Global and non-global solutions of a fractional reaction-diffusion equation perturbed by a fractional noise

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Finite time blowup of the Fujita equation with fractional Laplacian perturbed by fractional Brownian motion

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Abstract

We provide conditions implying finite-time blowup of positive weak solutions to the SPDE $du(t,x) = \left[\Delta_\alpha u(t,x) + Ku(t,x) + u^{1+\beta}(t,x)\right] dt + \mu u(t,x)\, dB_t^H$, $u(0,x) = f(x)$, $x \in \mathbb{R}^d$, $t \geq 0$, where $\alpha \in (0,2]$, $K \in \mathbb{R}$, $\beta > 0$, $\mu \geq 0$ and $H \in [\frac{1}{2}, 1)$ are constants, Δ_α is the fractional power $-(-\Delta)^{\alpha/2}$ of the Laplacian, (B_t^H) is a fractional Brownian motion with Hurst parameter H, and $f \geq 0$ is a bounded measurable function. To achieve this we investigate the growth of exponential functionals of the form $\int_{r_0}^T \frac{e^{\beta(Ks+\mu B_s^H)}}{s^{d\beta/\alpha}}\, ds$ as $T \to \infty$ with $r_0 > 0$.

1 Introduction

In this paper we find conditions under which nontrivial positive weak solutions to stochastic partial differential equations of the archetype

$$\begin{aligned} du(t,x) &= \left[\Delta_\alpha u(t,x) + Ku(t,x) + u^{1+\beta}(t,x)\right] dt + \mu u(t,x)\, dB_t^H, \qquad (1) \\ u(0,x) &= f(x),\ x \in \mathbb{R}^d, \end{aligned}$$

exhibit finite-time blowup, where Δ_α is the fractional power $-(-\Delta)^{\alpha/2}$ of the Laplacian, $\alpha \in (0,2]$, $K \in \mathbb{R}$, $\beta > 0$, $\mu \geq 0$ and $H \in [\frac{1}{2}, 1)$ are constants, the initial condition $f \geq 0$ is bounded and measurable, and $B^H \equiv \{B_t^H, t \geq 0\}$ is a one-dimensional fractional Brownian

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motion with Hurst parameter H (Brownian motion being included by $H = 1/2$). For $\alpha = 2$ the operator Δ_α is the usual Laplacian, and is the infinitesimal generator of d-dimensional Brownian motion with variance parameter 2. For $0 < \alpha < 2$,

$$\Delta_\alpha u(x) = \mathcal{A}(d,\alpha) PV \int_{\mathbb{R}^d} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}}\, dy, \quad u \in C_b^2(\mathbb{R}^d),$$

where $\mathcal{A}(d,\alpha)$ is a constant, and in this case Δ_α is the generator of the symmetric α-stable Lévy process $\{Z_t, t \geq 0\}$ on $\mathbb{R}^d$, which is characterized by $\mathbb{E}\left[\exp\left(\mathrm{i} u \cdot Z_t\right) | Z_0 = 0\right] = \exp\left(-t|u|^\alpha\right)$, $u \in \mathbb{R}^d$, $t \geq 0$; see e.g. [2]. Our motivation to consider the process $\{Z_t,\ t \geq 0\}$ and its generator Δ_α is that their properties are in a sense typical for a wide class of Lévy processes in $\mathbb{R}^d$. Due to the fact that the function $z \mapsto z^{1+\beta}$ is not Lipschitz when $\beta > 0$, the weak positive nontrivial solutions of (1) may eventually exhibit blowup in a finite time horizon T. In our previous work [6] (see also [3]), which addresses a special case of (1) on a bounded smooth domain $D \subset \mathbb{R}^d$ with $\alpha = 2$, and Dirichlet boundary conditions, we have proved that the probability $\mathbb{P}\{T = +\infty\}$ of non explosion in finite time of (1) can be estimated by the integral $\int_0^{\beta^{-1}\langle u(0),\psi\rangle^{-\beta}} h(y)\, dy$, where $\langle u(0), \psi\rangle = \int_D u(0,x)\psi(x)\, dx$, h is the density function of an inverse gamma distribution and ψ is the normalized eigenfunction corresponding to the smallest eigenvalue of Δ on D. A basic assumption in the articles quoted above is that the open domains on which the equations are defined are bounded and have a smooth boundary. This crucial fact enables to apply the eigenfunction method to produce sub- and super-solutions for such equations, which are useful to obtain bounds for the probability of explosion in finite time of their solutions, as well as for the explosion times.

In this paper we aim at obtaining similar asymptotic behaviors for equations of the form (1) which are defined on the entire Euclidean space $\mathbb{R}^d$. For such equations the eigenfunction method is not available, and a new approach to produce sub- and super-solutions of (1) is required. In contrast to the case investigated in our previous work, where the diffusion generated by Δ gets killed when it reaches the boundary of D, in the case we study here the motion with generator Δ_α propagates in the entire space, smearing out mass in the whole of $\mathbb{R}^d$ at a rate determined by the parameter α: the smaller the index α, the quicker the rate at which mass dissipates over the space. In this case it is to be expected a complex interaction between the reaction term represented by $\beta > 0$, the mobility of the diffusion determined by the index $\alpha \in (0, 2]$ and the spatial dimension d, and the random multiplicative perturbation represented by μB_t^H. As a matter of fact, we will show that the explosion regimes of Equation (1) are determined by the growth of the random functional

$$F_t = \int^t \frac{e^{\beta(Ks+\mu B_s^H)}}{s^{d\beta/\alpha}}\,ds \quad \text{as} \quad t\to\infty. \tag{2}$$

To our knowledge the asymptotics (2), which will be very useful to determine conditions for finite-time explosion of Eq. (1), are not know even for the case $H = 1/2$. Therefore, under the assumption $1/2 \le H < 1$ we prove in Section 2 that functionals of the form $\int_0^\infty e^{B_s^H - \nu s} ds$ are a.s. finite or infinite according to $\nu > 0$ or $\nu < 0$ respectively. With these tools we show that a nontrivial weak solution of (1) with initial value $f \ge 0$ blows up on the event $[\theta < \infty]$, with

$$\theta = \inf\left\{ t > r_0 : \int_{r_0}^t \frac{e^{\beta(Ks+\mu B_s^H)}}{s^{\beta d/\alpha}}\,ds \ge \frac{2^{d(2\beta+(1+\beta)/\alpha)}}{\beta p^\beta(1,0)\mathbb{E}^\beta\left[f\left(Z\left(2^{-\alpha}r_0\right)\right)\right]} \right\}, \tag{3}$$

(where r_0 and $p(1,0)$ are positive constants defined at the beginning of Section 3). Moreover, we obtain bounds for the explosion times of (1) in the cases $K < 0$ and $K > 0$. We remark that our present approach cannot cover the case of $K = 0$ and $1/2 < H < 1$. Nonetheless, for $H = 1/2$ we are able to deal with any value of $K \in \mathbb{R}$. In this case we find lower estimates for F_t which diverge to ∞ as $t \to \infty$; see Section 4 for details. In Section 5 we provide sufficient conditions for the existence of a global solution of (1), and we estimate the probability that the solution does not blow up in finite time. The behaviour of F_t as $t \to \infty$ is again studied for the case $H = 1/2$, and a qualitative behaviour of the solution that is different from the behavior in the deterministic case is found.

