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Global aspects of the space of 6D $\mathcal{N} = 1$ supergravities

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Abstract: We perform a global analysis of the space of consistent 6D quantum gravity theories with $\mathcal{N} = 1$ supersymmetry, including models with multiple tensor multiplets. We prove that for theories with fewer than $T = 9$ tensor multiplets, a finite number of distinct gauge groups and matter content are possible. We find infinite families of field combinations satisfying anomaly cancellation and admitting physical gauge kinetic terms for $T > 8$. We find an integral lattice associated with each apparently-consistent supergravity theory; this lattice is determined by the form of the anomaly polynomial. For models which can be realized in F-theory, this anomaly lattice is related to the intersection form on the base of the F-theory elliptic fibration. The condition that a supergravity model have an F-theory realization imposes constraints which can be expressed in terms of this lattice. The analysis of models which satisfy known low-energy consistency conditions and yet violate F-theory constraints suggests possible novel constraints on low-energy supergravity theories.

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1 Introduction

Six-dimensional $\mathcal{N} = 1$ supergravity is a useful domain for studying fundamental questions about the space of string vacua. On the one hand, the space of such theories is strongly constrained by gravitational, gauge, and mixed anomalies [1, 2]. On the other hand, 6D supergravity includes a rich variety of models with different gauge groups and matter representations. Given the difficulty of attaining a systematic classification of string vacua in four dimensions, and our limited knowledge at this point regarding constraints on the

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space of low-energy 4D field theories which can be consistently coupled to quantum gravity, it is desirable to find a context in which we can develop tools and experience for addressing global questions of this nature. 6D $\mathcal{N} = 1$ supergravity promises to provide a tractable framework in which we can address questions such as the global extent of the space of quantum-consistent low-energy theories, and begin to map how different string vacuum constructions fill regions in this space of low-energy theories.

In [3–5] we began a systematic analysis of the space of low-energy 6D $\mathcal{N} = 1$ supergravity theories. We showed in [4] that for theories with one tensor field (those models which admit a Lagrangian), there are a finite number of distinct nonabelian gauge groups and matter representations possible in models which do not suffer from clear quantum inconsistencies from anomaly violation or wrong-sign kinetic terms. In [5] we gave an explicit map from the set of such models to topological data for F-theory constructions. For most apparently-consistent low-energy models this map appears to give good string vacua through F-theory constructions. In some cases, however, the image of the map was found to exhibit some pathology; in such cases the models do not correspond to any known F-theory vacuum.

In this paper, we extend this global analysis to theories with multiple tensor fields. The Lagrangian describing gauge groups and matter fields for 6D $\mathcal{N} = 1$ supergravity theories with one tensor field was originally developed in [6, 7]. Field equations for the (non-Lagrangian) models with multiple tensor fields were analyzed by Romans [8]. In a theory with T antisymmetric 2-form tensor fields, the associated scalars parameterize a coset space $\mathrm{SO}(1,T)/\mathrm{SO}(T)$. We use the underlying $\mathrm{SO}(1,T)$ structure to give a fairly simple proof that for models with $T < 9$, there are a finite number of possible nonabelian gauge groups and matter representations consistent with anomaly cancellation and physical gauge kinetic terms. We find that for any T, each apparently-consistent low-energy supergravity theory can be associated with an integral lattice Λ. We use the structure of the lattice Λ to connect with F-theory, and identify constraints on low-energy theories associated with the existence of an F-theory vacuum construction. Thus, the results of this paper generalize and subsume many of the results of [4, 5]; the general structure developed for models with arbitrary T clarifies in some ways the more specific arguments previously given at $T = 1$.

In section 2 we review the structure of anomaly cancellation in general 6D $\mathcal{N} = 1$ supergravity theories, and demonstrate that each anomaly-free theory can be associated with an integral lattice. In section 3 we prove that the number of gauge groups and matter representations for consistent theories is finite for $T < 9$. We give explicit examples of infinite families where the theorem breaks down at $T \geq 9$ in section 4. In section 5 we use the integral lattice for each theory to construct topological data which would be associated with any corresponding F-theory construction, and describe constraints on the class of models which can be realized through F-theory. Section 6 contains some examples of F-theory embeddings of supergravity models, illustrating how geometrical constraints from F-theory rule out the possibility of F-theory realizations of some apparently-consistent low-energy models. Section 7 contains some discussion of global aspects of the 6D supergravity landscape, and section 8 contains a summary of the conclusions.

Note that the analysis in sections 2, 3, and 4 depends only upon the structure of low-energy supergravity and is independent of any structure associated with string theory.

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	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
λ	1	2	1	2	6	12	60	6	2

Table 1. Normalization factors for the simple groups.

2 Anomalies and lattices

We consider 6D supergravity theories with semi-simple gauge group $\mathcal{G} = \prod_i G_i$. The analysis of abelian factors will appear elsewhere [9]; such abelian factors have little effect on the structure of the nonabelian part of the theory. We include matter hypermultiplets which transform in a general representation of $\mathcal{G}$, and T tensor multiplets. For theories with multiple tensor multiplets, there is a generalized Green-Schwarz mechanism (described by Sagnotti [10]), which allows for a larger class of gauge anomalies to be cancelled. In this section we review this mechanism using the notation of [11], and show that the anomaly cancellation conditions imply the existence of an integral lattice associated with any consistent 6D $\mathcal{N} = 1$ theory.

2.1 Anomaly cancellation

Anomalies can be cancelled by the Green-Schwarz-Sagnotti mechanism if the anomaly polynomial I_8 can be written in the form

$$I_8(R, F) = \frac{1}{2}\Omega_{\alpha\beta} X_4^\alpha X_4^\beta \tag{2.1}$$

Here X_4^α is a 4-form constructed from the curvatures of the Yang-Mills and spin connections

$$X_4^\alpha = \frac{1}{2} a^\alpha \mathrm{tr} R^2 + \sum_i b_i^\alpha \left(\frac{2}{\lambda_i} \mathrm{tr} F_i^2 \right) \tag{2.2}$$

where a^α, b_i^α are vectors in the space $\mathbb{R}^{1,T}$ and $\Omega_{\alpha\beta}$ is a natural metric (symmetric bilinear form) on this space. The tr here refers to the trace in an appropriate "fundamental" representation of the group G_i, and λ_i is a normalization factor. These normalization factors are fixed by demanding that the smallest topological charge of an embedded SU(2) instanton is 1, as explained in [12]. These factors are listed in table 1 for all the simple groups. $I_8(R, F)$ is completely specified by the multiplets in the low-energy theory and can be computed using the formulae in [13, 14].

When $T \neq 1$, one cannot write down a Lorentz covariant Lagrangian. One can construct a partition function, however, along the lines of [15, 16] by coupling the extra (anti-self-dual) tensor fields to an auxillary 3-form gauge potential. The anomaly can be cancelled by a local counterterm of the form

$$\delta \mathcal{L}_{GSS} = -\Omega_{\alpha\beta} B^\alpha \wedge X_4^\beta \tag{2.3}$$

The 2-form field B^α has an anomalous gauge transformation, and the above term makes the tree-level Lagrangian gauge-variant in exactly the right way to cancel the one-loop

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anomaly. The gauge invariant 3-form field strength is defined as

$$H^\alpha = dB^\alpha + \frac{1}{2}a^\alpha \omega_{3L} + 2\sum_i \frac{1}{\lambda_i} b_i^\alpha \omega_{3Y}^i \tag{2.4}$$

where ω_{3L} and ω_{3Y}^i are Chern-Simons 3-forms of the spin connection and gauge field respectively.

The only $(1,0)$ supersymmetry multiplets that contain scalars are the hypermultiplets (4 real scalars) and the tensor multiplet (1 real scalar). The hypermultiplet moduli space is a quaternionic Kähler manifold, analogous to the 4D $\mathcal{N}=2$ case. The tensor multiplet scalar moduli space locally takes the form of the coset space $\mathrm{SO}(1,T)/\mathrm{SO}(T)$ [8], and can be parameterized by a vector j^α in the space $\mathbb{R}^{1,T}$ of norm $\Omega_{\alpha\beta} j^\alpha j^\beta = +1$. From the viewpoint of the low-energy theory, the above space is just one possible solution to the requirements imposed by SUSY, although no others are known. In section 5.2 we will see that the space $\mathrm{SO}(1,T)/\mathrm{SO}(T)$ arises naturally as a coset space containing the Teichmuller space of Kähler metrics in F-theory compactifications as an open set.

The anomaly polynomial does not specify the vectors a^α, b_i^β, but only constrains the $\mathrm{SO}(1,T)$ invariant quantities

$$\Omega_{\alpha\beta} a^\alpha a^\beta, \quad \Omega_{\alpha\beta} a^\alpha b_i^\beta, \quad \Omega_{\alpha\beta} b_i^\alpha b_j^\beta \tag{2.5}$$

We will use the notation $x \cdot y$ to denote the $\mathrm{SO}(1,T)$ invariant product $\Omega_{\alpha\beta} x^\alpha y^\beta$. Vanishing of the $\mathrm{tr}R^4$ anomaly implies that

$$H - V = 273 - 29T \tag{2.6}$$

where H, V, T denote the number of hyper, vector, and tensor multiplets respectively. The $\mathrm{tr}F^4$ contribution to the total anomaly must also cancel for the anomaly to factorize in the form (2.1). This gives the condition

$$B_{\mathrm{adj}}^i = \sum_R x_R^i B_R^i \tag{2.7}$$

where the coefficients A_R, B_R, C_R are defined as

$$\mathrm{tr}_R F^2 = A_R \mathrm{tr} F^2 \tag{2.8}$$

$$\mathrm{tr}_R F^4 = B_R \mathrm{tr} F^4 + C_R (\mathrm{tr} F^2)^2 \tag{2.9}$$

and x_R^i denotes the number of hypermultiplets transforming in representation R under gauge group factor G_i. We will similarly denote by x_{RS}^{ij} the number of hypermultiplets transforming in representations R, S under the factors G_i, G_j.

The remaining anomaly factorization conditions relate inner products between the vectors a, b_i to group theory coefficients and the representations of matter fields

$$a \cdot a = 9 - T \tag{2.10}$$

$$a \cdot b_i = \frac{1}{6}\lambda_i \left(A_{\mathrm{adj}}^i - \sum_R x_R^i A_R^i \right) \tag{2.11}$$

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$$b_i \cdot b_i = -\frac{1}{3}\lambda_i^2 \left(C^i_{\text{adj}} - \sum_R x^i_R C^i_R \right) \tag{2.12}$$

$$b_i \cdot b_j = \lambda_i \lambda_j \sum_{RS} x^{ij}_{RS} A^i_R A^j_S. \tag{2.13}$$

We demonstrate in the following section that these inner products are all integers, so that a, b_i can be used to define an integral lattice.

In the case when $T = 1$, which was studied in [2, 4, 17], the Green-Schwarz mechanism requires that the anomaly polynomial factorize as a simple product of polynomials. To relate this familiar case to the general formalism, we can choose the bilinear form on $\mathrm{SO}(1,1)$ as

$$\Omega_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.14}$$

For $T = 1$ we have $a^2 = 8$. The form (2.14) and the anomaly polynomial (2.1) are invariant under the rescaling

$$(X^1_4, X^2_4) \to (\mu\ X^1_4, \mu^{-1}\ X^2_4) \tag{2.15}$$

We can use this scale degree of freedom to set $a^\alpha \equiv (a^1, a^2) = (-2, -2)$. We can then identify b_i in this basis with parameters $\alpha_i, \tilde{\alpha}_i$ through

$$b_i = \frac{\lambda_i}{2}(\alpha_i, \tilde{\alpha}_i)\,. \tag{2.16}$$

Once this basis is chosen for $\Omega_{\alpha\beta}, a$, and b_i, the anomaly polynomial takes the familiar, friendly form used in [5] and most literature on $T = 1$ models

$$I_8 = X^1 X^2 = \left(\mathrm{tr}R^2 - \sum_i \alpha_i \mathrm{tr}F_i^2 \right) \left(\mathrm{tr}R^2 - \sum_i \tilde{\alpha}_i \mathrm{tr}F_i^2 \right) \tag{2.17}$$

The only symmetry unfixed is a Z_2 symmetry of the bilinear form that exchanges $X^1 \leftrightarrow X^2$.

