i is some integer. However, as in the earlier proof, the only possible limit is $\kappa_n \sum d_i \delta_{\tilde{a}_i}$, proving the theorem. □

Remark. Clearly, a version of the same result holds on a bounded open set $U \subset \mathbb{R}^n$ where we now define

$$r = \tfrac{1}{4}\min\{|a_i - a_j| \text{ for } i \neq j, \operatorname{dist}(a_i, \partial U)\}.$$

The proof uses only the local structure theorem and the fact that $[Ju^\epsilon] \to 0$ in M^1 away from the singularities; this last fact does not depend on any special properties of the torus $\mathbb{T}^n$.

5. Some extensions

In this section we present a couple of extensions of the above results.

We first show that an appropriate version of Theorem 1.4.3 holds also for the Ginzburg–Landau functional used to model the behavior of certain superconductors. After this, we present a brief discussion that ilustrates that the techniques used above can be modified very easily to work on manifolds.

These results are not used anywhere in this paper, but we expect that they may be useful in other contexts.

Magnetic field We define the functional

$$I^\epsilon_{\text{mag}}[u, A] := \int E^\epsilon_{\text{mag}}(u, A)dx,$$

where

$$E^\epsilon_{\text{mag}}(u, A) := \frac{1}{2}|\nabla_A u|^2 + \frac{1}{2}|\nabla \times A|^2 + \frac{1}{4\epsilon^2}(1 - |u|^2)^2.$$

We now think of u as taking values in the complex plane $\mathbb{C}$, and $A = A_1 dx_1 + A_2 dx_2$ is a 1-form with coefficients $A_i \in H^1(U)$. We will identify A with the function $(A_1, A_2) \in H^1(U; \mathcal{R}^2)$. We define $\nabla \times A := A_{2,x_1} - A_{1,x_2}$ and $\nabla_A u := (\nabla - iA)u$,

In physical models of superconductivity, A represents the magnetic potential, so that $\nabla \times A$ is the magnetic field.

We assume throughout this discussion that $u \in H^1(B_r, \mathbb{C})$ and $A \in H^1(B_r; \mathbb{R}^2)$.

We will prove

Theorem 3.5.1. *Suppose that*

$$\| \, [Ju] - \pi d\delta_0 \, \|_{\mathcal{M}^1(B_r)} \leq \gamma_0 r, \tag{3.5.1}$$

where $d = \pm 1$ and γ_0 is a constant which will be fixed below. Assume also that

$$\int_{B_r} E^\epsilon_{mag}(u, A)\, dx \leq \pi \ln\left(\frac{r}{\epsilon}\right) + \gamma_1 \tag{3.5.2}$$

for some γ_1. *Then there exists a point* $\xi \in B_{r/2}$ *and a constant* $C_1 = C_1(\gamma_1) > 0$ *such that*

$$\int_{B_\sigma(\xi)} E^\epsilon_{mag}(u, A)\, dx \geq \pi \log\left(\frac{\sigma}{\epsilon}\right) - C_1 \tag{3.5.3}$$

for every $\sigma \in [0, r/2]$. *Moreover,*

$$||\mu^\epsilon_u - \pi\delta_\xi||_{\mathcal{M}^1(B_r)} \leq \frac{C_1}{|\ln \epsilon|}, \tag{3.5.4}$$

$$||[Ju] - \pi\delta_\xi||_{\mathcal{M}^1(B_r)} \leq o_{\gamma_1}(1). \tag{3.5.5}$$

In addition, for any $p \in [1, 2)$, *there exists some* $C_p = C_p(\gamma_1)$ *such that*

$$||Du||_{L^p(B_r)} \leq C_p. \tag{3.5.6}$$

Finally,

$$||\nabla \times A||_{L^2(B_r)} \leq C(\gamma_1).$$

The idea is as follows: suppose we are given (u, A) as above, and let

$$\beta := ||\nabla \times A||_{L^2(B_r)} \tag{3.5.7}$$

and

$$F^\epsilon_{\text{mag}}(u, A) := \frac{1}{2}|\nabla_A u|^2 + \frac{1}{4\epsilon^2}\left(1 - |u|^2\right)^2,$$

so that

$$\int_{B_r} E^\epsilon_{\text{mag}}(u, A) = \frac{1}{2}\beta^2 + \int_{B_r} F^\epsilon_{\text{mag}}(u, A).$$

If $\beta \leq C(\gamma_1)$, then F^ϵ_{mag} is a small perturbation of the energy density E^ϵ studied in previous sections; this follows from Lemma 3.5.1 below. In this case, we thus expect to be able to prove the same sorts of results as before.

To show that $\beta \leq C(\gamma_1)$, we follow our earlier arguments to establish lower bounds for F^ϵ_{mag}, in which β appears as a parameter. By examining the dependence of these bounds on β, we find that (3.5.2) forces β to be $O(1)$.

We start by quoting some lemmas we will need from [12]. These are counterparts, for this modified functional, of the lower bounds given in Section 1 of this chapter.

Define

$$\lambda^\epsilon_\beta(r) := \min_{m \in [0,1]} \left\{ \frac{m^2}{r}\left[\left(\sqrt{\pi} - \frac{r\beta}{2}\right)^+\right]^2 + \frac{1}{C^*\epsilon}|1 - m|^N \right\}$$

where C^* and N are universal constants. Note that in the case $n = 2$, λ^ϵ coincides with λ^ϵ_β for $\beta = 0$.

Further define

$$\Lambda^\epsilon_\beta(s) := \int_0^s \lambda^\epsilon_\beta(r) \wedge \frac{c_0}{\epsilon}\, dr$$

for some sufficiently small c_0.

We define the set S_E, the essential degree dg, and so on exactly as before.

Similar to Lemma 3.1.4, we have

Lemma 3.5.1. *If $u \in C \cap H^1(U;\mathbb{R}^2)$ and $\mathrm{dg}(u;\partial B_r) \neq 0$ for $B_r \subset U$ with $r \geq \epsilon$, then*

$$\int_{\partial B_r} F^\epsilon_{\mathrm{mag}}\, dH^{n-1} \geq \lambda^\epsilon_\beta(r) \wedge \frac{c_0}{\epsilon}. \tag{3.5.8}$$

The main point is that the integrand differs from that in Lemma 3.1.6 essentially by a term of the form $\int_{\partial B_r} A\cdot\tau$, where τ is the tangent to ∂B_r. By Stokes' Theorem and Hölder's inequality, this can be estimated by β. This leads to the new term involving β in the definition of λ^ϵ_β.