In what follows we will work with mild solutions, rather than with weak solutions. The equivalence of these two notions of solution is discussed in the next section, where other preliminary results are given. Our results on finite-time explosion of (1) are presented in sections 3 and 4, those on the existence of a global solution in Section 5.

2 Preliminaries

The following theorem is well known in the case $H = 1/2$; see e.g. [4, 5, 12] or [17].

Lemma 1 *Let $1/2 \le H < 1$, and let $B^H \equiv \{B_r^H,\ r \ge 0\}$ be a fractional Brownian motion with Hurst parameter H defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If $\nu > 0$, then $\mathbb{P}\left(\int_0^\infty e^{B_s^H - \nu s} ds < \infty\right) = 1$. If $\nu < 0$, then $\mathbb{P}\left(\int_0^\infty e^{B_s^H - \nu s} ds = \infty\right) = 1$.*

Proof: The assertion is a rather direct consequence of the law of the iterated logarithm for fractional Brownian motion (see Orey, [9, Thm. 1.1]). It states that

$$\limsup_{t\to\infty} \frac{B_t^H}{t^H\sqrt{2\log\log t}} = 1 \text{ a.s.}$$

This means that for some $t_0 = t_0(\omega) > e$ and all $t \geq t_0$, $B_t^H \leq 2t^H\sqrt{2\log\log t}$. Assume $\nu > 0$. Since $H < 1$ there exists $t_1 = t_1(\omega) > t_0(\omega)$ such that for each $t \geq t_1$, $2t^H\sqrt{2\log\log t} < \nu t/2$. Therefore, for all $t \geq t_1$,

$$\int_t^\infty e^{B_s^H - \nu s} ds < \int_t^\infty e^{-\frac{\nu s}{2}} ds < \infty \text{ a.s,}$$

and since B^H has continuous paths it follows that $\int_0^\infty e^{H_s^H - \nu s} ds < \infty$ a.s.

In the case of $\nu < 0$, using the fact that $B_t^H = -B_t^H$ in distribution, it follows that $B_t^H \geq -2t^H\sqrt{2\log\log t}$ for $t > t_0$ a.s. Taking $t_1 > t_0$ such that $-\frac{\nu t}{2} > 2t^H\sqrt{2\log\log t}$ for all $t > t_1$, we obtain that a.s.

$$\int_0^\infty e^{B_t^H - \nu t} dt > \int_{t_1}^\infty e^{B_t^H - \nu t} dt > \int_{t_1}^\infty e^{-2t^H\sqrt{2\log\log t} - \nu t} dt > \int_{t_1}^\infty e^{-\frac{\nu t}{2}} dt = \infty.$$

This finishes the proof. ■

In the remaining part of this section we assume that $\frac{1}{2} < H < 1$.

By means of the transformation $v(t,x) = e^{-\mu B_t^H} u(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$, a weak solution u of (1) yields a weak solution v of the random PDE

$$\frac{\partial v(t,x)}{\partial t} = (\Delta_\alpha + K)\, v(t,x) + e^{\mu\beta B_t^H} v^{1+\beta}(t,x), \quad v(0,x) = f(x); \tag{4}$$

see [6]. Notice that $v(t,\cdot)$ is non-negative for each $t \geq 0$, which follows e.g. from the Feynman-Kac representation of semi-linear equations of the kind of (4), and from the positivity of f; see [1]. Hence $u(t,\cdot) = \exp\{\mu B_t^H\} v(t,\cdot)$ is also non-negative for each $t \geq 0$. Moreover, due to a.s. path continuity of B^H, it is clear that if τ is the blowup time of equation (1), then τ is also the blowup time of (4). Notice also that $\{\mathcal{T}(t) = e^{Kt}\mathcal{S}(t),\, t \geq 0\}$ is a semigroup of bounded linear operators having $\Delta_\alpha + K$ as its infinitesimal generator, where $\{\mathcal{S}(t),\, t \geq 0\}$ is the α-stable semigroup with generator Δ_α. We denote by $\{p(t,x),\, t > 0,\, x \in \mathbb{R}^d\}$ the family of spherically symmetric α-stable transition densities. Hence, for any $t \geq 0$ and all bounded measurable $f : \mathbb{R}^d \to \mathbb{R}$,

$$\mathcal{S}(t)f(x) = \mathbb{E}\left[f\left(Z_t\right)|Z_0 = x\right] = \int_{\mathbb{R}^d} p(t, y - x)f(y)\,dy, \quad x \in \mathbb{R}^d.$$

From classical results in semigroup theory, for any bounded measurable initial value $f \geq 0$ there exists a unique positive local mild solution v of the random PDE (4). Namely, there exists a number $0 < T_0 \leq \infty$ such that v satisfies the integral equation

$$v(t) = \mathcal{T}(t)f + \int_0^t \mathcal{T}(t-s)v^{1+\beta}(s)e^{\mu\beta B_s^H}\,ds \tag{5}$$

on $[0, t]$ for each $0 \leq t < T_0$, and $\|v(t)\|_\infty \uparrow \infty$ as $t \uparrow T_0$ when $T_0 < \infty$; see [10, Chapter 6]. Let $C_c^2(\mathbb{R}^d)$ be the space of continuous functions $\phi : \mathbb{R}^d \to \mathbb{R}$ with compact support and having two continuous derivatives. Recall that v is a weak solution of (4) provided that for any $\phi \in C_c^2(\mathbb{R}^d)$ there holds

$$\begin{aligned}\int_{\mathbb{R}^d} v(t,x)\phi(x)\,dx &= \int_{\mathbb{R}^d} v(0,x)\phi(x)\,dx + \int_0^t \int_{\mathbb{R}^d} \Big[v(s,x)\Delta_\alpha\phi(x) + Kv(s,x)\phi(x) \\ &\quad + e^{\mu\beta B_s^H} v^{1+\beta}(s,x)\phi(x)\Big]\,dx\,ds,\end{aligned}$$

whereas v is a mild solution of (4) if it satisfies (5) or, equivalently,

$$v(t,x) = \mathcal{S}(t)f(x) + \int_0^t \mathcal{S}(t-s)\left(Kv(s,\cdot) + e^{\mu\beta B_s^H}v^{1+\beta}(s,\cdot)\right)(x)\,ds. \tag{6}$$

Lemma 2 *v is a weak solution of (4) if and only if v is a mild solution of (4).*

Proof: Assume that v is a weak solution of (4). Let $h \in C^1(\mathbb{R})$, $\varphi \in C_b^2(\mathbb{R}^d)$, and let $g(s,x) = Kv(s,x) + \exp\{\mu\beta B_s^H\}v^{1+\beta}(s,x)$. Denote by $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\mathbb{R}^d)$. Using the integration by parts formula which is possible since $h \in C^1(\mathbb{R})$, it follows that

$$\begin{aligned}\langle h(t)\varphi(\cdot), v(t,\cdot)\rangle &= \langle h(0)\varphi(\cdot), v(0,\cdot)\rangle + \int_0^t \left\langle \frac{d}{ds}h(s)\varphi(\cdot), v(s,\cdot)\right\rangle ds \\ &\quad + \int_0^t \langle h(s)\Delta_\alpha\varphi(\cdot), v(s,\cdot)\rangle ds + \int_0^t \langle h(s)\varphi(\cdot), g(s,\cdot)\rangle\,ds.\end{aligned}$$