2.2 Proof of integrality

The inner products of the vectors $a^\alpha,\ b^\alpha_i$ are related to group invariants $(A_R,\ C_R)$ and the number of charged hypermultiplets in various representations. In this section we show that these inner products are quantized in $\mathbb{Z}$ when the normalization factors λ_i are chosen as in table 1. (2.11)–(2.13) imply that the inner products $a \cdot b_i,\ b_i \cdot b_i,\ b_i \cdot b_j$ are all quantized in integers if

$$\begin{aligned} \lambda_i \frac{\sum_R x^i_R A^i_R - A^i_{\text{adj}}}{6} &\in \mathbb{Z} \\ \lambda_i^2 \frac{\sum_R x^i_R C^i_R - C^i_{\text{adj}}}{3} &\in \mathbb{Z} \\ \lambda_i A^i_R &\in \mathbb{Z} \end{aligned} \tag{2.18}$$

We will prove the above statements for each of the simple groups, case by case. For the $\mathrm{SU}(N)$ (and $\mathrm{Sp}(N)$) series, this is easily proved using properties of Young diagrams,

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which are in one-to-one correspondence with the irreducible representations of SU(N) (and Sp(N)). For an arbitrary group G not of the form SU(N) or Sp(N), we find a sequence of maximal subgroups that terminates in SU(N) or Sp(N), e.g. $E_8 \supset$ SU(9), SO(8) $\supset$ SO(7) $\supset$ SO(6) $\cong$ SU(4). Then, by computing the branching of representations of G we can show integrality for G.

We start with SU(N), $N \geq 4$; the coefficients A_R, B_R, C_R can be easily calculated using two diagonal generators $T_{12},\ T_{34}$ which, in the fundamental representation, take the form

$$(T_{12})_{ab} = \delta_{a1}\delta_{b1} - \delta_{a2}\delta_{b2} \tag{2.19}$$

$$(T_{34})_{ab} = \delta_{a3}\delta_{b3} - \delta_{a4}\delta_{b4} \tag{2.20}$$

as in the appendix of [4]. The group theory factors $A_R,\ B_R,\ C_R$ can be computed in terms of traces of these generators.

$$A_R = \frac{1}{2}\mathrm{tr}_R T_{12}^2 \tag{2.21}$$

$$B_R + 2C_R = \frac{1}{2}\mathrm{tr}_R T_{12}^4 \tag{2.22}$$

$$C_R = \frac{3}{4}\mathrm{tr}_R T_{12}^2 T_{34}^2 \tag{2.23}$$

In the traces above, we sum over all states in the representation R, which can be represented in terms of the associated Young diagram D_R. In (2.23), let $|s\rangle$ denote a state in the representation normalized to $\langle s|s\rangle = 1$. A basis of such states corresponds to the set of Young tableaux, given by the Young diagram D_R labelled using integers from $1, 2, \cdots, N$. Let π_{ij} denote the operation that switches the labels $i \leftrightarrow j$ in a Young tableau. The states $\{|s\rangle, \pi_{12}|s\rangle, \pi_{34}|s\rangle, \pi_{12} \circ \pi_{34}|s\rangle\}$ all give equal contributions to the trace. (Note that these states need not be distinct, but when they are not all distinct, the contribution vanishes, since either the number of appearances of 1 and 2 are equal or the number of appearances of 3 and 4 are equal.) This makes $\mathrm{tr}_R T_{12}^2 T_{34}^2$ a multiple of 4, and therefore C_R is an integer divisible by 3 for every representation of SU(N), $N \geq 4$. This shows that for SU(N), $N \geq 4$,

$$\frac{1}{3}\left(\sum_R x_R C_R - C_{\mathrm{adj}}\right) \in \mathbb{Z} \tag{2.24}$$

Anomaly cancellation requires the vanishing of the $\mathrm{tr}F^4$ term, which sets $\sum_R x_R B_R - B_{\mathrm{adj}} = 0$. If we define $E_R := \frac{1}{2}\mathrm{tr}_R T_{12}^4$, then (2.22), (2.24) and the F^4 condition together imply that $\sum_R x_R E_R - E_{\mathrm{adj}}$ is divisible by 6. From the same kind of argument as above we know that in a representation R, the states $|s\rangle,\ \pi_{12}|s\rangle$ would together contribute $n(s)^2$ to A_R and $n(s)^4$ to E_R, with $n(s) := \langle s|T_{12}|s\rangle \in \mathbb{Z}$. Since $n(s)^2 \equiv n(s)^4 \pmod 6$, we have $E_R \equiv A_R \pmod 6$. Therefore,

$$\frac{1}{6}\left(\sum_R x_R A_R - A_{\mathrm{adj}}\right) \in \mathbb{Z}\,. \tag{2.25}$$

Since the normalization factor for SU(N) in table 1 is 1, this shows that for an SU($N \geq 4$) factor in the gauge group all the conditions in (2.18) are satisfied.

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For the other subgroups of $GL(N)$, we can again use the Young diagram approach (see [21] for details on Young diagrams for the other classical groups). The irreducible representations of $\mathrm{Sp}(N)$ are in one-to-one correspondence with Young diagrams with all possible contractions with the ϵ symbol subtracted. If we choose appropriate generators from the Cartan subalgebra of $\mathrm{Sp}(N)$ as in [4], whose squares are exactly equal to the squares of the $\mathrm{SU}(N)$ generators we chose above, the above proof for $\mathrm{SU}(N)$ carries through unchanged. In the case of $\mathrm{SO}(N)$, the set of Young diagrams only gives the representations with integer weight; these exclude the spinor representations. With the right choice of generators, the above proof shows that for the integer weight representations, we have integral inner products. If we include spinor representations, the inner products are actually quantized in $\frac{1}{2}\mathbb{Z}$. Including the factor of 2 in the normalization of $\mathrm{SO}(N)$ in table 1, however, we find that the inner products are again integral. An alternate proof of integrality for the $\mathrm{SO}(N)$ representations involves using a sequence of maximal subgroups. We will discuss this method in general below, and then apply it to the $\mathrm{SO}(N)$ case and also to the exceptional groups.

Consider a general simple group G. We wish to show that whenever $\sum_R x_R B_R - B_{\mathrm{adj}} = 0$, the conditions (2.18) are satisfied. Let H be a simple, maximal (proper) subgroup of the simple Lie group G. For a given representation R of G, we can compute the decomposition of R into irreducible representations S_i of the maximal subgroup $H \subset G$: $R = \oplus_i n(R)_i S_i$ where $n(R)_i$ denotes the multiplicity of representation S_i.

$$\mathrm{tr}_R F_G^2 = \sum_i n(R)_i \mathrm{tr}_{S_i} F_H^2 = \left(\sum_i n(R)_i A_{S_i}(H)\right) \mathrm{tr} F_H^2 \tag{2.26}$$

$$\mathrm{tr}_R F_G^4 = \sum_i n(R)_i \mathrm{tr}_{S_i} F_H^4 = \left(\sum_i n(R)_i B_{S_i}(H)\right) \mathrm{tr} F_H^4 + \left(\sum_i n(R)_i C_{S_i}(H)\right) (\mathrm{tr} F_H^2)^2 \tag{2.27}$$

Here tr denotes the trace in the fundamental representation of H. This gives us a way of computing $A_R(G)$ for an arbitrary group G using its maximal subgroup [14].

$$A_R(G) = \frac{\mathrm{tr}_R F_G^2}{\mathrm{tr}_f F_G^2} = \frac{\sum_i n(R)_i A_{S_i}(H)}{\sum_i n(f)_i A_{S_i}(H)} \,. \tag{2.28}$$

Here, for clarity, tr_f explicitly denotes the trace in the (suitably normalized) fundamental representation f of G with $f = \oplus_i n(f)_i S_i$ under $H \subset G$.

Anomaly cancellation for G implies that $\sum_R x_R B_R(G) - B_{\mathrm{adj}}(G) = 0$. Then,

$$\begin{aligned} &\sum_R x_R \mathrm{tr}_R F_G^4 - \mathrm{tr}_{\mathrm{adj}} F_G^4 = \left(\sum_R x_R C_R(G) - C_{\mathrm{adj}}(G)\right) (\mathrm{tr}_f F_G^2)^2 \\ \Rightarrow &\sum_R x_R C_R(G) - C_{\mathrm{adj}}(G) = \frac{\sum_R x_R \sum_i n(R)_i C_{S_i}(H) - \sum_i n(adj)_i C_{S_i}(H)}{(\sum_i n(f)_i A_{S_i}(H))^2} \,. \end{aligned} \tag{2.29}$$

If we have integrality for the group H, i.e. for every $\{y_S \in \mathbb{Z}\}$ satisfying

$$\sum_S y_S B_S(H) = 0 \Rightarrow \begin{cases} \lambda_H \sum_S y_S A_S(H) \in 6\mathbb{Z} \\ \lambda_H^2 \sum_S y_S C_S(H) \in 3\mathbb{Z} \end{cases} \tag{2.30}$$

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then, using $y_{S_i} = \sum_R x_R\, n(R)_i - n(\mathrm{adj})_i$ we have

$$\lambda_H \left(\sum_i n(f)_i A_{S_i}(H) \right) \frac{\sum_R x_R A_R(G) - A_{\mathrm{adj}}(G)}{6} \in \mathbb{Z} \tag{2.31}$$

$$\lambda_H^2 \left(\sum_i n(f)_i A_{S_i}(H) \right)^2 \frac{\sum_R x_R C_R(G) - C_{\mathrm{adj}}(G)}{3} \in \mathbb{Z} \tag{2.32}$$

The conditions (2.18) are then all satisfied for a group G, if we prove that

$$\lambda_G = \lambda_H \sum_i n(f)_i A_{S_i}(H) \tag{2.33}$$

for a maximal subgroup H.

We first consider the particular case of $G = E_8$ and the maximal subgroup $\mathrm{SU}(9) \subset E_8$ [14]. The coefficients A_R, B_R, C_R for E_8 are defined using the adjoint representation as the fundamental.

$$\begin{aligned} E_8 &\supset \mathrm{SU}(9) \\ 248 &= 84 \ \oplus \ \overline{84} \ \oplus \ \mathrm{adj} \end{aligned} \tag{2.34}$$

Using $A_{84}(\mathrm{SU}(9)) = 21,\ A_{\mathrm{adj}}(\mathrm{SU}(9)) = 18$, we have integrality for E_8, because $\lambda_{E_8} = 60$ and $\lambda_{\mathrm{SU}(9)} = 1$, and therefore relation (2.33) is satisfied

$$\lambda_{E_8} = \lambda_{\mathrm{SU}(9)}(2A_{84} + A_{\mathrm{adj}}) = 60\,. \tag{2.35}$$

In fact, the branching rule in (2.34) makes clear the origin of the normalization factors. Since

$$\mathrm{tr}_{\mathrm{adj}} F_{E_8}^2 = 2\ \mathrm{tr}_{84} F_{\mathrm{SU}(9)}^2 + \mathrm{tr}_{\mathrm{adj}} F_{\mathrm{SU}(9)}^2 = 60\ \mathrm{tr}_9 F_{\mathrm{SU}(9)}^2, \tag{2.36}$$

including a normalization factor of $1/60$ for the E_8 group trace, relative to the $\mathrm{SU}(9)$ trace, ensures that the minimum instanton number of any configuration of E_8 gauge fields is 1.

For the $\mathrm{SO}(N)$ series, we can show that (2.33) is satisfied by induction. For $N = 6$, $\mathrm{SO}(6) \cong \mathrm{SU}(4)$ we have

$$\mathrm{tr}_6 F_{\mathrm{SO}(6)}^2 = \mathrm{tr}_6 F_{\mathrm{SU}(4)}^2 = 2\,\mathrm{tr} F_{\mathrm{SU}(4)}^2 \tag{2.37}$$

Equation (2.33) is again satisfied since $\lambda_{\mathrm{SO}(6)} = 2, \lambda_{\mathrm{SU}(4)} = 1$. For the inductive step, since $\mathrm{SO}(N-1)$ is a maximal subgroup of $\mathrm{SO}(N)$,

$$\mathrm{tr}_N F_{\mathrm{SO}(N)}^2 = \mathrm{tr}_{N-1} F_{\mathrm{SO}(N-1)}^2 \tag{2.38}$$

Thus, integrality for $\mathrm{SO}(N-1)$ implies integrality for $\mathrm{SO}(N)$, and the inductive step is proved.

Similarly, we can prove integrality for the groups $E_6,\ E_7,\ F_4$ using the maximal subgroups $\mathrm{Sp}(4),\ \mathrm{SU}(8),\ \mathrm{SO}(9)$ respectively. In each case we find that relation (2.33) is satisfied. We have thus shown that the inner products $a \cdot b_i,\ b_i \cdot b_i,\ b_i \cdot b_j$ are all integral if all simple factors in the gauge group are drawn from the list

$$\{\mathrm{SU}(N \geq 4),\ \mathrm{SO}(N \geq 7),\ \mathrm{Sp}(N \geq 2),\ E_{6,7,8}, F_4\} \tag{2.39}$$

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and suitable scaling factors are applied to the anomaly coefficients. The groups $\mathrm{SU}(2), \mathrm{SU}(3)$ and G_2 are conspicuously absent from this list. The normalization factors for these groups are 1, 1 and 2 respectively, but in these cases, the condition that local anomalies are absent does not constrain the inner products to be integral. There is a more subtle anomaly, however, first discussed in [22], where the partition function is invariant under local gauge transformations (gauge current is conserved quantum mechanically), but not invariant under "large" gauge transformations. The analysis of such "global anomalies" in six dimensions was carried out in [23], and more thoroughly in [24]. Using their results, we find that the inner products in question are non-integral for $\mathrm{SU}(2), \mathrm{SU}(3)$ and G_2 precisely when the low-energy theory is plagued by a global anomaly, which renders these theories inconsistent.