We next state some estimates which correspond to Lemma 3.1.5. These too are proven in [12].

Lemma 3.5.2. *$\Lambda^\epsilon(\cdot)$ is increasing, and moreover*

$$\Lambda^\epsilon_\beta(r+s) \leq \Lambda^\epsilon_\beta(r) + \Lambda^\epsilon_\beta(s) \quad \forall r,s \geq 0; \tag{3.5.9}$$

$$\Lambda^\epsilon_\beta(r) \geq \pi\left[\ln\left(\frac{1}{\epsilon}\right) + \ln\left(r \wedge \frac{1}{\beta}\right)\right] - C \quad \forall r \geq 0. \tag{3.5.10}$$

Next, along the lines of Lemma 3.1.6 we have

Lemma 3.5.3. *If $u \in C \cap H^1(U;\mathbb{R}^2)$, $\epsilon \leq r_0 \leq r_1$, and $\mathrm{dg}(u;\partial B_s) \neq 0$ for all $s \in [r_0, r_1]$, then*

$$\int_{B_{r_1}\setminus B_{r_0}} F^\epsilon_{\mathrm{mag}}\, dx \geq \Lambda^\epsilon_\beta(r_1) - \Lambda^\epsilon_\beta(r_0). \tag{3.5.11}$$

Finally,

Lemma 3.5.4. *Suppose that $u \in C \cap H^1(U;\mathbb{R}^2)$ and $|u| \geq 1/2$ on ∂U. Assume also that*

$$\beta \leq C\epsilon^{-1/2},$$

where $\beta = ||\nabla \times A||_{L^2(B_r)}$ as above. Then there is a collection of closed, pairwise disjoint balls $\{B_i\}_{i=1}^k$ with radii r_i such that

$$S_E \subset \bigcup_{i=1}^{k} B_i, \tag{3.5.12}$$

$$r_i \geq \epsilon \quad \forall i, \tag{3.5.13}$$

$$B_i \cap S_E \neq \emptyset \quad \text{for each } i, \tag{3.5.14}$$

$$\int_{B_i \cap U} F^\epsilon_{mag}\, dx \;\geq\; \frac{c_0}{\epsilon} r_i \;\geq\; \Lambda^\epsilon_\beta(r_i). \tag{3.5.15}$$

We now give the proof of Theorem 3.5.1.

Proof.

1. The main new point is to prove that $\beta \leq C(\gamma_1)$. This is done as follows.

As in the proof of Theorem 3.2.1, we may assume that u is continuous, $|u| > \frac{1}{2}$ on ∂B_r, and

$$\mathcal{L}\left(S_d(u) \cap [\epsilon, r]\right) \geq 3r/4. \tag{3.5.16}$$

The first covering argument, Lemma 3.3.1, uses only properties of $\Lambda^\epsilon(\cdot)$ that are shared by $\Lambda^\epsilon_\beta(\cdot)$, such as subadditivity (3.5.9) and the fact that Λ^ϵ_β provides a lower bound for F^ϵ_{mag} on annuli (3.5.11). By the first covering argument, we can thus find a collection of balls $\{B_i\}$ with disjoint interiors satisfying (3.3.2), (3.3.3), and

$$\int_{B_i \cap B_r} F^\epsilon_{\text{mag}}(u, A) dx \geq \Lambda^\epsilon_\beta(r_i) \quad \text{for all } i.$$

As in Step 3 of the proof of Theorem 3.2.1, we deduce from (3.5.16) that $\sum r_i \geq 3r/8$. So

$$\begin{aligned}\int E^\epsilon_{\text{mag}}(u, A) dx &\geq \frac{1}{2}\beta^2 + \Lambda^\epsilon_\beta\left(\frac{3r}{8}\right) \\ &\geq \frac{1}{2}\beta^2 + \pi\left[\ln\left(\frac{1}{\epsilon}\right) + \ln\left(r \wedge \frac{1}{\beta}\right)\right]\end{aligned}$$

by (3.5.8). Comparing this with (3.5.2), we easily deduce that $\beta \leq C(\gamma_1)$.

2. Once we know that $\beta \leq C$, we can establish a version of Lemma 3.3.2 for Λ^ϵ_β. After that, $\Lambda^\epsilon_\beta(\cdot)$ has all the properties that were used to prove the covering Lemma 3.2.3. Everything else follows essentially without change from the proof of Theorem 1.4.3. □

Estimates on manifolds Finally, we demonstrate that the methods used above work equally well on manifolds. Instead of stating a general result, we discuss a simple, concrete example that is used in [13]. It will be clear from our discussion that one could go on to establish more elaborate results on more general Riemannian manifolds, in higher dimensions, etc.

Suppose that M is a 2-dimensional Lipschitz submanifold of some $\mathbb{R}^n$, equipped with the induced metric and with standard 2-dimensional Hausdorff measure, which we will write simply as dx. M can have a boundary and need not be compact.

Given a sufficiently differentiable function u on M, we write Du to indicate the tangential gradient. We are only assuming that M is Lipschitz, so there may be a subset of M (of measure 0) on which tangent planes do not exist; on such a subset clearly Du is not defined in general. Nonetheless we can talk about Sobolev spaces such as $H^1(M)$.

Suppose $x_0 \in M$, and let $R > 0$ be a number such that $\text{dist}(x_0, \partial M) \geq R$, and such that $B_R(x_0) := \{y \in M : \text{dist}(x, y) < R\}$ is a topological disk.

Given $u \in C \cap H^1(M; \mathbb{R}^2)$, we define as usual the set $S_E \subset M$, the essential degree dg, and so on.