Using a density argument as in [11] (see also [13]) we get for all $\psi \in C_b^{1,2}(\mathbb{R} \times \mathbb{R}^d)$

$$\begin{aligned}\langle \psi(t,\cdot), v(t,\cdot) \rangle &= \langle \psi(0,\cdot), v(0,\cdot) \rangle + \int_0^t \left\langle \frac{d}{ds}\psi(s,\cdot) + \Delta_\alpha \psi(s,\cdot), v(s,\cdot) \right\rangle ds \\ &\quad + \int_0^t \langle \psi(s,\cdot), g(s,\cdot) \rangle \, ds. \end{aligned} \tag{7}$$

Let us now fix $t > 0$ and a smooth test function ϕ, and let ψ be given by

$$\psi(s,x) = \mathcal{S}(t-s)\phi(x) = \begin{cases} \langle p(t-s, \cdot - x), \phi(\cdot) \rangle & \text{if } s < t, \\ \phi(x) & \text{if } s = t. \end{cases}$$

It follows from the equality above and the smoothness of p and ϕ that ψ is a smooth function, hence (7) holds. It reads

$$\begin{aligned}\langle \phi(\cdot), v(t,\cdot) \rangle &= \langle \phi(\cdot), \mathcal{S}(t) f(\cdot) \rangle + \int_0^t \left\langle \frac{\partial}{\partial s}\psi(s,\cdot) + \Delta_\alpha \psi(s,\cdot), v(s,\cdot) \right\rangle ds \\ &\quad + \int_0^t \langle \mathcal{S}(t-s)\phi(\cdot), g(s,\cdot) \rangle ds \\ &= \langle \phi(\cdot), \mathcal{S}(t) f(\cdot) \rangle + \int_0^t \langle \phi(\cdot), \mathcal{S}(t-s) g(s,\cdot) \rangle ds \end{aligned}$$

since ψ satisfies Kolmogorov's forward equation and $\mathcal{S}$ is symmetric. Since v and $\mathcal{S}(t)v$ are locally integrable, and the equality above holds for any $\phi \in C_b^2(\mathbb{R}^d)$, it follows that v solves (6). Hence v is a mild solution of (4).

To prove the converse let v be a mild solution of (4), and let $\varphi \in C_b^2(\mathbb{R}^d)$. Then, by the

Fubini theorem,

$$
\begin{aligned}
&\int_0^t \langle \Delta_\alpha \varphi(\cdot), v(s,\cdot)\rangle \, ds \\
&= \int_0^t \langle \Delta_\alpha \varphi(\cdot), \mathcal{S}(s) f(\cdot)\rangle \, ds + \int_0^t \left\langle \Delta_\alpha \varphi(\cdot), \int_0^t \chi_{[0,s]}(r)\mathcal{S}(s-r) g(r,\cdot)\, dr \right\rangle ds \\
&= \int_0^t \langle \mathcal{S}(s)\Delta_\alpha \varphi(\cdot), f(\cdot)\rangle \, ds + \int_0^t \left\langle \int_r^t \mathcal{S}(s-r)\Delta_\alpha \varphi(\cdot)\, ds, g(r,\cdot) \right\rangle dr \\
&= \int_0^t \left\langle \frac{d}{ds}\mathcal{S}(s)\varphi(\cdot), f(\cdot)\right\rangle ds + \int_0^t \left\langle \int_r^t \frac{d}{ds}\mathcal{S}(s-r)\varphi(\cdot)\, ds, g(r,\cdot) \right\rangle dr \\
&= \langle \mathcal{S}(t)\varphi(\cdot) - \varphi(\cdot), f(\cdot)\rangle + \int_0^t \langle \mathcal{S}(t-r)\varphi(\cdot) - \varphi(\cdot), g(r,\cdot)\rangle \, dr \\
&= \langle \varphi(\cdot), \mathcal{S}(t) f(\cdot)\rangle - \langle \varphi(\cdot), f(\cdot)\rangle + \int_0^t \langle \varphi(\cdot), \mathcal{S}(t-r) g(r,\cdot)\rangle \, dr - \int_0^t \langle \varphi(\cdot), g(r,\cdot)\rangle \, dr.
\end{aligned}
$$

It follows that $\int_0^t \langle \Delta_\alpha \varphi(\cdot), v(s,\cdot)\rangle \, ds = \langle \varphi(\cdot), v(t,\cdot)\rangle - \langle \varphi(\cdot), f(\cdot)\rangle - \int_0^t \langle \varphi(\cdot), g(r,\cdot)\rangle \, dr$, or

$$
\langle \varphi(\cdot), v(t,\cdot)\rangle = \langle \varphi(\cdot), f(\cdot)\rangle + \int_0^t \langle \Delta_\alpha \varphi(\cdot), v(s,\cdot)\rangle \, ds + \int_0^t \langle \varphi(\cdot), g(r,\cdot)\rangle \, dr.
$$

■

3 Finite-time explosion of positive solutions

We start by recalling the following useful properties of $p(t,x)$; see e.g. [14, Section 2] or [7, Appendix].

Lemma 3 *For any $t > 0$ the function $p(t,\cdot)$ is continuous and strictly positive. Moreover, for any $t > 0$ and $x \in \mathbb{R}^d$,*

A). $p(ts,x) = t^{-d/\alpha} p(s, t^{-1/\alpha}x)$ for any $s > 0$.

B). $p(t,0) \geq p(t,x)$. More generally, $p(t,y) \geq p(t,x)$ if $\|x\| \geq \|y\|$.

C). $p(t,x) \geq \left(\frac{s}{t}\right)^{d/\alpha} p(s,x)$ for all $t \geq s$.

D). $p\left(t, \frac{1}{r}(x-y)\right) \geq p(t,x)p(t,y)$, $x, y \in \mathbb{R}^d$, provided that $p(t,0) \leq 1$ and $r \geq 2$.

Henceforth $r_0 > 0$ will denote a fixed constant such that $p(r_0, 0) = r_0^{-d/\alpha} p(1, 0) < 1$.

Lemma 4 *Let v be the mild solution of (4), and let*

$$m(t) = \mathbb{E}\left[v(t, Z_t)\right] = \int_{\mathbb{R}^d} p(t, y) v(t, y)\, dy = \mathcal{S}(t) v(t, 0),\ t \geq 0,$$

where $\{Z_t,\ t \geq 0\}$ is the spherically symmetric α-stable process in $\mathbb{R}^d$ starting from 0. If m explodes in a finite time τ_m, then v also blows up in a finite time τ_v. Moreover,

$$\tau_v \leq (r_0 + \tau_m)\left(1 + 2^\alpha\right). \tag{8}$$

Proof: Let $\tau_m < \infty$ be the explosion time of m, and let $t \geq \tau_m + 2^\alpha (r_0 + \tau_m)$. Then, for $s \in [0, \tau_m]$,

$$t - s \geq t - \tau_m \geq 2^\alpha (r_0 + \tau_m) \geq 2^\alpha (r_0 + s),$$

so that

$$\varrho := ((t - s)/(r_0 + s))^{1/\alpha} \geq 2. \tag{9}$$

Since $p(r_0, 0) < 1$ we get

$$p(r_0 + s, 0) = ((r_0 + s)/r_0)^{-d/\alpha} p(r_0, 0) < 1 \text{ for all } s \in [0, \tau_m]. \tag{10}$$

