Open access • Journal Article • DOI:10.1007/BF01048261

Global analysis of the phase portrait for the Kuramoto-Sivashinsky equation

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Published on: 01 Oct 1992 - Journal of Dynamics and Differential Equations (Kluwer Academic Publishers-Plenum Publishers)

Topics: Stochastic partial differential equation, Differential equation, First-order partial differential equation, Hyperbolic partial differential equation and Separable partial differential equation

GLOBAL ANALYSIS OF THE PHASE PORTRAIT FOR THE KURAMOTO–SIVASHINSKY EQUATION

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IMA Preprint Series # 665

July 1990

GLOBAL ANALYSIS OF THE PHASE PORTRAIT FOR THE KURAMOTO-SIVASHINSKY EQUATION

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Abstract. The global behavior of the Kuramoto-Sivashinsky equation is studied. The existence of an absorbing ball in every Sobolev norm is proved. The transition of energy from low modes to high ones is observed. An upper estimate for the Hausdorff dimension of the attractor is given. The main tool is to use the methods of the theory of ordinary differential equations in the investigation of partial differential equations.

§1. Main results and intrinsic relations.

1.1. Statement of results. The paper deals with the so called Kuramoto-Sivashinsky equation

$$u_t = -((u_x)^2 + u_x^2 + \nu u_{x^4}), \nu > 0, u_{x^n} = \partial^n u/\partial x^n$$

with periodic boundary conditions: $u(t, x + 2\pi) = u(t, x)$. This equation occurs in various physical problems [1,2,3]. The numerical studies show chaotic behavior of its solutions [4]. The phase space of this equation is the set of functions defined on the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. The average of the right hand side is negative, if $u \not\equiv$ const. The difference between the function f and its average $\bar{f}$ will be called the "centralized function f". The main results of this paper are the following (details are given below).

(1) The centralized solutions of the Kuramoto-Sivashinsky equation, as well as all their derivatives, are bounded on the time interval $[0, +\infty)$.

(2) The transition of energy from low harmonics to high ones.

(3) The existence and Liapunov stability of the attractor, and an upper estimate of its Hausdorff dimension.

1.2. The equation for the centralized solutions and heuristics. Denote by P the operator which maps each function f to Pf (the centralized function). P is the projection of the space $L^2(S^1)$ along the vector $f_0 \equiv 1$ onto the orthogonal complement of the vector f_0. The right hand side of the Kuramoto-Sivashinsky equation depends only on Pu. Thus the centralized solution $w = Pu$ of this equation satisfies the equation

$$w_t = -P((w_x)^2 + w_{x^2} + \nu w_{x^4}).$$

Note that $P(w_x) = w_x$. Thus, after replacing w by u, we obtain

$$u_t = -(P(u_x)^2 + u_{x^2} + \nu u_{x^4}), \bar{u} = 0 \tag{1}$$

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From now on we will consider only equation (1). Its right hand side may be decomposed to give the following two equations:

$$u_t = -(u_{x^2} + \nu u_{x^4}),$$

(HJ)
$$u_t = -P(u_x)^2$$

We will call the second equation the Hamilton-Jacobi equation, though this is not completely accurate: only the centralized solutions of the actual Hamilton-Jacobi equation $u_t = -u_x^2$ satisfy (HJ). The linear equation has the singular point $0 : u \equiv 0$. For $\nu < 1$ this is an infinite dimensional saddle. The corresponding operator has the eigenvectors e^{ikx} with eigenvalues $\lambda_k = k^2 - \nu k^4$. For $k \leq \nu^{-1/2}, \lambda_k \leq 0$ (such k correspond to low modes); for $k \geq \nu^{-1/2}, \lambda_k \geq 0$ (high harmonics). V.I. Arnol'd, who attracted my attention to the Kuramoto-Sivashinsky equation, noticed that "the linear part causes the low harmonics to increase, and the high ones to decrease, while the nonlinear part mixes the low and high harmonics". The formalization of this heuristic picture takes up the main part of this paper: §3-5. We now pass to the exact formulation.

1.3. The bounds for the solutions. The phase space for equation (1) is the space $X = \overset{\circ}{C}{}^{\infty}(S^1)$ of infinitely smooth functions with zero average on S^1. It is endowed with different scalar products and corresponding norms; it is not complete in any of them. Let

$$f = \sum_{-\infty}^{\infty} a_k e^{ikx}, a_k = \bar{a}_{-k}, a_0 = 0$$

For any real s, the s-th Sobolev norm n_s of f is defined as follows:

$$n_s(f) = ||f||_s = \left(\sum k^{2s}|a_k|^2\right)^{1/2}$$

For natural $s = m$, the m-norm $n_m(f)$ is equal to the L^2-norm of the m-th derivative $f^{(m)}$:

$$||f||_m = ||f^{(m)}||_0.$$

Consider also a C^m norm in X:

$$||f||_{C^m} = \max_{S^1} |f^{(m)}|$$

Since $\bar{f} = 0$, this norm is equivalent to the usual C^m norm. From now on by "smooth" we will mean "infinitely differentiable".

THEOREM 1. *For any s there is a positive R_s such that all solutions of the equation (1) with a smooth initial condition enter the ball $n_s \le R_s$ after some positive time and never leave it thereafter.*

1.4. Properties of the norm and universal constants. . For any fixed $u \in \overset{\circ}{C}{}^{\infty}(S^1)$, the function $s \mapsto n_s(u)$ is monotone and logarithmically convex, that is, its logarithm is a convex function [11]. Consequently, the functional n_s has the following properties:

$$\frac{n_s}{n_{s-ka}} \le \left(\frac{n_s}{n_{s-a}}\right)^k \text{ for } a > 0; \quad n_s^2 \le n_{s-a} n_{s+a}$$

The Sobolev imbedding theorem implies that, for any natural m and any $s > m + 1/2$, there is a constant C such that $||u||_{c^m} \le C||u||_s$. For convenience we abuse the notation by writing $A \lesssim B$, where the left hand is a function or functional, and the right hand may be a constant. This notation means that for some $C > 0$ the inequality $A \le CB$ holds everywhere. For instance,

$$||u||_{C^m} \lesssim ||u||_s \text{ for } s > m + 1/2.$$

1.5. The transition of energy. In this subsection (and only here) the square of any Sobolev norm is called the energy. If $f = A \sin x$, then all the energy is concentrated in the first mode; if $f = \sum a_k e^{ikx}, g = \sum_{|k|>100} a_k e^{ikx}$ and $||g||_s^2 \ge ||f||_s^2/2$ - that is, more than half the energy is concentrated in the modes of order higher than 100.

To state the transition theorem exactly, let us denote by E_N the space of trigonometrical polynomials of degree not higher than N and with zero average. Denote by P_N the orthogonal projection operator $L^2(S^1) \to E_N$ with kernel $E_N^{\perp}$ (obtained by eliminating the higher order terms of the Fourier series), and let $P_N^{\perp} = id - P_N$ (this is obtained by eliminating the terms of order lower than $N + 1$).

THEOREM 2. (TRANSITION OF ENERGY). *For every N and every $\lambda \in (0, 1)$ there exists an $R = R(N, \lambda, \nu)$ such that for any function $\varphi \in X$ lying outside the ball $n_1 \le R$, there exists a time T_φ such that the solution of the equation (1) with the initial condition φ takes the value ψ at time T_φ, where ψ satisfies $||P_N^{\perp}\psi||_3^2 \ge \lambda||\psi||_3^2$.*

For example, if $\lambda = 0.99, N = 100$, the theorem asserts that more than 99% of the energy n_3^2 for the function ψ is concentrated in the modes of order higher than 100. For example, φ might be taken as $A \sin x$ for $A > R$ and thus have no high modes in its Fourier expansion.

1.6. Generalized dissipative systems and their attractors. I failed to find for the equation (1) an absorbing domain in the classical sense, that is a domain on whose boundary the vector field points inward (such systems are called dissipative). Such a domain probably does not exist at all. Only a "generalized absorbing domain" was found,

which is "no worse" than an absorbing one: an equation with such a domain has just the same properties as if it were dissipative.

DEFINITION 1. Let $\{g^t | t \geq 0\}$ be a semiflow in the space X with continuous dependence on the initial conditions. Let B and $\widetilde{B}$ be two sets in X with $B \subset \widetilde{B}$. We say that the domain B is globally absorbing for the semiflow $\{g^t\}$ with periphery $\widetilde{B}$ and time delay T, if

1°. The orbit of any point $\varphi \in B$ returns to the domain B after at most time T and meanwhile does not leave the domain $\widetilde{B}$.

2°. The orbit of each initial point φ enters the domain B after some positive time.

A system which defines a semiflow with a generalized globally absorbing domain which is compact, together with its periphery, is called "generalized dissipative".

We say that the orbit γ of a semiflow is defined on the whole time axis if there exists a map $g : \mathbb{R} \to x$ such that $g\mathbb{R}_+ = \gamma$ and, for any $s > 0, g(t+s) = g^t(g(s))$

DEFINITION 2. The maximal attractor of a generalized dissipative system is the union of all those orbits which are defined on the whole time axis and lie in the periphery.

It turns out that the attractor above may be defined by the same formula as for a dissipative system, and it will be also Liapunov stable.

PROPOSITION 1. *Let $\{g^t | t \geq 0\}$ be a generalized dissipative semiflow with generalized globally absorbing domain B with periphery $\widetilde{B}$and time delay T, the closures of B and $\widetilde{B}$ being compact, and let A be its attractor in the sense of Definition 2. Then*

$$A = \bigcap_{n>0} g^{Tn}\widetilde{B}.$$

Furthermore, the attractor A is nonempty and Liapunov stable: for any neighbourhood U of A there exists a positive $t(U)$ such that $g^t\widetilde{B} \subset U$ for any $t > t(U)$.