As in the discussion of the functional with magnetic field, to use our earlier arguments, it suffices to verify that we can define functions, say $\tilde{\lambda}^\epsilon$ and $\tilde{\Lambda}^\epsilon$, that can be used to give lower bounds on circles and on balls and annuli, respectively, and to check that these functions have certain properties such as subadditivity.

The only point about which we need to be careful is that, given $x \in M$ and $r > 0$, in general $H^1(\partial B_r(x)) \neq 2\pi r$. In order to deal with this, we define

$$l(r) = \inf\{H^1(\partial B_r(x)) : x \in B_R(x_0), \ \ \text{dist}(x, \partial M) < r\},$$

$$L(r) = \sup\{H^1(\partial B_r(x)) : x \in B_R(x_0), \ \ \text{dist}(x, \partial M) < r\}.$$

We assume that

$$C^{-1} 2\pi r \leq l(r) \leq L(r) \leq C 2\pi r \quad \forall r \in (0, R). \tag{3.5.17}$$

We also assume that l and L are strictly increasing functions for $r \in [0, R]$.

Since M is Lipschitz, given any $x_0 \in M$ we can always find an R such that these assumptions are satisfied.

We first remark that a version of Lemma 3.1.3 remains true in this context.

Lemma 3.5.5. *Suppose that $u \in H^1(M; B^2)$, and let $\rho := |u|$. Then there exist constants C, N such that, if $x \in B_R(x_0)$, $\epsilon \leq r < \text{dist}(x, \partial M)$, and*

$$\gamma_{x,r} := \int_{\partial B_r(x)} \frac{1}{2}|D\rho|^2 + \frac{1}{4\epsilon^2}(\rho^2 - 1)^2 dH^1,$$

then

$$\|1-\rho\|_{L^\infty(\partial B_r(x))} \leq (C\epsilon\gamma_{x,r})^{1/N}.$$

The lemma is quite easy to prove directly. It also follows from Lemma 3.1.3, since $\partial B_r(x_0)$ is an isometric embedding of a standard Euclidean circle in the plane, of radius $C^{-1}r$, for some constant C that can be bounded uniformly for $x \in B_R(x_0)$ as a result of (3.5.17).

Next we define

$$\tilde{\lambda}^\epsilon(r) := \min_{m\in[0,1]} \left[m^2\frac{2\pi^2}{L(r)} + \frac{1}{C^*\epsilon}(1-m)^N\right] \tag{3.5.18}$$

for a suitable constant C^* (which will be selected below) and N as above.

The main point of our present arguments is that, once we modify λ^ϵ by replacing $2\pi r$ by $L(r)$, all our proofs follow exactly as before.

Lemma 3.5.6. *Suppose that* $u \in C \cap H^1(M;\mathbb{R}^2)$, $x \in B_R(x_0)$ *with* $\epsilon \leq r <$ $\operatorname{dist}(x, \partial M)$. *If* $\operatorname{dg}(u; \partial B_r(x)) \neq 0$ *then*

$$\int_{\partial B_r(x)} \frac{1}{2}|Du|^2 + \frac{1}{4\epsilon^2}(|u|^2-1)^2 dH^1 \geq \tilde{\lambda}^\epsilon(r) \wedge \frac{c_0}{\epsilon}.$$

Proof. Fix x as above, and let $m := \inf_{\partial B_r(x_0)} |u|$. If $m \leq 1/2$ then

$$\int_{\partial B_r(x)} \frac{1}{2}|Du|^2 + \frac{1}{4\epsilon^2}(|u|^2-1)^2 dH^1 \geq \frac{c_0}{\epsilon}$$

for appropriately small c_0, as a result of Lemma 3.5.5.

If $m \geq 1/2$, write $u = \rho e^{i\phi}$. Note that $|Du|^2 = |D\rho|^2 + \rho^2|D\phi|^2$, so that

$$\int_{\partial B_r(x)} \frac{1}{2}|Du|^2 + \frac{1}{4\epsilon^2}(|u|^2-1)^2 dH^1 \geq \gamma_{x,r} + m^2 \int_{\partial B_r(x)} |D\phi|^2.$$

Also, the assumption that $\operatorname{dg}(u; \partial B_r(x)) \neq 0$ implies that

$$\begin{aligned} 2\pi^2 &\leq \frac{1}{2}\left|\int_{\partial B_r(x)} D\phi \cdot \tau \, dH^1\right|^2 \\ &\leq L(r) \int_{\partial B_r(x)} \frac{1}{2}|D\phi|^2 dH^1. \end{aligned}$$

The last two equations and Lemma 3.5.5 give the result for suitable C^*. □

We now define

$$\tilde{\Lambda}^\epsilon(s) := \int_0^s \tilde{\lambda}^\epsilon(r) \wedge \frac{c_0}{\epsilon}\, dr.$$

Now Lemmas 3.1.5, 3.1.6 and 3.1.7 follow exactly as before, except that the lower bound (3.1.16) in Lemma 3.1.5 is replaced by

$$\lim_{\epsilon\to 0} |\ln\epsilon|^{-1}\tilde{\Lambda}^\epsilon(r) \geq Q\pi,$$

for

$$Q := \liminf_{r\to 0} \frac{2\pi r}{L(r)}.$$

At this point we can easily begin to recover some of our earlier results. For example, suppose $u \in C \cap H^1(M;\mathbb{R}^2)$, with x_0, R as above, and define

$$S_d(u) := \{r \in (0,R) : \mathrm{dg}(u;\partial B_r) = d\}.$$

We close this discussion with the following proposition, which is used in [13].

Proposition 3.5.1. *Suppose there exists some σ such that*

$$\mathcal{L}^1(S_d(u)) \geq \sigma.$$

Then

$$\int_{B_R(x_0)} E^\epsilon(u)\,dx \geq \tilde{\Lambda}^\epsilon\left(\frac{\sigma}{2}\right).$$

In particular, if

$$\liminf_{r\to 0} \frac{2\pi r}{H^1(\partial B_r(x))} \geq 1$$

for all $x \in B_R(x_0)$, then

$$|\ln\epsilon|^{-1}\int_{B_R(x_0)} E^\epsilon(u)\,dx \geq \pi + o(1)$$

as $\epsilon \to 0$.