Therefore, for any $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
p(t - s, y - x) &= p\left(\frac{t - s}{r_0 + s}(r_0 + s), y - x\right) \\
&= \left(\frac{t - s}{r_0 + s}\right)^{-d/\alpha} p\left(r_0 + s, \left(\frac{t - s}{r_0 + s}\right)^{-1/\alpha} (y - x)\right) \\
&= \varrho^{-d} p\left(r_0 + s, \frac{y - x}{\varrho}\right) \\
&\geq \varrho^{-d} p(r_0 + s, x) p(r_0 + s, y),
\end{aligned} \tag{11}$$

where we have used (9), (10) and Lemma 3.D to obtain the last inequality. From (5) and (11) it follows that for each $t \geq \tau_m + 2^\alpha (r_0 + \tau_m)$ and all $x \in \mathbb{R}^d$,

$$\begin{aligned} v(r_0+t,x) &= e^{Kt}\int_{\mathbb{R}^d} p(t,x-y)v(r_0,y)\,dy \\ &\quad + \int_0^t \int_{\mathbb{R}^d} e^{K(t-s)}p(t-s,x-y)v^{1+\beta}(r_0+s,y)e^{\mu\beta B^H_{r_0+s}}ds\,dy \qquad (12)\\ &\geq \varrho^{-d}\int_0^{\tau_m}\int_{\mathbb{R}^d} e^{K(t-s)+\mu\beta B^H_{r_0+s}}p(r_0+s,x)p(r_0+s,y)v^{1+\beta}(r_0+s,y)\,dy\,ds \\ &\geq \varrho^{-d}\int_0^{\tau_m} e^{K(t-s)+\mu\beta B^H_{r_0+s}}p(r_0+s,x)m^{1+\beta}(r_0+s)\,ds, \end{aligned}$$

where we have used Jensen's inequality to obtain the last inequality. Since $r_0 > 0$, the RHS of the above inequality is infinite. Hence $\tau_v \leq r_0 + \tau_m + 2^\alpha (r_0 + \tau_m) = (r_0 + \tau_m)(1 + 2^\alpha)$. This finishes the proof. ■

Proposition 5 *Let $r \geq 0$ and let $w, g : [r, \infty) \to [0, \infty)$ be nonnegative continuous functions such that*

$$w(t) \geq \sigma + \int_r^t g(s)w^{1+\beta}(s)\,ds, \quad t \geq r,$$

where $\sigma > 0$ is a constant. If

$$\int_r^A g(s)\,ds > \frac{1}{\beta\sigma^\beta} \text{ for some } A > r, \qquad (13)$$

then w blows up in a finite time τ_w. An upper bound for τ_w is given by

$$\theta = \inf\left\{t > r : \int_r^t g(s)\,ds \geq \frac{1}{\beta\sigma^\beta}\right\}.$$

Proof: Arguing by comparison (see e.g. [15, Thm 1.3]) we obtain that $w \geq \xi$, where $\xi(t) = \sigma + \int_r^t g(s)\xi^{1+\beta}(s)\,ds$, $t \geq r$. Moreover, $\xi(t) = (\sigma^{-\beta} - \beta\int_r^t g(s)\,ds)^{-1/\beta}$, which blows up at θ. ■

The following theorem provides an upper bound for the blowup time of equation (1).

Theorem 6 *Let $\frac{1}{2} < H < 1$, and let u be a weak solution of (1) such that $u(0,\cdot) = f$, where f is a nonnegative bounded measurable function which does not almost everywhere vanish. Then*

u blows up in finite time on the event $[\theta < \infty]$, where

$$\theta = \inf\left\{ t > r_0 : \int_{r_0}^{t} \frac{e^{\beta\left(Ks+\mu B_s^H\right)}}{s^{\beta d/\alpha}} ds \geq \frac{2^{d(2\beta+(1+\beta)/\alpha)}}{\beta p^{\beta}(1,0)\mathbb{E}^{\beta}\left[f\left(Z\left(2^{-\alpha}r_0\right)\right)\right]} \right\}. \tag{14}$$

If τ denotes the blowup time of (1), then $\tau \leq (r_0+\theta)(1+2^{\alpha})$ on $[\theta < \infty]$.

Remark 7 If α is small, the dissipation in (1) gets more important, and it is well known that this in favor of the existence of global solution, or at least to postpone the time of its blowup. Formula (14) clearly reflects this fact since, when α decreases, the left side decreases (for $t >> 1$) and the numerator on the right side increases. On the other hand, for α fixed, a high dimension d is in favor of transience of the process and of dissipation. An increasing value of d has therefore the same influence on the qualitative behavior of the solution of (1) as a decreasing value of α, which is again confirmed by (14).

Proof: Due to the semigroup property (12) of v,

$$v(t+r_0) = \mathcal{T}(t)v(r_0) + \int_0^t \mathcal{T}(t-s)v^{1+\beta}(s+r_0)e^{\mu\beta B^H_{r_0+s}}\,ds, \quad t \geq 0,$$

hence

$$\begin{aligned} & m(t+r_0) \\ = \;& \mathcal{S}(t+r_0)v(t+r_0)(0) \\ = \;& \mathcal{S}(t+r_0)\left(e^{Kt}\mathcal{S}(t)v(r_0) + \int_0^t e^{K(t-s)}\mathcal{S}(t-s)v^{1+\beta}(r_0+s)e^{\mu\beta B^H_{r_0+s}}ds\right)(0) \\ = \;& e^{Kt}\mathcal{S}(2t+r_0)v(r_0,0) + \int_0^t e^{K(t-s)}\mathcal{S}(2t+r_0-s)v^{1+\beta}(r_0+s,0)e^{\mu\beta B^H_{r_0+s}}ds, \end{aligned} \tag{15}$$

where, due to Lemma 3.C and Jensen's inequality,

$$\begin{aligned} \mathcal{S}(2t+r_0-s)v^{1+\beta}(r_0+s,0) \;&\geq\; \left(\frac{r_0+s}{2t+r_0-s}\right)^{d/\alpha} \mathcal{S}(r_0+s)v^{1+\beta}(r_0+s,0) \\ &\geq\; \left(\frac{r_0+s}{2t+2r_0}\right)^{d/\alpha} \left(\mathcal{S}(r_0+s)v(r_0+s,0)\right)^{1+\beta} \\ &=\; \left(\frac{r_0+s}{2t+2r_0}\right)^{d/\alpha} m^{1+\beta}(r_0+s), \quad 0 \leq s \leq t. \end{aligned} \tag{16}$$