Therefore, imposing that the low-energy theory is free of local and global anomalies, we have shown that the anomaly coefficients define an integral lattice Λ. This lattice Λ will play a crucial role in defining possible embeddings into F-theory.

2.3 Integral lattices and dyonic strings

Since the inner products $a \cdot a, a \cdot b_i, b_i \cdot b_j$ compatible with the anomaly cancellation equations are all integral, we can use this inner product structure to form an integral lattice

$$\Lambda = \begin{pmatrix} a^2 & -a \cdot b_1 & -a \cdot b_2 & \cdots \\ -a \cdot b_1 & b_1^2 & b_1 \cdot b_2 & \cdots \\ -a \cdot b_2 & b_1 \cdot b_2 & b_2^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.40}$$

Note that this lattice may be degenerate; in some cases there can be linear relations between the vectors a, b_i. We choose to define the lattice in terms of $-a$ rather than a since generally $-a$ is a positive vector in the sense that $-a \cdot j > 0$ for those models with F-theory descriptions, as we discuss in section 5.

For models which have a consistent quantum UV completion, there is a natural interpretation of the lattice Λ in terms of the charge lattice of BPS states. The BPS states of the 6D $\mathcal{N} = (1,0)$ SUSY algebra are extended string-like states, known as dyonic strings, with arbitrary charges under the $(1,T)$ multiplet of two-form fields B^α [25]. For theories with nonabelian gauge fields, there are BPS states known as gauge dyonic strings, where the gauge field has an instanton profile in the directions transverse to the string [26]. For every nonabelian factor G_i in the gauge group, there is a corresponding gauge dyonic string with conserved 2-form charge given by the vector b_i. (The dyonic string is obtained from the gauge dyonic string by taking the instanton size $\to 0$ limit.). In a consistent quantum theory, just as the product of electric and magnetic monopole charges is quantized in standard 4D electromagnetic theory due to the single-valued nature of the electron wave function, the inner product $b \cdot b'$ of dyonic string charges is quantized in the 6D theory [27]. Thus, we expect that in a consistent quantum theory, if there are quantum excitations associated with the solitonic dyonic strings, these states must live in an integral lattice $\tilde{\Lambda}$ of signature $(1,T)$ of which Λ is a sublattice. It is interesting and perhaps suggestive that

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the integrality of the lattice Λ follows directly from the anomaly cancellation conditions, with no further assumptions about the quantum consistency of the theory or the existence of quantum string states.

3 Finite bound for fixed $T < 9$

In this section we prove that for fixed $T < 9$ there are a finite number of distinct possible combinations of nonabelian gauge group and matter representations. This analysis is purely based on aspects of the low-energy supergravity theory, and is independent of string theory or any other specific UV completion.

The finite range of possible gauge groups and matter representations for $T = 1$ was proven in [4]. We give a similar proof here for any fixed T between 0 and 8, using the $\mathrm{SO}(1,T)$ invariant inner product structure on the vectors a, b_i, j. As in [4], we ignore abelian factors; such factors do not affect the anomaly cancellation conditions (2.7)–(2.13) on the nonabelian gauge group factors. The constraint on infinite families breaks down at $T = 9$ due to the change of sign of $a^2 = 9 - T$. When a has positive norm, it places stronger constraints on the range of allowed models. We give explicit examples of infinite families of anomaly-free models with acceptable gauge kinetic terms at $T = 9$ and greater in section 4.

The proof for $T < 9$ proceeds by contradiction. We assume that there is an infinite family of models $\{\mathcal{M}_{(\gamma)}\} = \{\mathcal{M}_{(1)}, \mathcal{M}_{(2)}, \ldots\}$ with nonabelian gauge groups $\{\mathcal{G}_{(\gamma)}\}$. There are a finite number of (semi-simple) groups $\mathcal{G}$ with dimension below any fixed bound. For each fixed $\mathcal{G}$, there are a finite number of representations whose dimension is below the bound (2.6) on the number of hypermultiplets. Thus, as argued in [4], any infinite family $\{\mathcal{G}_{(\gamma)}\}$ must include gauge groups of arbitrarily large dimension. For any given model in the family we decompose the semi-simple gauge group into a product of simple group factors $\mathcal{G}_{(\gamma)} = G_1^{(\gamma)} \times G_2^{(\gamma)} \times \cdots \times G_{k(\gamma)}^{(\gamma)}$. We divide the possibilities into two cases as in [4].

1. The dimension of the simple factors in the groups $\mathcal{G}_{(\gamma)}$ is bounded across all γ, that is $\dim(G_i^{(\gamma)}) \leq D$ for all $1 \leq i \leq k(\gamma)$ for every theory $\mathcal{M}_{(\gamma)}$. In this case, the number of simple factors is unbounded over the family.

2. The dimension of at least one simple factor in $\mathcal{G}_{(\gamma)}$ is unbounded. For example, the gauge group is of the form $\mathcal{G}_{(\gamma)} = \mathrm{SU}(N_{(\gamma)}) \times \tilde{\mathcal{G}}_{(\gamma)}$, where $N_{(\gamma)} \to \infty$.

Case 1. In this case we can rule out infinite families for arbitrary but fixed T. In case 1 there are an unbounded number of simple factors, but the dimension of each factor $G_i^{(\gamma)}$ is bounded by $\dim G_i^{(\gamma)} \leq D$. Assume that we have an infinite sequence of models whose gauge groups have $N_{(\gamma)}$ factors, with $N_{(\gamma)}$ unbounded. To simplify notation, we drop the subscript γ which indexes theories in the family $\{\mathcal{M}_{(\gamma)}\}$. We consider one model $\mathcal{M}$ in this infinite sequence, with N factors. We divide the factors G_i into 3 classes:

1. **Type Z**: $b_i^2 = 0$

2. **Type N**: $b_i^2 < 0$

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3. **Type P**: $b_i^2 > 0$

We begin by recapitulating some simple arguments from [4]. Since the dimension of each factor is bounded, the contribution to $H - V$ from $-V$ is bounded below by $-ND$. For fixed T the total number of hypermultiplets is then bounded by

$$H \leq 273 - 29T + ND \equiv B \sim \mathcal{O}(N)\,. \tag{3.1}$$

This means that the dimension of any given representation is bounded by the same value B. The number of gauge group factors λ under which any matter field can transform nontrivially is then bounded by $2^\lambda \leq B$, so $\lambda \leq \mathcal{O}(\ln N)$.

Now, consider the different types of factors. Denote the number of type N, Z, P factors by $N_{N,Z,P}$, where

$$N = N_N + N_Z + N_P\,. \tag{3.2}$$

We can write the b_i's in a (not necessarily integral) basis where $\Omega = \mathrm{diag}(+1, -1, -1, \ldots)$ as

$$b_i = (x_i, \vec{y}_i)\,. \tag{3.3}$$

For any type P factor, $|x_i| > |\vec{y}_i|$, so $b_i \cdot b_j > 0$ for any pair of type P factors. Thus, there are hypermultiplets charged under both gauge groups for every pair of type P factors. A hypermultiplet charged under $\lambda \geq 2$ gauge group factors appears in $\lambda(\lambda - 1)$ (ordered) pairs, and contributes at least 2^λ to the total number of hypermultiplets H. Each ordered pair under which this hypermultiplet is charged then contributes at least

$$\frac{2^\lambda}{\lambda(\lambda - 1)} \geq 1 \tag{3.4}$$

to the total number of hypermultiplets H. It follows that the $N_P(N_P - 1)$ type P pairs, under which at least one hypermultiplet is charged, contribute at least $N_P(N_P - 1)$ to H, so

$$N_P(N_P - 1) \leq B \tag{3.5}$$

Thus,

$$N_P \leq \sqrt{B} + 1 \sim \mathcal{O}(\sqrt{N}) \tag{3.6}$$

which is much smaller than N for large N. So most of the $b_i's$ associated with gauge group factors in any infinite family must be type Z or type N.

Now consider type N factors. Any set of r mutually orthogonal type N vectors defines an r-dimensional negative-definite subspace of $\mathbb{R}^{1,T}$. This means, in particular, that we cannot have $T + 1$ mutually orthogonal type N vectors. If we have N_N type N vectors, we can define a graph whose nodes are the type N vectors, where an edge connects every two nodes associated with perpendicular vectors. Turán's theorem [28] states that the maximum number of edges on any graph with n vertices which does not contain a subset of $T + 1$ completely connected vertices is

$$(1 - 1/T)n^2/2 \tag{3.7}$$

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where the total number of possible edges is $n(n-1)/2$. Thus, applying this theorem to the graph described above on nodes associated with type N vectors, the number of ordered pairs with charged hypermultiplets must be at least

$$\frac{N_N^2}{T} - N_N \,. \tag{3.8}$$

It then follows that

$$N_N \leq \sqrt{TB} + T \sim \mathcal{O}(\sqrt{N}) \,. \tag{3.9}$$

Finally, consider type Z factors. Vectors b_i, b_j of the form (3.3) associated with two type Z factors each have $|x_i| = |\vec{y}_i|$ and have a positive inner product unless they are parallel, in which case $b_i \cdot b_j = 0$. Denote by μ the size of the largest collection of parallel type Z vectors. Each type Z vector is perpendicular to fewer than μ other type Z vectors, so there are at least $N_Z(N_Z - \mu)$ pairs of type Z factors under which there are charged hypers. We must then have

$$N_Z(N_Z - \mu) = (N_Z - \mu)(N - N_P - N_N) \leq B \,. \tag{3.10}$$

But from (3.6), (3.9) this means that $N_Z - \mu$ is of order at most $\mathcal{O}(1)$ (and is bounded by D as $N \to \infty$), while N_Z is of order $\mathcal{O}(N)$. Thus, all but a fraction of order $1/N$ of the type Z factors have vectors in a common parallel direction. In [4], we showed that all group + matter configurations which give type Z factors have a positive value for $H - V$. The total contribution to $H - V$ is then bounded by

$$H - V > \mu - [(N_Z - \mu) + N_P + N_N]\,(D) \sim \mathcal{O}(N) \tag{3.11}$$

which exceeds the bound $H - V \leq 273 - 29T$ for sufficiently large N. Thus, we have ruled out case 1 by contradiction for all $T > 0$.

Case 2. In [4] we proved that there are no infinite families with factors of unbounded size for $T = 1$. A very similar proof works up to $T = 8$; we outline this proof using the inner product structure and vectors a, b_i, j, making use of results from [4]. As discussed in [4], for SU(N) the F^4 anomaly cancellation condition

$$B_{\text{Adj}} = 2N = \sum_R x_R B_R \tag{3.12}$$

can only be satisfied at large N when the number of multiplets x_R vanishes in all representations other than the fundamental, adjoint, and two-index antisymmetric and symmetric representations. For these representations, indexed in that order, (3.12) becomes

$$2N = x_1 + 2Nx_2 + (N-8)x_3 + (N+8)x_4 \,. \tag{3.13}$$

Note that we do not distinguish here between representations and their conjugates, which give equal anomaly contributions. The solutions to (3.13) at large N, along with the corresponding solutions for the other classical groups SO(N), Sp(N) are listed in table 2. We discard solutions $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ and $(0, 0, 1, 1)$, where $a \cdot b_i = b_i^2 = 0$ since

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Group	Matter content	$H-V$	$a\cdot b$	b^2
SU(N)	$2N\ \square$	N^2+1	0	-2
	$(N+8)\ \square+1\ \begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\frac{1}{2}N^2+\frac{15}{2}N+1$	1	-1
	$(N-8)\ \square+1\ \square\square$	$\frac{1}{2}N^2-\frac{15}{2}N+1$	-1	-1
	$16\ \square+2\ \begin{smallmatrix}\square\\\square\end{smallmatrix}$	$15N+1$	2	0
SO(N)	$(N-8)\ \square$	$\frac{1}{2}N^2-\frac{7}{2}N$	-1	-1
Sp($N/2$)	$(N+8)\ \square$	$\frac{1}{2}N^2+\frac{7}{2}N$	1	-1
	$16\ \square+1\ \begin{smallmatrix}\square\\\square\end{smallmatrix}$	$15N-1$	2	0

Table 2. Allowed charged matter for an infinite family of models with gauge group $H(N)$. The last two columns give the values of $a\cdot b, b^2$ in the factorized anomaly polynomial.

for $T<9$ these relations combined with $a^2>0$ imply that $b_i=0$ and therefore that the kinetic term for the gauge field is identically zero. The contribution to $H-V$ from each of the group and matter combinations in table 2 diverges as $N\to\infty$. This cannot be cancelled by contributions to $-V$ from an infinite number of factors, for the same reasons which rule out case 1. Thus, any infinite family must have an infinite sub-family, with gauge group of the form $\hat{H}(M)\times H(N)\times\tilde{G}_{M,N}$, with both $M,N\to\infty$. For any factors G_i,G_j with $a\cdot b_i, a\cdot b_j\neq 0$, in a (non-integral) basis where $\Omega=\mathrm{diag}(+1,-1,-1,\ldots)$, and $a=(\sqrt{a^2},0,0,\ldots)$ writing

$$b_i=(x_i,\vec{y}_i) \tag{3.14}$$

with $x_i=a\cdot b_i/\sqrt{a^2}$ we have

$$x_ix_j=(a\cdot b_i)(a\cdot b_j)/a^2\geq b_i\cdot b_j=\sum_{R,S}x_{RS}A_RA_S\,. \tag{3.15}$$

Since x_ix_j can be taken to be constant for the infinite family of pairs $\hat{H}(M),H(N)$, while A_R grows for all representations besides the fundamental, the only possible fields charged under more than one of the infinite factors in table 2 are bifundamentals.