Proof. The first covering argument Lemma 3.3.1 can be used with $\tilde{\Lambda}^\epsilon$ exactly as before. By the argument from Steps 2 and 3 of the proof of Theorem 3.2.1, we verify that the balls produced by this procedure have radii that sum to at least $\sigma/2$. Thus by subadditivity the total energy in the balls is at least $\tilde{\Lambda}^\epsilon(\sigma/2)$. The proposition follows. □

CHAPTER 4.

AUXILIARY RESULTS ON RENORMALIZED ENERGY

1. A technical lemma

In this section and the next we give the proof of Theorem 1.4.5, which asserts that, if a sequence of functions u^ϵ is close to minimizing the Ginzburg–Landau energy I^ϵ, then the functions are close to energy minimizers. Note that the condition that a sequence be nearly energy minimizing depends on the limiting vortex configuration. This dependence is expressed through the renormalized energy.

This may be thought of as a version of the elementary fact that, if $u \in H^1(U)$ solves $\Delta u = 0$ in $U \subset \mathbb{R}^n$, and if $v \in H^1(U)$ is a function such that $v = u$ on ∂U, then

$$\|Dv - Du\|_{L^2}^2 = \|Dv\|_{L^2}^2 - \|Du\|_{L^2}^2.$$

The lemma in this section gives a sharp lower bound for the energy of vortex cores, under quite weak assumptions. The proof of Theorem 1.4.5 essentially combines this lower bound with some results of Bethuel, Brezis and Hélein [2], which are valid away from the vortex cores.

Lemma 4.1.1. *Suppose $u^\epsilon \in H^1(B_\rho)$ and*

$$[Ju^\epsilon] \rightharpoonup \pi\delta_0$$

weak- in $\mathcal{M}(B_\rho)$. Then*

$$\int_{B_\rho} E^\epsilon(u^\epsilon)\, dx \ \geq I(\epsilon,\rho) + o(1)$$

as $\epsilon \to 0$.

Remark. Recall that $I(\epsilon,\rho)$ is defined as

$$I(\epsilon,\rho) = \min\left\{\int_{B_\rho} E^\epsilon(u)dx \ : \ u \in H^1(B_\rho), u(x) = \frac{x}{|x|} \text{ for } x \in \partial B_\rho\right\}.$$

It is clear that rotating the boundary data through some constant angle α has no effect on the minimum, so that for any fixed α,

$$I(\epsilon,\rho) = \min\left\{\int_{B_\rho} E^\epsilon(u)dx \ : \ u \in H^1(B_\rho), u(x) = e^{i\alpha}\frac{x}{|x|} \text{ for } x \in \partial B_\rho\right\}.$$

Proof.

1. Fix $\rho > 0$ and suppose that u^ϵ is a sequence of functions as in the hypotheses. We may assume that each u^ϵ is smooth.

Suppose some small $\delta > 0$ is given. It suffices to show that there exists some $\epsilon_0 > 0$ such that

$$\int_{B_\rho} E^\epsilon(u^\epsilon)dx \geq I(\epsilon, \rho) - \delta \tag{4.1.1}$$

for all $\epsilon < \epsilon_0$. Since $I(\epsilon, \rho) \leq \pi \ln(\rho/\epsilon) + C$, we may assume that

$$\begin{aligned} \int_{B_\rho} E^\epsilon(u^\epsilon)dx &\leq I(\epsilon, \rho) \\ &\leq \pi \ln(\rho/\epsilon) + C_1 \quad \text{by (1.3.12)}, \end{aligned} \tag{4.1.2}$$

as the conclusion is otherwise immediate.

First we will show, in Claim 1 below, that if ϵ is sufficiently small, we can find a radius $r < \frac{1}{2}\rho$ on which u^ϵ has certain good properties, and moreover that this radius is bounded away from zero.

In Claim 2, we use these good properties to show that we can construct a function $\widetilde{u^\epsilon}$ which agrees with u^ϵ on B_r and equals $e^{i\alpha}\frac{x}{|x|}$ on ∂B_ρ for some α, and such that

$$\int_{B_\rho \setminus B_r} E^\epsilon(\widetilde{u^\epsilon}) \leq \ln(\rho/r) + \delta/2. \tag{4.1.3}$$

Finally, we check in Claim 3 that for ϵ sufficiently small,

$$\int_{B_\rho \setminus B_r} E^\epsilon(u^\epsilon) \geq \ln(\rho/r) - \delta/2.$$

This will establish (4.1.1), since

$$\begin{aligned} I(\epsilon, \rho) &\leq \int_{B_\rho} E^\epsilon(\widetilde{u^\epsilon})\, dx \\ &= \int_{B_\rho} E^\epsilon(u^\epsilon)\, dx + \int_{B_\rho \setminus B_r} \left[E^\epsilon(\widetilde{u^\epsilon}) - E^\epsilon(u^\epsilon)\right]\, dx. \end{aligned}$$

Claim 1. Given $\delta_1 > 0$, if ϵ is sufficiently small we can find $r < \rho/2$ such that

$$\int_{\partial B_r} E^\epsilon(u^\epsilon)\, dH^1 \leq \frac{1}{r}(\pi + \delta_1) \tag{4.1.4}$$

and

$$\deg(u^\epsilon; \partial B_r) = 1. \tag{4.1.5}$$

Moreover, $r \geq r_0$, where r_0 depends only on δ_1.

2. **Proof of Claim 1.** By taking ϵ_0 sufficiently small, we may assume that

$$||[Ju^\epsilon] - \delta_0||_{\mathcal{M}^1(B_\rho)} \leq \gamma_0 r_0$$

for all $\epsilon < \epsilon_0 = \epsilon_0(r_0)$, where $r_0(\delta_1)$ will be chosen below. Lower bounds (see (3.2.4)) thus yield, for small ϵ,

$$\int_{B_{r_0}} E^\epsilon(u^\epsilon)\,dx \geq \pi \log\left(\frac{r_0}{\epsilon}\right) - C_2. \tag{4.1.6}$$

By combining (4.1.2) and (4.1.6), we see that

$$\int_{B_{\rho/2}\setminus B_{r_0}} E^\epsilon(u^\epsilon)\,dx \leq \pi \log\left(\frac{\rho/2}{r_0}\right) + C_3 \tag{4.1.7}$$

for all $\epsilon \leq \epsilon_0(r_0)$, but with a constant C_3 which is independent of r_0.