In order to estimate the first summand in the RHS of (15) notice that $v(r_0,y) \geq e^{Kr_0}\int_{\mathbb{R}^d} p(r_0, x-y)f(x)\,dx$ for all $y \in \mathbb{R}^d$, where due to $p(r_0,0)<1$ and Lemma 3,

$$
\begin{aligned}
e^{Kr_0}\int_{\mathbb{R}^d} p\left(r_0, \frac{2x-2y}{2}\right) f(x)\,dx &\geq e^{Kr_0}p(r_0,2y)\int_{\mathbb{R}^d} p(r_0,2x)f(x)\,dx \\
&= 2^{-2d}e^{Kr_0}p(2^{-\alpha}r_0,y)\int_{\mathbb{R}^d} p(2^{-\alpha}r_0,x)f(x)\,dx \\
&= 2^{-2d}e^{Kr_0}\mathbb{E}\left[f\left(Z\left(2^{-\alpha}r_0\right)\right)\right]p(2^{-\alpha}r_0,y) \\
&= c_1\cdot p(2^{-\alpha}r_0,y),
\end{aligned}
$$

with $c_1 = 2^{-2d}e^{Kr_0}\mathbb{E}\left[f\left(Z\left(2^{-\alpha}r_0\right)\right)\right]$. Therefore, by Lemma 3.A,

$$
\begin{aligned}
\mathcal{S}(2t+r_0)v(r_0,0) &= \int_{\mathbb{R}^d} p(2t+r_0,y)v(r_0,y)\,dy \geq c_1 p(2t+(1+2^{-\alpha})r_0,0) \\
&\geq c_1(2t+2r_0)^{-d/\alpha}p(1,0) = c_2(2t+2r_0)^{-d/\alpha}, \qquad (17)
\end{aligned}
$$

where

$$
c_2 := c_1 p(1,0) = p(1,0)2^{-2d}e^{Kr_0}\mathbb{E}\left[f\left(Z\left(2^{-\alpha}r_0\right)\right)\right]. \qquad (18)
$$

Plugging (17) and (16) into (15), and multiplying both sides of (15) by $e^{-K(r_0+t)}(2t+2r_0)^{d/\alpha}$, renders

$$
e^{-K(r_0+t)}(2t+2r_0)^{d/\alpha}m(t+r_0) \geq c_2e^{-Kr_0} + \int_0^t e^{-K(r_0+s)}(r_0+s)^{d/\alpha}m^{1+\beta}(r_0+s)e^{\mu\beta B^H_{r_0+s}}\,ds.
$$

Let

$$
w(r) = e^{-Kr}(2r)^{d/\alpha}m(r) \quad \text{and} \quad g(r) = 2^{-(1+\beta)d/\alpha}e^{K\beta r+\mu\beta B^H_r}r^{-\beta d/\alpha}, \quad r \geq 0. \qquad (19)
$$

We have shown that $w(r_0+t) \geq c_2e^{-Kr_0} + \int_0^t g(r_0+s)w^{1+\beta}(r_0+s)\,ds = c_2e^{-Kr_0} + \int_{r_0}^{r_0+t} g(r)w^{1+\beta}(r)\,dr$, or

$$
w(t) \geq c_2e^{-Kr_0} + \int_{r_0}^t g(r)w^{1+\beta}(r)\,dr, \quad t \geq r_0.
$$

Notice that m and w explode at the same (finite) time due to (19), and so do v and u. Moreover, from Proposition 5,

$$
\tau_m = \tau_w \leq \theta = \inf\left\{t > r_0 : \int_{r_0}^t \frac{e^{\beta\left(Ks+\mu B^H_s\right)}}{s^{\beta d/\alpha}}ds \geq \frac{2^{(1+\beta)d/\alpha}e^{Kr_0\beta}}{\beta c_2^\beta}\right\},
$$

with $\left(2^{(1+\beta)d/\alpha}e^{Kr_0\beta}\right)\left(\beta c_2^{\beta}\right)^{-1} = \left(2^{d(2\beta+(1+\beta)/\alpha)}\right)\left(\beta p^{\beta}(1,0)\mathbb{E}^{\beta}\left[f\left(Z\left(2^{-\alpha}r_0\right)\right)\right]\right)^{-1}$. Hence $\tau_m < \infty$ provided that $\theta < \infty$, and if τ is the blowup time of u, then from Lemma 4 it follows that

$$\tau \leq (r_0 + \tau_m)(1 + 2^{\alpha}) \leq (r_0 + \theta)(1 + 2^{\alpha}).$$

This finishes the proof. ■

In the following theorem we give conditions which imply finite-time blowup of positive solutions u of (1). We also give upper bounds for the probability of non explosion of u before a given time $t > 0$ for sufficiently large t.

Theorem 8 *Let $\frac{1}{2} < H < 1$, and let τ be the blowup time of equation (1) with initial value $u(0) = f \geq 0$. If $K > 0$ then any positive nontrivial solution of (1) is almost surely local, i.e. $\mathbb{P}[\tau < \infty] = 1$ for any bounded measurable $f \geq 0$ which does not almost everywhere vanish. If $K < 0$ there exists $\eta > r_0$ such that for all $t > 5(\eta + r_0)$, the probability of non explosion of Eq. (1) before t is upper bounded by $\mathbb{P}\left[\int_{\eta}^{\frac{t}{1+2^{\alpha}}-r_0} e^{\beta\left(2Ks+\mu B_s^H\right)}ds < \frac{2^{d(2\beta+(1+\beta)/\alpha)}}{\beta p^{\beta}(1,0)\mathbb{E}^{\beta}[f(Z(2^{-\alpha}r_0))]}\right]$.*

Remark 9 In most situations decreasing the initial condition f postpones or even prevents blowup of the solution. This holds true also for the solution of (1) as shows the formula above for the upper bound of the probability for non explosion of the solution before t. The case $K = 0$ will be treated in the next sections.

Proof: Let us assume that $K > 0$. We can choose $t_1 > r_0$ such that for all $s > t_1$, $s^{-\beta d/\alpha}\exp\{K\beta s\} > \exp\{K\beta s/2\}$. Therefore,

$$\int_{r_0}^{t} e^{\beta\left(Ks+\mu B_s^H\right)} s^{-\beta d/\alpha}\, ds > \int_{t_1}^{t} e^{\frac{Ks\beta}{2}+\mu\beta B_s^H}\, ds \to \infty \text{ as } t \to \infty \text{ a.s.},$$

where we have used Lemma 1 of Section 2. Hence $\mathbb{P}[\tau < \infty] = 1$ due to Theorem 6 because $\theta < \infty$ a.s. under our assumptions. Hence, for any nontrivial, positive initial condition, Equation (4) exhibits finite-time blowup almost surely. Since the functions $t \mapsto e^{\mu B_t^H}$ are bounded on bounded intervals, the assertion follows from the equality $u(t,x) = e^{\mu B_t^H} v(t,x)$. In case of $K < 0$ we choose $\eta > r_0$ such that $s^{-\beta d/\alpha}\exp\{K\beta s\} > \exp\{2K\beta s\}$ for all $s > \eta$.

Therefore, due to Theorem 6 we get that for any $t \geq 5(\eta + r_0)$,

$$\begin{aligned}\mathbb{P}[\tau \geq t] \leq \mathbb{P}\left[\theta > \frac{t}{1+2^\alpha} - r_0\right] &= \mathbb{P}\left[\int_{r_0}^{\frac{t}{1+2^\alpha}-r_0} e^{\beta\left(Ks+\mu B_s^H\right)} s^{-\beta d/\alpha}\, ds < \frac{2^{(1+\beta)d/\alpha} e^{Kr_0\beta}}{\beta c_2^\beta}\right] \\ &\leq \mathbb{P}\left[\int_{\eta}^{\frac{t}{1+2^\alpha}-r_0} e^{\beta\left(2Ks+\mu B_s^H\right)}\, ds < \frac{2^{(1+\beta)d/\alpha} e^{Kr_0\beta}}{\beta c_2^\beta}\right], \quad (20)\end{aligned}$$

where c_2 is given by (18). ■

Remark 10 **(a)** Notice that the estimate (20) is informative only for those $t > 5(\eta + r_0)$ for which the event $[\tau \geq t]$ has positive probability.