We now consider all possible infinite families built from products of groups and representations in table 2 with bounded $H-V$. There are 5 such combinations with two factors. These combinations were enumerated in [4], and are listed in table 4 in that paper. These combinations include two infinite families shown to satisfy anomaly factorization by Schwarz [29], as well as three other similar families. In [4] it was shown that for $T=1$ the models in all five of these infinite families are unacceptable because the gauge kinetic term for the two factors are opposite in sign and therefore one is always unphysical. The same consequence follows as long as $T<9$, where $a^2>0$. This can be shown as follows: For each two-factor infinite family we have two vectors b_1,b_2 which satisfy $a\cdot(b_1+b_2)=0$ and $(b_1+b_2)^2=0$. But these conditions imply $b_1+b_2=0$, so that $j\cdot b_1$ and $j\cdot b_2$ cannot both be positive. For example, for the theory found by Schwarz with gauge group SU(N)$\times$SU(N) with two bifundamental fields, we have $a\cdot b_1=a\cdot b_2=0, b_1^2=b_2^2=-2, b_1\cdot b_2=2$, from which it follows that $b_1=-b_2$. This proof breaks down when $a^2\leq 0$, since then $a\cdot b=b^2=0$ is not sufficient to prove $b=0$. In the following section we give an infinite family of examples which has no clear inconsistency at $T=9$.

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Note that while for $T = 1$ there are no infinite families with more than two large gauge factors, at larger T there are families with three large gauge factors. For example, there is an infinite family of models with

$$G = \mathrm{SU}(N-8) \times \mathrm{SU}(N) \times \mathrm{SU}(N+8) \tag{3.16}$$

with bifundamental matter in the $(N-8, N, 1)$ and $(1, N, N+8)$ representations. (This model cannot occur at $T = 1$ since then it is not possible to have $b_1 \cdot b_3 = 0$ when $a \cdot b_i = 0, b_i^2 = -2$). For this model, and for the similar models with the first and/or last factor replaced with $\mathrm{Sp}(N/2-4)$ and/or $\mathrm{SO}(N+8)$, a similar argument to that used to rule out the two-factor infinite families shows that $b_1 + b_2 + b_3 = 0$ when $T < 9$ so that we cannot have $j \cdot b_i > 0$ for all three gauge group factors.

This proves case 2 of the analysis. So we have proven that for $T < 9$ there are a finite number of distinct gauge groups and matter content which satisfy anomaly cancellation with physical kinetic terms for all gauge field factors. We have ruled out infinite families with unbounded numbers of gauge group factors at any finite T, though we show that such infinite families exist when T is unbounded in section 4. We have not ruled out infinite families with a finite number of gauge group factors which become unbounded at finite $T > 8$. Indeed, we give an explicit construction of such a family in section 4.

A systematic enumeration of the finite set of possible gauge groups and matter content compatible with a fixed $T < 9$ is in principle possible. One approach to the enumeration is to break the gauge group into blocks associated with simple factors and their associated matter content, and then to combine blocks in such a way that the gravitational anomaly $H - V$ is not exceeded. This approach was discussed and applied for some classes of models in [5]. With more tensor fields, the limit $H - V = 273 - 29T$ more strictly constrains the range of possible matter representations, although the increased dimensionality of the space $\mathbb{R}^{1,T}$ allows blocks to be combined more freely. One could proceed with a systematic enumeration by sequentially classifying all models with matter transforming under at most λ distinct gauge group factors for increasing values of λ. It is easy to see that for any given gauge group there are a finite number of matter representations such that $H/\lambda - V < 273 - 29T$. This bound is a useful guide in constructing all allowed models, though care must be taken since for any fixed T there can be a finite number of type N blocks which contribute negatively to $H - V$. We leave a complete and systematic enumeration of the finite set of possible $T < 9$ models for future work.

4 Infinite families for $T \geq 9$

In this section we give some examples of infinite families of models which satisfy anomaly cancellation and admit correct-sign kinetic terms, when $T \geq 9$.

4.1 Example: infinite families at fixed $T \geq 9$

From the way in which the finiteness proof breaks down at $T = 9$, it is fairly straightforward to construct an infinite class of apparently-consistent supergravity models at $T = 9$. We consider again the infinite class of models found by Schwarz with gauge group $G = \mathrm{SU}(N) \times$

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SU(N) and two bifundamental matter fields, but now with $T > 8$. For this gauge group and matter representation, at $T = 9$ we need vectors $-a, b_1, b_2$ with inner product matrix

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix} . \tag{4.1}$$

In a basis with $\Omega = \mathrm{diag}(+1, -1, -1, \ldots)$, this can be realized through the vectors

$$\begin{aligned} -a &= (3,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1) \\ b_1 &= (1,\ -1,\ -1,\ -1,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0) \\ b_2 &= (2,\ \ 0,\ \ 0,\ \ 0,\ -1,\ -1,\ -1,\ -1,\ -1,\ -1) \end{aligned} \tag{4.2}$$

This choice of vectors satisfies the correct gauge kinetic term sign conditions $j \cdot b_i > 0$ for $j = (1, 0, 0, \ldots)$. It is straightforward to construct similar examples for $T > 9$ by simply adding additional 1's in additional columns for a.

We will show in section 6 that at $T = 9$ these models are incompatible with F-theory for large enough values of N and thus do not have any known string realization.

4.2 Example: infinite families with unbounded numbers of factors

Although we proved that for any fixed T there are no infinite families with unbounded numbers of factors (case 1), this restriction does not hold when T itself is unbounded. From the gravitational anomaly condition (2.6) it would seem that a family with increasing T is difficult to construct, as the upper bound on $H - V$ becomes increasingly negative. By choosing gauge factors with minimal matter, however, we can find anomaly-free models with arbitrarily many gauge group factors. This is not possible for most types of gauge group factors. For example, for factors of the form SU(N), as noted in [5], for any number of antisymmetric tensor representations the F^4 condition fixes the number of fundamental representations so that $H/2 - V$ is positive. Thus, for factors of this form we cannot build combinations with arbitrarily negative total $H - V$ with matter transforming under at most two gauge group factors.

To minimize the total $H - V$ we can consider gauge group factors such as SO(8) and E_8, for which no charged matter is needed to satisfy the anomaly equations. For a pure SO(8) factor, we have $V = 28$. Since each such factor is associated with a type N vector with $b_i^2 = -4$, we need an additional tensor in T to accommodate a type N vector perpendicular to the b vectors from all other factors for each SO(8) factor. Adding one SO(8) factor and one tensor to a model contributes a total of

$$\Delta(H - V + 29T) = 1 \tag{4.3}$$

to the total gravitational anomaly, so the number of such factors which can be added is large but bounded.

The same is not true for E_8 factors. Each such factor has $V = 248$. Considering only the constraints from anomaly cancellation and gauge kinetic term sign conditions, we can construct a family of models with gauge group $G = E_8^k$ and no charged matter. The

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associated vectors $\{-a,\ b_i\}$ satisfy $-a\cdot b_i = -10, b_i^2 = -12, b_i\cdot b_j = 0, i \neq j$. For sufficiently large T such vectors can be found. For example, when $T = 9 + 8k$, a representation with the inner product $\Omega = \mathrm{diag}(+1, -1, -1, \ldots)$ is given by

$$\begin{aligned}
-a &= (\ \ 3,\ -1,\ (-1)_4\ \ ,\ (-1)_4\ \ ,\ -1,\ \ -1,\ -1,\ -1,\ \cdots) \\
b_1 &= (\ -1,\ -1,\ (-1)_3,\ -3,\ \ (0)_4\ \ ,\ \ 0,\ \ \ 0,\ \ 0,\ \ 0,\ \cdots) \\
b_2 &= (\ -1,\ -1,\ \ \ 0_4\ \ ,\ (-1)_3,\ -3,\ \ 0,\ \ \ 0,\ \ 0,\ \ 0,\ \cdots) \\
&\vdots \\
b_k &= (\ -1,\ -1,\ \ \ 0_4\ \ ,\ \ \cdots\ \ ,\ 0_4,\ (-1)_3,\ -3,\ \ 0,\ \cdots)
\end{aligned} \tag{4.4}$$

The notation x_n indicates that the entry x repeats n times. Note that the last $4k + 8$ entries of all the vectors $b_1,\ b_2, \cdots, b_k$ are all zero. This represents an infinite family of models satisfying anomaly cancellation. There exists a choice of j such that gauge kinetic terms for all factors have the correct sign,

$$j = (-|j_0|, 0, 0, \cdots, 0, 1, 1, \cdots, 1), \quad |j_0| > \sqrt{4k+8}\,, \tag{4.5}$$

where the last $4k+8$ entries in j are 1. By choosing $(4k+8)/3 > |j_0|$, we can also arrange for $-a\cdot j > 0$. As we show in section 6, this class of models is nevertheless not compatible with F-theory and has no known string realization for large enough k.

5 Supergravity models in F-theory

Based only on the structure of the low-energy supergravity theory, we have now shown that there are a finite number of possible gauge groups and matter representations for models with $T < 9$. Every consistent supergravity model, furthermore, is characterized by an integral lattice Λ. The next question we would like to address is which of these models can be realized in string theory. By determining the subset of apparently-consistent low-energy theories which can be realized through each of the known approaches to string compactification, we can hope to chart the full space of 6D supergravity theories. Identifying characteristic features of models which cannot be realized through any existing string construction may lead to the identification of new string vacua, or new constraints on the low-energy theories.

We focus here on identifying F-theory constructions of low-energy supergravity models. In subsection 5.1 we show that the structure of anomaly-free $\mathcal{N} = 1$ 6D supergravity theories is closely related to that of F-theory compactifications, allowing us to map the discrete data of the 6D supergravity theory to topological data for an F-theory construction. This generalizes the analysis of [5], in which the map from low-energy supergravity to F-theory topological data was described for models with $T = 1$.

In subsection 5.2 we examine some of the constraints on low-energy theories which must be satisfied for an F-theory realization to exist. While, as discussed in [5], a large fraction of the apparently-consistent supergravity models at $T = 1$ seem consistent with F-theory, the constraints imposed by F-theory limit the range of possible models substantially as T increases. For $T > 8$, F-theory reduces the infinite number of apparently-consistent

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models to a finite number. We do not attempt to give a complete and definitive analysis of the constraints from F-theory on low-energy theories here, but we identify a number of general constraints on the structure of the low-energy theory imposed by F-theory. We give some specific examples of these F-theory constraints on apparently-consistent models with various values of T in section 6.

5.1 Mapping to F-theory

F-theory[1] [31–33] is a limit of string theory which generalizes type IIB string theory by allowing the axiodilaton τ to vary over a d-dimensional compactification space B. This can be thought of as describing an auxiliary 2-torus whose complex structure depends upon the axiodilaton, giving an elliptic fibration over B. The elliptic fiber degenerates at complex codimension one loci in the base B, which correspond to 7-branes. Specific types of singularities of the elliptic fibration structure on divisors ξ_i in the base give rise to corresponding nonabelian gauge group factors G_i in the resulting low-energy gravity theory. When B is a (complex) 2-dimensional space, the F-theory construction is characterized by an elliptically fibered Calabi-Yau 3-fold with section. As shown in [31–33], F-theory can be used to describe nonperturbative string vacua which are inaccessible by direct supergravity compactification, including 6D models with multiple tensor fields. F-theory is the most general approach developed so far to construct compactifications of string theory to six dimensions. There are heterotic compactifications on K3 with certain kinds of bifundamental matter fields, however, which are not described in the standard F-theory approach. We discuss this further in section 7; we are not aware of any other string constructions of $\mathcal{N} = 1$ 6D supergravity models which do not also have F-theory descriptions.