We prove that A is nonempty. Let $\widetilde{\widetilde{B}} = \bigcup_{t>0} g^t B$. Evidently, $\widetilde{\widetilde{B}} \subset \widetilde{B}$. Definition 1 implies that any point $x \in \widetilde{B}$ enters the domain $\widetilde{\widetilde{B}}$ in time T and never leaves it. Thus $g^T\widetilde{B} \subset \widetilde{B}$. Let

$$A_1 = \cap_{n>0} g^{Tn}\widetilde{B}.$$

This is a countable intersection of nested compact sets, hence it is nonempty. We claim that it coincides with the attractor in the sense of Definition 2. Indeed, A_1 is the union of all the orbits which are defined on the whole time axis and lie in $\widetilde{B}$. Moreover, A_1 is closed. Thus $A = A_1$ is nonempty.

We next prove the Liapunov stability of the attractor. The function "exit time from $\widetilde{B}$ in reverse time" is defined as follows:

$$t(x) = \max\{t \geq 0 | \exists y \in \widetilde{B} : g^t y = x, g^\tau y \in \widetilde{B} \text{ for } \tau \in [0,t]\}.$$

Obviously, $t(x) = \infty$ iff the orbit of x is defined on the whole time axis, that is to say, iff $x \in A$. The function is upper semicontinuous: if $x = \lim x_n$, then $t(x) \geq \limsup t(x_n)$. This follows from the continuous dependence on the initial conditions for the orbits of the semiflow and the fact that $\widetilde{B}$ is closed. Suppose now that A is not Liapunov stable. Then for any n there is a point $x_n \in \widetilde{B} \backslash U$ such that $t(x_n) \geq n$. The compactness of $\widetilde{B}$ implies that $\{x_n\}$ may be assumed convergent: $x_n \to x, x \notin U$. The semicontinuity of the function t then gives $t(x) = \infty$. Thus $x \in A \subset U$, a contradiction.

1.7. The upper bound for the dimension of the attractor. As usual, $\overset{\circ}{H}_s$ denotes the Sobolev space, obtained as the completion of $\overset{\circ}{C}^\infty$ in the norm n_s. It is proved below that in each space $\overset{\circ}{H}_s$ with $s \geq 4$ equation (1) defines a generalized dissipative system. The standard part of the proof is contained in subsection 1.8; the more sophisticated part in sections 2-5. Hence equation (1) possesses an attractor A_s in any $\overset{\circ}{H}_s$. Equation (1), just as the heat equation, smoothens the initial conditions. Thus, all the attractors A_s consist of smooth functions and coincide. We may thus replace A_s by A in what follows.

THEOREM 3. *Consider the L_2-metric in the space $\overset{\circ}{C}^\infty$. The corresponding Hausdorff dimension of the attractor A is finite. Moreover,* $\dim_H A \leq \nu^{-36.5}$.

1.8. Main theorem and reductions.

THEOREM 1A. *The Kuramoto-Sivashinsky equation defines a semiflow which has a generalized absorbing domain B, given by the inequality $n_1 \leq R$, where R depends on ν : $R \leq \nu^{-72}$.*

This is the main theorem, to be proved below in full detail. The scheme of reduction of theorems 1 and 3 to theorem 1A is given in the remaining part of §1. The rest of the reduction may be found in [3]. Theorem 2 is also proved below in full detail.

In this section the idea of the reduction of theorem 1 to 1A is given. For any natural m, we prove the existence of an R_m such that the following domain B_m is a generalized absorbing domain for equation (1):

$$B_1 = \{n_1 \leq R_1\}; B_3 = B_1 \bigcap \{n_3 \leq R_3\}$$
$$B_m = B_{m-1} \bigcap \{n_m \leq R_m\}, m > 3.$$

Consider the case $n = 3$. Let $R_1 = R$, with R as in theorem 1A. The domain B_1 is globally absorbing, according to theorem 1A. We prove that, in the domain $B_3 \backslash B_1$, the functional n_3 decreases along solutions. In section 4.3 the following inequality is proved:

$$\frac{d(n_3^2)}{dt} \lesssim n_3^3 + n_4^2 - \nu n_5^2;$$

here d/dt is the derivative with respect to equation (1). By using the logarithmic convexity of the norm n_s, we obtain the following inequalities in the ball $n_1 < R_1$:

$$n_3^3 \le n_1^{3/2} n_5^{3/2} \le R_1^{3/2} n_5^{3/2}; \quad n_4^2 \le n_1^{1/2} n_5^{3/2} \le R_1^{1/2} n_5^{3/2}$$

thus

$$\frac{d(n_3^2)}{dt} \lesssim 2R_1^{3/2} n_1^{3/2} - \nu n_5^2 < 0 \text{ for } n_5 > 4R_1^3 \nu^{-2}, R_1 > 1$$

If $R_3 > 4R_1^3\nu^{-2}$, then $n_5 \ge n_3 > 4R_1^3\nu^{-2}$in $B_1 \backslash B_3$. Thus in this domain $dn_3/dt < 0$.

Now we prove that, if $R_3 > 4\nu^{-1}R_1$, the orbits of equation (1) may leave the domain $B_1 \backslash B_3$ only by entering the ball B_3. This follows from Corollary 2.3 of subsection 2.3 below, which asserts that on the intersection of the sphere $n_1 = R_1$ and the domain $R_3 > 2\nu^{-1}R_1$ the energy n_1 decreases along the orbits of equation (1). Thus, by Theorem 1A, all the orbits of (1) enter the ball B_1, and afterwards, as has just been proved, they enter $B_3 = B_1 \cap \{n_3 \le R_3\}$.

The case $m > 3$ is treated in an analogous way.

1.9. The upper estimate of the Hausdorff dimension of the attractor. Here we give the main ideas involved in the reduction of Theorem 3 to theorem 1A. The reasoning below gives an estimate for the Galiorkin approximation to equation (1) for a sufficiently large dimension N of the truncated phase space; this estimate does not depend on N. To obtain the same estimate for the attractor of the whole equation (1), it is sufficient to proceed in the standard way [5,6,7,8].

According to [9,10], consider the quadratic form F_u on the periphery $\widetilde{B}$ of the generalized absorbing domain of (1) (the results in these papers are proved for dissipative systems, but by using Proposition 1 one may check their validity in the case of generalized dissipative systems).

$$F_u(\xi) = (d\mathcal{W}_u(\xi), \xi);$$

here $\mathcal{W}_u$ is the right hand side of equation (1). Then one must find a quadratic form G such that $G(\xi) \ge F_u(\xi)$ for any $u \in \widetilde{B}$. Let $\lambda_1 \ge \cdots \ge \lambda_j \ge \cdots$ be the eigenvalues of the form G. If the number k is such that $\sum_1^k \lambda_j < 0$, then $\dim_H A \le k$.

The form F_u is given by:

$$F_u(\xi) = (2u_x\xi_x, \xi)_{L^2} + ||\xi||_1^2 - \nu||\xi||_2^2$$

An upper estimate for this form in the ball $||u||_1 \le 2R$ follows from the inequalities:

$$\begin{aligned} |(u_x\xi_x, \xi)| &\lesssim ||u||_1 ||\xi||_1^2 \lesssim R_1 ||\xi||_1^2 \le R_1 ||\xi||_0 ||\xi||_2 \\ &\le 3\nu^{-1} R_1^2 ||\xi||_0^2 + \frac{\nu}{3}||\xi||_2^2; \\ ||\xi||_1^2 &\le 3\nu^{-1}||\xi||_0^2 + \frac{\nu}{3}||\xi||_2^2 \end{aligned}$$

The form G may be taken as follows:

$$G(\xi) = 3\nu^{-1}(R_1^2+1)||\xi||_0^2 - \frac{\nu}{3}||\xi||_2^2$$

The eigenvalues of this form are $\lambda_j = \mu_k - \nu j^4/3$ with $\mu = 3\nu^{-1}(R_1^2+1)$. Their sum is $\sum_k = \sum_{j=1}^k \lambda_j \approx \mu k - \nu k^5/5$. For $k \gtrsim (\mu\nu^{-1})^{1/4}$ one has $\sum_k < 0$. Theorem 1A implies that $\mu \lesssim \nu^{-145}$. Thus $\dim_H A \lesssim \nu^{-36.5}$. E. Titi showed that this exponent may be lowered to 29.

1.10. Previous results. The boundeness of the solutions and the dimension of the global attractor of the Kuramoto-Sivashinsky equation were investigated in [3]. In this paper the upper estimate for the dimension is given under the assumption that all the orbits enter the ball $n_1 \leq R$; it is also proved that this is the case for even initial conditions. Our methods, based on ideas from ordinary differential equations, and applied to the case of partial differential equations, allow us to get similar estimates for *all* initial conditions, not only even ones. The estimate [3] for the even case is better than ours: the exponent is $-13/8$ instead of -36.5. It seems that the method developed below may be used to get similar results for equations in a much wider class than (1). The main results of the paper were announced in [12].

During his visit to Moscow in June, 1989, G. Sell kindly communicated to me that the uniform estimates for the solutions of Kuramoto Sivashinsky equation given by Theorem 1A imply the existence of a finite-dimensional inertial manifold [14]. This existence was proved in [14] only for even initial conditions. Now it is proved for arbitrary initial conditions. Thus the Kuramoto-Sivashinsky equation can be reduced to an ordinary differential equation.

It is a pleasure to thank Professors V.I. Arnol'd, N.D. Vvedenskaya, L.R. Volevich, Ju. B. Radvogin, A.V. Fursikov, G. Sell for fruitful discussions, and E. Titi and M. Taboada, who read an English version of the text and made many useful comments.

§2. Global analysis of the phase portrait.

2.1. Pumping and dissipation of energy. From now on, the energy will be defined to be the norm of the functional $E = n_1^2$. The Hamilton-Jacobi equation $u_t = -P(u_x^2)$ preserves the energy:

$$\frac{d}{dt_{HJ}}E = 2(u_x, (-Pu_x x^2)_x) = 2(u_{xx}, u_x^2) = 2\oint u_x^2 u_{xx} = \frac{2}{3}\oint (u_x^3)_x = 0.$$

Here and below $d/dt_{HJ}(resp. d/dt_{KS})$ denotes differentiation along the orbits of the equation (HJ) (respectively (1)); $\oint f$ is the integral of the function f on S^1 with respect to the Lebesgue measure. The derivative of E along the orbits of (1) is

$$\frac{dE}{dt_{KS}} = n_2^2 - \nu n_3^2,$$

For $\nu > 1$ the energy n_1^2 is a Liapunov function and the point 0 is the global attractor. From now on we assume that $\nu \leq 1$.