3. Fix r_0 so small that

$$\delta_1 \ln\left(\frac{\rho/2}{r_0}\right) \geq 2C_3, \tag{4.1.8}$$

where C_3 is the constant from (4.1.7). Then (4.1.7) readily implies that

$$T_1 := \left\{r \in \left[r_0, \frac{\rho}{2}\right] \;:\; (4.1.4) \text{ holds}\right\}$$

has measure strictly greater than zero, say

$$\mathcal{L}^1(T_1) \geq C_4^{-1} > 0$$

for some large constant C_4.

We have shown in (3.2.25) that if

$$||[Ju] - \pi\delta_0||_{\mathcal{M}^1(B_r)} \leq h \tag{4.1.9}$$

and (4.1.2) holds, then

$$\mathcal{L}^1(T_2) \geq \tfrac{1}{2}\rho - r_0 - O(h) - o(1),$$

as $\epsilon \to 0$, where

$$T_2 := \{r \in [r_0, \rho/2] \;:\; (4.1.5) \text{ holds}\} = S_1(u) \cap [r_0, \rho/2],$$

in the notation of Lemma 3.2.2.

If ϵ is sufficiently small, then the h in (4.1.9) can be made so small that

$$\mathcal{L}^1(T_1) + \mathcal{L}^1(T_2) > \tfrac{1}{2}\rho - r_0.$$

It follows that for such ϵ, there is some $r \in T_1 \cap T_2 \subset [r_0, \rho/2]$. This is Claim 1.

Claim 2. There exists some constant C such that, if (4.1.4) and (4.1.5) hold for some $r \leq \rho/2$ and ϵ_0 is sufficiently small, then there is a function $\widetilde{u^\epsilon} \in H^1(B_\rho)$ such that $u^\epsilon = \widetilde{u^\epsilon}$ in B_r,

$$u^\epsilon = e^{i\alpha}\frac{x}{|x|} \quad \text{on } \partial B_\rho \quad \text{for some } \alpha,$$

and

$$\int_{B_\rho \setminus B_r} E^\epsilon(\widetilde{u^\epsilon}) \leq \ln\left(\frac{\rho}{r}\right)(1 + C\delta_1).$$

Since we can select δ_1 in (4.1.4) to be as small as we like and ρ/r is bounded, this will prove (4.1.3).

4. **Proof of Claim 2.** We need only define $\widetilde{u^\epsilon}$ in the annulus $B_\rho \setminus B_r$.

First, from Lemma 3.1.3 and (4.1.4) we see that

$$||u^\epsilon| - 1| \leq (C\epsilon)^{1/N} \tag{4.1.10}$$

on ∂B_r, for some fixed C, N.

Let $r_\epsilon := r + (C\epsilon)^{1/N}$ for these values of C, N. We define $\widetilde{u^\epsilon}$ in the annulus $\{x \mid r \leq |x| \leq r_\epsilon\}$ by stipulating that $\widetilde{u^\epsilon}/|\widetilde{u^\epsilon}|$ is constant in the radial direction, and $|\widetilde{u^\epsilon}|$ is linear in the radial direction, with $|\widetilde{u^\epsilon}| = |u^\epsilon|$ when $|x| = r$, and $|\widetilde{u^\epsilon}| = 1$ when $|x| = r_\epsilon$.

One can easily check, using (4.1.4) and (4.1.10), that

$$\int_{B_{r_\epsilon} \setminus B_r} E^\epsilon(\widetilde{u^\epsilon})dx = o(1) \tag{4.1.11}$$

as $\epsilon \to 0$, and that

$$\int_{\partial B_{r_\epsilon}} E^\epsilon(\widetilde{u^\epsilon})\, dH^1 \leq (1 + o(1))\frac{1}{r}(\pi + \delta_1). \tag{4.1.12}$$

5. We will eventually define u^ϵ in the annulus $B_\rho \setminus B_{r_\epsilon}$. First, since $\deg(\widetilde{u^\epsilon}; \partial B_{r_\epsilon}) = \deg(\widetilde{u^\epsilon}; \partial B_r) = 1$, we can write

$$\widetilde{u^\epsilon}\,\Big|_{\partial B_{r_\epsilon}} = e^{i(\alpha + \theta(x) + \phi(x))},$$

where α is a constant, θ satisfies $e^{i\theta(x)} = x/|x|$ and ϕ is single-valued with

$$\int_{\partial B_r} \phi\, dH^1 = 0.$$

We record some properties of ϕ: first we have defined ϕ only on ∂B_{r_ϵ}; we extend it to a function on $\mathbb{R}^2 \setminus \{0\}$ which is homogeneous of degree zero, still denoted ϕ.

Note that since ϕ is single-valued, integrating by parts yields

$$\int_{\partial B_{r_\epsilon}} D\theta \cdot D\phi \, dH^1 = \int_{\partial B_{r_\epsilon}} (D_\tau\theta)(D_\tau\phi)\, dH^1$$

$$= \frac{1}{r_\epsilon}\int_{\partial B_{r_\epsilon}} D_\tau\phi \, dH^1 = 0. \tag{4.1.13}$$

Next, we assert that

$$\int_{\partial B_{r_\epsilon}} |D_\tau\phi|^2 dH^1 \leq 4\delta_1, \tag{4.1.14}$$

say, for all ϵ sufficiently small. This follows from (4.1.12) by a short argument which uses (4.1.13) and the explicit computation $|D\theta(x)| = 1/|x|$.

By homogeneity and (4.1.14), one may compute that

$$\int_{\partial B_s} |D\phi|^2 \, dH^1 \leq C\frac{s}{r_\epsilon}\delta_1. \tag{4.1.15}$$

Similarly, using homogeneity and Poincaré's inequality (which holds since ϕ has integral zero) one can verify that

$$\int_{\partial B_s} \phi^2 \, dH^1 \leq Csr_\epsilon\delta_1. \tag{4.1.16}$$

6. Next we define

$$\widetilde{u^\epsilon}(x) = e^{i(\alpha+\theta(x)+\lambda(|x|)\phi(x))}$$

for $x \in B_\rho \setminus B_{r_\epsilon}$, where λ is some function which will be chosen below, with $\lambda(r_\epsilon) = 1$ and $\lambda(\rho) = 0$.