In low-energy theories that arise from F-theory compactifications, various aspects of the low-energy physics including the gauge group and matter content are controlled by the geometry of the elliptic 3-fold. Much of the work on F-theory has focused on understanding the consequences for the low-energy theory of specific geometric structures in the Calabi-Yau compactification space. We would like to turn this around and ask — Given the low-energy theory, what are the necessary conditions for the existence of a UV-completion in the form of an F-theory compactification?

The structure of the integral lattice Λ determined by the vectors a, b_i is closely related to the cohomology lattice of a two-dimensional F-theory base B. For example, the anomaly conditions (2.13) relate the inner product $b_i \cdot b_j$ to the number of hypermultiplets simultaneously charged under $G_i \times G_j$. In F-theory, the number of such hypermultiplets is related to the intersection product of divisors ξ_i in the base B associated with the nonabelian gauge group factors G_i. Analysis of anomaly cancellation in F-theory constructions shows that these inner products can be identified [11, 34, 35]

$$b_i \cdot b_j = \xi_i \cdot \xi_j \,. \tag{5.1}$$

In fact, as discussed in [5], we can associate the $\mathrm{SO}(1,T)$ vector b_i for each factor G_i with a corresponding divisor ξ_i in the base B. This furthermore leads us to interpret

[1] See [30] for a good general review and introduction to F-theory.

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the bilinear form $\Omega_{\alpha\beta}$ as the intersection product in $H^2(B,\mathbb{Z})$. The vector a with norm $a \cdot a = 9 - T$ is naturally identified with the canonical divisor of the base K_B, which also satisfies $K_B \cdot K_B = 9 - T$. The inner products between a and b_i also agree with the corresponding inner products in F-theory, $-a \cdot b_i = -K_B \cdot \xi_i$ [11, 34, 35]. The requirement of physical gauge kinetic terms requires that there exist an $\mathrm{SO}(1,T)$ vector j satisfying $j \cdot j = 1$ with $j \cdot b_i > 0$ for all i. This vector corresponds in F-theory to the Kähler form J on the base B, and the condition $J \cdot \xi_i > 0$ is the requirement that the curves wrapped by 7-branes have positive volume. This successfully identifies all the parameters of the low-energy theory, up to the two-derivative level,[2] with geometric quantities in the F-theory compactification. To summarize, for any supergravity model with an F-theory realization, we must have a lattice embedding

$$\Lambda \hookrightarrow H^2(B,\mathbb{Z}) \tag{5.2}$$

which can be associated with an explicit map from the vectors a, b_i, j into divisor classes in B so that

$$a \to K_B \tag{5.3}$$

$$b_i \to \xi_i \tag{5.4}$$

$$j \to J \tag{5.5}$$

where K_B is the canonical divisor, the ξ_i are *effective*, *irreducible* curves, and J is a Kähler class on the base.

Example: $T = 1$. In [5] we gave an explicit formulation of the map (5.2) for the case $T = 1$. In that case, the F-theory base manifold is restricted to be a Hirzebruch surface $\mathbb{F}_m$, whose second cohomology admits a basis (integral for even m)

$$e_1 = D_v + \frac{m}{2} D_s \tag{5.6}$$

$$e_2 = D_s \tag{5.7}$$

in which the intersection form takes the form (2.14). In terms of the coordinates $\alpha_i, \tilde{\alpha}_i$ for b_i, the divisor associated with each vector b_i then becomes

$$b_i \to \frac{\lambda_i}{2} \left(\alpha_i e_1 + \tilde{\alpha}_i e_2\right) . \tag{5.8}$$

The correspondence described through this map gives us an explicit construction of the topological F-theory data for any supergravity model which can be realized in F-theory. Not all gauge groups and matter representations associated with anomaly-free supergravity models, however, have valid F-theory realizations. A complete description of the necessary and sufficient conditions on the low-energy supergravity data which guarantee the existence of an F-theory construction is somewhat complicated and is left for future work. We now describe, however, some of the simple constraints necessary for a model to have an F-theory realization. As we discuss in section 6, these constraints are sufficient to rule out a number of apparently-consistent models at $T < 9$ and all infinite families of apparently-consistent models at $T \geq 9$.

[2] Except for the metric on the hypermultiplet moduli space.

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5.2 F-theory constraints on low-energy supergravity

Lattice embedding. The first condition which is clearly necessary to realize a model in F-theory is the embedding condition (5.2), which states that the lattice Λ must admit an embedding into $H^2(B,\mathbb{Z})$ for some F-theory base B. The space $H^2(B,\mathbb{Z})$, as we discuss in detail below, has the structure of a unimodular lattice. The embedding condition (5.2) thus implies the existence of a lattice embedding of Λ into a unimodular lattice.

More specific constraints on which models can be mapped to F-theory can be determined by giving a complete categorization of the cohomology groups of complex surfaces B which are acceptable base manifolds for an F-theory compactification. To this end we now discuss in slightly more detail the geometry of the base B, which is a general complex, Kähler, 2-dimensional surface with an effective anti-pluricanonical divisor. (The existence of such a divisor is a weak form of "positive curvature".) The space $H^2(B,\mathbb{Z})$ has the structure of a free $\mathbb{Z}$-module (without torsion) of rank $b_2(B)$, and the intersection product defines a symmetric inner product, making this into a "lattice". Poincaré duality further implies that the lattice with the inner product is self-dual, or equivalently unimodular. The signature of the lattice is $(2h^{2,0}+1, h^{1,1}-1)$ by the Hodge index theorem, where $h^{i,j}$ denote the Hodge numbers. If the base B had any holomorphic 1-forms or holomorphic 2-forms, then the total space of the elliptic fibration would also have holomorphic 1-forms of holomorphic 2-forms, and so it would have enhanced supersymmetry (and necessarily be of the form $(K3\times T^2)/G$ or T^6/G). Thus, to ensure that the F-theory model has exactly $\mathcal{N}=1$ supersymmetry, we must assume that $h^{1,0}(B)=h^{2,0}(B)=0$; it follows that the lattice has signature $(1, h^{1,1}-1)$.

There are two key properties of the base B which lead to a complete classification: first, the line bundles $\mathcal{O}(-4K_B)$ and $\mathcal{O}(-6K_B)$ have sections f and g (which serve as coefficients in the Weierstrass equation of the F-theory model), and second, $h^{1,0}(B)=h^{2,0}(B)=0$. It then follows from the classification of algebraic surfaces (see for example [36]) that either B is an Enriques surface, $B=\mathbb{P}^2$, or B is the blowup of a Hirzebruch surface $\mathbb{F}_m$ in $k\geq 0$ points. A third property — that there is no curve in B along which f vanishes at least 4 times and g vanishes at least 6 times (the "minimal" property of a Weierstrass equation)—guarantees that $|m|\leq 12$ [32, 33].

The number of tensor multiplets in the low-energy theory is $T=h^{1,1}-1$ [32, 33], and so $H^2(B,\mathbb{Z})$ is a $T+1$ dimensional unimodular lattice of signature $(1,T)$. If the lattice is even, then by Milnor's theorem [37] we must have $T\equiv 1 \pmod 8$ and the lattice is isomorphic (as a $\mathbb{Z}$-module) to $U\oplus E_8(-1)^{\oplus k}$, where

$$U=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{5.9}$$

When the lattice is odd, then the lattice is isomorphic to $\mathbb{Z}^{T+1}$ with the inner product $\mathrm{diag}\{+1,-1,-1,\cdots,-1\}$. The only bases with even lattices are the Hirzebruch surfaces $\mathbb{F}_m$ for even m, with lattice U, and the Enriques surface with lattice $U\oplus E_8(-1)$. All other possible bases are blowups of Hirzebruch surfaces $\mathbb{F}_m, |m|\leq 12$, and $\mathbb{P}^2$, all of which lead to odd lattices.

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To summarize, given a low-energy theory with T tensor multiplets characterized by the lattice Λ, an F-theory realization can only exist if Λ embeds, as a lattice, into a signature $(1, T)$ unimodular lattice.

In F-theory, the lattice $H_2(B, \mathbb{Z}) \cong H^2(B, \mathbb{Z})$ corresponds to the charge lattice of BPS states obtained by wrapping D3 branes on curves in B. This is precisely the charge lattice of BPS dyonic string states $\tilde{\Lambda}$, discussed in section 2.3, into which there must be a lattice embedding $\Lambda \hookrightarrow \tilde{\Lambda}$. A similar unimodularity condition arises in the compactification of the heterotic string on the torus, from modular invariance of the world-sheet string theory. It is interesting to speculate whether this kind of unimodularity condition may be a general consistency condition for any quantum 6D supergravity theory. It may be, for example, that such a condition is necessary for unitarity of the theory. We leave further investigation of this question to future work.

Constraint on canonical class and singular divisors. F-theory imposes strong constraints on the possible values of a. From (5.3), a maps to the canonical class K_B of B. For those surfaces with $H^2(B, \mathbb{Z}) \cong U$, we can always choose a basis so that $a \to K_B = (-2, -2)$ as discussed in section 2. For the Enriques surface, $K_B = a = 0$. For all the remaining surfaces, we can choose a basis with respect to which $H^2(B, \mathbb{Z})$ has inner product $\mathrm{diag}\{+1, -1, -1, \cdots, -1\}$, and such that K_B takes the form

$$-K_B = (3, -1, -1, \ldots, -1)\,. \tag{5.10}$$

This imposes substantial constraints on the choice of a. In particular, a is *primitive*[3] in all cases with odd lattices.

The geometry of elliptic fibrations implies that vectors b_i must map to *effective irreducible divisors* ξ_i in any F-theory realization under the map in (5.4). This constraint has various consequences, an example of which is the following:

Claim. If $b^2 < 0$, then the vector b must be primitive in any F-theory realization.
We prove this by contradiction; assume that b maps to an irreducible, effective divisor ξ, and that $b^2 < 0$ with b a non-primitive vector. Then, there exists an integer $n > 1$ such that the class $\xi' := \xi/n$ is integral and, therefore, an effective divisor. Now, $\xi' \cdot \xi < 0$, and since ξ is an irreducible, effective divisor this implies that ξ' must contain ξ as a component. This is impossible because it would require the class $(1/n - k)\xi$ to be effective, for integers $k \geq 1,\ n \geq 2$.

These conditions on a, b_i impose further constraints on which supergravity models can be compatible with F-theory.

Positivity conditions and the Kähler and Mori cones. As noted above, the supergravity constraint that all gauge fields have kinetic terms of the correct sign, $j \cdot b_i > 0$, has a corresponding interpretation in F-theory. In F-theory, the divisors ξ_i supporting the singularity giving rise to nonabelian gauge group factors must all be effective and irreducible, from which it follows that $J \cdot \xi_i > 0$ where J is the Kähler form of B. This is an example of

[3] A lattice vector v is *primitive* if $\frac{1}{d}v$ is not a lattice vector $\forall\ d \in \mathbb{Z},\ |d| \neq 1$.

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an F-theory constraint with a clear analogue in the low-energy theory. In F-theory there is a similar constraint on the (negative of the) canonical class $-K_B$, so that for all F-theory compactifications $-K_B \cdot J > 0$. This constraint has no obvious counterpart in supergravity. Note, however, that just as supersymmetry constrains the action so that the gauge kinetic term is proportional to $-j \cdot b_i \operatorname{tr} F_i^2$ [10], a similar argument suggests that the action should have a higher-derivative term proportional to $j \cdot a \operatorname{tr} R^2$. Such higher-derivative terms can have sign constraints from causality [38]; we leave a further exploration of this possible constraint on low-energy models for further work.

To understand the F-theory constraints on ξ_i and J more clearly, it is helpful to describe in more detail the structure of the Kähler cone and dual Mori cone.

As discussed in section 2, the low-energy theories have an $\mathrm{SO}(1,T)/\mathrm{SO}(T)$ moduli space of tensor multiplet scalars and a hypermultiplet moduli space. In the F-theory compactifications we are considering, the $h^{1,1}(B) = T + 1$ Kähler moduli of the base correspond to the tensor moduli space (except the overall volume of B, which is in a hypermultiplet), while the complex structure moduli of the elliptic fibration correspond to the hypermultiplet moduli space of the low-energy theory. The Kähler metric is completely specified by a choice of Kähler form $J \in H^{1,1}(B,\mathbb{R}) \cong H^2(B,\mathbb{Z}) \otimes \mathbb{R}$. Therefore, J is a vector in the space $\mathbb{R}^{1,T}$ which can be normalized to satisfy $\mathrm{Vol(B)} = 1 = \frac{1}{2}\mathrm{J} \cdot \mathrm{J}$. To study the structure of the moduli space, we can imagine starting with a fixed vector J and looking at the transformations of $\mathbb{R}^{1,T}$ that generate inequivalent Kähler forms. The total space of such transformations is $\mathrm{SO}(1,T)$; $\mathrm{SO}(T)$ transformations in the transverse space orthogonal to J do not change the metric, and so the moduli space of inequivalent metrics is (locally) parameterized by $\mathrm{SO}(1,T)/\mathrm{SO}(T)$.