DEFINITION 1. *The cone of pumping K_p and the cones of dissipation K_d, of slow dissipation K_{sd}, and of rapid dissipation K_{rd} are defined as follows:*

$$K_p : n_2^2 \geq \nu n_3^2; K_d : n_2^2 \leq \nu n_3^2;$$
$$K_{sd} : n_2^2 \geq \frac{1}{2}\nu n_3^2; K_{rd} : n_2^2 \leq \frac{\nu}{2}\nu_3^2.$$

The derivative of the energy along the orbits of (1) is positive iff this point belongs to the cone of pumping.

2.2. The global behavior of solutions.

THEOREM 4. *There exist positive constants T, c and a with the following property: for any initial condition φ, lying in the cone of slow dissipation and outside the ball $B : n_1 \leq R(\nu) = c\nu^a$, the corresponding solution of equation (1) will enter the cone of rapid dissipation in time $T_\varphi \in [0, T]$. The arc of the orbit*

$$(*) \qquad \Gamma = \{g^t\varphi | t \in [0, T_\varphi]\}$$

belongs to the ball $n_1 \leq 2||\varphi||_1$. Its endpoint ψ has the property: $||\psi||_1 \leq ||\varphi||_1 - 1$. One may take $T = \pi^2/R(\nu), a = 72$.

We next deduce Theorem 1A from Theorem 4. We prove that B is a generalized global absorbing ball for system (1) with periphery $\widetilde{B} : n_1 \leq 2R(\nu)$ and time delay T. The evolution from the initial to the final point of the arc Γ in Theorem 4 will be called a step. Take an arbitrary initial condition φ. If it belongs to the cone of rapid dissipation or to the ball B, one must study possibilities A or B below. If $\varphi \in K_{sd} \backslash B$ then, according to theorem 4, the corresponding orbit enters the cone of rapid dissipation after one step and loses energy by at least by 1. Three cases are possible (Figure 1):

A. The solution remains forever in the cone of rapid dissipation. In this case the energy will decrease, and the solution will tend to the saddle point. This case has probability zero: the initial condition belongs to the stable manifold of the saddle, which has finite codimension.

B and C. The solution leaves the cone of rapid dissipation at the point φ_1; in case B, $\varphi_1 \in B$; in case C, $\varphi_1 \notin B$.

B. By Theorem 4, the solution will either remain in the ball B or leave it, but will remain in it for no longer than a time interval T, while not escaping the ball $\widetilde{B}$.

C. By Theorem 4, after one step the solution will pass from the point φ_1 to $\psi_1 \in K_{sd}$, with its energy decreasing by at least 1. Thus, after a finite number of steps, it will enter the ball B and will never leave $\widetilde{B}$.

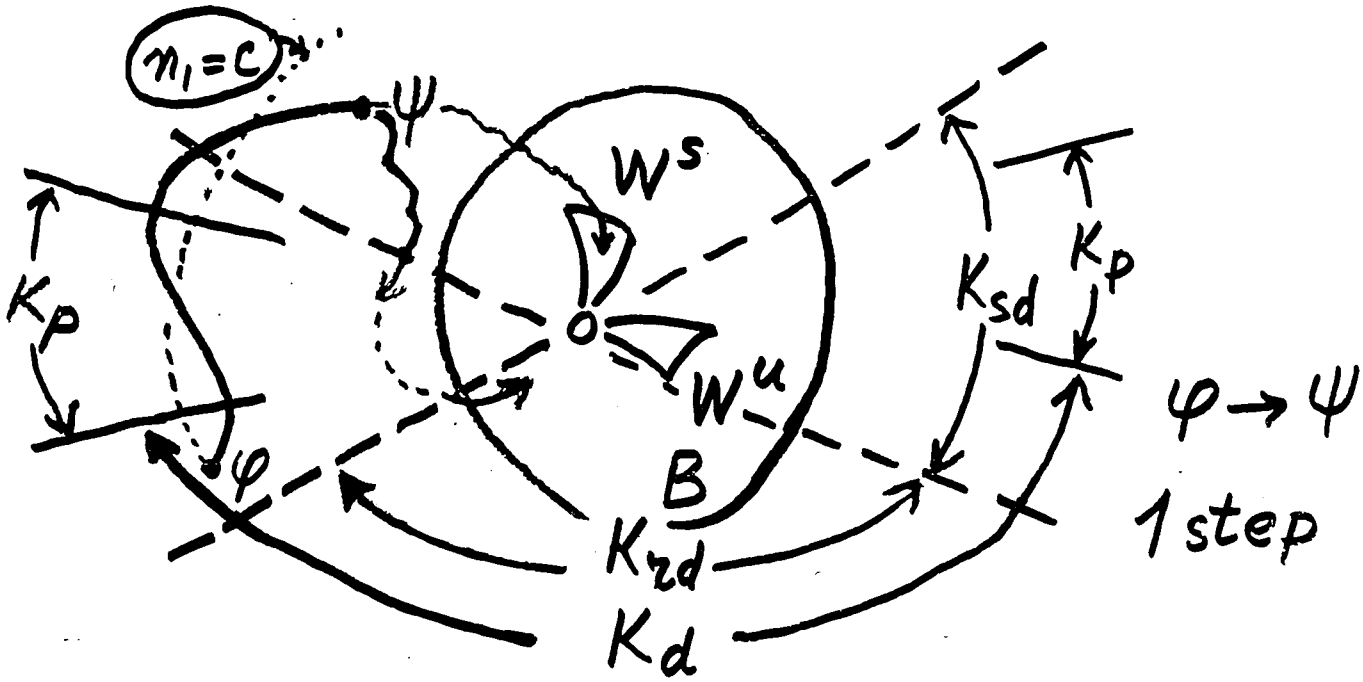

Figure 1.

2.3. The Exit and Loss of Energy lemmas. Theorem 4 follows from two lemmas, which are proved in sections 3 and 4.

LEMMA 1 (EXIT LEMMA). *There exists a positive constant T and for any $\rho > \nu^{-1}$ there exists $R = R(\nu, \rho)$ with the following property. For each initial condition φ, lying in the cone of slow dissipation and outside the ball $n_1 \le R$, there is a time $T_\varphi < T$ such that the arc $(*)$ is in the ball $n_1 \le 2||\varphi||_1$ and has an endpoint ψ for which*

$$||\psi||_3 \ge \rho||\varphi||_1.$$

COROLLARY. *$\psi \in K_{rd}$ for $\rho \ge 4\nu^{-1}$.*

One must verify that ψ satisfies the inequality which defines the cone of rapid dissipation. Namely,

$$||\psi||_2^2 \le ||\psi||_1 ||\psi||_3 \le 2||\psi||_3^2/\rho \le \frac{1}{2}\nu||\psi||_3^2.$$

The first inequality follows from logarithmic convexity of the norm $n_s(u)$ as a function of s; the second, from the conclusion of the Lemma: $||\psi||_1 \le 2||\psi||_1 \le 2||\psi||_3/\rho$; the third, from the lower estimate for ρ.

LEMMA 2 (ENERGY LOSS LEMMA). *Let φ, ψ, ρ be as in Lemma 1. If $\rho > \nu^{-2}$ and c is sufficiently large, then*

$$||\psi||_1 \le ||\varphi||_1 - 1.$$

2.4. The transition of energy. Theorem 2 is a direct consequence of Lemma 1.

Let φ and ψ be as in Lemma 1. Let us suppose that $||P_n^{\perp}\psi||_3^2 \leq \lambda||\psi||_3^2$ and obtain a contradiction for sufficiently large ρ. The inverse inequality will give Theorem 2. The previous assumption gives

$$||P_N\psi||_3^2 \geq (1-\lambda)||\psi||_3^2.$$

The definition of the Sobolev norms by Fourier coefficients implies that

$$||P_N\psi||_3 \leq N^2||\psi||_1 - 1.$$

Thus

$$||\psi||_3 \leq (1-\lambda)^{-1/2}N^2||\psi||_1 \leq 2(1-\lambda)^{-1/2}N^2||\psi||_1.$$

For $\rho > 2(1-\lambda)^{-1/2}N^2$ this contradicts Lemma 1.

The analogues of Lemmas 1 and 2 may be proved for N-th Galiorkin approximations to the Kuramoto-Sivashinsky equations with estimates independent of N. This is done in the next three sections, which form the main part of the paper. Lemmas 1 and 2 are then proved by letting N tend to infinity.

§3. Galiorkin approximations to the Kuramoto-Sivashinsky equation as small perturbations of the Hamilton-Jacobi equation.

3.1. Rescaling. Outside the large ball the quadratic terms of equation (1) dominate the linear ones. Let us change the scale, so as to transform the ball of radius ε^{-1} into the unit ball while preserving the quadratic terms of (1): $w = \varepsilon u, \tau = \varepsilon^{-1}t$. Equation (1) will take the form:

$$w_\tau = -(P(w_x^2) + \varepsilon(w_{x^2} + \nu w_{x^4})).$$

Replace w by u and τ by t:

$$u_t = -(P(u_x^2) + \varepsilon(u_{x^2} + \nu u_{x^4})). \tag{1_ε}$$

The phase flow transformation of equation (1_ε) will be denoted by g_ε^t.

LEMMA 1A. *For any $\rho > \nu^{-1}$ there exists a positive $\varepsilon_0 = \varepsilon_0(\nu,\rho)$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and for any initial condition φ lying in the cone of slow dissipation and the sphere $n_1 = 1$, there exists a t_φ such that the arc*

$$\Gamma_{\varepsilon,\varphi} = \{g_\varepsilon^t\varphi | t \in [0,t_\varphi]\}$$

belongs to the ball $n_1 \leq 2$ and reaches the sphere $n_3 = \rho$.

Lemma 1 follows from Lemma 1A. Take $R(\nu,\rho) = \varepsilon_0^{-1}$. Let, in Lemma 1, $||\varphi|| = R > R(\nu,\rho)$. Rescale the equation (1) as before with $\varepsilon = R^{-1}$. Then the arc $(*)$ from Lemma 1 will be transformed into the arc $\Gamma_{\varepsilon,\varphi}$ of Lemma 1A, and the inequalities of Lemma 1A will give the inequalities of Lemma 1 with $T = \pi^2/R(\nu,\rho)$.