(b) The upper bound for the probability of non explosion before t in Theorem 8 is in terms of an exponential functional of fractional Brownian motion with drift. At our best knowledge the exact values of these probabilities are known only for the case $H = 1/2$, i.e. for Brownian motion. We consider therefore this case in more detail in the next section.

4 The case $H = 1/2$

The transformation $v(t) = e^{-\mu B_t^{1/2}} u(t)$ leading to equation (4) brings, in the case of $H = 1/2$, an additional second order term coming from Itô's formula, and the associated random PDE therefore reads (we write $W(t)$ instead of $B_t^{1/2}$)

$$\frac{\partial v(t,x)}{\partial t} = \left(\Delta_\alpha + K - \frac{\mu^2}{2}\right) v(t,x) + e^{\mu\beta W(t)} v^{1+\beta}(t,x), \quad v(0,x) = f(x); \qquad (21)$$

see [3]. Let $\mathcal{K} = K - \frac{\mu^2}{2}$. Notice that our analysis in the previous sections remains valid for the random PDE (21). It suffices to replace the constant K by $\mathcal{K}$ in the appropriate formulas. For $\mathcal{K} \neq 0$ our results are similar to the case of $H > 1/2$. However, when $H = 1/2$ we obtain a finite-time explosion result if $\mathcal{K} = 0$, as the following theorem shows.

Theorem 11 *Let $\{W(t),\ t \geq 0\}$ be a one-dimensional standard Brownian motion, and let u be a weak solution of (1) such that $u(0,\cdot) = f$, where f is a nonnegative bounded measurable function which does not almost everywhere vanish. Then u blows up in finite time on the event $[\theta < \infty]$, where*

$$\theta = \inf\left\{t > r_0 : \int_{r_0}^{t} \frac{e^{\beta(\mathcal{K}s+\mu W(s))}}{s^{\beta d/\alpha}} ds \geq \frac{2^{d(2\beta+(1+\beta)/\alpha)}}{\beta p^\beta(1,0)\mathbb{E}^\beta\left[f\left(Z\left(2^{-\alpha} r_0\right)\right)\right]}\right\}. \qquad (22)$$

If τ denotes the blowup time of (1), then $\tau \leq (r_0 + \theta)(1 + 2^\alpha)$ on $[\theta < \infty]$. Moreover:

1. *If $\mathcal{K} > 0$ then $\mathbb{P}[\tau < \infty] = 1$ and therefore u exhibits blowup in finite time a.s.*

2. *If $\mathcal{K} < 0$ there exists $\eta > r_0$ such that for all $t > 5(\eta + r_0)$,*

$$\mathbb{P}[\tau > t] \leq \mathbb{P}\left[\int_\eta^{\frac{t}{1+2^\alpha} - r_0} e^{\beta(2\mathcal{K}s + \mu W(s))} ds < \frac{2^{d(2\beta + (1+\beta)/\alpha)}}{\beta p^\beta(1,0)\mathbb{E}^\beta\left[f\left(Z\left(2^{-\alpha} r_0\right)\right)\right]}\right].$$

   In particular,

$$\mathbb{P}[\tau = \infty] \leq \mathbb{P}\left[\int_\eta^{\infty} e^{\beta(2\mathcal{K}s + \mu W(s))} ds < \frac{2^{d(2\beta + (1+\beta)/\alpha)}}{\beta p^\beta(1,0)\mathbb{E}^\beta\left[f\left(Z\left(2^{-\alpha} r_0\right)\right)\right]}\right], \qquad (23)$$

   i.e. the probability that u is a nontrivial global solution is upper bounded by the RHS of (23).

3. *Assume $\mathcal{K} = 0$. In dimension $d \leq \alpha/\beta$ any non-trivial positive solution of (24) exhibits finite-time blowup.*

Remark 12 We refer to the Dufresne and Yor formulas ([17] for $t < \infty$, [4] for $t = \infty$) for the explicit calculation of the upper bounds of $\mathbb{P}[\tau > t]$ and of $\mathbb{P}[\tau = \infty]$ in the case $\mathcal{K} < 0$.

We will prove only part 3 of Theorem 11; the proof of parts 1 and 2 uses the same arguments as in the previous section and will be omitted. Henceforth we will assume that $\mathcal{K} = 0$, and therefore consider the random PDE

$$\frac{\partial v(t,x)}{\partial t} = \Delta_\alpha v(t,x) + e^{\mu\beta W(t)} v^{1+\beta}(t,x), \quad v(0,x) = f(x). \qquad (24)$$

In this setting the integral functional figuring in (22) reduces to

$$I_0(T) = \int_{r_0}^{T} \frac{e^{\mu\beta W(s)}}{s^{d\beta/\alpha}}\, ds, \quad 0 < r_0 \leq T. \qquad (25)$$

We need the following two lemmas for the proof of part 3 of Theorem 11.

Lemma 13 *Let $W \equiv \{W(t), 0 \leq t \leq T\}$ be a standard Brownian motion defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}\right)$. The process*

$$X \equiv \left\{X(t) = e^{-t/2} W(e^t - 1), \quad 0 \leq t \leq T\right\} \qquad (26)$$

is a Brownian motion which is equivalent to W, i.e. there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ having the same null sets as $\mathbb{P}$ and such that $(X(t), \mathcal{M}_t, \mathbb{Q})$ is a Wiener process, where $\mathcal{M}_t = \sigma\left\{\sigma\{X(s),\ s \le t\} \cup \mathcal{N}\right\}$, $0 \le t \le T$, with $\mathcal{N} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$.

Proof: For $s \ge u$ let $h(s,u) = \frac{1}{2}e^{(u-s)/2}$. Then

$$Y \equiv \left\{ Y(t) = \int_0^t 2h(t,u)\, dW(u), \quad 0 \le t \le T \right\} \tag{27}$$

is a centered Gaussian process whose covariance function $\mathbb{E}\left[Y(t)Y(s)\right] = e^{-\frac{t}{2}+\frac{s}{2}} - e^{-\frac{t}{2}-\frac{s}{2}}$, $s \le t$, is the same as the covariance function of X, and therefore X and Y are equivalent. We are going to show that Y and W are equivalent as well. In fact a simple calculation shows that for $0 \le t \le T$,

$$Y(t) = W(t) - \int_0^t \left(\int_0^s h(s,u)\, dW(u) \right) ds. \tag{28}$$

For $0 \le t \le T$ let

$$M(t) = \exp\left\{ \int_0^t \int_0^s h(s,u)\, dW(u)\, dW(s) - \frac{1}{2}\int_0^t \left(\int_0^s h(s,u)\, dW(u) \right)^2 ds \right\}. \tag{29}$$