In general, B has an automorphism group $\widetilde{\Gamma}$ and some quotient Γ of $\widetilde{\Gamma}$ acts faithfully on $H^2(B,\mathbb{Z})$. For example, if $B = \mathbb{F}_0$ or B is the blowup of $\mathbb{F}_0$ at a single point, then $\Gamma \cong \mathbb{Z}_2$, while if B is the blowup of $\mathbb{F}_0$ at two distinct points then Γ is the dihedral group of order 12. Note that Γ must be a subgroup of $\mathrm{Aut}(H^2(B,\mathbb{Z}))$, the automorphism group of the lattice (which leaves the inner product invariant). The subgroup $\Gamma \subset \mathrm{Aut}(H^2(B,\mathbb{Z}))$ induced by automorphisms of B introduces a further identification on the moduli space, since these just correspond to large diffeomorphisms. The moduli space of fixed volume, Kähler metrics is then

$$\mathcal{M}_K \subset \Gamma \backslash \mathrm{SO}(1,T)/\mathrm{SO}(T) \tag{5.11}$$

This locally agrees with the structure of the moduli space we see from the low-energy theory. The discrete group $\Gamma \subset \mathrm{Aut}(H^2(B,\mathbb{Z}))$ corresponds to the S-duality group; in the $T = 1$ cases this was discussed in [39].

The constraint on J from the F-theory side is that J must lie within the Kähler cone of the base surface. Since $h^{2,0}(B) = 0$, the *Kleiman criterion* [40] characterizes the Kähler cone as the set of those J such that (1) $J \cdot J > 0$ and, (2) for all effective divisors D, $J \cdot D > 0$. If we normalize the volume to 1, the first condition simply states that J lies in $\mathrm{SO}(1,T)/\mathrm{SO}(T)$ as above. To analyze the second condition, it is useful to work with the dual of the Kähler cone, called the Mori cone [41], which is the set of linear combinations $\sum r_i D_i$ of effective divisors D_i using nonnegative real coefficients r_i. A Kähler class J is

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outside the Kähler cone if $J \cdot D < 0$ for D an effective divisor. For such Kähler classes, there is no known F-theory vacuum construction. Note that the Kähler cone is thus essentially defined in terms of the Mori cone.

The F-theory constraint that ξ_i is effective and irreducible implies that ξ_i lies within the Mori cone for B. Thus, it follows from combining this constraint with the definition of the Kähler cone that $J \cdot \xi_i > 0$ for all i, though the Mori cone and Kähler cone conditions taken together are a stronger set of constraints than this inequality which we can understand from the low-energy theory.

As an example of the Mori cone and dual Kähler cone, consider the bases $\mathbb{F}_m$, which lead to $T = 1$. For these bases B, the set of effective divisors is generated by D_v and D_s, with intersection pairings

$$D_v \cdot D_v = -m, \qquad D_v \cdot D_s = 1, \qquad D_s \cdot D_s = 0\,. \tag{5.12}$$

Effective divisors corresponding to irreducible curves are given by

$$D_v, \quad aD_v + bD_s,\ a \geq 0, b \geq ma\,. \tag{5.13}$$

In this case, the Mori cone occupies the first quadrant $\{aD_v + bD_s \mid a \geq 0, b \geq 0\}$, and the dual Kähler cone can be described as $\{aD_v + bD_s \mid a \geq 0, b \geq ma\}$. The volume one classes in the Kähler cone are those $aD_v + bD_s$ with $a > 0$ for which

$$b = \frac{ma^2 + 1}{2a}. \tag{5.14}$$

As another example of a Kähler cone, we consider the blowup of $\mathbb{F}_1$ in a single point away from D_v (which coincides with the blowup of $\mathbb{P}^2$ at two distinct points). We can take as a basis for $H^{1,1}(B)$ the exceptional divisor E as well as curves D_v and $D_s - E$, where D_v and D_s are pulled back from $\mathbb{F}_1$. (Note that $D_s - E$ is the proper transform of that fiber on $\mathbb{F}_1$ which passed through the point which was blown up.) In this basis, if we write a putative Kähler class as $J = aE + bD_v + c(D_s - E)$ then

$$J^2 = -a^2 - b^2 + 2ac + 2bc - c^2 \tag{5.15}$$

so if we set the volume to one we can solve for c:

$$c = a + b \pm \sqrt{2ab - 1}. \tag{5.16}$$

We learn that $ab \geq \frac{1}{2}$; also, since $J \cdot D_s = b$ and D_s is effective, a and b must be positive. The set of such volume one classes can be represented as a double cover of the semi-hyperbola $ab \geq \frac{1}{2}$ branched on the boundary.

We now investigate the conditions on the Kähler cone imposed by the various effective divisors. We have

$$J \cdot E = -a + c = b \pm \sqrt{2ab - 1} \tag{5.17}$$

$$J \cdot D_v = -b + c = a \pm \sqrt{2ab - 1} \tag{5.18}$$

$$J \cdot (D_s - E) = a + b - c = \mp\sqrt{2ab - 1}. \tag{5.19}$$

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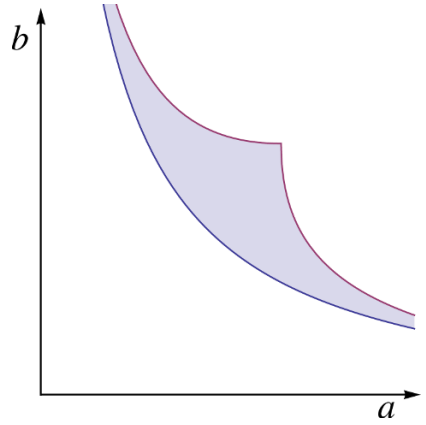

Figure 1. Kähler cone for the one point blowup of $\mathbb{F}_1$ away from D_v.

Since all of these must be nonnegative, we must have $c = a + b - \sqrt{2ab - 1}$ (which selects one of the two branches of the double cover), as well as $b \geq \sqrt{2ab - 1}$ and $a \geq \sqrt{2ab - 1}$. These latter two conditions are additional semi-hyperbolas in the $a - b$ plane, and the region they define is illustrated in figure 1 above.

Note that the $\mathbb{Z}_2$ automorphism in this example exchanges the curves D_v and E, and so acts to exchange a and b in the equations and figure above.

Kodaira condition. The Kodaira condition corresponds to the mathematical requirement that the elliptic fibration over B with singularities on the divisors ξ_i associated with nonabelian group factors gives a Calabi-Yau manifold. This condition states that

$$-12K_B = \sum_i \nu_i \xi_i + Y \tag{5.20}$$

where K_B is the canonical divisor, ν_i are multiplicities associated with different singularity types (e.g. N for SU(N), 6 for SO(8), 10 for E_8, etc.), and Y is a residual divisor, which must be a sum of effective divisors and satisfies $J \cdot Y > 0$ unless $Y = 0$. Pulling this relation back to the low-energy theory, this condition becomes

$$j \cdot \left(-12a - \sum_i \nu_i b_i \right) \geq 0. \tag{5.21}$$

This condition is also helpful as a simple guide in determining which low-energy theories admit an F-theory realization.

Weierstrass model. As far as we know, the existence of a map (5.2) to a prospective F-theory base does not guarantee that there is an F-theory construction for a given model, even when all the topological and Kähler cone conditions just listed are satisfied. An explicit elliptic fibration must be constructed with correct singularity types for the desired matter content; this is generally accomplished using Weierstrass models, the Kodaira singularity classification, and the Tate algorithm [32, 33, 42]. We do not know any general way to prove from the topological data that there exists a Weierstrass model with the desired

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properties. In [5] we gave explicit constructions of Weierstrass models for some simple low-energy gauge group and matter combinations with $T = 1$. We found that in these cases there is a precise correspondence between the number of degrees of freedom needed in the Weierstrass model to construct given gauge and matter content and the associated value of $H - V$ in the low-energy model. Based on this correspondence we conjectured (by a simple degree of freedom counting) that it will always be possible to construct Weierstrass models precisely when the gravitational anomaly bound on $H - V$ is not exceeded. It would be nice to have either a proof of this conjecture, or an explicit counterexample. Note, however, that there are various matter combinations found in [5] which do not currently have known F-theory realizations through explicit local Weierstrass constructions. A proof of the general conjecture that Weierstrass models exist for any model admitting a map (5.2) satisfying the various topological F-theory conditions would presumably require a more complete understanding of the range of possible matter content which can be produced by local singularities, even in the case $T = 1$.

6 Examples

In the previous section we have described some features of the geometrical data underlying any F-theory construction. We do not have a completely determined set of rules which can be used to identify the subset of low-energy supergravity theories which do have F-theory realizations. What does seem to be the case, however, is that when no map of the form (5.2) exists from Λ, a, b_i, j into the geometry associated with any possible F-theory base B for a given low-energy model, or one of the previously listed constraints such as the Kodaira condition is violated, there is no known string construction of that low-energy model. Models which do not admit such a map must at the present time be regarded as lying in the "swampland" [43–45] consisting of theories which cannot be ruled out from low-energy considerations and yet which cannot be realized in string theory. In this section we describe some explicit examples of such low-energy models with no known string theory realization.

We focus in this section on explicit examples of low-energy theories which illustrate various features of the associated lattice and F-theory map. We begin with several examples encountered in the previous paper [5] on $T = 1$ models. We first describe a large class of simple models which do apparently admit F-theory realizations, and then describe models where the F-theory map does not lead to acceptable divisors. The lattice embedding language clarifies the issues involved in these cases. We then return to the infinite families described in section 4, and show how these families violate some of the conditions for F-theory constructions.

6.1 Examples with F-theory realizations: SU(N) product models at $T = 1$

In [5] we systematically analyzed a simple general class of supergravity models, considering all models with gauge group factors SU(N), $N \geq 4$ and matter in fundamental, bifundamental, and antisymmetric tensor representations. We identified all 16,418 models of this type which satisfy anomaly cancellation at $T = 1$. We performed some basic checks which

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indicated that the topological conditions such as the Kodaira condition are satisfied automatically for all these models as a consequence of the anomaly cancellation conditions. We developed Weierstrass models for a few of these theories and found that generally the number of degrees of freedom fixed in the Weierstrass polynomial description matched precisely with the contribution to $H-V$ from each part of the model. This is expected, as the number of unfixed degrees of freedom corresponding to moduli in the model should correspond to the number of uncharged hypermultiplets. This agreement makes it seem plausible that all the models in this class, which satisfy all the F-theory topological conditions, have true F-theory realizations through Weierstrass models. As we discuss below, the close agreement between supergravity anomaly constraints and topological F-theory constraints seems particularly strong at $T=1$; for larger values of T, we find more apparently satisfactory low-energy supergravity models which cannot be realized in F-theory; examples of this type even arise among the class of models with $\mathrm{SU}(N)$ gauge group factors restricted to matter in fundamental, bifundamental, and antisymmetric tensor representations.

6.2 Example: embedding failure

In [5] we found a number of models at $T=1$ which do not seem to have acceptable F-theory counterparts. It is illuminating to consider these models from the point of view of the lattice embedding $\Lambda \to H^2(B,\mathbb{Z})$. One problematic model encountered in [5] has the following structure:

$$G = \mathrm{SU}(4) \qquad \text{matter} = 1 \times \text{adjoint} + 10 \times \begin{smallmatrix}\square\\\square\end{smallmatrix} + 40 \times \square \tag{6.1}$$

$$\Lambda = \begin{pmatrix} 8 & 10 \\ 10 & 10 \end{pmatrix} \tag{6.2}$$

$$H - V = 220 \tag{6.3}$$

It is not hard to check that this lattice Λ cannot be integrally embedded in any unimodular $\mathrm{SO}(1,1)$ lattice. In particular, there is no embedding in the lattice U, where we can choose $-a=(2,2)$; then in terms of (2.16) we have $\alpha,\tilde{\alpha}=5\pm\sqrt{5}$. There is also no integral embedding in the lattice diag $(+1,-1)$. Choosing $-a=(3,-1)$, we have $b=(x,y)$ with $3x+y=10, x^2-y^2=10$, with no integer solutions.

Thus, this model seems not to have a realization in F-theory. A similar situation arises for the same group with one adjoint, 11 antisymmetric and 44 fundamental representations, with $H-V=242$. These models are the only ones we have encountered explicitly at $T=1$ which do not have a unimodular lattice embedding. It would be very interesting to understand whether there is a novel string construction which could lead to models such as these, or if the absence of a unimodular embedding can be related to a breakdown of consistency for a general quantum gravity theory.