3.2 Galiorkin approximations. Let E_N be the space of real trigonometrical polynomicals and the projection operators P_N and $P_N^{\perp}$ be those defined in subsection 1.5. The N-th Galiorkin approximation is constructed as follows. Its phase space is E_N, and the corresponding vector field is the projection P_N onto E_N of the right hand side of (1_ε), restricted to E_N. This projection commutes with differentiation, so the N-th Galiorkin approximation has the form:

$$u_t = -(P_N(u_x^2) + \varepsilon(u_{x^2} + \nu u_{x^4})), u \in E_N(KS_{N\varepsilon}).$$

Here $KS_{N\varepsilon}$ denotes the N-th Galiorkin approximation to the ε-rescaled Kuramoto-Sivashinsky equation.

LEMMA 1B. *For any $\rho > \nu^{-1}$ there exists an $\varepsilon_0 > 0$, depending on ν and ρ and independent of N, such that for any $\varepsilon \in (0, \varepsilon_0)$ and any initial condition φ with the properties $\varphi \in E_N \cap K_{sd}, ||\varphi||_1 = 1$, there exists $t_\varphi \in (0, \pi^2)$ such that the arc*

$$\Gamma = \{g_{N\varepsilon}^t \varphi | t \in [0, t_\varphi]\}$$

is contained in the ball $n_1 \leq 2$ and reaches the sphere $n_3 = \rho$. Here $g_{N\varepsilon}^t$ denotes the phase flow transformation of the equation $(KS_{N\varepsilon})$, corresponding to time t.

Lemma 1A may be deduced from Lemma 1B by letting N tend to ∞. The Galiorkin approximations will be considered as small perturbations of the Hamilton-Jacobi equation in the ball $n_3 < \rho$. To begin with, we prove a general perturbation lemma for a differential equation in a finite or infinite dimensional space.

3.3 The perturbation Lemma. Consider two differential equations in the pre Hilbert space X:

$$\dot{x} = v(x) \text{ and } \dot{x} = v(x) + w(x).$$

If X is finite dimensional, it is supposed to be an Euclidian space. If X is infinite dimensional, it is supposed to have a scalar product $< \cdot, \cdot >$ and a corresponding norm $|| \cdot ||$, perhaps being incomplete in this norm. The most common estimate of the distance between the orbits of the vector fields v and $v + w$ uses the Gronwall inequality, which contains the Lipshitz constant of the field v. Yet in the typical infinite dimensional situation the field v is not Lipshitzian. Nevertheless, the distance between the orbits may be estimated, if the unperturbed equation $\dot{x} = v(x)$ is solved and an upper bound is found for the norm of the Fréchet derivative of the corresponding phase flow.

We recall next the definition of a flow box (with a flat bottom).

Let $\mathcal{D}$ be a closed subset of a hyperplane Λ in X, at all points of which the vector field is transversal to the hyperplane. The flow box of the field v with bottom $\mathcal{D}$ and time interval T is the set

$$\Omega = \{g_v^t x | x \in \mathcal{D}, t \in [0, T]\}.$$

The sets

$$\mathcal{D}^T = g_v^T \mathcal{D}, \{g_v^t \partial\mathcal{D} | t \in [0,T]\}, \mathcal{D}^t = g_v^t \mathcal{D}$$

are called the top, the side and the t-floor of the flow box, respectively. The t-floor is defined for any $t \in [0,T]$. Denote by Π_v the projection map of the flow box onto its bottom along the orbits of v: for any $x \in \mathcal{D}$ and $t \in [0,T]$ one has $\Pi_v(g_v^t x) = x$. From now on g_v^t will denote the phase flow transformation corresponding to the flow (or semiflow) of v at time t. The time function t is defined by everywhere on Ω by $g_v^t x \mapsto t$.

LEMMA 3. *Let v and w be vector fields in the pre-Hilbert space X. If X is finite dimensional the fields v and w are supposed to be smooth. If X is infinite dimensional, then we shall make the following smoothness assumption: the map*

$$\phi : (t,\tau) \mapsto g_v^t \circ g_{v+w}^\tau \varphi$$

is smooth (wherever defined). Let Γ be the arc of the orbit of the field $v+w$ defined by :

$$\Gamma = \{\Gamma(\tau) | \tau \in [0,\tau_0]\}, \Gamma(\tau) = g_{v+w}^\tau \varphi,$$

Let Ω be the flow box of the unperturbed field v with bottom $\mathcal{D}$ in the hyperplane Λ: $\langle \psi - \varphi, v(\varphi)\rangle = 0$ and time interval T. Assume that $\Gamma \subset \Omega$, and also:

(1) *$||w|| < \alpha$ everywhere on Γ.*

(2) *If $y \in \Gamma, g_v^t y \in \Omega$, then*

$$||dg_v^t(y)|| \leq F.$$

(3) *$||v|| \geq C$ everywhere on $\mathcal{D}$.*

(4) *The angle between the vectors $v(\psi)$ and $v(\varphi)$, for any $\psi \in \mathcal{D}$, does not exceed a fixed value, say $\pi/3$.*

Then

(1) *The length of the projection $\Pi_v \Gamma$ does not exceed $2\alpha FT$.*

(2) *$|(t \circ \Gamma)'_\tau - 1| \leq 3\alpha F/C$.*

We begin with the finite dimensional case. In this case the smoothness of the functions, which is analyzed below, follows from the smoothness of the vector fields v and w.

PROPOSITION 1. *Under the assumptions of Lemma 3*

$$||d\Pi_v(y)|| \leq 2F \text{ for any } y \in \Gamma,$$

(this proposition is proved below).

Set $\gamma(t) = \Pi_v \Gamma(\tau)$. Then

$$\dot{\gamma}(\tau) = d\Pi_v \dot{\Gamma}(\tau) = d\Pi_v(v \circ \Gamma(\tau) + w \circ \Gamma(\tau)) = d\Pi_v w \circ \Gamma(\tau).$$

Condition 1 of the Lemma, together with Proposition 1 implies

$$||\dot{\gamma}(\tau)|| \leq 2\alpha F.$$

This gives the first assertion of the Lemma.

2. The time function $t : \Omega \to \mathbb{R}$ was defined above as

$$t : g_v^t x \mapsto t \text{ for any } t \in [0, T], x \in \mathcal{D}.$$

The form dt is the invariant of the action of the phase flow v. Thus, letting $y = \Gamma(\tau)$:

$$\frac{d}{dt}(t \circ \Gamma(\tau)) = (dt, (v + w)(y)) = (1 + (dt, w(y)).$$

Here (w, ξ) is the value of the 1-form w on the vector ξ. Let $x = \Pi_v y$, and bring the vector $w(y)$ to the point x by the phase flow of v; let w_1 the vector thus obtained: $w_1 \in T_x X$. Condition 2 of the lemma implies that $||w_1|| \leq F\alpha$. Decompose the vector w_1 as the sum $w_1 = w' + w'', w' = \beta v(x)$, $w'' \in T_x\Lambda, \beta \in \mathbb{R}$. Since $dt = 0$ on $T_x\Lambda$ and $(dt, v(x)) = 1$, one has

$$(dt, w_1) = (dt, w') = \beta.$$

Moreover, $w' = w_1 - w'', w'' = d\Pi_v(x)w_1$. These elementary geometrical considerations developed in the proof of Proposition 1 below, imply that $||d\Pi_v(x)|| \leq 2$. Thus $||w''|| < 2||w_1||$ and

$$|\beta| = ||w'||/||v(x)|| \leq 3||w_1||/||v(x)|| \leq 3\alpha F/C.$$

This proves assertion 2.

3. In the infinite dimensional case, the proof of assertion 1 is the same as in the finite dimensional one. In order to transfer the proof of assertion 2 to the infinite dimensional case, one must check the smoothness of the restriction of the function t to the surface $\sum = \{\phi(t, \tau)\}$. This smoothness follows from the smoothness of the map ϕ and the transversality of the field v to the bottom. After this, all the previous reasons will remain true.

4. To finish the proof of Lemma 3 we must prove Proposition 1.

Proposition 1 follows from assumptions 3,4 of Lemma 3 and the inequality

$$||d\Pi_v(y)|| \leq ||g_v^{-t}(y)||/\cos \angle(v(x), v(\varphi)) \text{ where } y = g_v^t x.$$

Let us prove this inequality. It is obvious that

$$d\Pi_v(y) = d\Pi_v(x) \circ dg_v^{-t}(y) :$$

in order to find a projection of the vector onto the bottom of the box along the orbits it is sufficient to bring it by the flow to some point on the bottom, and then to take a projection of the vector thus obtained onto the bottom along the field v. The norm of the projection operator : $T_xX \to T_x\Lambda$ along the vector $v(x)$ does not exceed one times the cosine of the angle between the vectors $v(x)$ and $v(\varphi)$, the latter being the normal to Λ. In the case $\dim X = 2, \dim \Lambda = 1$ this is just elementary geometry; the general case is reduced to the two dimensional one by considering the plane containing the image and the preimage of the projection.

3.4. Galiorkin approximations to the rescaled Kuramoto-Sivashinsky equation. These are considered as small perturbations of the Hamilton-Jacobi equation. The proof of Lemma 1B uses the following two ideas.

1. Each solution of the Hamilton-Jacobi equation survives only for a finite time, after which blow up occurs: the higher derivatives (particularly, those of order 3) become infinite.

2. While the solution of the Hamilton-Jacobi equation lies in the ball $n_3 \leq \rho$, the solution of the $(KS_{N\varepsilon})$ equation with the same initial condition for sufficiently large N and small ε stays close to the former. The first solution reaches the sphere $n_3 \neq \rho$ and goes outside; thus the second solution reaches the same sphere. After this the two solutions "go different ways": the Hamilton-Jacobi solution blows up, while the solution of the $(KS_{N\varepsilon})$ enters the cone of rapid dissipation and its high modes decrease.

The detailed exposition is as follows. The perturbation Lemma 3 will be used for the unperturbed field of the (HJ) equation and the perturbed field of the $(KS_{N\varepsilon})$ equation. The phase space X will be the space of all smooth functions over S^1 with zero average, with the H_{-1} scalar product. This space is not complete (pre-Hilbert, but not Hilbert).