If we show that $\{M(t),\ 0 \le t \le T\}$ is a martingale, then, by Girsanov's theorem applied to (28), it will follow that Y and W are equivalent under the measure $\mathbb{Q}$ determined by $d\mathbb{Q} = M(t)d\mathbb{P}$. To show that (29) is a martingale we use Itô's formula and get that $dM(t) = M(t)(\int_0^t h(t,u)\, dW(u))dW(t)$. It follows that $\{M(t), 0 \le t \le T\}$ is a local martingale, hence there exists an increasing sequence of stopping times $\{T_n\}$ such that $T_n \uparrow T$ and $\{M^n(t) := M(t \wedge T_n), 0 \le t \le T\}$ is a $\mathbb{P}$-martingale for each $n \ge 1$. Let

$$Y^n(t) = W(t) - \int_0^{t \wedge T_n} \left(\int_0^s h(s,u)\, dW(u) \right) ds, \quad n \ge 1.$$

Due to Girsanov's theorem, $\{Y_t^n,\ 0 \le t \le T\}$ is a Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{Q}_T^n)$, where $d\mathbb{Q}_T^n(\omega) = M^n(T)(\omega)d\mathbb{P}(\omega)$. Let us denote by $\mathbb{E}_{\mathbb{Q}}^n$ the expectation with respect to $\mathbb{Q}_T^n$.

We get that

$$\begin{aligned}
\ln M^n(t) &= \int_0^{t\wedge T_n}\int_0^s h(s,u)\,dW(u)\,dW(s) - \frac{1}{2}\int_0^{t\wedge T_n}\left(\int_0^s h(s,u)\,dW(u)\right)^2 ds \\
&= \int_0^{t\wedge T_n}\left(\int_0^s h(s,u)\,dW(u)\right) dY(s) + \frac{1}{2}\int_0^{t\wedge T_n}\left(\int_0^s h(s,u)\,dW(u)\right)^2 ds \\
&= \frac{1}{2}\int_0^{t\wedge T_n}\left(\int_0^s dY(u)\right) dY(s) + \frac{1}{2}\int_0^{t\wedge T_n}\left(\int_0^s \frac{1}{2}dY(u)\right)^2 ds \\
&= \int_0^{t\wedge T_n}\left(\int_0^s \frac{1}{2}dY^n(u)\right) dY^n(s) + \frac{1}{2}\int_0^{t\wedge T_n}\left(\int_0^s \frac{1}{2}dY^n(u)\right)^2 ds, \qquad (30)
\end{aligned}$$

where we have used (28) and (27) to obtain respectively the second and third equalities above. Using now that $\{Y^n(t), 0 \le t \le T\}$ is a Wiener process under $\mathbb{Q}^n_T$, from (30) we obtain that

$$\mathbb{E}^n_{\mathbb{Q}}\left[\ln M^n(T)\right] = \frac{1}{8}\mathbb{E}^n_{\mathbb{Q}}\left[\int_0^{T\wedge T_n}\left(Y^n(s)\right)^2 ds\right] \le \frac{1}{8}\int_0^T \mathbb{E}^n_{\mathbb{Q}}\left[\left(Y^n(s)\right)^2\right] ds = \frac{1}{8}\int_0^T s\,ds = \frac{T^2}{16}.$$

Moreover, by Jensen's inequality

$$\mathbb{E}^n_{\mathbb{Q}}\left[\ln M^n(T)\right] = \mathbb{E}\left[M^n(T)\ln M^n(T)\right] \ge \left(\mathbb{E}\left[M^n(T)\right]\right)\ln\left(\mathbb{E}\left[M^n(T)\right]\right),$$

from which we infer that $\sup_{n\ge 1}\mathbb{E}\left[M^n(T)\right] < \infty$. This implies that $\{M^n(t),\ 0 \le t \le T\}$ is a uniformly integrable martingale for each n, hence $M(t) = M(t\wedge T) = \lim_{n\to\infty} M^n(t)$, which proves that $\{M(t), 0 \le t \le T\}$ is a martingale. This finishes the proof of the lemma. ■

Lemma 14 *Let X be given by (26). Then*

$$\int_{\ln(r_0+1)}^{\ln(T+1)} e^{\mu\beta X(t)}\mathbf{1}_{\{X(t)\ge 0\}}\,dt \to \infty \quad \textit{as} \quad T\to\infty. \qquad (31)$$

Proof: Recall the identity

$$\mathbb{E}\left[\exp\left(-\frac{m^2}{2}A^+_{g(s_\zeta)}\right)\right] = \frac{2\zeta K_\zeta(m)}{mK_{\zeta+1}(m)}, \quad m \ge 0, \qquad (32)$$

(see [17, pag. 132]), where K_ζ is the Macdonald's function [16, 6·22], also known as modified Bessel function of the second kind, which is given for any complex number ν by

$$K_\nu(z) = \frac{1}{2}\left(\frac{1}{2}z\right)^\nu \int_0^\infty r^{-\nu-1}e^{-r-z^2/4r}\,dr, \quad \mathrm{Re}(z^2) > 0,$$

and $A_t^+ = \int_0^t \exp(2X(s)) \mathbf{1}_{\{2X(s)\geq 0\}} \, ds$. Moreover, $g(t) = \sup\{s \leq t : X(s) = 0\}$ and s_ζ is an exponentially distributed random variable with parameter $\zeta^2/2$, independent of the process X. Letting $\zeta \to 0$ in (32) renders, for each fixed m,

$$\mathbb{E}\left[\exp\left(-\frac{m^2}{2}A_\infty^+\right)\right] = \lim_{\zeta\to 0}\mathbb{E}\left[\exp\left(-\frac{m^2}{2}A_{g(s_\zeta)}^+\right)\right] = \lim_{\zeta\to 0}\frac{2\zeta K_\zeta(m)}{mK_{\zeta+1}(m)} = 0.$$

It follows that $A_\infty^+ := \lim_{t\to\infty} A_t^+ = \int_0^\infty \exp(2X(s))\mathbf{1}_{\{2X(s)\geq 0\}}ds = \infty$ a.s., from which we infer the a.s. divergence of the integral in (31). ■

Proof of Theorem 11.3 We will show that the integral $I_0(T)$ given by (25) a.s. diverges as $T \to \infty$. Using the change of variables $s = e^t - 1$ and Lemma 13, we obtain

$$\begin{aligned}
\int_{r_0}^{T} \frac{e^{\mu\beta W(s)}}{s^{d\beta/\alpha}}\,ds &= \int_{\ln(r_0+1)}^{\ln(T+1)} \frac{e^{\mu\beta W(e^t-1)}}{(e^t-1)^{d\beta/\alpha}}e^t\,dt \\
&= \int_{\ln(r_0+1)}^{\ln(T+1)} \frac{e^{\mu\beta e^{t/2}X(t)+t}}{(e^t-1)^{d\beta/\alpha}}\,dt \qquad (33)\\
&\geq \int_{\ln(r_0+1)}^{\ln(T+1)} \frac{e^{\mu\beta X(t)+t}}{e^{td\beta/\alpha}}\mathbf{1}_{\{X(t)\geq 0\}}\,dt \\
&= \int_{\ln(r_0+1)}^{\ln(T+1)} e^{\mu\beta X(t)-t(d\beta/\alpha-1)}\mathbf{1}_{\{X(t)\geq 0\}}dt \\
&\geq \int_{\ln(r_0+1)}^{\ln(T+1)} e^{\mu\beta X(t)}\mathbf{1}_{\{X(t)\geq 0\}}\,dt \qquad (34)
\end{aligned}$$