6.3 Example: outside the Mori cone

Another class of models we found in [5] which does not appear to admit an F-theory realization has less extreme problems. For example, consider a model with

$$G = \mathrm{SU}(N) \qquad \text{matter} = 1 \times \square\square + (N-8) \times \square \tag{6.4}$$

$$\Lambda = \begin{pmatrix} 8 & -1 \\ -1 & -1 \end{pmatrix} \tag{6.5}$$

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This model has an embedding into the lattice diag $(+1,-1)$ realized through the vectors $-a = (3,-1), b = (0,-1)$. If we identify the lattice diag $(+1,-1)$ with the second cohomology of $\mathbb{F}_1$ by using the standard basis (D_u, D_v) with $D_u = D_v + D_s$ (which satisfies $D_u^2 = 1, D_v^2 = -1, D_v \cdot D_u = 0$), then the class $-a$ maps to $3D_u - D_v = 2D_v + 3D_s$ which coincides with $-K_{\mathbb{F}_1}$. However, b maps to $-D_v$, which is not an effective class, so this embedding does not send the vector b into the Mori cone. In fact, further checking shows that *no* embedding in this case maps a to $K_{\mathbb{F}_1}$ while sending b into the Mori cone. Thus, there is no F-theory realization of these models.

There are a number of other low-energy models which do not admit an F-theory realization in the analysis of [5], and which suffer from similar "cone" problems when a lattice embedding is found. For example, we encountered in [46] a model with structure

$$G = \mathrm{SU}(24) \times \mathrm{SO}(8) \qquad \text{matter} = 3 \times (\square, 1) \tag{6.6}$$

$$\Lambda = \begin{pmatrix} 8 & 3 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \tag{6.7}$$

$$H - V = 225 \tag{6.8}$$

This lattice is degenerate and admits an embedding into diag $(+1,-1)$ through

$$-a = (3,-1) \tag{6.9}$$

$$b_1 = (1,0) \tag{6.10}$$

$$b_2 = (0,-1) \tag{6.11}$$

Just as above, this is compatible with the intersection form on $\mathbb{F}_1$, but where $b_2 \to -D_v$, giving a class outside the Mori cone.

It would be interesting to investigate whether there is some residue of the Mori cone constraint in low-energy supergravity. One can imagine that a class of apparently-consistent low-energy theories with no F-theory realization may be given by a novel stringy construction analogous to F-theory outside the Mori cone (or its dual, the Kähler cone). It is familiar from type II compactification that passing outside the Kähler cone may give valid orbifold or non-geometric Landau-Ginzburg phases of string compactifications [47]. This seems harder to understand in the F-theory context, where the moduli are real,[4] than in the type II context, where the moduli are complex and it is easier to continuously deform a model to a region outside the Kähler cone. In any case, it would be interesting to explore this phenomenon in F-theory, or through dual constructions.

6.4 Example: new possibilities and constraints for $T > 1$

When the number of tensor multiplets T increases, the constraint from the gravitational anomaly becomes stronger as $H - V \leq 273 - 29T$. This would seem to more strongly constrain the number of possible models. In general, models which are possible at $T = 1$

[4] Note that in another context with real Kähler moduli — compactifications of M theory to five dimensions — some of the walls of the Kähler cone serve as obstructions to further deformation [48, 49].

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continue to be acceptable until T is so large that the number of charged hypermultiplets exceeds the constraint from the gravitational anomaly; in most cases this can be realized by simply adding an additional unit to $-a$ in the extra negative-definite dimension provided by each additional tensor multiplet. Because the dimensionality of the space in which a, b_i are embedded increases as T increases, however, new possibilities for combinations of gauge group factors arise. Although this increases the number of apparently-consistent supergravity models, F-theory realizations of these new possibilities at larger values of T are more strongly constrained by the condition of embeddability in a unimodular lattice.

SU(N) factors at $T = 2$. A simple example of how the situation changes at increased T is given by the class of models having gauge group factors SU(N) with matter only in the fundamental representation. Anomaly cancellation conditions require that the number of fundamentals for each such factor is $f = 2N$. At $T = 1$, models with a single SU(N) gauge group factor are possible up to $N = 15$ by the gravitational anomaly condition. It seems that these models all admit F-theory realizations. The associated divisor in $\mathbb{F}_2$ for these models is D_v, and $-K = 2D_v + 4D_s$, so all topological constraints are satisfied for each of these models. We constructed explicit Weierstrass models for this family of models up to SU(14) in [5], and believe that such a model exists for SU(15), though the algebra needed to explicitly construct such a solution becomes complicated.

Now consider models with multiple SU(N) factors under each of which all matter transforms under either the trivial or fundamental representation. The vector b associated with an SU(N) factor with $f = 2N$ fundamental matter fields must satisfy $a \cdot b = 0, b^2 = -2$. At $T = 1$, we cannot have a model with more than one such SU(N) factor and only fundamental matter, since $a \cdot b = 0$ gives a unique vector b up a scale factor. When we consider analogous models at $T = 2$, however, the situation is rather different. Consider a group $G = \mathrm{SU}(N) \times \mathrm{SU}(N)$ where each of the matter fields transforms in the fundamental representation of at most one SU(N) factor (we assume for simplicity that there are no bifundamental matter fields). For $T = 1$, as just mentioned, there is no consistent model with physical gauge kinetic terms. On the other hand, when $T > 1$, we have the same inner products $a \cdot b_i, b_i \cdot b_j$, but the vectors b_1, b_2 need not be linearly dependent. The associated lattice is

$$\Lambda = \begin{pmatrix} 9-T & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{6.12}$$

From the low-energy point of view this seems like a perfectly acceptable model. There is, however, no embedding of this lattice into a unimodular lattice for $T = 2$. In the canonical form of the SO(1,2) metric diag $(+1, -1, -1)$, where $-a = (3, -1, -1)$, if we write $b = (x, y, z)$ for integral x, y, z then we must have

$$3x + y + z = 0 \tag{6.13}$$

$$x^2 - y^2 - z^2 = -2\,. \tag{6.14}$$

The only solutions to these Diophantine equations are

$$b = (0, 1, -1), \qquad b = (0, -1, 1)\,, \tag{6.15}$$

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so we cannot have two vectors b_1, b_2 in a (1, 2) unimodular lattice which are perpendicular both to one another and to a, where both have norm -2. Thus, though at $T = 2$ this model seems perfectly acceptable from the low-energy point of view, it cannot be realized through F-theory by any known mechanism.

We see from this example that as T increases, the constraints on which gauge factors and matter content can be combined in the low-energy theory are reduced, and the unimodular embedding constraint becomes a stronger constraint. This seems to be a general feature of models at $T > 1$. Thus, the fraction of models in the swampland seems to increase at larger T, even though the gravitational anomaly constraint becomes more stringent.

6.5 Example: infinite families at $T \geq 9$

In section (4.1) we described an infinite family of models at $T \geq 9$ analogous to the $\mathrm{SU}(N) \times \mathrm{SU}(N)$ models shown by Schwarz to satisfy anomaly factorization at $T = 1$ in [29]. It is clear from the above discussion that these models violate the consistency conditions needed for an F-theory construction. Note that the a and b_i's from (4.2) satisfy $-a = b_1 + b_2$ at $T = 9$. This means that $Y = -12a - N(b_1 + b_2)$ must have $j \cdot Y < 0$ at $N > 12$, violating the Kodaira condition (5.21). The vectors listed in (4.2) are not uniquely determined, so one may suspect that there might exist another choice of vectors that satisfy the necessary conditions for an F-theory embedding. This is not possible, however. The preceding argument can be strengthened and generalized along the following lines to show that no infinite family of models can be realized in F-theory at $T = 9$. For any such model we have, as discussed in case 2 in the proof in section 3, $a \cdot (b_1 + b_2) = 0$ and $(b_1 + b_2)^2 = 0$. For $T = 9$, $a^2 = 0$ so a is a null (type Z) vector. It follows that $b_1 + b_2 = -xa$. Moreover, since $-a, b_1, b_2$ are effective, we have $x > 0$. The vector a must be primitive in F-theory, and can be put in the form $-a = (3, -1, -1, \cdots)$, so x is an integer. So, $j \cdot Y = j \cdot (-12a)(1 - a_N x)$, where a_N is $\frac{N}{12}$ for $\mathrm{SU}(N)$, $\frac{N+6}{12}$ for $\mathrm{SO}(2N)$, $\frac{N}{12}$ for $\mathrm{Sp}(\frac{N}{2})$ [32, 33]. Since a_N grows with N in all three cases, at large enough N we must have $j \cdot Y < 0$. It follows that the Kodaira condition is violated at large enough N. This bounds the range of N at $T = 9$ for any of the infinite classes of models considered in case 2 in the proof of finiteness. Note that in [50, 51], the $\mathrm{SU}(N) \times \mathrm{SU}(N)$ model with 9 tensor multiplets was identified for $N = 8$.

We also described in section 4.2 an infinite family of models with gauge group E_8^k and $T = 9 + 8k$ for arbitrary k. For the vectors $-a$, b_i listed in (4.4), there is no F-theory realization. This is because the residual locus in such an embedding is not effective. It is easily checked that

$$\begin{aligned} j \cdot Y &= 12 j \cdot (-a) - 10 \sum_{i=1}^{k} j \cdot b_i \\ &= 12(4k+8) - |j_0|(10k+36) \\ &\quad < 12(4k+8) - (10k+36)\sqrt{4k+8} < 0, \text{ for } k \geq 1 \end{aligned} \tag{6.16}$$

This conclusion depends upon the choice of vectors in 4.4. Note that the $k = 2$ member of this family can be realized in heterotic-M-theory compactified on $K3 \times S^1/\mathbb{Z}_2$, as outlined

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in [39]. The fact that this infinite family of models cannot be realized in F-theory also follows, however, from a more general argument which we now give.

In fact, there is a uniform bound on the rank of the total gauge group which holds for all F-theory models (although we do not know precisely what the bound is), and this excludes the infinite family with gauge group E_8^k as well as all other infinite families. To see that there is such a bound, note that, as discussed previously, the base B of any F-theory model must admit a map to a minimal surface $B_{\min}$ which is either an Enriques surface, $\mathbb{P}^2$, or one of the Hirzebruch surfaces $\mathbb{F}_m$ with $|m| \leq 12$. The coefficients f and g in the Weierstrass equation of the F-theory model push forward to sections $\bar{f} \in H^0(-4K_{B_{\min}})$ and $\bar{g} \in H^0(-6K_{B_{\min}})$, with an induced discriminant $\bar{\Delta} = \{4\bar{f}^3 + 27\bar{g}^2 = 0\} \in |-12K_{B_{\min}}|$. Now each component of the gauge group is either associated to a component of $\bar{\Delta}$ (with its rank determined by the multiplicity) or to a singular point of $\bar{\Delta}$ which is blown up by the map $B \to B_{\min}$. The multiplicities of the components of $\bar{\Delta}$ are uniformly bounded (since $-12K_{B_{\min}}$ can be written as a sum of effective divisors in only finitely many ways), so we only need to show that the total ranks of gauge groups coming from singular points are bounded. Note that when $B_{\min}$ is an Enriques surface, $\bar{\Delta}$ is empty so there is nothing to check.

For each fixed $B_{\min}$ which is not an Enriques surface, we consider the set of all possible pairs $(\bar{f}, \bar{g}) \subset H^0(-4K_{B_{\min}}) \oplus H^0(-6K_{B_{\min}})$. We can stratify this set according to the types of singularities of $\bar{\Delta}$ which appear, and each stratum is a locally closed algebraic subset. Moreover, each stratum has a unique associated gauge group, and so there is a specific rank which is associated to it.

But the Hilbert basis theorem implies that any stratification of an affine algebraic variety into locally closed algebraic subsets has only finitely many strata. Thus, there are only finitely many possible different gauge groups which can occur, so in particular, their ranks must be bounded. And since there are only finitely many possibilities for $B_{\min}$ (other than Enriques surfaces), there is a uniform bound for all F-theory models. As in the proof of finiteness from section 3, this also shows that there is a finite number of distinct gauge groups and matter representations which can be realized through F-theory.

7 Global picture of the 6D $\mathcal{N} = 1$ landscape

The strong constraints which anomalies place on nonabelian gauge group structure and matter content in $\mathcal{N} = 1$ 6D theories have given us a global outline of which of these supergravity theories have the potential to describe consistent quantum theories. Explicit knowledge of this set of theories gives us a powerful tool for exploring the connection between string theory and low-energy physics. We can in principle make a list of the finite number of possible theories with $T < 9$. For each of the various approaches to string compactification (heterotic, F-theory, ...) we can then determine which subset of these possible theories can be realized through each class of construction. For $T > 8$, there are infinite families of models satisfying the known low-energy consistency conditions, and our analytic control of the total space is weaker. In this section we summarize some of our knowledge regarding the extent and connectivity of the 6D $\mathcal{N} = 1$ supergravity landscape.