PROPOSITION 2. *For any pair of positive (α, ρ) there exist N and ε such that the right hand sides of equations (HJ) and $(KS_{N\varepsilon})$ in the intersection of the balls $\widetilde{B} : n_1 \leq 2$ and $Q : n_3 \leq \rho$ differ by no more than α in the H_1 norm.*

Remark. It is sufficient to take N and ε such that

$$\alpha \geq \rho(2C/N + \nu\varepsilon) + 2\varepsilon,$$

C being a universal constant.

We have to prove the inequality

$$||P_N^{\perp}(u_x^2) + \varepsilon(u_{x^2} + \nu u_{x^4})||_{-1} \leq \alpha$$

if $||u||_3 \leq \rho, ||u_1|| < 2$, N and ε are sufficiently large and small, respectively. The following inequalities hold:

$$||P_N^{\perp}(u_x^2)||_{-1} \overset{(1)}{\leq} \frac{1}{N}||u_x^2||_0 \overset{(2)}{\leq} \frac{2\pi}{N}||u_x||_{W_4^1}^2 \overset{(3)}{\leq}$$
$$\overset{(3)}{\leq} \frac{C}{N}||u||_{3/2}^2 \overset{(4)}{\leq} \frac{C}{N}||u||_1||u||_3,$$

C being a universal constant. The first inequality is a corollary of the definition of the H_{-1} norm by means of the Fourier coefficients; the second one, of the Cauchy inequality; the third one, of the Sobolev imbedding theorem; the fourth one, of the monotonicity and logarithmic convexity in s of the norm $n_s(u)$. On the other hand,

$$||\varepsilon(u_{x^2} + \nu u_{x^4})||_{-1} \leq \varepsilon(||u||_1 + \nu||u||_3).$$

The desired inequality is a consequence of the last two estimates.

3.5. Properties of the Hamilton-Jacobi solutions. The properties of the solution of the Hamilton-Jacobi equation are stated here and will be proved in section 5. In this section they will be used to prove Lemma 1B. The phase flow transformation of (HJ), corresponding to the time t, is denoted by g^t_{HJ}.

1°. **The blow-up time.** The solution of the (HJ) equation blows up either in the future or in the past. The blow-up times are the values T^+_φ and T^-_φ with the following properties: $T^-_\varphi < 0 < T^+_\varphi$; the solution $g^t_{HJ}\varphi$ is smooth for all values of $T \in (T^-_\varphi, T^+_\varphi)$ and ceases to be smooth for $t = T^-_\varphi$ and T^+_φ. Furthermore, one has

$$T^+_\varphi = -\frac{1}{2\min\varphi''}, T^-_\varphi = -\frac{1}{2\max\varphi''}, \leq \frac{1}{2||\varphi||_{C^2}}|T^\pm_\varphi| \leq \frac{\pi^2}{||\varphi||_1}.$$

2°. **The conservation law (see 2.1).** The function n_1^2 is preserved by the (HJ) equation.

3°. **Upper and lower estimates of the n_3 norm near the blow-up time.** For some positive C,

$$\frac{C|T^\pm_\varphi|}{|T^\pm_\varphi - t|^3} \leq ||g^t_{HJ}\varphi||^2_3 \leq \frac{|T^\pm_\varphi||^5||\varphi||^2_3}{|T^\pm_\varphi - t|^5}.$$

4°. **Variation of the initial conditions.** For some positive C and any $t \in (T^-_\varphi, T^+_\varphi)$

$$||dg^t_{HJ}(\varphi)||_{-1} \leq C||\varphi||_3^{3/2}||\varphi||_1^{-1}.$$

Note. Near the blow-up time the norms n_s, for $s > 1$, tend to infinity along the (HJ)-solutions, while the n_1 norm is preserved. Thus it is not surprising that the n_{-1} norm of the variation of the initial condition "does not notice" the approaching to the blow-up time.

3.6. Construction of the flow box for the (HJ) equation. We now pass to the proof of Lemma 1B. Consider two vector fields v and $v + w$ corresponding respectively to the (HJ) and $(SK_{N\varepsilon})$ equations in the phase space X defined in subsection 3.4. Let φ be the initial condition from Lemma 1B: $||\varphi||_1 = 1$, $\varphi \in K_{sd}$. The hyperplane Λ is defined as follows

$$\langle \psi - \varphi, v(\varphi)\rangle_{-1} = 0, v(\varphi) = -P(u_x^2).$$

The bottom of the flow box of the field v will be chosen so as to fulfill the conditions 3 and 4 of Lemma 3, namely, the upper estimate of the n_{-1} norm of the field v on the bottom and lower estimate of the angle between the vectors of the field v at different points of the bottom. Moreover, the blow-up will be almost simultaneous for all the initial conditions at the bottom. The time interval T will be chosen so as to obtain an inequality $n_3 > \rho$ on the top of the box. Thus the ball $Q : n_3 \leq \rho$ does not intersect the top. Let us prove that such a choice of the bottom and the time interval is possible.

PROPOSITION 3. *For some positive C_1, C_2 and any φ, ψ, satisfying the conditions*

$$||\varphi||_1 = 1, \varphi \in K_{sd}, ||\varphi - \psi||_{C^2} \leq \sigma, \sigma \leq C_1 \nu^{1.2}$$

the following inequalities hold:

$$||P(\psi_x^2)||_{-1} \geq C_2 \nu^{1.2}, |\angle(P(\varphi_x^2), P(\psi_x^2))| \leq \pi/3.$$

The proof is given in 5.5.

PROPOSITION 4. *If $||\varphi||_1 = 1, \varphi \in K_{sd}$, then*

$$||\varphi||_3 \leq 2\nu^{-1}, ||\varphi||_{C^2} \lesssim \nu^{-4/5}.$$

The logarithmic convexity of $n_s(\varphi)$ is s implies

$$||\varphi||_3 / ||\varphi||_1 \leq ||\varphi||_3^2 / ||\varphi||_2^2$$

The first assertion of the Proposition is now a consequence of the definition of the cone of slow dissipation. The second one follows from the first one and the Sobolev imbedding theorem.

The bottom of the box now will be defined as follows:

$$\mathcal{D} = \{\psi \in \Lambda | \ ||\varphi - \psi||_{C^2} \leq \sigma\}.$$

The time interval T will be taken close to $T_\varphi : T = T_\varphi - \delta$, with δ small.

PROPOSITION 5. *If $\rho \geq 2\nu^{-1}$, and $\delta = \delta(\nu, \rho), \sigma = \sigma(\nu, \rho)$ are sufficiently small, then for any $t \in [T - \delta, T]$ one has $n_3 > \rho$ everywhere on the t-floor of the flow box Ω. For sufficiently small β_1, β_2 the values $\delta = \beta_1 \rho^{-2/3} \nu^{4/15}, \sigma = \beta_2 \delta \nu^{8/5} = \beta_1 \beta_2 \rho^{-2/3} \nu^{28/15}$ will be adequate.*

We next estimate the difference $|T_\varphi^+ - T_\varphi^-|$ for $||\varphi - \psi||_{C^2} \leq \sigma$. The formula of subsection 3.5 for T_φ^+ implies, for $\sigma < T_\varphi^+/2$, the inequality

$$|T_\varphi^+ - T_\psi^+| \lesssim \sigma (T_\varphi^+)^{-2}.$$

Proposition 4 gives $(T_\varphi^+)^{-2} \lesssim \nu^{-8/5}$. Thus for β_2 sufficiently small and $\sigma = \beta_2 \delta \nu^{8/5}$ one has

$$|T_\varphi^+ - T_\psi^+| < \delta/2.$$

Let $y = g_{HJ}^t \psi$ be an arbitrary point of the t-floor of Ω for $t \in [T-\delta, T]$. Then for any $\psi \in \mathcal{D}$

$$0 < T_\psi^+ - t < 2\delta.$$

Property 3° of the (HJ)-equation and Proposition 4 imply that, for some positive C_1, C_2 and sufficiently small $\delta = \delta(\nu, \rho)$, one has

$$n_3^2(y) \geq C_1 T_\psi^+ \delta^{-3} \geq C_2 \nu^{4/5} \delta^{-3} \geq \rho^2.$$

The first part of the Proposition is thus proved. The second, containing the explicit formulas for δ and σ is proved by directly checking the inequalities (i) $\sigma < T_\varphi^+/2$: $T_\varphi^+/2 \geq \beta \nu^{4/5} \geq \beta_1 \beta_2 \nu^{2/3+28/15} \geq \sigma$; the last inequality follows from the condition $\rho \geq \nu^{-1}.(ii) C_2 \nu^{4/5} \delta^{-3} \geq \rho^2$, which is obtained by an appropriate choice of β_1.

Note. For $\rho > \nu^{-1}$ and β_1, β_2 sufficiently small the constant σ from Proposition 4 satisfies the inequality of Proposition 3.

The construction of the flow box Ω is thus finished.

3.7. The a priori estimate. Let φ be as in Subsection 3.6. Property 1° of the (HJ)-equation implies that $T_\varphi^+ < \pi^2$. Consider the arc $\widetilde{\Gamma}$ of the orbit of the equation $(KS_{N\varepsilon})$ with initial point φ and time length $\tau_0 = T - \delta$. Denote by Γ the part of $\widetilde{\Gamma}$ located in the intersection $\Omega \cap Q (Q = \{n_3 \leq \rho\})$ from the beginning to the end of Γ or to the first exit point from $\Omega \cap Q$. It is proved below that such an exit point exists, but for now the possibility $\Gamma = \widetilde{\Gamma}$ is not excluded. Note that $\varphi \in \Omega \cap Q$ because $||\varphi||_3 \leq 2\nu^{-1} \leq \rho$; the first inequality follows from Proposition 4, the second one from the definition of ρ.

PROPOSITION 6. *For $\rho \geq 2\nu^{-1}, \varepsilon$ and N sufficiently small and large, respectively, one has $1/2 < n_1|_\Gamma < 2$.*

Recall that $\tau_0 < T < \pi^2$. Thus it is sufficient to prove that the function $|dn_1^2/dt_{N\varepsilon}|$ is small over Γ : $dn_1^2/dt_{N\varepsilon} = \varepsilon(n_2^2 - \nu n_3^2) \leq \varepsilon \rho^2 \ll 1$. The first inequality follows from $n_2 \leq n_3 \leq \rho$ in Q.