provided that $d\beta/\alpha \leq 1$, where the equality (33) is in law. From Lemma 14 we know that the integral (34) a.s. diverges as $T \to \infty$. Hence, defining the events

$$A_T^M = \left[\int_{\ln(r_0+1)}^{\ln(T+1)} e^{\mu\beta X(t)}\mathbf{1}_{\{X(t)\geq 0\}}\,dt > M\right]$$

for any $M > 0$ and all $T \geq r_0$, we see that

$$1 = \mathbb{Q}\left[\liminf_{T\to\infty} A_T^M\right] \leq \mathbb{Q}\left[\liminf_{T\to\infty}\left[\int_{\ln(r_0+1)}^{\ln(T+1)} \frac{e^{\mu\beta e^{t/2}X(t)+t}}{(e^t-1)^{d\beta/\alpha}}\,dt > M\right]\right],$$

and it follows that $\mathbb{P}\left[\liminf_{T\to\infty}\left[I_0(T) > M\right]\right] = 1$ because $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures due to Lemma 13. In this way we conclude that in dimensions $d \leq \alpha/\beta$, Eq. (24) exhibits finite-time blowup for any nontrivial positive initial value. ■

5 Existence of global solutions

In this last section we give conditions which are sufficient for the existence, globally in time, of the weak solution of (1). Let

$$\mathcal{K} = \begin{cases} K \ \text{ if } H > 1/2, \\ K - \mu^2/2 \ \text{ if } H = 1/2. \end{cases}$$

Theorem 15 *Let* $1/2 \leq H < 1$. *If*

$$\beta \int_0^\infty e^{\mu\beta B_r^H} \| e^{\mathcal{K}t} \mathcal{S}(r) f \|_\infty^\beta \, dr < 1, \tag{35}$$

then the weak solution u *of (1) is global in the sense that* $|u(t,x)| < \infty$ *for all* $(t,x) \in [0,\infty) \times \mathbb{R}^d$. *Moreover,*

$$u(t,x) \leq \frac{e^{\mu\beta B_t^H + \mathcal{K}t} \mathcal{S}(t) f(x)}{\left(1 - \beta \int_0^\infty e^{\mu\beta B_r^H + \mathcal{K}\beta r} \|\mathcal{S}(r) f\|_\infty^\beta \, dr\right)^{1/\beta}}, \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

The proof of Theorem 15 is a straightforward adaptation of the proof of [3, Thm. 5] and will not be given here.

Remark 16 Theorems 8 and 15 may be applied to provide upper and lower bounds for the probability that u does not blowup in finite time. The upper bound follows directly from Theorem 8:

$$\begin{aligned} \mathbb{P}(\tau = +\infty) &= \mathbb{P}(\tau > t \textit{ for all } t) \leq \mathbb{P}(\theta > \frac{t}{1+2^\alpha} - r_0 \textit{ for all } t) \\ &= \mathbb{P}\left(\int_{r_0}^\infty \exp\left[\beta(\mathcal{K}r + \mu B_r^H)\right] r^{-d\beta/\alpha} \, dr < c'\beta^{-1} \right), \textit{ where} \\ c' &= \frac{2^{d(2\beta + (1+\beta)/\alpha)}}{p^\beta(1,0) E^\beta[f(Z(2^{-\alpha} r_0)]}. \end{aligned}$$

For the lower bound we get by Lemma 3,

$$\|\mathcal{S}(r) f\|_\infty^\beta = r^{-d\beta/\alpha} \left(\sup_x \int p(1, r^{-1/\alpha}(y-x)) f(y) \, dy \right)^\beta \leq r^{-d\beta/\alpha} p(1,0)^\beta \|f\|_1^\beta,$$

and therefore

$$\int_0^\infty \exp\left[\beta(\mathcal{K}r+\mu B_r^H)\right]\|\mathcal{S}(r)f\|_\infty^\beta\,dr \le p(1,0)^\beta\|f\|_1^\beta\int_0^\infty \exp\left[\beta(\mathcal{K}r+\mu B_r^H)\right]r^{-d\beta/\alpha}\,dr.$$

By (35)

$$\begin{aligned}\mathbb{P}(\tau=+\infty) &\ge \mathbb{P}\left(\int_0^\infty \exp\left[\beta(\mathcal{K}r+\mu B_r^H)\right]\|\mathcal{S}(r)f\|_\infty^\beta\,dr<\beta^{-1}\right)\\ &\ge \mathbb{P}\left(\int_0^\infty \exp\left[\beta(\mathcal{K}r+\mu B_r^H)\right]r^{-d\beta/\alpha}\,dr<\frac{1}{\beta p(1,0)^\beta\|f\|_1^\beta}\right).\end{aligned}$$

We notice that the upper and the lower bounds for $\mathbb{P}(\tau=+\infty)$ are in terms of the same integral, with, however, different lower bounds of the integration interval.

Remark 17 **(a)** In the classical deterministic case $K=\mu=0$ it is well known that under the assumptions

$$d>\alpha/\beta \quad \text{and} \quad 0\le f(x)\le m_1p(\xi,x), \quad x\in\mathbb{R}^d, \tag{36}$$

where $\xi>0$ and $m_1>0$ are suitably chosen constants, the mild solution of the Fujita equation

$$\frac{\partial u(t,x)}{\partial t}=\Delta_\alpha u(t,x)+u^{1+\beta}(t,x), \quad u(0,x)=f(x), \quad x\in\mathbb{R}^d,$$

is global and satisfies $0\le u(t,x)\le m_2p(t+\xi,x)$, $t\ge0$, $x\in\mathbb{R}^d$, for a certain positive constant m_2, see e.g. [8, Theorem 3.5]. Now assume that $H=1/2$ and $K=\mu^2/2\neq0$, so that $\mathcal{K}=0$. In this case a sufficient condition for u to be a global solution is given by Theorem 15. Under assumption (36), the verification of condition (35) leads to show that the improper integral $\int_\epsilon^\infty \frac{e^{\mu\beta W(s)}}{s^{d\beta/\alpha}}\,ds$ is convergent for some $\epsilon>0$. However, it turns out that this integral is divergent for any given $\epsilon>0$, even if $d>\alpha/\beta$. This shows that the stochastic perturbation in equation (1) qualitatively affects the behavior of the solution. The reason is that Δ_α has not enough dissipativity. A possible way to increase dissipativity is to replace Δ_α by a non autonomous differential operator of the type $e^t\Delta_\alpha$. By considering the two-parameter semigroup associated to this operator, a direct calculation shows that (35) can be satisfied trajectorywise for suitably chosen initial conditions f as in (36). For such a choice, the solution does not blow up in finite time.

(b) For $\mathcal{K}<0$ the probability that (35) holds in the case $d>\alpha/\beta$ can again be estimated by the Dufresne and Yor formulas ([17], [4], case $H=1/2$) for initial conditions f as in (36). If (35) is satisfied, the solution is global.

Acknowledgment The second- and third-named authors acknowledge the hospitality of Institut Élie Cartan de Lorraine, where part of this work was done. This research was partially supported by CONACyT (Mexico), Grant No. 257867.

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