We then compute

$$D\widetilde{u^\epsilon} = \left(D\theta + \lambda'(|x|)\frac{x}{|x|}\phi(x) + \lambda(|x|)D\phi(x)\right) ie^{i(\dots)}.$$

Since

$$\frac{x}{|x|}\cdot D\theta = \frac{x}{|x|}\cdot D\phi = 0,$$

we use (4.1.13), (4.1.15) and (4.1.16) to compute

$$\begin{aligned}\int_{B_\rho\setminus B_{r_\epsilon}} E^\epsilon(\widetilde{u^\epsilon})\, dx &= \frac{1}{2}\int_{r_\epsilon}^{\rho}\int_{\partial B_s} |D\theta|^2 + \lambda^2|D\phi|^2 + (\lambda')^2\phi^2 \, dH^1\, ds \\ &= \pi\ln\left(\frac{\rho}{r_\epsilon}\right) + \frac{1}{2}\int_{r_\epsilon}^{\rho}\int_{\partial B_s} (\lambda^2|D\phi|^2 + (\lambda')^2\phi^2)\, dH^1\, ds \\ &\leq \pi\ln\left(\frac{\rho}{r_\epsilon}\right) + C\delta_1 r_\epsilon\int_{r_\epsilon}^{\rho}\left(s\lambda(s)^2 + \frac{1}{s}\lambda'(s)^2\right) ds.\end{aligned}$$

Let λ minimize the above integral, subject to the conditions $\lambda(r_\epsilon) = 1$ and $\lambda(\rho) = 0$. Since r_ϵ is bounded away from zero and ρ, the integral can be estimated independent of ϵ to yield

$$\int_{B_\rho \setminus B_{r_\epsilon}} E^\epsilon(\tilde{u^\epsilon})dx \leq \pi \ln\left(\frac{\rho}{r_\epsilon}\right) + C\delta_1.$$

Together with (4.1.11), this proves Claim 2, since $r_\epsilon \to r$ as $\epsilon \to 0$.

Claim 3.

$$\int_{B_\rho \setminus B_r} E^\epsilon(u^\epsilon) \geq \ln\left(\frac{\rho}{r}\right) - \delta/2$$

for all ϵ sufficiently small.

Proof of Claim 3. We use the machinery for proving lower bounds developed in Section 2 of Chapter 3.

Recall that $||[Ju^\epsilon] - \pi\delta_0||_{\mathcal{M}^1(B_\rho)} \to 0$ by hypothesis. We therefore deduce from (3.2.25) that

$$\mathcal{L}^1(\mathcal{S}) \to \rho - r,$$

where

$$\mathcal{S} := \{s \in [r, \rho] \mid \mathrm{dg}(u; \partial B_s) = 1\}.$$

Then

$$\int_{B_\rho \setminus B_r} E^\epsilon(u^\epsilon)\, dx = \int_r^\rho \int_{\partial B_s} E^\epsilon(u^\epsilon)\, dH^1(x)\, ds$$
$$\geq \int_{s \in \mathcal{S}} \lambda^\epsilon(s) \wedge \frac{c_0}{\epsilon}\, ds$$

by Lemma 3.1.4. Since $\lambda^\epsilon(\cdot)$ is nonincreasing, the right-hand side is estimated from below by

$$\int_{\rho - \mathcal{L}^1(\mathcal{S})}^{\rho} \lambda^\epsilon(s) \wedge \frac{c_0}{\epsilon}\, ds.$$

One easily verifies from the definition (3.1.12) of λ^ϵ that as $\epsilon \to 0$, $\lambda^\epsilon(s) \to \pi/s$ uniformly for s bounded away from zero. Claim 3 follows. □

2. A variational result

Harmonic maps into S^1 Suppose we are given a collection of points $a_1, ..., a_m \in \mathbb{T}^2$ and nonzero integers $d_1, ..., d_m$ such that $\sum_i d_i = 0$. For $\rho > 0$ such that

$$\rho < \tfrac{1}{2} \min_{i \neq j} |a_i - a_j|,$$

we define

$$\mathbb{T}^2_\rho := \mathbb{T}^2 \setminus \bigcup_i B_\rho(a_i).$$

Let

$$\mathcal{A}_\rho := \{v \in H^1(\mathbb{T}^2_\rho; S^1) \ : \ \deg(v; \partial B_\rho(a_i)) = d_i \ \forall i\}.$$

In order to describe Dirichlet energy minimizers in $\mathcal{A}_\rho$, we introduce an auxiliary problem. Let $\Phi_\rho : \mathbb{T}^2_\rho \to \mathbb{R}$ solve

$$\begin{aligned} \Delta\Phi_\rho &= 0 && \text{in } \mathbb{T}^2_\rho, \\ \Phi_\rho &= \text{const.} && \text{on } \partial B_\rho(a_i), \quad i = 1, ..., m, \\ \int_{\partial B_\rho(a_i)} \frac{\partial \Phi_\rho}{\partial \nu} &= 2\pi d_i, && i = 1, ..., m. \end{aligned}$$

Such a function can be constructed by solving an appropriate minimization problem; see [2] and the citations therein. We will occasionally write $\Phi_\rho(x; a, d)$ to explicitly indicate the dependence of Φ_ρ on the various parameters.

The following proposition is essentially proven in Bethuel, Brezis and Hélein [2]. The arguments there are given on a domain with boundary, but they can easily be adapted to the periodic setting.