3.8. Reaching the sphere $n_3 = \rho$. Here the proof of the Lemma 1B is finished. Let Γ be the same arc as in subsection 3.7. Suppose that it does not reach the sphere $n_3 = \rho$. Then three possibilities arise (Figure 2.), each of them leading to a contradiction. In all three cases $\Gamma \subset Q$ and one of the assertion holds:

(A.) Γ reaches the bottom of the flow box Ω.

(B.) Γ reaches the side of Ω.

(C.) Γ belongs to Ω.

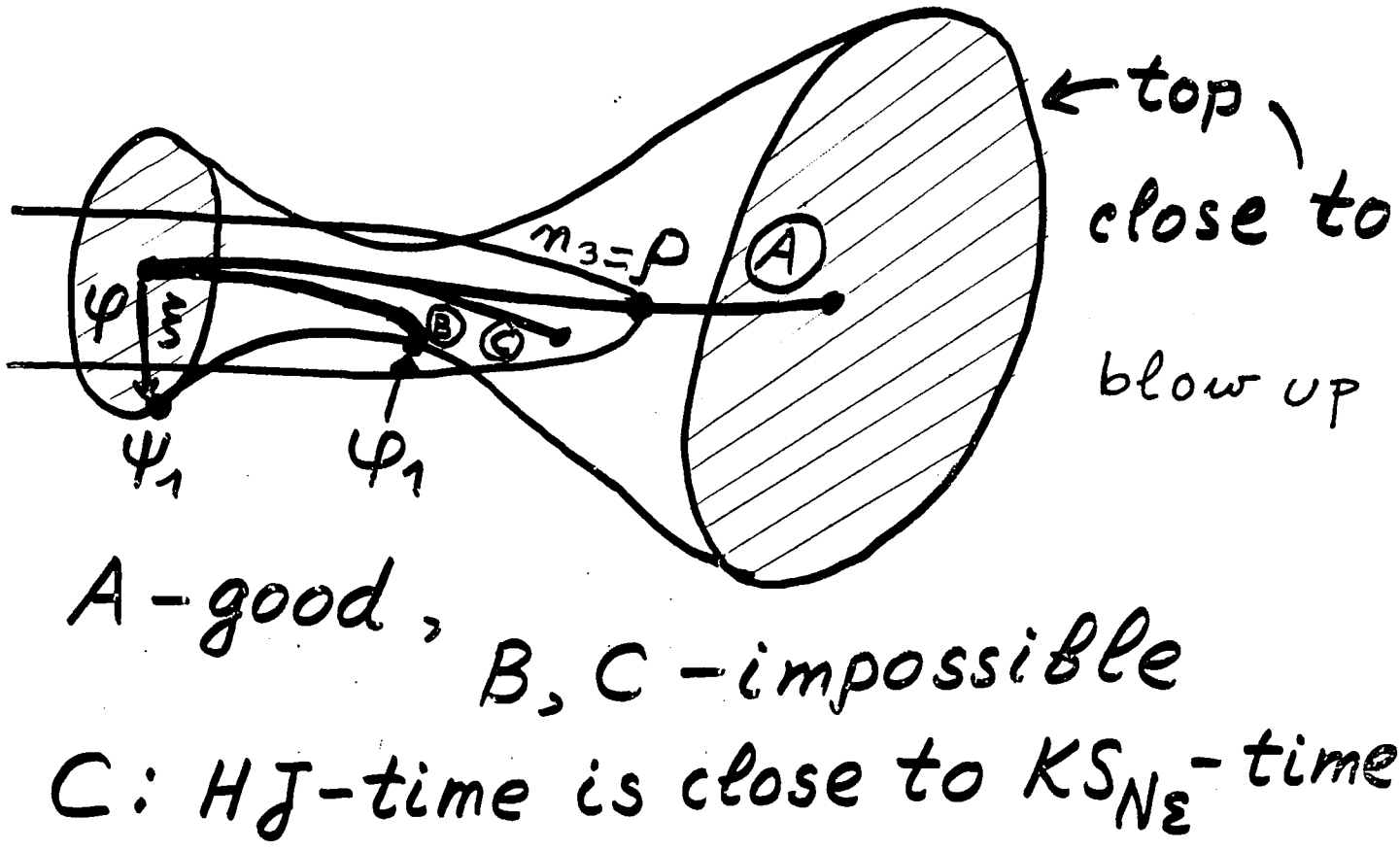

Figure 2.

The impossibility of the case A has just been proved: the ball Q has an empty intersection with the top of Ω, according to Proposition 5.

The impossibility of cases B and C will be deduced from Lemma 3. We check the conditions of this Lemma for the fields v and w, the flow box Ω, the arc Γ and the space $X = \mathring{C}^\infty$ with the H_{-1} norm.

Condition 1 is a consequence of Propositions 2 and 6:

$$||w|_\Gamma||_{-1} \leq \alpha \text{ for } \alpha = \rho(2C/N + \nu\varepsilon) + 2\varepsilon \lesssim \varepsilon\nu\rho.$$

The last inequality is valid for N sufficiently large and $\rho > 2\nu^{-1}$. Condition 2 is a consequence of Proposition 6 and Property 4 of the (HJ) equation:

$$\max ||dg^t_{HJ}(u)||_{-1} \stackrel{\text{def}}{=} F; F < \rho^{3/2} \text{ for } ||u||_3 \leq \rho, ||u||_1 \geq 1/2, u \in \Gamma$$

Conditions 3 and 4 follow from Proposition 3. The value of the constant in the condition 4 may be chosen to be $C\nu^{1.2}$.

The smoothness assumption of Lemma 3 is evidently fulfilled. In fact, the initial condition $u(\tau) = g^\tau_{v+w}\varphi, u(\tau) \in \Gamma$ depends smoothly on τ; the solution of the (HJ)-equation with the initial condition $u(\tau)$ also depends smoothly on τ.

We now use the perturbation Lemma to prove the impossibility of cases B and C. Case C is impossible, because the arc Γ reaches the t-floor of Ω for a value of t close to $T - \varphi^+$. This follows from assertion 2 of Lemma 3. The difference $|T_\varphi^+ - t|$ is so small, and the time T_φ^+ is so close to the blow-up time of all the solutions in the box, that on the t-floor the inequality $n_3 > \rho$ holds. (see Proposition 5).

Case B is impossible because of the logarithmic convexity in s of the norm n_s with respect to s. More specifically, let Γ reach the side of the box Ω at the point φ_1, remaining in the ball $Q : n_3 \leq \rho$ (Figure 2). Let $\psi_1 = \Pi_v \varphi_1, \xi = \psi - \varphi$. (Figure 2). The following properties of ξ lead to contradiction:

$$1°) \quad ||\xi||_{-1} \leq 2\alpha FT \quad 2°) \quad ||\xi||_{C^2} = \sigma \quad 3°) \quad ||\xi||_3 \lesssim \rho\nu^{-2}.$$

We prove the first of these properties. Property 1° follows from assertion 1 of Lemma 3. Property 2° follows from the definition of the bottom of the box. Property 3° follows from property 3° of (HJ)-solutions; the detailed proof is as follows. In subsection 3.6 a lower estimate for the blow-up time in the past is given for the (HJ)-solutions beginning at the bottom of the box. Namely, there is a positive β such that for any $\psi \in \mathcal{D}$ one has $\beta\nu^{-4/5} \leq |T_\psi^-|$. Property 1° of the (HJ)-equation implies that $|T_\psi^-| \leq \pi^2$. Thus, by using the rightmost inequality in property 3° of (HJ)-equation, one obtains

$$||\psi_1||_3 \lesssim \rho\nu^{-2}.$$

Since $||\varphi||_3 \leq 2\nu^{-1}$, the vector $\xi = \psi_1 - \psi$ has property 3°. Thus all three properties of the vector ξ are proved. We will deduce a contradiction for ε sufficiently small.

The Sobolev embedding theorem implies $||\xi||_{C^2} \lesssim ||\xi||_{2.6}$. The logarithmic convexity of n_s gives then

$$\left(\frac{||\xi||_3}{||\xi||_{2.6}}\right)^{10} \geq \frac{||\xi||_3}{||\xi||_{-1}} \text{ or } ||\xi||_{-1} \geq ||\xi||_3^9 ||\xi||_{2.6}^{10} \geq C\rho^{-g}\nu^{18}\sigma^{16}$$

for sufficiently small C. Since α is arbitrary small (together with ε) the last inequality contradicts the property 1° of ξ. This ends the proof of Lemma 1B.

Note. The previous inequality implies that case B is impossible for $\varepsilon = C\rho^{-11.5}\nu^{17}\sigma^{10}$, if C is small. Here the following inequalities are used: $||\xi||_{-1} \leq 2\alpha FT$, $T = 0(1); \alpha \lesssim \varepsilon\nu\rho, F \lesssim \rho^{3/2}$. Let $\rho = C\nu^{-2}$ with C large enough. Then the assertion of Lemma 1B holds for $\varepsilon = c\nu^{72}$ with c small enough.

§4. Loss of energy.

In this section we prove Lemma 2 of Subsection 2.3.