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Note that the term "landscape" is often used to denote a space of effective theories containing discrete points with no massless moduli (flat directions). Generally such a landscape includes supersymmetric AdS vacua in addition to metastable dS vacua with broken supersymmetry. In the case of six-dimensional vacua, it is actually impossible to stabilize all the moduli while preserving supersymmetry. In Minkowski space, the gravitational anomaly condition $H - V + 29T = 273$ makes it impossible to avoid moduli. Since the tensor multiplet contains massless scalars, to avoid tensor moduli we must have $T = 0$. This implies that $H \geq 273$ and therefore we have hypermultiplet moduli. In six dimensions, there are also no $\mathcal{N} = 1$ supersymmetric AdS vacua. This can be seen from the fact that there are no AdS_6 superalgebras with 8 supercharges in the Nahm classification [52]. The "landscape" in this paper refers to the complete moduli space of 6D gravity theories with one supersymmetry, although all these vacua are Minkowski and have massless scalar moduli.

7.1 Extent of the space of 6D supergravities

For $T = 1$, our understanding of the space of theories, while still incomplete in many respects, seems to suggest a simple global picture. There are a finite number of low-energy nonabelian gauge groups and matter content which are compatible with anomaly cancellation and physical constraints on gauge kinetic terms. We have an explicit approach to embedding these models into F-theory, which seems successful for almost all acceptable gauge groups and matter content. There are a few exotic combinations of gauge groups and matter representations which give lattices which cannot be embedded into any unimodular lattice associated with an acceptable F-theory base. For the moment, these models live in the swampland. In the $T = 1$ supergravity landscape there are also some models which we have found whose realization in F-theory would require divisors outside the Mori cone. This may correspond to a new class of orbifold or non-geometric F-theory construction whose precise implementation remains to be elucidated. For the vast majority of $T = 1$ models, we have an explicit map from each model to a set of divisor classes in a given F-theory base. Degree of freedom counting suggests that this topological data can be completed to a full F-theory construction through a Weierstrass model, though a proof of this assertion in a general context remains to be found. We have not addressed here the question of U(1) factors in the gauge group and their associated charges. The anomaly constraints on U(1) factors are more complicated [14] and lead to systems of Diophantine equations, whose analysis will be discussed elsewhere [9].

We have shown in this paper that the situation for $T < 9$ is similar to that for $T = 1$ as a^2 is positive. Again, there are a finite number of distinct nonabelian groups and matter representations possible for this class of models. As T increases, it seems to be easier to construct models which violate the conditions outlined in section 5.2 and which, therefore, have no F-theory realization. Thus, the apparent swampland increases as T increases.

For $T \geq 9$, however, the situation changes dramatically. As a^2 becomes negative, infinite families of apparently-consistent low-energy theories appear. Some infinite families arise at fixed T, such as an infinite family we have explicitly constructed with gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N)$ at $T \geq 9$. The infinite families at finite T must contain a bounded

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number of gauge group factors, as shown in section 3. Other infinite families of models extend to arbitrarily large values of T, and the number of simple factors in the gauge group can become unbounded as $T \to \infty$. The infinite families described in section 4 satisfy the unimodular embedding constraint, indicating that this constraint is not a strong constraint for the existence of an F-theory construction. As shown in section 6.5, however, the infinite families we have explicitly constructed are not compatible with other constraints from F-theory, such as the Kodaira constraint.

The bounded rank argument in section 6.5 shows that the number of 6D $\mathcal{N} = 1$ supergravity models compatible with F-theory must be finite. A heuristic version of this argument is as follows: each of the nonabelian gauge group factors G_i and associated matter fields are realized in F-theory by tuning the coefficients of polynomials f, g in the Weierstrass model

$$y^2 = x^3 + fx + g\,. \tag{7.1}$$

For any F-theory model on a blowup of a space $\mathbb{F}_m$, f, g can be thought of as sections of $-4K, -6K$ respectively over (a generally singular) $\mathbb{F}_m$, and have a fixed number of total coefficients available for tuning (roughly $244 = 273-29$). Since each gauge group factor and associated matter require tuning additional coefficients to achieve the desired singularity type for the fibration (which may be at a singular point in the base which needs to be blown up), only a finite number of distinct combinations of gauge groups and matter content can be realized in this fashion.

This argument can be translated into the language of the low-energy supergravity theory. As described in [5], in any 6D $\mathcal{N} = 1$ supergravity theory the set of fields can be decomposed into "blocks" associated with the simple factors in the gauge group and associated matter representations. The finiteness argument above for F-theory models suggests that, for models consistent with F-theory, each block added to a model must contribute positively to $H-V+29T$. This is certainly the case for most supergravity blocks associated with F-theory singularities which we understand. We leave a more general and rigorous proof of these assertions for future work, but this argument suggests that by classifying individual blocks which contribute positive values $H-V+29T$ we should in principle be able to enumerate all of the finite set of supergravity models with possible F-theory realizations at any fixed T, even when exotic matter fields such as those encountered in [5] are included for which the F-theory singularity type is not yet classified. Including the transitions we discuss in the following subsection which change the value of T, corresponding to blowing up singular points on the F-theory base, would in principle make it possible to connect the entire finite set of F-theory vacua in terms of the low-energy structure.

It was recently shown that all 10D supergravity theories not realized in string theory are inconsistent as quantum theories [53]. It was conjectured in [3] that this "string universality" property holds for 6D $\mathcal{N} = 1$ supergravities. Whether or not this is true, it is interesting to speculate that the constraints associated with F-theory constructions may have some shadow in the low-energy theory which can lead to new quantum consistency conditions for 6D supergravities. For example, the constraint that Λ be embeddable in a unimodular lattice, or the sign constraint $-a \cdot j > 0$ may be realizable in some sim-

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ple way as consistency conditions on any low-energy supergravity theory, as discussed in section 5. For other constraints, such as the detailed constraints on the form of a or the Kähler/Mori cone, it is harder to see how such conditions can arise directly from the supergravity description. It may be that these conditions can best be understood in terms of the BPS string states underlying the anomaly lattice Λ; we hope to return to this question in future work.

7.2 Charting the space of supergravities with string constructions

One important question is whether the constraints imposed by F-theory are satisfied by all 6D models arising from string constructions, or whether they are just signatures of an F-theory "corner" of the 6D supergravity landscape. We have looked at some examples of 6D theories realized through other string constructions, including CFTs and Gepner models [54, 55], orientifold models [50, 51, 56, 57] intersecting brane models [58, 59], heterotic constructions [2, 46], and non-geometric string vacua [60]. In general, at least from a limited sampling, it seems that most of the low-energy theories associated with these constructions can be mapped to acceptable data for an F-theory construction, so F-theory seems to cover a large fraction of the space of 6D low-energy theories which can be realized through any string construction. It would, however, clearly be desirable to explore the other branches of string theory more completely, to develop a more systematic understanding of how the sets of low-energy theories arising from other string constructions intersect with those coming from F-theory, and to ascertain whether the constraints described in 5.2 are truly universal stringy constraints for 6D $\mathcal{N} = 1$ theories.

One exception we have encountered to the general existence of F-theory constructions is given by a class of heterotic line bundle constructions described in [46]. In that paper we analyzed a class of low-energy models arising from heterotic string compactifications which are also characterized by lattices, but in a slightly different fashion than those models considered here. For a fixed gauge group containing factors $\mathrm{U}(N) \times \mathrm{U}(M)$, the models examined in [46] can have some number of bifundamental matter fields in the $(N, \bar{M})+(\bar{N}, M)$ representation and some number of bifundamental matter fields in the $(N, M) + (\bar{N}, \bar{M})$ representation. Each type of field contributes in the same way to the anomaly polynomials, so models with the same total number of bifundamental fields map through (5.2) to identical F-theory constructions. We do not know of any mechanism in F-theory as it currently exists to construct models with different distributions of bifundamental matter fields of these two types. Thus, in this class of models the heterotic theory generates models which cannot be realized in F-theory, although these models do not violate any of the constraints described in section 5.2. It would be nice to no whether there is a generalization of F-theory which would capture these heterotic models with mixed classes of bifundamental fields.

One class of models which we have not yet considered is the class of gauged supergravity models [24, 61]. It may be possible to perform a similar analysis of general 6D gauged supergravity models, although the significance of such supergravity theories is unclear as they do not give rise to stable Minkowski vacua. We leave this for future work.

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7.3 Connectivity of the space of 6D supergravities

In most of this paper, and in the preceding discussion, we have referred to supergravity models with distinct gauge groups and matter content as distinct "theories" or "models". This is not quite correct. In fact, each consistent model with a fixed gauge group and matter content has a moduli space of vacua. At certain limits in the moduli space, the theory can develop a singularity and the field content can change in a discrete fashion. The simplest example of this is the phenomenon of Higgsing, familiar from the standard model and basic quantum field theory, in which a vacuum expectation value of scalar fields breaks a gauge symmetry, giving mass to a formerly massless gauge field while removing one or more scalar fields from the spectrum. Such transitions in the full space of six-dimensional supergravity theories connect different branches while preserving the total $H - V$. More exotic transitions, studied in [32, 39, 63], arise at singular points where strings become massless, associated with points in the low-energy theory where $j \cdot b_i$ vanishes. Passing through such transition points can change the number of tensor multiplets in the theory, still preserving $H - V + 29T$. In this way, the set of six-dimensional supergravity models is really a highly-connected space with many branches of different dimensionality. The tools we have developed in this and preceding papers may be useful in exploring the global structure of this space. For example, the anomaly constraints can be used to characterize allowed transitions in terms of the field content of the low-energy theory. With a better understanding of what kinds of transitions between branches are allowed, it may be possible to prove that the space of acceptable models is connected into a single moduli space. Thus, we may be able to probe the validity of at least the elliptically-fibered version of the Reid fantasy [64] by analysis of the connectivity structure of the countable set of apparently-consistent low-energy $\mathcal{N} = 1$ 6D supergravity theories.

If in fact it could be shown that the set of consistent low-energy 6D theories is connected, it would provide a picture in which there is a single consistent $\mathcal{N} = 1$ supergravity theory, with many connected branches having different gauge groups and matter content.

If it could be shown that either the entire space of theories, or a connected subset thereof, corresponds to the set of theories which can be realized by F-theory or other string constructions, it would give a very simple picture of string theory as a single unified theory underlying quantum gravity in 6D. Indeed, it seems likely that this can be realized for the set of models which can be realized through F-theory, since the various singularity types realized by tuning the coefficients of f, g in (7.1) are all connected continuously in the space of coefficients. Thus, at least the space of F-theory compactifications should form a connected moduli space, with various branches associated with 6D supergravity theories with various gauge groups and matter content.

8 Conclusions

In this paper we have addressed some global questions regarding the space of consistent supergravity theories in six dimensions. We have focused on theories with $\mathcal{N} = 1$ supersymmetry and nonabelian gauge groups. We have extended our previous analysis of such theories to incorporate multiple tensor multiplets. We have shown that when the number

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of tensor multiplets T is less than 9, there are a finite number of possible gauge groups and matter representations possible for such theories. We have identified infinite families of models at $T = 9$ and greater which satisfy anomaly cancellation and which have proper signs for all gauge kinetic terms.

We have shown that every consistent 6D supergravity theory can be associated with an integral lattice, associated with the coefficients in the anomaly polynomial. This lattice can be used to construct topological data for an F-theory compactification whenever one exists. We have found a variety of low-energy supergravity models which do not violate any known consistency conditions from the low-energy point of view but which have no embedding in F-theory. The geometrical constraints of F-theory provide criteria for identifying such low-energy models from the associated lattice structure, and suggest possible new low-energy consistency conditions for quantum supergravity theories.

The overall picture is that, while for Lagrangian models with only one tensor field most apparently-consistent supergravity theories have realizations in F-theory, for models with more tensor fields the vast majority of apparently-consistent models have no known string realization through F-theory or any other string vacuum construction. Thus, most of these models lie in the "swampland". If these models cannot be realized through some novel string construction, it will indicate that string theory imposes strong constraints on 6D $\mathcal{N} = 1$ supergravity theories beyond the known stringent anomaly cancellation and gauge kinetic term sign constraints. If these additional constraints can be understood in terms of new quantum consistency conditions on the set of low-energy effective theories, it will provide a new window on general theories of quantum gravity; if not, it will indicate the existence of stringy constraints which may distinguish string theory from other possible UV-complete quantum gravity theories.

The perspective and tools developed in this work provide a framework in which it may be possible to carry out a systematic mapping of the landscape of 6D supergravity theories. Identifying which subsets of this landscape are associated with the different classes of string vacuum construction, and understanding how the many branches of this landscape are connected through Higgs and tensionless string transitions, promises to lead to a richer understanding of how the different string constructions are related, and of the nature of the landscape and the swampland. Such understanding in the simpler case of six dimensions will hopefully teach us some new lessons which may be relevant in the more complicated and physically relevant case of four dimensions.

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