Proposition 4.2.1. *There exists a function u_ρ which minimizes the Dirichlet energy in $\mathcal{A}_\rho$. This function is unique up to a phase, and satisfies*

$$j(u_\rho) \ = \ -\nabla \times \Phi_\rho.$$

In particular,

$$\frac{1}{2}\int_{\mathbb{T}^2_\rho} |Du_\rho|^2 = \frac{1}{2}\int_{\mathbb{T}^2_\rho} |D\Phi_\rho|^2 = m\pi \ln\left(\frac{1}{\rho}\right) + W(a, d) + O(\rho). \tag{4.2.1}$$

If $v \in \mathcal{A}_\rho$, then

$$||j(v) - j(u_\rho)||^2_{L^2(\mathbb{T}^2_\rho)} = ||Dv||^2_{L^2(\mathbb{T}^2_\rho)} - ||Du_\rho||^2_{L^2(\mathbb{T}^2_\rho)}. \tag{4.2.2}$$

Finally,

$$||j(u_\rho) - j(H)||^2_{L^2(\mathbb{T}^2_\rho)} = O(\rho). \tag{4.2.3}$$

These assertions are proved in [2], Theorems I.1, I.6 and I.7.

We will prove Theorem 1.4.5.

Recall. It is an easy and well-known fact that, if $u^\epsilon \to u$ strongly in some space and $v^\epsilon \rightharpoonup v$ weak-* converges in the dual space, then $u^\epsilon v^\epsilon \to uv$ in the sense of distributions. We will use this several times in the upcoming proof.

Proof.

1. Fix some $0 < \rho < \min_{i\neq j} \frac{1}{2}|a_i - a_j|$. The hypothesis (1.4.29) implies that

$$[Ju^\epsilon] \rightharpoonup d_i \pi \delta_{a_i}$$

weak-* in $\mathcal{M}(B_\rho(a_i))$, for $i = 1, ..., m$. Lemma 4.1.1 and the assumed upper bounds (1.4.30) thus imply that

$$\limsup_{\epsilon\to 0} \int_{\mathbb{T}^2_\rho} E^\epsilon(u^\epsilon)\, dx \;\le\; m\pi \ln\left(\frac{1}{\rho}\right) + W(a,d) + C\rho + \gamma_2. \tag{4.2.4}$$

By (1.4.28) this immediately implies that the functions $\frac{1}{|u^\epsilon|} j(u^\epsilon)$ are uniformly bounded in $L^2(\mathbb{T}^2_\rho)$.

After passing to a subsequence, which for convenience we still denote u^ϵ, we may assume that

$$u^\epsilon \to \bar{u} \quad \text{strongly in } L^p(\mathbb{T}^2_\rho),$$

for every $p < \infty$. It is clear that we must have $|\bar{u}| = 1$ a.e, so that

$$|u^\epsilon| \to 1 \quad \text{strongly in } L^p(\mathbb{T}^2_\rho)$$

for every $p < \infty$. In addition,

$$Du^\epsilon \rightharpoonup D\bar{u} \quad \text{weakly in } L^2(\mathbb{T}^2_\rho);$$

$$\frac{1}{|u^\epsilon|} j(u^\epsilon) \rightharpoonup \text{some limit, say } \tilde{j} \quad \text{weakly in } L^2(\mathbb{T}^2_\rho).$$

Finally, since u^ϵ converges strongly and Du^ϵ converges weakly in the appropriate spaces, we also have

$$j(u^\epsilon) \rightharpoonup j(\bar{u}) \quad \text{weakly in } L^p(\mathbb{T}^2_\rho) \text{ for every } p \in [1,2).$$

Note also that $[J\bar{u}] = \pi \sum d_i \delta_{a_i}$, on account of (1.4.29).

2. We now claim that $\tilde{j} = j(\bar{u})$. This is not hard:

$$\begin{aligned} j(\bar{u}) &= \text{weak } L^1 \lim_{\epsilon\to 0} j(u^\epsilon) \\ &= \text{weak } L^1 \lim_{\epsilon\to 0} \left(|u^\epsilon| \frac{j(u^\epsilon)}{|u^\epsilon|}\right) \\ &= (\text{strong } L^2 \lim_{\epsilon\to 0} |u^\epsilon|)\left(\text{weak } L^2 \lim_{\epsilon\to 0} \frac{j(u^\epsilon)}{|u^\epsilon|}\right) \\ &= \tilde{j}. \end{aligned}$$

3. We next verify that $\bar{u}$ belongs to the class of functions A_ρ defined earlier. We expect this to follow from (1.4.29), and indeed it does:

Fix some a_i and let $\phi(x) = f(|x - a_i|)$ for some smooth nonincreasing function f such that

$$f(r) = \begin{cases} 1 & \text{if } r \leq \rho, \\ 0 & \text{if } r \geq 2\rho. \end{cases}$$

Then (1.4.29) implies that

$$\begin{aligned} \pi d_i &= \lim_{\epsilon \to 0} \int \phi[Ju^\epsilon] \\ &= \frac{1}{2} \int \nabla \times \phi \cdot j(\bar{u}) \\ &= \frac{1}{2} \int \mathbb{J} D\phi \cdot j(\bar{u}) \\ &= \int_\rho^{2\rho} f'(r) \int_{\partial B_r} \mathbb{J}\nu \cdot j(\bar{u})\, dH^1\, dr \\ &= -\int_\rho^{2\rho} f'(r) \int_{\partial B_r} \tau \cdot j(\bar{u})\, dH^1\, dr. \end{aligned} \tag{4.2.5}$$

Also, for any $r \in (\rho, 2\rho)$,

$$\begin{aligned} 0 &= \int_{B_r \setminus B_\rho} Ju\, dx \\ &= \frac{1}{2} \int_{\partial B_r} \tau \cdot j(\bar{u}) - \frac{1}{2} \int_{\partial B_\rho} \tau \cdot j(\bar{u}). \end{aligned}$$

Thus (4.2.5) becomes

$$d_i = \frac{1}{\pi} \int_{\partial B_\rho} \tau \cdot j(\bar{u}),$$

which is what we want to prove; compare with the definition of degree given in (3.1.5).