4.1. An upper estimate for the energy. Let Γ be the same arc as in Subsection 3.8. We can estimate the maximum of the energy n_1^2 on Γ by a value depending on ε and ν. We have

$$\frac{dn_1^2}{dt_{KS_{N\varepsilon}}} = \varepsilon(n_2^2 - \nu n_3^2).$$

Let Γ' be the part of Γ which belongs to the cone of pumping. According to Subsection 3.7, $n_1|_\Gamma \leq 2$. Together with Proposition 4 of 3.6 this implies that $n_3|_{\Gamma'} \leq 2\nu^{-1}$. The logarithmic convexity of n_s gives $n_2^2|_{\Gamma'} \leq 4\nu^{-1}$. The time length corresponding to Γ' is less than or equal to the time length of Γ and does not exceed a universal constant. Everywhere on $\Gamma\backslash\Gamma'$ the derivative $dn_1^2/dt_{\Gamma_{n\varepsilon}}$ is negative since $n_1^2(\varphi) = 1$. Thus

$$\max_\Gamma \; n_1^2 \leq 1 + 4\varepsilon\nu^{-1}\tau.$$

Consequently

$$\max_\Gamma \; n_1 \leq 1 + 2\varepsilon\nu^{-1}\tau.$$

4.2 Dissipation of energy. The energy decreases in the domain $K_{sd}\backslash K_p$. By definition of the cone of rapid dissipation we have, everywhere in K_{rd}, the inequality

$$n_2^2 - \nu n_3^2 \leq -\nu n_3^2/2.$$

This implies that the energy n_1^2 decreases rapidly in K_{rd} along the $KS_{N\varepsilon}$-orbits. In this subsection we show that for $\rho \geq C\nu^{-2}$ and sufficiently large C the arc Γ stays in K_{rd} for such a long time that the energy will decrease substantially along this arc.

Denote by Γ'' a subset of Γ consisting of a finite number of arcs, ordered according to the time orientation on Γ and having the following property: n_3 takes the same value at the initial point of each arc as at the end point of the previous one; at the beginning of the first arc $n_3 = 4\nu^{-1}$; at the end of the last, $n_3 = C\nu^{-2} = \rho$ (Figure 3). This set of arcs exists because of the Exit Lemma; it is finite because the right hand side of $(KS_{N\varepsilon})$-equation is analytic. A simple calculation gives

$$\frac{dn_3^2}{dt_{KS_{N\varepsilon}}} = \frac{dn_3^2}{dt_{HJ}} + \varepsilon(n_4^2 - \nu n_5^2)$$

Because of the logarithmic convexity of n_3 with respect to s one has: $n_5/n_4 \geq n_3/n_2$. Thus $n_4^2 - \nu n_5^2 \leq 0$ everywhere in K_{rd} (and even in K_{sd}). We now estimate dn_3^2/dt_{HJ}.

$$\begin{aligned}\frac{1}{2}dn_3^2/dt_{HJ} &= -((Pu_x^2)_{x^3}, u_{x^3}) = (P(u_x^2), u_{x^3}) \\ &= (u_x^2, Pu_x^6) = (u_x^2, u_{x^6}) = -((u_x^2)_{x^3}, u_{x^3}).\end{aligned}$$

$$|dn_3^2/dt_{HJ}| \lesssim |\oint u_{x^2}u_{x^3}^2| + |\oint u_x u_{x^3} u_x^4|$$

But

$$\oint u_x u_{x^3} u_{x^4} = -\oint u_{x^2}u_{x^3}^2/2; |u_{x^2}| \lesssim n_3$$

Thus

$$|dn_3^2/dt_{HJ}| \lesssim |\oint u_{x^2}u_{x^3}^2| \lesssim n_3^3, |dn_3/dt_{HJ}| \lesssim n_3^2$$

4.3. Change of variables. We begin with the following general result:

PROPOSITION 1. *Let x, a smooth function of t, satisfy the inequality $0 \le dx/dt < f(x)$, where g is a nonnegative function on the interval*

$$t \in [a,b]; x(a) = A, x(b) = B.$$

Then

$$\int_a^b g(x(t)dt \ge \int_A^B \frac{g(x)}{f(x)}dx.$$

This proposition is a direct consequence of the change of variables formula.

4.4. Estimate for the energy dissipation. Let $\rho \ge C\nu^{-2}$. Suppose that Γ'' is a single arc (in the general case, Proposition 1 may be applied to each arc of the set Γ'' separately). Take $g = n_1|_{\Gamma''}, x = n_c|_{\Gamma''}$, $t = t_{KS_{N\varepsilon}}$ = the time corresponding to the $(KS_{N\varepsilon})$ equation. Let A and B be the values of t corresponding to the beginning and the end of the arc Γ''; thus $A = 4\nu^{-1}; B = C\nu^{-2}$. In subsection 4.2 we proved the inequality $dx/dt \lesssim x^2$. For sufficiently large C, Proposition 1 implies that

$$n_1(b) - n_1(a) = \int_a^b \frac{dn_1}{dt}dt \le -\frac{\varepsilon\nu}{2}\int_a^b n_3^2 dt \le -\int_A^B \frac{\varepsilon\nu}{2}dn_3 \le -\frac{\varepsilon C}{2}$$

Also, if C is large, the growth of the energy along Γ' is less than the loss of energy along Γ''. This proves Lemma 2.

§5. Properties of the Hamilton-Jacobi equation.

In this section we prove the assertions of Subsection 3.5 and Proposition 3 of 3.6.

5.1. Explicit formulas for solutions of (HJ). Consider the two initial problems:

$$u_t = -Pu_x^2, u|_{t=0} = \varphi, \bar{u} = 0;$$
$$w_t = -w_x^2, w|_{t=0} = \varphi.$$

Denote the restriction of the solution of the first problem to the circle $\{t\} \times S^1$ by $u^t = g^t\varphi$ and denote by w^t the corresponding restriction for the second problem. In

Subsection 1.1 the equality $u^t = Pw^t$ was stated. The solution w^t is found in the standard way [13, §8]:

$$w^t = v^t \circ t^{[-1]} \text{ where } v^t = \psi + t\varphi^2, \chi^t = id + 2\varphi' t.$$

The exponent in the composition is enclosed in square brackets in order to avoid confusion with an algebraic exponent; with this notation, $\chi^{[-1]}$ is the inverse of χ.

5.2. The blow-up time. The solutions $g^t\varphi$ are defined and smooth for all t for which χ^t is a diffeomorphism. This smoothness is lost for the values $t = T_\varphi^+ > 0$ and $t = T_\varphi^- < 0$, which can be obtained directly by solving the equation

$$\chi^{t'} \equiv 1 + 2\varphi'' t = 0.$$

Thus

$$T_\varphi^+ = -1/2 \min \varphi'', T_\varphi^- = -1/2 \max \varphi'', |T_\varphi^\pm| \geq 1/2||\varphi||_{c^2}.$$

If the value $\max \varphi''$ or $|\min \varphi''|$ is small, then the norm $||\varphi||_1$ is also small. This leads in a simple way to the inequality $|T_\varphi^\pm| \leq \pi^2/||\varphi||_1$. This proves Property 1° of 3.5.

5.3. Estimates for the H_3-norm along solutions of (HJ). We prove that for any $\varphi \not\equiv 0$ and $t \in (T_\varphi^+/2, T_\varphi^+)$ there exists a positive C such that

$$(*) \qquad ||g^t\varphi||_3^2 = \oint (\varphi^{(3)^2}/(1 + 2\varphi'' t)^5) d\xi$$

$$(**) \qquad CT_\varphi^+/(T_\varphi^+ - t)^3 \leq ||g^t\varphi||_3^2 \leq (T_\varphi^{+5}/(T_\varphi^+ - t)^5)||\varphi||_3^2$$

Similar inequalities hold for $t \in (T_\varphi^+/2, T_\varphi^-)$. The first one is valid also for $t \in (0, T_\varphi^+)$.

A calculation with some mysterious simplifications gives:

$$(g^t\varphi)_{x^3} \circ \chi^t = \varphi^{(3)}(\chi^{t'})^{-3}.$$

Formula $(*)$ can be derived from here by using the change of variable $\xi = \chi^t(x)$ in the integral $\oint [(g^t\varphi)_{x^3}]^2 dx$.

We next prove the rightmost inequality $(**)$ for $t \in (0, T_\varphi^+)$. Set $\psi = \chi^{t'} = 1 + 2\varphi'' t$. Then formula $(*)$ will take the form: $||g^t\varphi||_3^2 = \oint (\varphi^{(3)})^2/\psi^5$. For $t \in (0, T_\varphi^+)$ one has

$$(***) \qquad \min \psi = 1 + 2 \min \varphi'' t = 1 - t/T_\varphi^+ = (T_\varphi^+ - t)/T_\varphi^+.$$

Together with $(*)$, this implies the desired inequality.

Let us prove the leftmost inequality $(**)$ for $t \in (T_\varphi^+/2, T_\varphi^+)$. For the same ψ as above one has $\psi''' = \psi'/2t$. By $(*)$

$$||g_\varphi^t||_3^2 = (4t^2)^{-1} \oint (\psi^{-2}/\psi^5) = (9t^2)^{-1} \oint (\psi^{-3/2})^{-2} \geq$$

$$\overset{(1)}{\geq} (18\pi T_\varphi^{+2})^{-1} (\oint |(\psi^{-3/2})'| d\xi)^2 = C_0 (T_\varphi^+)^{-2} (\text{var} \psi^{-3/2})^2 \geq$$

$$\overset{(2)}{\geq} C_0 (T_\varphi^+)^{-2} [(T_\varphi^+ - t)^{-3/2} T_\varphi^{+3/2} - 1]^2, C_0 = (18\pi)^{-1}.$$

Here $\mathrm{var} f$ is the variation of the function f; inequality (1) is a consequence of the Cauchy inequality; (2) follows from the equality $\min \psi = (T_\varphi^+ - t)/T_\varphi^+$ (see $(***)$) and the fact that $\psi = 1$ whenever $\psi'' = 0$. Moreover, $t \in (T_\varphi^+/2, T_\varphi^+)$. Thus

$$[(T_\varphi^+ - t)^{-3/2}(T_\varphi^+)^{3/2} - 1]^2 \geq (T_\varphi^+)^3/8(T_\varphi^+ - t)^3.$$

This implies the desired inequality for sufficiently small c:

$$||g^t\varphi||_3^2 \geq cT_\varphi^+/(T_\varphi^+ - t)^3$$

5.4. The variation of the initial conditions. Here we prove property 4° from 3.5. Another calculation with mysterious simplifications gives the formulas

$$(dg^t)(\varphi)\psi \stackrel{\text{def}}{=} \frac{d}{d\varepsilon} g^t(\varphi + \varepsilon\psi)|_{\varepsilon=0} = \psi \circ \chi_\varphi^{t[-1]}.$$

To estimate the norm of this Fréchet derivative it is convenient to pass from the space $\overset{\circ}{H}_{-1}$ to $\overset{\circ}{H}_0$. To do this consider the operator I, defined on the space of centralized functions by $I : f \mapsto F, F' = f, \bar{F} = 0$. The equivalent definition of the n_{-1} norm is given by the equality $||f||_{-1} = ||If||_0$. Any bounded operator $A : \overset{\circ}{H}_{-1} \to \overset{\circ}{H}_{-1}$, defined on an everywhere dense subset $\overset{\circ}{H}_0 \subset \overset{\circ}{H}_{-1}$, generates an operator $B : \overset{\circ}{H}_0 \to \overset{\circ}{H}_0$ defined on an everywhere dense subset $\overset{\circ}{H}_1 \subset \overset{\circ}{H}_0$ and having the same norm: $||A||_{-1} = ||B||_0$. In fact $B = I \circ A \circ I^{[-1]}$. The coincidence of the norms is obvious, because I preserves the length of each vector.