4. In light of Step 3, it follows from (4.2.4) that

$$0 \leq ||D(\bar{u})||^2_{L^2(\mathbb{T}^2_\rho)} - ||D(u_\rho)||^2_{L^2(\mathbb{T}^2_\rho)} \leq 2(\gamma_2 + C\rho)$$

and hence, by Proposition 4.2.1, that

$$||j(\bar{u}) - j(u_\rho)||^2_{L^2(\mathbb{T}^2_\rho)} \leq 2(\gamma_2 + C\rho). \tag{4.2.6}$$

5. Let

$$p^\epsilon := \frac{1}{|u^\epsilon|} j(u^\epsilon) - j(\bar{u}).$$

Using (4.2.4) and Proposition 4.2.1, we compute

$$\|j(\bar{u})\|^2_{L^2(\mathbb{T}^2_\rho)} + C(\gamma_2 + \rho) \geq \limsup_{\epsilon \to 0} \left\| \frac{1}{|u^\epsilon|} j(u^\epsilon) \right\|^2_{L^2(\mathbb{T}^2_\rho)}$$
$$= \limsup_{\epsilon \to 0} \|j(\bar{u}) + p^\epsilon\|^2_{L^2(\mathbb{T}^2_\rho)}.$$

Since $p^\epsilon \rightharpoonup 0$ weakly in L^2, as we have verified in Steps 1 and 2, this gives

$$\limsup_{\epsilon \to 0} \|p^\epsilon\|^2_{L^2(\mathbb{T}^2_\rho)} \leq C(\gamma_2 + \rho).$$

This fact, with (4.2.6) and (4.2.3), immediately imply that

$$\limsup_{\epsilon \to 0} \left\| \frac{1}{|u^\epsilon|} j(u^\epsilon) - j(H) \right\|^2_{L^2(\mathbb{T}^2_\rho)} \leq C(\gamma_2 + \rho).$$

With (1.4.28) and (4.2.4) this implies that

$$\limsup_{\epsilon \to 0} \| \, D|u^\epsilon| \, \|^2_{L^2(\mathbb{T}^2_\rho)} \leq C(\gamma_2 + \rho).$$

6. Now fix some $\hat{\rho} \in (0, \rho)$. Clearly $T^2_\rho \subset T^2_{\hat{\rho}}$, so

$$\limsup_{\epsilon \to 0} \left\| \frac{1}{|u^\epsilon|} j(u^\epsilon) - j(H) \right\|^2_{L^2(\mathbb{T}^2_\rho)} \leq \limsup_{\epsilon \to 0} \left\| \frac{1}{|u^\epsilon|} j(u^\epsilon) - j(H) \right\|^2_{L^2(\mathbb{T}^2_{\hat{\rho}})}$$
$$\leq C(\gamma_2 + \hat{\rho}).$$

Letting $\hat{\rho}$ go to zero, we obtain (1.4.31).

By the same argument we deduce (1.4.32). □

REFERENCES

[1] G. Alberti, R. L. Jerrard, and H. M. Soner, *Functions of bounded n-variation*, in preparation.

[2] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg–Landau Vortices*, Birkhäuser, Boston, 1994.

[3] F. Bethuel and X. Zheng, *Density of smooth functions between two manifolds in Sobolev spaces*, J. Funct. Anal. **80** (1988), 60–75.

[4] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I,II*, Geom. Funct. Anal. **3** (1993), 107–156, 209–262.

[5] H. Brézis, J. M. Coron and E. Lieb, *Harmonic maps with defects*, Comm. Math. Phys. **107** (1986), 649–705.

[6] H. Brezis and L. Nirenberg, *Degree theory and BMO: Part i: compact manifolds without boundaries*, Selecta Math. (N.S.) **1** (1995), 197–263.

[7] J. E. Colliander and R. L. Jerrard, *Vortex dynamics for the Ginzburg–Landau-Schrödinger equation*, Internat. Math. Res. Notices (1998), 333–358.

[8] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer-Verlag, Berlin, 1989.

[9] L. C. Evans and W. Gangbo, *Differential equation methods for the Monge–Kantorovitch mass transfer problem*, to appear.

[10] V. L. Ginzburg and L. P. Pitaevskii, *On the theory of superfluidity*, Soviet Physics JETP **34** (7)(5) (1958), 858–861.

[11] E. P. Gross, *Dynamics of interacting bosons*, in *Physics of Many Particle Systems* (E. Meeron, ed.), Gordon and Breech, New York, 1966, pp. 231–406.

[12] R. L. Jerrard, *Lower bounds for generalized Ginzburg–Landau functionals*, SIAM Math. Anal., to appear.

[13] R. L. Jerrard, *Vortex dynamics for the Ginzburg–Landau wave equation*, Calc. Var. Partial Differential Equations, to appear.

[14] R. L. Jerrard and H. M. Soner, *Dynamics of Ginzburg–Landau vortices*, Arch. Rational Mech. Anal. **142** (1998), 99–125.

[15] F-H. Lin, *Vortex dynamics for the nonlinear wave equations*, preprint, 1997.

[16] F. H. Lin, *Solutions of Ginzburg–Landau equations and critical points of the renormalized energy*, Ann. Inst. H. Poincaré Anal. Non Linéaire **12** (1995), 599–622.

[17] F. H. Lin, *Some dynamical properties of Ginzburg–Landau vortices*, Comm. Pure. Appl. Math. **49** (1996), 323–359.

[18] F. H. Lin, *A remark on the previous paper: "Some dynamical properties of Ginzburg–Landau vortices"*, Comm. Pure. Appl. Math, **49** (1996), 361–364.

[19] C. Marchioro and M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Springer, Berlin, 1994.

[20] S. Müller, *Det = det. A remark on the distributional determinant*, Comptes Rendus Acad. Sci. Paris **311** (1) (1990), 13–17.

[21] L. P. Pitaevskii, *Vortex lines in an imperfect Bose gas*, Soviet Physics JETP **13** (2) (1961), 451–454.

[22] E. Sandier, *Lower bounds for the energy of unit vector fields and applications*, preprint, 1997.

[23] M. Struwe, *On the asymptotic behavior of minimizers of the Ginzburg–Landau model in 2 dimensions*, Differential Integral Equations **7** (1994), 1613–1624.

[24] D. R. Tilley and J. Tilley, *Superfluidity and Superconductivity*, Adam Hilger Ltd., Bristol and Boston, 1986.

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