In the case under consideration

$$A\psi = \psi \circ \chi^{t[-1]}, Bf = I \circ (f' \circ \chi^{t[-1]}).$$

Let us find an upper estimate for the norm $||B||_0$. Let

$$f \in \overset{\circ}{H}_1 \not\equiv 0, Bf(x_0) = 0$$

$$g(x) = \int_{x_0}^{x} f' \circ \chi^{t[-1]} d\xi, x \in [x_0, x_0 + 2\pi], g = Bf + a(x - x_0).$$

Here a is the average of $f' \circ \chi^{t[-1]}$. Then $Bf = g - a(x - x_0)$.

We next prove the inequalities

$$\binom{*}{*} \qquad ||g||_0 \lesssim ||f||_0(||\varphi||_3 + ||\varphi||_{c^2}^{3/2})||\varphi||_1^{-1}, |a| \lesssim ||f||_0||\varphi||_3||\varphi||_1^{-1}$$

Together with the Sobolev imbedding theorem for C^2 this implies property 4°. To see this, let $\chi^t = \chi$. Then

$$g(x) = \int_{x0}^{x} f' \circ \chi^{[-1]} = \int_{x_0}^{x} (f \circ \chi^{[-1]})' \chi' \circ \chi^{[-1]} = f \circ \chi^{[-1]}(x) \chi' \chi^{[-1]}(x)$$
$$- \int_{x_0}^{x} f \circ \chi^{[-1]} \cdot \chi'' \circ \chi^{[-1]} \cdot (\chi^{[-1]})' = (f\chi') \circ \chi^{[-1]}(x) - \int_{\chi^{[-1]}(x_0)}^{\chi^{[-1]}(x)} f\chi''.$$

We now need the following general inequality: $||h \circ \chi^{[-1]}||_0 \leq ||h||_0 \sqrt{\max |\chi'|}$. In fact,

$$||h \circ \chi^{[-1]}||_0^2 = \oint h^2 \circ \chi^{[-1]} = \oint \frac{h^2 \circ \chi^{[-1]} (\chi^{[-1]})'}{(\chi^{[-1]})'} \leq ||h||_0^2 \max |\chi'|.$$

Thus

$$||f \circ \chi^{[-1]} \cdot \chi' \circ \chi^{[-1]}||_0 \leq ||f \circ \chi^{[-1]}||_0 ||\chi||_{C^1} \leq$$
$$\leq ||f||_0 ||\chi||_{C^1}^{3/2} \lesssim ||f||_0 ||\varphi||_{C^2}^{3/2} ||\varphi||_1^{-3/2}$$

The last inequality uses the expression for χ and the lower estimate for $|T_\varphi^{\pm}|$ of 5.1. Moreover, for any $\alpha, \beta \in S^1$,

$$|\int_{\alpha}^{\beta} f\chi''| \leq ||f||_0 ||\chi''||_0 \leq 2t ||f||_0 ||\varphi||_3 \lesssim ||f||_0 ||\varphi||_3 ||\varphi||_1^{-1}.$$

This gives the first of the inequalities $\binom{*}{*}$; it remains to prove the second. Using the 2π-periodicity of f and the definition of a, one obtains:

$$a = g(x_0 + 2\pi)/2\pi = \oint f\chi''/2\pi,$$
$$|a| \lesssim ||f||_0 ||\varphi||_3 ||\varphi||_1^{-1}.$$

5.5. Proof of Proposition 3 of 3.6. Let us recall the formulation.

PROPOSITION 3 OF 3.6. *For some positive C_1, C_2 and any φ, ψ satisfying the conditions*

$$||\varphi||_1 = 1, \varphi \in K_{sd}, ||\varphi - \psi||_{C^2} \leq \sigma, \sigma \leq C_1 \nu^{1.2}$$

the following inequalities hold

$$||P(\psi_x^2)||_{-1} \geq C_2 \nu^{1,2}, |\angle(P(\varphi_x^2), P(\psi_x^2))| \leq \frac{\pi}{3}.$$

Note. The condition $\varphi \in K_{sd}$ implies that the high modes in the Fourier expansion of φ do not dominate the low ones in the sense of all norms n_s for $s < 3$ (see the Note below and the remarks following it).

PROPOSITION 1. *Let the function $f \in H_1(S^1)$ have a zero, and* $\max|f| \geq C, |f'| \leq C_1$. *Then a) if* $\bar{f} = 0$, *then*

$$\max|If| \geq C^2/2C_1 \quad b) \oint |f| \geq C^2/C_1 \quad c)||f||_0 \geq C^{3/2}/\sqrt{C_1}.$$

The proof of assertion (a) is illustrated in Figure 4. Since it has a zero, the function f may be considered as a 2π-periodic function over $\mathbb{R}$. . Thus the shadowed triangle in Figure 4 with area C^2/C_1 may be placed under the graph of the function f and its base may be placed between two neighboring zeroes of f, and consequently, on the period interval. Thus the oscillation on S^1 of any indefinite integral of f is not less than C^2/C_1. Consequently, $If \geq C^2/C_1$. This gives assertion (a). Assertion (b) is proved similarly. Let us prove (c). We have: $\max|f^2| \geq C^2, |(f^2)'| \leq 2CC_1$. Thus, by (b), $\oint f^2 \geq C^4/2CC_1 = C^3/2C_1$.

PROPOSITION 2. *Let* $\bar{f} = 0, ||f||_0 = 1, |f'| < C_1$. *Then* $||Pf^2||_{-1} \gtrsim C_1^{3/2}$.

Let us first estimate first $\max|IPf^2|$ from below by using Proposition 1A. Then $||IPf^2||_0 = ||Pf^2||_{-1}$, using Proposition 1c. Let $\max|f| = C$. Then

$$\max|Pf^2| \geq C^2/2, |(Pf^2)'| \leq 2CC_1.$$

By applying Proposition 1A to the function Pf^2, one has

$$\max|IPf^2| \geq C^3/16C_1.$$

By applying Proposition 1C to the function IPf^2, which satisfies the previous estimate and also $|(IPf^2)'| = |Pf^2| \leq 2C^2$, one obtains

$$||IPf^2||_0 = ||Pf^2||_{-1} \gtrsim C^{7/2}C_1^{-3/2}.$$

Finally, the inequality $||f||_0 = 1$ implies $C \geq 1/\sqrt{2\pi}$. Thus

$$||Pf^2||_{-1} \gtrsim C_1^{-3/2}.$$

Note. The logarithmic convexity in s of the norm n_s and the conditions $||\varphi||_1 = 1, ||\varphi||_3 \leq 2\nu^{-1}$ imply the lower estimate on $||\varphi_x||_{-1}$. Yet it says nothing about the norm $||P\varphi_x^2||_{-1}$. For example, if the function φ_x is 1 on one half of S^1, and -1 on the other, then $P\varphi_x^2 = 0$.

We now prove the first assertion of Proposition 3 of 3.6: $||P\varphi_x^2||_{-1} \gtrsim \nu^{1.2}$. Set $\varphi_x = f$. By assumption, $||f||_0 = 1$. By Proposition 4 of 3.6 $|f'| \lesssim \nu^{-4/5}$. Proposition 2 now implies

$$||Pf^2||_{-1} \gtrsim \nu^{1.2}.$$

To prove the second statement, we note that

$$P(\varphi_x^2 - \psi_x^2) = P[(\varphi_x - \psi_x)(\varphi_x + \psi_x)]$$

The function $\varphi_x - \psi_x$ is small in H_{-1}, and the function $\varphi_x + \psi_x$ is bounded; yet multiplication by a bounded function followed by the projection P is not a bounded operator in H_{-1}. Thus the following Proposition is natural

PROPOSITION 3. *Let* $f \in H_{-1}$, $g \in H_1$. *Then* $||P(fg)||_{-1} \leq ||f||_{-1}||g||_1$.

The integration by parts formula in the space of functions with zero average over S^1 takes the form

$$I(P(fg)) = I[((If)g)' - P((If)g')] = \\ = P((If)g) - IP((If)g').$$

We can estimate the first term as follows:

$$||P((If)g)||_0 \leq ||(If)g||_0 \leq ||If||_0 \max|g| \lesssim ||f||_{-1}||g||_1.$$

In order to estimate the second term, we use the simple inequality

$$|IPh| \leq 2 \oint |h|$$

$$||IP((If)g')||_0 \lesssim \oint |(If)g'| \leq ||If||_0||g'||_0 = ||f||_{-1}||g||.$$

Let us now prove the second assertion of Proposition 3 of 3.6. Everywhere in $\mathcal{D}$ one has $n_1 \leq 2$ if σ is sufficiently small. Consequently $n_0 \leq 2$. Moreover, $||\varphi_x + \psi_x||_{-1} = ||\varphi + \psi||_0 \leq 4$. Thus

$$||\varphi_x - \psi_x||_1 = ||\varphi - \psi||_2 \lesssim ||\varphi - \psi||_{C^2} \leq \sigma \lesssim \nu^{1.2}.$$

Proposition 3 now implies that for any $\varphi, \psi \in \mathcal{D}$

$$||P(\varphi_x^2 - \psi_x^2)||_{-1} \lesssim \sigma \lesssim \nu^{1.2}.$$

Finally, if the difference of the two vectors is small compared to the length of each of them, then the modulus of the angle between them is less than $\pi/3$.

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